Sequential Equilibrium in Multi-Stage Games with Infinite Sets of Types and Actions

By

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Paper can be found at https://sites.google.com/site/philipjreny/home/research
Dynamic multi-stage game $\Gamma$

• Players move simultaneously at each date $k=1,\ldots,K$.

• Date $k$: Nature chooses $\theta_k \in \Theta_k$ according to $p_k(\cdot|\theta_{<k},a_{<k})$, then each player $i$ chooses simultaneously from his $A_{ik}$.

• $\Theta = \times_k \Theta_k$; $A = \times_i A_{ik}$; $\Theta \times A = \{ \text{outcomes of the game} \}$.

• $T = \times_i T_{ik}$; $T_{ik} = \{ i \text{'s types at date } k \}$. (topologized, and with a countable basis)

• $\tau_{ik}: \Theta_{\leq k} \times A_{<k} \rightarrow T_{ik}$ specifies $i$’s (information) type $t_{ik}$ at date $k$.

• $u_i: \Theta \times A \rightarrow \mathbb{R}$ is player $i$’s bounded vNM utility function.

• A strategy for $i$ at date $k$ is a function $s_{ik}: T_{ik} \rightarrow \Delta(A_{ik})$.

• A strategy for $i$ is $s_i = (s_{i1},\ldots,s_{iK}) \in S_i = \times_k S_{ik}$; $S = \times_i S_i$. (r.c.p, 23)
An existence problem/strategic entanglement (Harris-Reyn-Robson 1995)

**Example 1.** Date 1: Player 1 chooses $a_1$ from $[-1,1]$, player 2 chooses from \{L,R\}.  
Date 2: Players 3 and 4 observe the date 1 choices and each choose from \{L,R\}.

- For $i = 3,4$, player $i$’s payoff is $-a_1$ if $i$ chooses L and $a_1$ if $i$ chooses R.

- Player 2’s payoff depends on whether she matches 3’s choice.  
  If 2 chooses L then she gets 1 if player 3 chooses L but $-1$ if 3 chooses R; and  
  If 2 chooses R then she gets 2 if player 3 chooses R but $-2$ if 3 chooses L.

- Player 1’s payoff is the sum of three terms:  
  (First term) If 3 and 4 match he gets 10, if they mismatch he gets 0;  
  plus (second term) if 2 and 3 match he gets $-|a_1|$, if they mismatch he gets $|a_1|$;  
  plus (third term) he gets $-|a_1|^2$.

- There is no SPE. But for any $\varepsilon \in (0,1)$ there are $\varepsilon$-SPE. For example, player 1 chooses $\pm \varepsilon$ with probability $\frac{1}{2}$ each, player 2 chooses L and R each with probability $\frac{1}{2}$, and players $i = 3,4$ choose L if $a_1 \leq 0$ and choose R if $a_1 > 0$.

- The weak*-limit distribution over outcomes is $a_1 = 0$ and $a_i = 0.5[L] + 0.5[R]$ for all $i \in \{2,3,4\}$. But in this limit, 3’s and 4’s actions are perfectly correlated independently of 1’s and 2’s. So no strategy profile can produce this distribution and we may say that players 3 and 4 are strategically entangled in the limit.
Problems of spurious signaling in naïve finite approximations

Example 2. Nature: $\theta \in \{1,2\}$; $p(\theta) = \theta/3$.
Player 1: $t_1 = \emptyset$, $a_1 \in [0,1]$. Player 2: $t_2 = (a_1)^\theta$, $a_2 \in \{1,2\}$.

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<th>$a_2 = 1$</th>
<th>$a_2 = 2$</th>
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<td>$[1/3]: \theta = 1$</td>
<td>1,1</td>
<td>0,0</td>
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<tr>
<td>$[2/3]: \theta = 2$</td>
<td>1,0</td>
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1’s payoff is zero in any Nash equilibrium.

But if player 1 is restricted to any finite subset, $F \not\subseteq \{0,1\}$, of his action space $[0,1]$, he must obtain $u_1 \geq 1/3$ in any SPE since when $t_2$ is the highest action in $F$ less than 1, player 2 must respond with $a_2 = 1$ since the state must be $\theta = 1$. 
Problems of requiring sequential rationality tests with positive probability in all events

*Example 4.* (BoS) **Date 1**: Nature chooses $\theta \sim U[0,1]$ and player 1 chooses $a_1 = (\delta_1, \eta_1) \in \{L, R\} \times [0,1]$. **Date 2**: Player 2 observes $t_2 = \theta$ if $\delta_1 = L$ and $t_2 = \eta_1$ if $\delta_1 = R$, and chooses $a_2 \in \{L, R\}$.

<table>
<thead>
<tr>
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All BoS equilibria seem reasonable. 

- but if for “consistency of beliefs,” any observed event must be given positive probability when possible,…

- then, e.g., $t_2 = \pi/4$ will imply $a_2 = R$ since $t_2 = \pi/4$ has positive probability only when $a_1 = (\delta_1, \eta_1) = (R, \pi/7)$.

- but then all equilibria are such that $(\delta_1, a_2) = (R, R)$. 
Example 5. Nature chooses $\theta = (\omega_1, \omega_2)$, with $\omega_1, \omega_2 \sim U[-1,3]$.

Player 1 observes $t_1 = \omega_1$ and chooses $a_1 \in \{-1,1\}$.

Player 2 observes $t_2 = a_1$ and chooses $a_2 \in \{-1,1\}$.

Payoffs: $u_1(\omega_1, \omega_2, a_1, a_2) = a_1 a_2$; $u_2(\omega_1, \omega_2, a_1, a_2) = \omega_2 a_2$.

- Since no player observes $\omega_2$ and $E(\omega_2) > 0$, player 2 should always choose $a_2 = 1$, regardless of the action of player 1 that she observes.

- So player 1 should choose $a_1 = 1$ regardless of the $\omega_1$ that he observes. The only sensible equilibrium payoffs are $u_1 = u_2 = 1$.

- But consider $s_1(\omega_1) = -1$ iff $\omega_1 \neq -1$ and $s_2(a_1) = -a_1$. This yields the payoff vector $(u_1, u_2) = (-1,1)$

- These strategies are supported by perturbing nature to give small positive probability to the event $\{(\omega_1, \omega_2) = (-1,-1)\}$. 
Observable open sets and neighborhood bases

- Let $T^*_ik = \{\text{open } C \subseteq T^*_ik : \exists s \in S \text{ s.t. } \Pr(C|s) > 0\}$; \textit{observable} open sets.

- Assume for today’s talk that every open $C \subseteq T^*_ik$ is observable.

- Let $T^* = \bigcup_{ik} T^*_ik$ (disjoint union) be the collection of (observable) open sets.

- $\mathcal{B} \subseteq T^*$ is a \textit{neighborhood basis for the players’ types} iff $\forall i,k$

  $T^*_ik \in \mathcal{B}$, and, $\forall t^*_ik \in T^*_ik$, $\forall \text{open } C \ni t^*_ik$, $\exists B \in \mathcal{B}$ s.t. $t^*_ik \in B \subseteq C$. 
For any $\varepsilon > 0$, and for any $\mathcal{F} \subseteq \mathcal{T}^*$, say that $s \in S$ is an $(\varepsilon, \mathcal{F})$-sequential equilibrium of $\Gamma$ iff for all $i$ and $k$, and for every $C \in \mathcal{F} \cap \mathcal{T}_{ik}^*$,

1. $\Pr(C|s) > 0$, and
2. $U_i(r_i, s_{-i}|C) \leq U_i(s|C) + \varepsilon$, for any date-$k$ continuation $r_i$ of $s_i$.

Let $\mathcal{Y}$ be the set of measurable outcome events $Y \subseteq \Theta \times A$.

A mapping $\mu : \mathcal{Y} \times \mathcal{B} \to [0,1]$ is an open sequential equilibrium of $\Gamma$ iff $\mathcal{B}$ is a neighborhood basis for the players’ types, and, $\forall \varepsilon > 0$, $\forall$ finite $\mathcal{F} \subseteq \mathcal{B}$, and $\forall$ finite $\mathcal{G} \subseteq \mathcal{Y}$, $\exists (\varepsilon, \mathcal{F})$-sequential equilibrium, $s$, s.t.

$$|\Pr(Y|C,s) - \mu(Y|C)| < \varepsilon, \forall (Y, C) \in \mathcal{G} \times \mathcal{F}.$$
Subgame perfection can fail if payoffs are discontinuous

**Example 6.** 1 chooses $a_1 \in [0,1]$; 2 sees $t_2 = a_1$ and chooses $a_2 \in [0,1]$.

Payoffs: $u_1(a_1,a_2) = u_2(a_1,a_2) = a_2$, if $(a_1,a_2) \neq (1/2, 1/2)$,

$= 2$, if $(a_1,a_2) = (1/2, 1/2)$

- Unique SPE: $a_1 = 1/2$; $s_2(1/2) = 1/2$ and $s_2(a_1) = 1$ if $a_1 \neq 1/2$. So, $u_1 = u_2 = 2$.

- But, $a_1 \sim U[0,1]$ and $s_2(a_1) = 1 \ \forall a_1$ yields an open sequential equilibrium with payoffs $u_1 = u_2 = 1$.

- This failure of subgame perfection occurs because 2’s sequential rationality is not tested at the exact event \{a_1 = 1/2\}.

- This problem can arise even with continuous utilities when the behavior of future players is discontinuous. To guarantee subgame perfection, one needs a stronger solution concept, requiring sequential rationality at more than just open sets.
Regular projective games
Let $\Gamma=(\Theta,N,K,A,T,p,\tau,u)$ be a multi-stage game.
Regular projective games

Let \( \Gamma = (\Theta, N, K, A, T, p, \tau, u) \) be a multi-stage game. \( \Gamma \) is a **regular projective game** iff there is a finite index set \( J \) and sets \( \Theta_{kj} \) and \( A_{ikj} \) such that for every player \( i \) and date \( k \),
Regular projective games

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(R.1) $\quad A_{ik} = \times_{j \in J} A_{ikj}, \quad \Theta_k = \times_{j \in J} \Theta_{kj},$

(R.2) $\quad \Theta_{kj}$ and $A_{ikj}$ are nonempty compact metric spaces $\forall j \in J$, and all spaces, including products, are given their Borel sigma-algebras,

(R.3) $\quad u_i : \Theta \times A \to \mathbf{R}$ is continuous,
Regular projective games
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Regular projective games

Let $\Gamma=(\Theta,N,K,A,T,p,\tau,u)$ be a multi-stage game. $\Gamma$ is a *regular projective game* iff there is a finite index set $J$ and sets $\Theta_{kj}$ and $A_{ikj}$ such that for every player $i$ and date $k$,

(R.4) there is a nonnegative, continuous $f_k : \Theta_{\leq k} \times A_{<k} \to [0,\infty)$, and

\[ p_k(C|\theta_{<k},a_{<k}) = \int_{C} f_k(\theta_k|\theta_{<k},a_{<k}) \rho_k(d\theta_k), \]

\[ \forall (\theta_{<k},a_{<k}) \in \Theta_{<k} \times A_{<k} \text{ and } \forall \text{Borel subsets } C \subseteq \Theta_k, \]

where $\rho_k = \times_{j \in J} \rho_{kj}$ is a product measure on $\Theta_k = \times_{j \in J} \Theta_{kj}$,
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Regular projective games

Let $\Gamma=(\Theta,N,K,A,T,p,\tau,u)$ be a multi-stage game. $\Gamma$ is a regular projective game iff there is a finite index set $J$ and sets $\Theta_{kj}$ and $A_{ikj}$ such that for every player $i$ and date $k$,

(R.5) there exist sets $M_{1ik} \subseteq \{1,\ldots,k\} \times J$ and $M_{2ik} \subseteq N \times \{1,\ldots,k-1\} \times J$, s.t.

$$\tau_{ik}(\theta_{\leq k}, a_{<k}) = ((\theta_{hj})_{hj \in M_{1ik}}, (a_{hnj})_{hnj \in M_{2ik}}),$$

i.e., $i$'s type at $k$ is just a list of state coordinates and action coordinates from dates up to $k$. 
Existence of open sequential equilibria

**Theorem.** In regular projective games, the set of open sequential equilibria is nonempty and in all finite games it is equivalent to the set of Kreps-Wilson sequential equilibria.

**Remark.** Since distinct players can observe the same choice of Nature, regular projective games need not satisfy the information diffuseness condition of Milgrom-Weber (1985).
A mapping $\mu : Y \times B \to [0,1]$ is a subgame perfect open sequential equilibrium of $\Gamma$ iff $B$ is a neighborhood basis for the players’ types, and, $\forall \varepsilon > 0$, $\forall$ finite $F \subseteq B$, and $\forall$ finite $G \subseteq Y$, $\exists (\varepsilon, F)$-sequential equilibrium, $s$, s.t. $s$ is an $\varepsilon$-subgame perfect equilibrium and,

$$|\Pr(Y|C, s) - \mu(Y|C)| < \varepsilon, \forall (Y, C) \in G \times F.$$ 

**Theorem.** In regular projective games, the set of subgame perfect open sequential equilibria is nonempty.
Dynamic multi-stage game $\Gamma$

- Players move simultaneously at each date $k=1,\ldots,K$.

- Date $k$: Nature chooses $\theta_k \in \Theta_k$ according to $p_k(\cdot|\theta_{<k},a_{<k})$, then each player $i$ chooses simultaneously from his $A_{ik}$.

- $\Theta = \times_k \Theta_k$; $A = \times_i A_{ik}$; $\Theta \times A = \{\text{outcomes of the game}\}$.

- $T = \times_i T_{ik}$; $T_{ik} = \{i's \text{ types at date } k\}$. (topologized, and with a countable basis)

- $\tau_{ik} : \Theta_{\leq k} \times A_{<k} \rightarrow T_{ik}$ specifies $i$'s (information) type $t_{ik}$ at date $k$.

- Perfect recall: $\forall ik \in L, \forall m < k, \exists$ measurable $\phi_{ikm} : T_{ik} \rightarrow T_{im} \times A_{im}$ s.t.

$$\phi_{ikm}(\tau_{ik}(\theta_{\leq k},a_{<k})) = (\tau_{im}(\theta_{\leq k},a_{<m}),a_{im}) \ \forall \theta \in \Theta, \forall a \in A.$$
Dynamic multi-stage game $\Gamma$

- Players move simultaneously at each date $k=1,\ldots,K$.

- Date $k$: Nature chooses $\theta_k \in \Theta_k$ according to $p_k(\cdot|\theta_{<k},a_{<k})$, then each player $i$ chooses simultaneously from his $A_{ik}$. (endowed with sigma-algebras)

- $\Theta = \times_k \Theta_k$; $A = \times_i A_{ik}$; $\Theta \times A = \{\text{outcomes of the game}\}$.

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- $\tau_{ik}: \Theta_{\leq k} \times A_{<k} \rightarrow T_{ik}$ specifies $i$'s (information) type $t_{ik}$ at date $k$. (perfect recall, 22)

- $u_i: \Theta \times A \rightarrow \mathbb{R}$ is player $i$'s bounded vNM utility function. (measurable functions)

- A strategy for $i$ at date $k$ is a function $s_{ik}: T_{ik} \rightarrow \Delta(A_{ik})$. ($S_{ik} = \{\text{strategies for } i \text{ at date } k\}$)

- **RCP:** $s_{ik}(\cdot|t_{ik}) \in \Delta(A_{ik}) \ \forall t_{ik} \in T_{ik}$, and $s_{ik}(C|t_{ik})$ is mbl in $t_{ik} \ \forall \text{mbl } C \subset A_{ik}$. (return, 2)
Spurious miscoordination: the need for uniform sequential rationality across all events

Example 8.
- Players 1 and 2 each make a choice from [0,1] at date 1.
- After observing the date 1 choices, players 3 and 4 each make a choice from [0,1] at date 2.
- At date 3, nature determines with equal probability whether player 5 observes the choices of 1 and 3 or of 2 and 4, but player 5 does not know which pair of choices he observes.
- After the observation, player 5 chooses an action from {0,1}, which determines the common payoff of the two players whose choices he observed. All other players receive a payoff of zero, including player 5.

- It should not be possible for each of the two odd players to receive an expected payoff close to 1/2 without each of the two even players also being able to do so.
- But for any finite $\mathcal{F} \subseteq T^*$, if $x \neq y$ are actions for player 2 that are both in one of 4’s type sets in $\mathcal{F}$ and any set of 4’s in $\mathcal{F}$ that contains $x$ also contains $y$, then consider strategies that give positive probability to all sets in $\mathcal{F}$ and that also satisfy the following:
  - Players 1 and 3 place probability $1-\varepsilon^2$ on, respectively, $a_1=x$ and $s_3(t_3)=x \ \forall t_3$, and players 2 and 4 place probability $1-\varepsilon^2$ on, respectively, $a_2=y$ and $s_4(t_4)=y \ \forall t_4$. Player 5’s strategy chooses action 1 iff he observes $(x,x)$.
  - These (nonsensical) strategies reach every set in $\mathcal{F}$ with positive probability and, for $\varepsilon>0$ small enough, are $\varepsilon$-sequentially rational there. So they form an $(\varepsilon,\mathcal{F})$-sequential equilibrium.
  - To eliminate: sequential rationality should be imposed uniformly across all events. (But how?)