Discussion of “Financial Networks and Contagion”
Elliott, Golub, and Jackson (2013)

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Macro Financial Modeling and Macroeconomic Fragility Conference
October 2013
A model of interconnected organizations with claims on:

(i) some fundamental assets
(ii) each other.

Amplification mechanism: Discontinuous loss in productive value if an institution’s market value falls below a certain threshold.

**Key question**: how the nature of such interdependencies affect the stability of the system as a whole?

**Results**: “more interconnectivity” has a non-monotonic effect.
Model

- $n$ institutions/organizations
- $m$ assets
- $p_k$: price of asset $k$
- $D_{ik}$: share of asset $k$ held by institution $i$

Interconnectivity: cross-holding of shares
- $C_{ij}$: fraction of institution $j$ owned by organization $i$.
- $\hat{C}_{ii}$: fraction held by $i$’s outside shareholders.

$$\hat{C}_{ii} = 1 - \sum_{j \neq i} C_{ij}$$
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The cross-holdings define a network:
Integration and Diversification

- **Integration:** $C'$ more integrated than $C$ if
  \[ \hat{C}'_{ii} \leq \hat{C}_{ii} \quad \forall \ i \]
  captures the total level of exposure of organizations to each other.

- **Diversification:** $C'$ more diversified than $C$ if
  \[ C'_{ij} \leq C_{ij} \quad \forall \ i,j \text{ such that } C_{ij} > 0 \]
  \[ C'_{ij} > C_{ij} \quad \text{for some } i, j \text{ such that } C_{ij} = 0. \]
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captures how spread out cross-holdings are.
Book Values

- **Book value** of organization $i$:

$$V_i = \sum_{j \neq i} C_{ij} V_j + \sum_k D_{ik} p_k$$

$$V = (I - C)^{-1} Dp$$

- However,

$$\sum_i V_i > \sum_k p_k.$$ 

- In fact, if $\hat{C}_{ii} = \hat{c} < 1$ for all $i$:

$$\sum_i V_i = \frac{1}{\hat{c}} \sum_k p_k.$$ 

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Market Values

- **Market values**: the equity value of the organization held by its *outside* investors.

\[ v_i = \hat{C}_{ii} \cdot V_i. \]

- or in vector form:

\[ v = \hat{C}(I - C)^{-1}Dp. \]
**Contagion**: A drop in the value of the an asset held by \( j \) can lead to the fall in value of \( i \) even if it does not directly hold the asset.

Absent any amplification mechanism, however, the losses are simply reallocated across the network:

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\frac{\partial}{\partial p_k} \sum_i v_i = 1 \quad \text{for all networks}
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All network structures are alike.
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Failure costs:

\[ v = \hat{C}(I - C)^{-1} \left( Dp - \beta(p) \mathbf{1}_{\{v_i < \underline{v}_i\}} \right). \]

Discontinuous loss in productive value if an institution’s market value falls below a certain threshold \( \underline{v}_i \).

* inefficient use of the assets
* discontinuous jumps in the cost of capital
* bankruptcy and legal costs

Value of the organizations: fixed point \((v_1, \ldots, v_n)\).
Fix a matrix $\pi = [\pi_{ij}]$

$\pi_{ij}$: the fraction of nodes that have in-degree $i$ and out-degree $j$.

$G(\pi, n)$: the set of all networks on $n$ nodes with distribution $\pi$.

Draw a network $G \in G(\pi, n)$ uniformly at random.

$$C_{ij} = \frac{cG_{ij}}{d_{j,\text{out}}}$$

$$\hat{C}_{ii} = 1 - c$$

$c = \text{the extent of integration.}$

$d = \text{expected out-degree of the vertex at the end of a random edge}$
Limit Contagion

- Regularity assumptions:
  - $v_i = \bar{v}$ for all $i$
  - $D = I$: each bank holds a single asset.
  - $p = (1, 1, \ldots, 1)$
  - shock: $p_i \rightarrow 0$ uniformly at random.
  - $\beta_j = 1$: the asset value of a failing firm is completely wiped out.

- Fix $\pi$, and consider a sequence of networks growing in size $n$

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- **Limit contagion:**
  if a non-vanishing fraction of organizations fail as \( n \to \infty \).
Main Result: Integration

Proposition

\[ c(1 - c) < \alpha, \text{ then there is no limit contagion.} \]

Integration’s effect is non-monotonic:

- **low**: little exposure to others, failures do not trigger cascades
- **high**: difficult to get the first failure (drop in own assets does not trigger failure)
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Main Result: Diversification

Proposition

Suppose integration is neither too low nor too high.

(a) $d < 1$: no limit contagion.

(b) $1 < d_{\text{max}} < \alpha c (1 - c)$: limit contagion.

(c) $d_{\text{min}} > \alpha c (1 - c)$: no limit contagion.

Diversification’s effect is non-monotonic:

- **low**: fragmented network; no widespread contagion
- **high**: little exposure to any single org; failures do not spread
Most analytical results for asymptotically large random graphs:

- Even though the intuitions are clear from the current results, still valuable to solve the problem for deterministic, small structures (even if structures are simple).

- Possible to obtain results for the “first threshold of failure”? 
A model of value interdependencies $v_i = f_i(v_{-i})$.

Current interpretation:

- firms have (debt?) obligations to one another.
- if $i$’s value does not cover its obligations, firm $j$ gets $C_{ij}V_i$.
- outside owners get $v_i = \hat{C}_{ii}V_i$.
- If $v_i < v$, the owner stops the operations of the firm.

Discontinuities triggered by value and not the cash flow.

Margin requirements? Collateral/balance sheet constraints?

Fleshing out the micro-foundations in more detail.
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potentially depends on diversification and integration.
Very interesting and relevant paper, with clean insights (even though the problem may first look intractable).

- Fleshing out the micro-foundations in more detail
- There is still value in analyzing simple, non-random, “finite” size networks.
- Given the interpretation, comparative statics that focus on the non-distressed regime also valuable.