Dynamic Mechanisms without Money

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Features

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- Principal with commitment has the decision rights.
- No transfers to facilitate truth-telling.
- No hard (possibly statistical) evidence either.
Some Examples

- Patients want the nurses’ attention.
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- Managers want the go-ahead for their projects.
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- State departments want to expand.
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- Patients want the nurses’ attention.
- Managers want the go-ahead for their projects.
- State departments want to expand.
- Cities/States want more resources.
Objective

- Is there a “simple” optimal policy?
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- Is there a “simple” optimal policy?
- How does utility/inefficiency evolve over time?
Objective

- Is there a “simple” optimal policy?
- How does utility/inefficiency evolve over time?
- How does this depend on the lack of transfers?
1. **Linking incentives:**

2. **Dynamic mechanism design:**

3. **Virtual budgets/“Chip” Mechanisms:**

4. **Dynamic contracting (“Immizeration”):**
Related Literature

1. **Linking incentives:**

2. **Dynamic mechanism design:**

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4. **Dynamic contracting (“Immizeration”):**
The Model
Discrete time $n = 0, 1, \ldots$
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Agent’s type-value is $v_n \in \{l, h\}$. 
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Values follow a Markov chain, with:

$$\mathbf{P}[\nu_{n+1} = h \mid \nu_n = h] = 1 - \rho_h, \quad \mathbf{P}[\nu_{n+1} = l \mid \nu_n = l] = 1 - \rho_l.$$

Assume $1 - \rho_h \geq \rho_l$.

($h$ is more likely to be followed by $h$ than $l$ is.)
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($h$ is more likely to be followed by $h$ than $l$ is.) 

The (invariant) probability of $h$ and the (unconditional) expected value of the unit are

$$q := \rho_l/(\rho_h + \rho_l), \quad \mu := \mathbb{E}[\nu] = qh + (1 - q)l.$$
Discount factor $\delta \in (0, 1)$. 
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Agent’s **Utility**: $U = (1 - \delta) \sum_{n=0}^{\infty} \delta^n x_n v_n$, 

Principal’s **Payoff** (Welfare): $W = (1 - \delta) \sum_{n=0}^{\infty} \delta^n x_n (v_n - c)$,

where $x_n \in \{0, 1\}$ is the principal’s decision to supply in period $n$. 
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Revelation Principle $\Rightarrow m \in M := \{l, h\}$, and Agent tells the truth.

A **policy** is a map $x = (x_n)_{n \geq 0}$, $x_n : M^n \to \Delta(\{0, 1\}).$
Roadmap

1. i.i.d., Binary Values.
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i.i.d. Values
Wlog, the agent’s *ex ante* utility is a valid state variable.
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A policy is represented as a map $U \mapsto (p_m, U_m)$, $m = l, h$, with

$$p_m \in [0, 1], \quad U_m \in [0, \mu].$$

Note that all utilities are in $[0, \mu]$. 
The Optimization Program

The principal’s problem is a Markov decision problem.
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The optimality equation is, for any \( U \in [0, \mu] \),

\[
W(U) = \sup_{(p_m, U_m)} \left\{ (1 - \delta) (qp_h(h - c) + (1 - q)p_l(l - c)) + \delta (qW(U_h) + (1 - q)W(U_l)) \right\},
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s.t. (“Promise Keeping”)

$$U = (1 - \delta)(qp_hh + (1 - q)p_ll) + \delta (qU_h + (1 - q)U_l),$$
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s.t. (“Promise Keeping”)

\[
U = (1 - \delta) (qp_h h + (1 - q)p_l l) + \delta (qU_h + (1 - q)U_l),
\]

and, for \( m = l, h, m' \neq m \) (“Incentive Constraint-m”)

\[
(1 - \delta)p_m m + \delta U_m \geq (1 - \delta)p_{m'} m + \delta U_{m'}.
\]
Complete Information

Same program, without incentive constraints.
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If initial utility is freely chosen, \((p_h, p_l) = (1, 0)\) and \(U =qh\).
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Instead, taking \(U\) as given, stationary policy.

\[
\begin{cases}
  p_h = \frac{U}{qh}, & p_l = 0 \quad \text{if } U \in [0, qh], \\
  p_h = 1, & p_l = \frac{U - qh}{(1 - q)l} \quad \text{if } U \in [qh, \mu].
\end{cases}
\]

\[
\bar{W}(U) = \begin{cases}
(1 - \frac{c}{h}) U & \text{if } U \in [0, qh], \\
(1 - \frac{c}{l}) U + cq \left(\frac{h}{l} - 1\right) & \text{if } U \in (qh, \mu].
\end{cases}
\]
Figure: Complete information payoff, $(\delta, l, h, q, c) = (.95, .4, .6, .6, .5)$
Second-Best: Two Observations

1. Efficient allocation as long as possible.

   Caveat: efficient allocation infeasible if \( U, \mu - U \) “too small.”
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1. Efficient allocation as long as possible.
   
   Caveat: efficient allocation infeasible if $U, \mu - U$ "too small."

2. One IC always binds: $l$ pretending $h$.

   Differs from standard adverse selection model where $h$ mimicks $l$. 
Dynamics, I

Two equations, IC-1 and PK, and two unknowns, $U_h, U_I$: 
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The high type has \((1 - \delta)(h - l)\) excess utility over sending \(m = 1\):

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\[
U = (1 - \delta) q(h - l) + \delta U_l.
\]

Hence: \( U_l < U \) iff \( U < q(h - l) =: \underline{U} \): Utility is trapped below \( \underline{U} \).
Region $[0, \mathcal{U})$: transient, leading to $\{0\}$. 
Dynamics, II

Region $[0, U)$: transient, leading to $\{0\}$.

Region $[U, 1)$: transient, leading to either $[0, U)$ or $\{\mu\}$.
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Drift? Given by PK!

$$U = (1 - \delta)qh + \delta \left( qU_h + (1 - q)U_l \right).$$

$$E[U']$$
Dynamics, II

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Drift? Given by PK!

$$U = (1 - \delta)qh + \delta\left(qU_h + (1 - q)U_l\right).$$

$U$ drifts up/down according to $U \geq qh$. 
Formally

An optimal policy is:

\[ p_h(U) = \min \left\{ 1, \frac{U}{(1 - \delta)\mu} \right\}, \quad p_l(U) = \max \left\{ 0, 1 - \frac{\mu - U}{(1 - \delta)l} \right\}. \]
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Payoff \( W \) is:
- linear and equal to \( \bar{W} \) for \( U \leq U \);
- strictly concave and below \( \bar{W} \) for \( U \in (U, \mu) \);
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- $C^1$ over $(0, \mu)$, with $\lim_{U\uparrow\mu} W'(U) = \lim_{U\uparrow\mu} \bar{W}'(U)$.

Optimal choice of $U_0 = U^*$ solves $W'(U) = 0$. 

\[19 / 1\]
Second-Best

Figure: Payoff as a function of utility, \((\delta, l, h, q, c) = (0.95, 0.4, 0.6, 0.6, 0.5)\)
A Martingale

\[ W(U) = (1 - \delta) q(h - c) \]

\[ + \delta \left( q W \left( \frac{U - (1 - \delta) \mu}{\delta} \right) + (1 - q) W \left( \frac{U - (1 - \delta) U}{\delta} \right) \right), \]

and so by differentiation,

\[ W'(U_n) = \mathbb{E}[W'(U_{n+1})]. \]

Hence, probability \( \alpha \) of absorption at \( U = 0 \) solves

\[ \frac{W'(0)}{h} \alpha + \frac{W'(\mu)}{l} (1 - \alpha) = 0, \]

or

\[ \frac{\alpha}{1 - \alpha} = \frac{h/l}{(h - c)/(c - l)}. \]
Implementation

Let $f := (1 - \delta)U$, and $g := (1 - \delta)\mu - f$. 
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Give the agent a “budget” of $U^*$. 
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In each period:

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2. If he asks for the item, charge $g$ in addition;
3. Give him a yield at rate $r = \frac{1-\delta}{\delta}$.
A Comparison with Token Mechanisms as in Jackson and Sonnenschein (2007)

1. Is the agent a prophet or a forecaster? (Is the problem static or dynamic?)
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2. Are token mechanisms optimal?
A Comparison with Token Mechanisms as in Jackson and Sonnenschein (2007)

A prophetic agent (static)  A forecasting agent (dynamic)
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Token mechanisms
A Comparison with Token Mechanisms as in Jackson and Sonnenschein (2007)

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Lemma
It holds that

$$|\mathcal{W}(U^*) - q(h - c)| = \mathcal{O}(1 - \delta).$$

In the case of a prophetic agent, the average loss converges to zero at rate $\mathcal{O}(\sqrt{1 - \delta})$. 
### A Comparison with Token Mechanisms as in Jackson and Sonnenschein (2007)

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Markovian Values
Values follow a Markov chain, with:

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Set-Up

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\]

Assume \(1 - \rho_h > \rho_l\).

All else: the same.
Reduction to Dynamic Programming, I

Revelation Principle still applies.
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State variables:

1. A pair of promised interim utilities: $U_h, U_l$.

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Choice variables, given $(\phi, U_h, U_l)$:

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Choice variables, **given** $(\phi, U_h, U_l)$:

1. Supply decision: $p = (p_h, p_l) \in [0, 1]^2$.

2. Promised pair tomorrow: for $m = h, l$: $(U_h(m), U_l(m)) \in \mathbb{R}^2$. 
The optimality equation becomes

\[
W(U_h, U_l, \phi) = \sup \left\{ \phi \left( (1 - \delta)p_h (h - c) + \delta W(U_h(h), U_l(h), 1 - \rho_h) \right) \\
+ (1 - \phi) \left( (1 - \delta)p_l (l - c) + \delta W(U_h(l), U_l(l), \rho_l) \right) \right\},
\]

over \((p_h, p_l, U_h(h), U_l(h), U_h(l), U_l(l))\) s.t.

\[
U_h = (1 - \delta)p_h h + \delta (1 - \rho_h) U_h(h) + \delta \rho_h U_l(h) \\
\geq (1 - \delta)p_l h + \delta (1 - \rho_h) U_h(l) + \delta \rho_h U_l(l),
\]

and similarly for \(U_l\).
Incentive Feasibility

Let us ignore optimality and focus on incentives.
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Given the same $U_h$, the values of $U_l$ are not the same under front- and backloading. $U_l$ is higher under backloading.
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Front- and backloading define the boundaries of the IC-feasible set.
Figure: The set $V$ for parameters $(\delta, \rho_h, \rho_l, l, h) = (9/10, 1/3, 1/4, 1/4, 1)$. 
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Optimal Policy

Recall the policy is defined for all \((U_h, U_l) \in V, \phi \in \{\rho_l, 1 - \rho_h\}\).
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The optimal policy is “simple” and independent of beliefs.

I will focus on the reachable subset of states given the optimal \(U^*\).
The Lower Boundary

$U^*$ lies on the lower boundary (frontloading policies).
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\( U^* \) lies on the lower boundary (frontloading policies).

This does not imply that frontloading occurs:

Any policy s.t. IC-1 binds in every period, and s.t. \( p_h = 1 \) (“whenever possible”) yields utilities on this boundary.
The Lower Boundary

$U^*$ lies on the lower boundary (frontloading policies).

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**Any** policy s.t. IC-I binds in every period, and s.t. $p_h = 1$ ("whenever possible") yields utilities on this boundary.

They differ in terms of the principal’s payoff, and utility dynamics.
Choose efficient allocation

\[ p_h = \min \left\{ 1, \frac{U_h}{(1 - \delta) h} \right\}, \quad p_l = \max \left\{ 0, 1 - \frac{\mu_l - U_l}{(1 - \delta) l} \right\}, \]

and continuation utilities on the lower boundary s.t. IC-l binds.
It holds that $U(h) \leq U$. 
Dynamics

It holds that \( U(h) \leq U \).

\( U(l) \leq U \) if and only if \( U \) is low enough.
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$U$ is drifting up iff

$$\frac{\rho_h}{\rho_h + \rho_l} U_l + \frac{\rho_l}{\rho_h + \rho_l} U_h \geq \frac{\rho_l}{\rho_h + \rho_l} h.$$  

(“Long-run efficient utility below current long-run promise.”)
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(“Long-run efficient utility below current long-run promise.”)

Utility is eventually absorbed at $U \in \{0, \mu\}$. We know of no formula for absorption probability.
Implementation

The obvious “budget” unit is: \# of consecutive periods the agent can claim the unit, no questions asked: \((B_n, \gamma_n) \in \mathbf{N} \times [0, 1]\).
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Not asking for the unit leads to the revised promise

\[
\frac{U_l(B_n, \gamma_n)}{\delta} = E_I \left[ U_{v_{n+1}}(B_{n+1}, \gamma_{n+1}) \right].
\]

where

\[
E_I \left[ U_{v_{n+1}}(B_{n+1}, \gamma_{n+1}) \right] = (1 - \rho_l)U_l(B_{n+1}, \gamma_{n+1}) + \rho_l U_h(B_{n+1}, \gamma_{n+1})
\]

is the expected utility from tomorrow’s \((B_{n+1}, \gamma_{n+1})\) given \(I\) today.
Implementation

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is the expected utility from tomorrow’s \((B_{n+1}, \gamma_{n+1})\) given \(I\) today.

Asking for it leads to a payment \((1 - \delta)I\), as before:

\[
\frac{U_I(B_n, \gamma_n) - (1 - \delta)I}{\delta} = E_I \left[ U_{v_{n+1}}(B_{n+1}, \gamma_{n+1}) \right].
\]
Continuous-Time Limit

Lack of smoothness prevents easy comparative statics.
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Let \( \rho_h \approx \lambda_h \Delta, \rho_l \approx \lambda_l \Delta, \) \( r \approx (1 - \delta)\Delta \) and take \( \Delta \to 0. \)
Continuous-Time Limit

Lack of smoothness prevents easy comparative statics.

Let $\rho_h \cong \lambda_h \Delta$, $\rho_l \cong \lambda_l \Delta$, $r \cong (1 - \delta)\Delta$ and take $\Delta \to 0$.

Flow values evolve according to a two-state Markov chain with parameters $\lambda_l, \lambda_h$. 
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Two simplifications:

1. No “kinks:” lower boundary parameterized by $\tau \in \mathbb{R}_+$;
Continuous-Time Limit

Lack of smoothness prevents easy comparative statics.

Let $\rho_h \cong \lambda_h \Delta$, $\rho_I \cong \lambda_I \Delta$, $r \cong (1 - \delta) \Delta$ and take $\Delta \to 0$.

Flow values evolve according to a two-state Markov chain with parameters $\lambda_I, \lambda_h$.

Two simplifications:

1. No “kinks:” lower boundary parameterized by $\tau \in \mathbb{R}_+$;
2. Degenerate beliefs: $\phi \in \{0, 1\}$. 
Figure: Incentive-feasible set for \((r, \lambda_h, \lambda_l, l, h) = (1, 10/4, 1/4, 1/4, 1)\)
Payoff Dynamics

Payoffs (conditional on the point belief) satisfy the coupled ODE:

\[(r + \lambda_h)W_h(\tau) = r(h - c) + \lambda_h W_l(\tau) - W_h'(\tau),\]

and

\[(r + \lambda_l)W_l(\tau) = \lambda_l W_h(\tau) + \frac{g(\tau)}{\mu - q(h - l)e^{-(\lambda_h + \lambda_l)\tau}} W_l'(\tau),\]

where \(g(\tau) := q(h - l)e^{-(\lambda_h + \lambda_l)\tau} + le^{r\tau} - \mu\), and \(W(0) = 0\).
Proposition

The value function of the principal is given by

\[
\begin{cases}
\tilde{W}_1(\tau) & \\
\tilde{W}_1(\tau) - w_0(\tau) \frac{h-l}{h_l} c r \mu \left( \int_{\tau}^{\tilde{\tau}} e^{\int_{\tau}^{s} f(s) ds} \right) & \\
\tilde{W}_1(\tau) - w_0(\tau) \frac{h-l}{h_l} c \left( 1 + r \mu \left( \int_{\tau}^{\infty} e^{\int_{\tau}^{s} f(s) ds} \right) \right) &
\end{cases}
\]

\[
\begin{cases}
\text{if } \tau \in [0, \hat{\tau}), \\
\text{if } \tau \in [\hat{\tau}, \tau_0), \\
\text{if } \tau \geq \tau_0,
\end{cases}
\]

where \( \tilde{W}_1(\tau) := (1 - e^{-r\tau})(1 - c/h)\mu, w_0(\tau) := \mu e^{-r\tau} - (1 - q)l, \)

\[f(\tau) := r - (\lambda_h + \lambda_l) \frac{w_0(\tau)}{g(\tau)} e^{r\tau},\]

and \( \tau_0 \) is the positive root of \( w_0 \), and \( \hat{\tau} \) of \( g \).
Figure: Payoff; \((\lambda_l, \lambda_h, r, l, h, c) = (p/4, 10p/4, 1, 1/4, 1, 2/5), \ p = 1, 1/4\)
Persistence, Convergence

Lemma
The value $W(\tau)$ decreases pointwise in persistence $1/p$, where $\lambda_h = p\bar{\lambda}_h$, $\lambda_l = p\bar{\lambda}_l$, for some fixed $\bar{\lambda}_h$, $\bar{\lambda}_l$. 
Persistence, Convergence

Lemma
The value $W(\tau)$ decreases pointwise in persistence $1/p$, where $\lambda_h = p\lambda_h$, $\lambda_l = p\lambda_l$, for some fixed $\lambda_h, \lambda_l$.

Lemma
It holds that
\[
|\max_{\tau} W(\tau) - q(h - c)| = O(r).
\]
A Comparison with the Transfer Case

With transfers, efficiency is trivial (the agent pays $c$ for the unit).
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Logic is different:

1. With transfers, IC-$h$ becomes the problematic constraint (because transfers are used to extract surplus).
2. Information rents in period $n$ can be extracted in period 0 via higher prices (as the expected rent is insensitive to the initial value when $n$ is large).
A Comparison with the Transfer Case

Transfers have two benefits:
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A Comparison with the Transfer Case

Transfers have two benefits:

1. Promises made in period \( n \) can be cleared then.

2. Clearing earlier reduces cost of future information rents.
All That is Missing…

1. Statistical signals.
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2. More than one agent (allocation problems).
All That is Missing…

1. Statistical signals.

2. More than one agent (allocation problems).

3. More than one good (matching problems).
Thank You!
Figure: The set $V, \overline{V}, (\delta, \rho_h, \rho_l, l, h) = (9/10, 1/3, 1/4, 1/4, 1)$
Figure: Optimal policy for \((\delta, \rho_h, \rho_l, l, h) = (9/10, 1/3, 1/4, 1/4, 1)\)
Optimal Policy

Let $P_t, P_b$ denote the top and bottom boundary of $V$.
Let $V_t, V_b$ denote the regions of $V$ above (below) some polygonal chain $P$ (omitted here).
Optimal Policy

Let $P_t, P_b$ denote the top and bottom boundary of $V$.

Let $V_t, V_b$ denote the regions of $V$ above (below) some polygonal chain $P$ (omitted here).

For all $U = (U_h, U_l) \in V$, set

$$p_l = \max \left\{ 0, 1 - \frac{\mu_l - U_l}{(1 - \delta)l} \right\}, \quad p_h = \min \left\{ 1, \frac{U_h}{(1 - \delta)h} \right\},$$

and

$$U(h) \in P_b, \quad U(l) \in \begin{cases} P_b & \text{if } U \in V_b, \\ P_t & \text{if } U \in P_t. \end{cases}$$

Furthermore, if $U \in V_t$, $U(l)$ is chosen so that IC-$h$ binds.
General i.i.d. Distributions

What remains the same:

- Convergence to either 0 or \( \mu \) (so inefficiency is pushed back).
- Martingale property of \( W' \).
- Slopes match \( \bar{W} \) at the end points.

What changes:

- Strict concavity on all \([0, \mu]\), \( W < \bar{W} \) on \((0, \mu)\).
- Policy no longer efficient as long as possible; not even cut-off.
- \((F(v) = v^\alpha, \alpha \geq 1)\):
  - \( U \geq U^{**} \): \( \exists 0 < v_1 < v_2 < 1 \) such that
    \[
    p(v) = \begin{cases} 
    0 & \text{if } v \leq v_1, \\
    1 & \text{if } v \geq v_2,
    \end{cases}
    \]
    and continuously increasing on \([v_1, v_2]\).
  - \( U < U^{**} \): same, excepts \( v_2 = 1 \).
Complete Information

Efficient policy yields $v_m^*$:

$$\begin{align*}
v_h^* &= (1 - \delta)h + \delta(1 - \rho_h)v_h^* + \delta \rho_h v_l^*, \\
v_l^* &= \delta(1 - \rho_l)v_l^* + \delta \rho_l v_h^*.
\end{align*}$$

Policy and payoff are piecewise linear.

Increasing in each $U_m$ iff $U_m \leq v_m^*$. 
Figure: Impact of persistence
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