Risk, Ambiguity and Model Misspecification:

An Econometric Perspective

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Outline

I. Who is making the decisions?
II. Uncertainty components
III. Statistical decision theory (static)
IV. Dynamic decision theory and recursivity
V. Decision theory in practice
VI. Axiomatic defenses

I. Who is making the decisions?

A) econometricians -
construct estimators with good properties, produce posterior distributions, or bounds on posterior probabilities

B) economic agents -
forward-looking private agents confront uncertainty with implications for allocations and prices

C) policy makers -
design prudent policies in the face of uncertainty

While econometricians primary goal may be to summarize evidence as it pertains to alternative models, economic agents and policy makers take actions whose consequences may differ depending on which model is the correct one.

II. Uncertainty components

A) risk -
uncertainty within a model: uncertain outcomes with known probabilities

B) ambiguity -
uncertainty across models: unknown weights for alternative possible models

C) misspecification -
uncertainty about models: unknown flaws of approximating models

III. Statistical decision theory

- unknown parameter vector θ (could be infinite dimensional) that resides in a set Θ.
- a decision maker observes a random vector X with realized values x in a set $\mathcal{X}$ is described by a probability model $f(x|\theta)$, a density relative to a measure τ over $\mathcal{X}$
- A decision maker takes an action $a \in \mathcal{A}$ that can depend on x. A decision rule is a suitably measurable function $A : \mathcal{X} \to \mathcal{A}$
- Represent the decision maker's preferences in terms of a utility function $U(a, x, \theta)$. Integrate over x to construct expected utility conditioned on θ:

$$\overline{U}(A|\theta) = \int_{\mathcal{X}} U[A(x), x, \theta] f(x|\theta) \tau(dx)$$

$\overline{U}(A|\theta)$ is the risk function for decision rule A

Alternative approaches

- ▷ Wald - no probabilistic structure over Θ
- ▷ de Finetti and Savage - subjective prior $\pi(d\theta)$
- ▷ Gilboa-Schmeidler - family of priors π

Consider extensions of the second two of these based on more recent developments in decision theory

- ▷ smooth ambiguity preferences - different utility functions for risk conditioned on a model and ambiguity across models
- ▷ variational preferences - penalize search over alternative priors

Risk and Admissibility

Risk conditioned on a model:

$$\overline{U}(A|\theta) = \int_{\mathcal{X}} U[A(x), x, \theta] f(x|\theta) \tau(dx)$$

Only induces a partial ordering among decision rules

- decision rule is admissible if it cannot be strictly dominated in θ
- Bayes decision rule maximizes

$$\max_{A \in \mathcal{A}} \int_{\Theta} \overline{U}(A|\theta) \pi(d\theta)$$

  for pior π. Could solve a corresponding conditional problem on X using the implied posterior.
- complete class theorem - under mild conditions every admissible rule is a Bayes rule

Variational Preferences

Introduce a convex function C to assess a decision maker's response to ambiguity about the prior π. The decision maker solves:

$$\max_{A \in \mathcal{A}} \min_{\pi} \int_{\Theta} \overline{U}(A|\theta)\pi(d\theta) + C(\pi).$$

The cost function imposes a penalty on the choice of prior. Maxmin is a special case.

Suppose:

$$\max_{A \in \mathcal{A}} \min_{\pi} \int_{\Theta} \overline{U}(A|\theta)\pi(d\theta) + C(\pi) = \min_{\pi} \max_{A \in \mathcal{A}} \int_{\Theta} \overline{U}(A|\theta)\pi(d\theta) + C(\pi)$$

For the inner problem on the right, A solves a Bayesian problem for a given π. Outer problem determines worst-case prior.

Smooth Ambiguity Preferences

Consider a two-stage lottery

1) draw θ using $\pi(d\theta)$
2) draw X given θ using $f(x|\theta)\tau(dx)$

Introduce a concave function Φ for the first stage of the lottery

Decision problem:

$$\max_{A \in \mathcal{A}} \Phi^{-1} \left[\int_{\Theta} \Phi \left[\overline{U}(A|\theta) \right] \pi(d\theta) \right]$$

The Φ^{-1} converts the objective to a certainty equivalent.

Relative Entropy Example

Given a baseline prior $\pi_o(d\theta)$, consider alternative priors of the form $\pi(d\theta) = g(\theta)\pi_o(d\theta)$ where $\int g(\theta)\pi_o(d\theta) = 1$.

- ▷ confront prior uncertainty using the cost function

$$C(\pi) = \kappa \int_\Theta [\log g(\theta)]\, g(\theta)\pi_o(d\theta)$$

- ▷ solve

$$\min_\pi \int_\Theta \overline{U}(A|\theta)\pi(d\theta) + C(\pi) = \\ -\kappa \log \int_\Theta \exp\left[-\frac{1}{\kappa}\overline{U}(A|\theta)\right]\pi_o(d\theta)$$

In this special case variational and smooth ambiguity preferences coincide.

Exponential Tilting

- minimizing prior

$$g^*(\theta) \propto \exp\left[-\frac{1}{\kappa}\overline{U}(A|\theta)\right]$$

- Tilt towards low utility models. Depends on decision rule A.
- When the order of optimization can be reversed, construct a worst case prior for which the decision rule solves the Bayesian problem.

"Good Thinking" I. J. Good

More Notation

- Represent a future state as a random vector S with realized values s.
- Let $\psi^*(s \mid a, x, \theta)$ denote the density relative to a measure τ^* over alternative s's in $\mathcal{S}$ conditioned on the current period action a and observed data x.
- Introduce a next period utility function U^* that depends on (s, a).
- Expected utility integrate over s to construct:

$$U(a, x, \theta) = \int_{\mathcal{S}} U^*(s, a)\psi^*(s \mid a, x, \theta)\tau^*(ds).$$

Even when not directly payoff relevant, the θ dependence of U is induced by the dependence of ψ^* on θ.

Dynamics and recursivity I

Suppose $U_1^*(a,x)$ is period utility function, $U_2^*(s)$ is a value function and $\pi(d\theta \mid x)$ is the posterior conditioned on x.

- expected utility

$$\widehat{U}(a,x) = U_1^*(a,x) + \beta \int_\Theta \int_\mathcal{S} U_2^*(s)\psi^*(s \mid a,x,\theta)\tau^*(ds)\pi(d\theta|x)$$

- recursive smooth ambiguity

$$\begin{aligned}\widehat{U}(a,x) = {} & U_1^*(a,x) \\ & + \beta\Phi^{-1} \int_\Theta \Phi\left[\int_\mathcal{S} U_2^*(s)\psi^*(s \mid a,x,\theta)\tau^*(ds)\right]\pi(d\theta|x)\end{aligned}$$

Choose a to maximize $\widehat{U}(a,x)$ as a function of x.

Dynamics and recursivity II

Suppose $U_1^*(a,x)$ is period utility function and $U_2^*(s)$ where is a value function

- expected utility

$$\widehat{U}(a,x) = U_1^*(a,x) + \beta \int_\Theta \int_\mathcal{S} U_2^*(s)\psi^*(s \mid a,x,\theta)\tau^*(ds)\pi(d\theta|x)$$

- recursive variational

$$\begin{aligned}\widehat{U}(a,x) &= U_1^*(a,x) \\ &+ \beta \min_\pi \int_\mathcal{S} U_2^*(s)\psi^*(s \mid a,x,\theta)\tau^*(ds)\pi(d\theta|x) + \widehat{C}(\pi \mid x)\end{aligned}$$

Choose a to maximize $\widehat{U}(a,x)$ as a function of x.

Misspecification I

- let ψ_o^* be a baseline conditional probability density
- introduce:

$$\psi^*(s \mid a, x, \theta) = \theta(s \mid a, x)\psi_o^*(s \mid a, x)$$

for $\theta \geq 0$ satisfying

$$\int \theta(s \mid a, x)\psi_o^*(s \mid a, x)\tau^*(ds) = 1.$$

notice that θ is a relative density and Θ is convex.

- introduce a convex cost function

$$C^*(\theta) = \kappa \int_{\mathcal{S}} [\log \theta(s \mid a, x)]\, \theta(s \mid a, x)\psi_o^*(s \mid a, x)\tau^*(ds)$$

Solve

$$\widehat{U}(a, x) = U_1^*(a)$$
$$+ \beta \min_{\theta} \int_{\mathcal{S}} U_2^*(s, a)\psi^*(s \mid a, x, \theta)\tau^*(ds) + C^*(\theta|a, x)$$

Misspecification II

- Problem

$$\widehat{U}(a,x) = U_1^*(a) + \beta \min_{\theta} \int_{\mathcal{S}} U_2^*(s)\psi^*(s \mid a,x,\theta)\tau^*(ds) + C^*(\theta|a,x)$$

- Solution

$$\widehat{U}(a,x) = U_1^*(a) - \beta\kappa \log \left(\int_{\mathcal{S}} \exp\left[-\frac{1}{\kappa} U_2^*(s) \right] \psi_o^*(s \mid a,x,\theta)\tau^*(ds) \right)$$

Kreps-Porteus style recursion justified differently

V. Decision theory in practice

A. econometric applications
B. implications for decentralized economies
C. macro policy analysis

Econometricians

uncertainty outside a family of economic models

- summarizes evidence - “noninformative priors” or bounds on posterior probabilities, classical confidence sets
- poses and solves decision problems using loss functions over parameters
- provides needed input into a decision problem

Economic agents

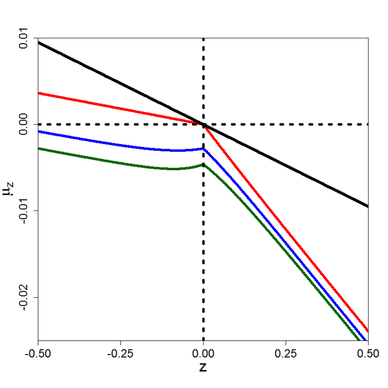
uncertainty inside economic models

- ▷ Apply Bayesian decision theory under rational expectations or rational learning
- ▷ Confront uncertainty more broadly conceived such ambiguity across models or potential model misspecification

Consequences

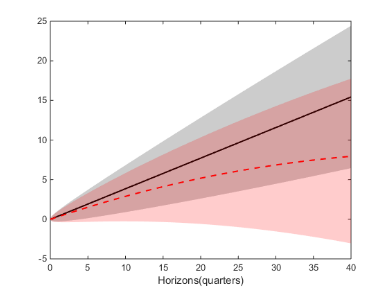
- ▷ alters resource allocation - shift away from uncertainty investment opportunities
- ▷ modifies prices that clear security markets - augment “risk prices” with uncertainty components that reflect, for instance, the (exponential) tilting probabilities toward adverse utility outcomes

Macroeconomic growth uncertainty



Local growth evolution. Left panels: larger baseline entropy. Right panels: smaller baseline entropy. **Black**: baseline model; **red**: worst-case model without misspecification concerns.

Macroeconomic growth uncertainty



Distribution of $Y_t - Y_0$ under the baseline model and worst-case model. The **black** solid line depicts the baseline median and gray shaded region are includes .1- .9 deciles. The red dashed line is the median under the worst-case model and the red shaded region includes the .1-.9 deciles.

Policy makers

When making decisions policy makers seek influence the private sector. Posed formally as a Ramsey problem or as a Stackelberg game

- how should policy makers confront uncertainty
- how do policy makers take account of how the private sector responds to uncertainty
- how does uncertainty alter policy makers incentives to experiment

Private and public sector uncertainty

- ▷ use a Ramsey planner as a stand-in for the benevolent policy maker.
- ▷ robustness concerns from private and/or public sectors alter the design of good policies.
- ▷ policy maker engages in managing or monitoring expectations and may be cautious because of model uncertainty.

VI. Axiomatic defenses

A. smooth ambiguity models
B. recursive multiple priors and rectangularity
C. dynamic variational preferences