Leverage and Disagreement*

François Geerolf †
UCLA
June 2, 2015

Abstract

I build a model of the cross section of leverage ratios for borrowers based on heterogenous beliefs about future asset returns and endogenous collateral constraints. In equilibrium, borrowers and lenders are matched in an assortative way according to their relative optimism through hedonic interest rates, which are disconnected from risk aversion or expected default probabilities. Under minimal assumptions on the underlying distribution of beliefs, the leverage ratio distribution of borrowers is Pareto in the upper tail, a prediction verified in micro-level administrative data for homeowners, publicly available data for entrepreneurs and in the TASS database for hedge funds. Expected and realized returns to levered portfolios are very skewed and fat tailed, even when heterogeneity in beliefs vanishes. The market features a high degree of customization and fragmentation, as many real world financial markets which are organized over-the-counter. Pyramiding lending arrangements (equivalently tranching) result from the desire of lenders to leverage themselves into these allocative interest rates; extended with short-sales, the model provides an equilibrium characterization of short interest and rebate rates. Finally, the leverage ratio distribution of borrowers provides useful information on the buildup of risk, an insight I illustrate using data for the US housing market between 1987 and 2012.

Keywords: Leverage, Over-The-Counter markets, Entrepreneurship

JEL classification: G01, G21, E3, E44

*Thanks in particular to Emmanuel Farhi, Christian Hellwig, Philippe Martin and Jean Tirole for continuous support and advice. I also thank Andy Atkeson, Adrien Auclert, John Campbell, Nicolas Coeurdacier, Eduardo Davila, Xavier Gabaix, John Geanakoplos, Gary Gorton, Ben Hebert, Edward Kung, Hashem Pesaran, Andrei Shleifer, Alp Simsek, Robert Ulbricht, Pierre-Olivier Weill, and seminar participants at the 10th Cowles Annual Conference on GE and its Applications, INET/IMF 3rd Conference on Macroeconomic Externalities, SED Meetings in Toronto, Top Finance Graduate Award Conference at CBS, CREI, the Board, Harvard, HEC Paris, MIT Sloan, NYU, Sciences-Po, TSE, UCLA, USC and Wisconsin-Madison for useful comments. I am also grateful to Harvard, MIT, and TSE, where part of this research was carried out, for their hospitality. All remaining errors are mine.

†E-mail: fgeerolf@econ.ucla.edu. This paper previously circulated under the title "A Theory of Power Law Distributions for the Returns to Capital and of the Credit Spread Puzzle". Comments are welcomed.
Introduction

Belief disagreement has been blamed for the importance and severity of the last financial crisis. Optimistic about the future evolution of house prices, many homeowners bought houses using leverage, and some banks leveraged into Mortgage-Backed Securities that would only pay off if house prices continued to rise. When house prices decreased, Lehman Brothers filed for bankruptcy, and other major banks and insurance companies would have gone bankrupt without government intervention. Why did banks take on such a bet on future house price values? Can regulators devise early-warning systems alerting in real time on the buildup of risk? Given the recurrence and severity of credit driven financial crises (Schularick and Taylor (2012)), new insights on this question are deeply in need.

In this paper, I develop a model of investment driven by heterogeneous beliefs about the future payoff of a risky asset. Under minimal assumptions on the underlying distribution of beliefs, the leverage ratio distribution of borrowers is Pareto in the upper tail, a prediction verified in Dataquick for homeowners, publicly available data for entrepreneurs and in the TASS database for hedge funds. The measured Pareto tail coefficient is informative of the number of levels of pyramiding lending arrangements driven by heterogeneous beliefs, and therefore of the relative optimism of the marginal investor (hence of the degree of asset price overvaluation). Using Dataquick data from 1987 to 2012, I show through the lens of the model how the evolution of the leverage ratio distribution of US homeowners was suggestive of the buildup of risk in the financial system between 2000 and 2006, with increased optimism of the marginal investor, and higher expected returns at each level of the financial intermediation chain. Mortgage rates, as well as repo rates on Mortgage-Backed Securities, are higher than the risk-free rate in the model, even though they are considered completely safe by agents trading them. Investment in high yield fixed income securities is thus not necessarily evidence of risk-shifting, but may be interpreted as a sign of belief disagreement.

More precisely, I construct a two-period model along the lines of Geanakoplos (1997) where a continuum of risk-neutral (or risk-averse but certain of what will happen) agents have heterogeneous beliefs about the payoff of a risky asset. In the simplest environment considered in the paper, which rules out short-sales and pyramiding lending arrangements, agents can invest their endowment in three ways: they can buy the risky asset potentially borrowing against it as collateral, lend to investors in the risky asset, or invest in a storage technology. In equilibrium, the agents’ population partitions itself into three groups endogenously based on their level of optimism: the most optimistic do leveraged investing, agents with intermediate levels of optimism do collateralized lending, and the least optimistic invest in the storage technology. Several implications of this model can then be traced back to the following key result: in equilibrium, there is positive sorting between borrowers and lenders, as more optimistic lenders lend to more optimistic borrowers. More optimistic lenders indeed allow borrowers to achieve a higher leverage ratio, because they agree to lend more for each unit of risky asset in collateral, and that buying more units of the risky asset is relatively worth more to relatively more optimistic borrowers:
borrowers’ and lenders’ beliefs are thus complementary.

The positive sorting result allows to solve in a very tractable way for the competitive equilibrium of what then becomes an assignment model in the tradition of Sattinger (1975). Lenders choose loans which are fully secured according to their beliefs. The leverage ratios of these loans are higher when they are relatively more optimistic. Those higher leverage loans would be attractive to all borrowers if implicit returns on loan contracts were the same, because borrowers would ideally like to leverage themselves into the asset as much possible, agents being either risk neutral or certain of what will happen. For markets to clear, interest rates must therefore rise when leverage increases, so that only more optimistic borrowers are willing to leverage more. Interest rates are hedonic prices, and are higher than the returns to cash, even though loans are perfectly safe according to lenders and borrowers trading them. Moreover, each lender is effectively lending to one particular borrower using a contract with a different leverage ratio and a different interest rate, and the sorting occurs through the loan contracts that both lenders and borrowers endogenously choose. The market therefore features a high degree of customization and fragmentation, as in real world collateralized debt markets: for example, it is well known that repurchase agreements markets are organized Over-The-Counter (OTC).

When disagreement goes to zero, the leverage of the most optimistic borrowers goes to infinity because lenders are less and less worried about the collateral. This effect is stronger than the diminished room for speculation allowed by lower disagreement. To the limit, very few borrowers end up borrowing almost all agents’ wealth. A key contribution of this paper is to show that Pareto distributions for leverage ratios obtain in the cross-section, which allow to recover elements of the belief density function. This Pareto distribution for leverage ratios obtains in this model through a microfounded and static mechanism, unlike in existing models of random growth. I show in particular that the Pareto coefficient of the leverage ratio distribution decreases as pyramiding lending arrangements develop, and asset prices reflect more and more the opinion of the most extreme optimists. Therefore, the model provides a quantitative framework for evaluating the buildup of risk in the financial system, one which goes beyond the aggregate value for credit quantities, but instead looks at their distribution. Using Dataquick data, I show how monitoring the leverage ratio distribution could have allowed to infer in real time that very marginally optimistic investors were more and more determining the price of US housing from 2000 to 2006. The leverage ratio distribution thus may have then been used as an early-warning signal to alert on the buildup of risk in the financial system.

This model thus yields four novel insights. First, an economy with disagreement is shown to generate very large amounts of equilibrium leverage, even from vanishingly small heterogeneity in beliefs. The often used benchmark in which every trader invests in financial markets in a similar way does not in fact obtain by continuity, in the limit where disagreement between investors goes to zero. Instead of a no-trade outcome, there is a natural tendency for financial markets to be characterized by agents taking very large and heterogenous positions, and for the intermediation
of savings by large financial institutions.¹

Second, the model provides a new driving force for mortgage rates, repo rates, and credit spreads on collateralized bonds, which are disconnected from expected default probabilities, unlike in textbook finance theory. In the model, interest rates are hedonic prices which result from an equilibrium positive sorting between lenders and borrowers. To the best of my knowledge, the present model is the first to find this new determinant for credit spreads on bonds, an asset pricing factor which is not related to default risk. This insight can be potentially important given the importance of fixed income securities in modern financial markets. As we shall see later, it is also crucial for understanding the existence of pyramiding lending arrangements. On the normative front, it shows that financial institutions buying high yield fixed income securities are not necessarily engaging in risk shifting, and thus that more "skin in the game", such as higher capital requirements for banks, might not make them less prone to crises.

Third, even though markets are anonymous ex-ante, and trading is assumed to be competitive, there is equilibrium sorting between lenders and borrowers, who are effectively matched in an assortative manner. In the model, borrowers choose over contracts and not lenders. But everything happens "as if" a particular lender was lending to a particular borrower, with a different interest rate, and a different leverage ratio. A qualitative insight from the model, which is likely to carry through in more general environments, is that speculation requires very specific contracts, tailored to the particular beliefs of every financial market participant. This can help explain why even heavily traded instruments tend to be traded Over-The-Counter (OTC), a puzzle for search theory when it is applied to financial markets.² While the search literature following Duffie et al. (2005) has assumed an exogenous OTC structure for derivative or repo markets, the model can rationalize this microstructure as resulting from a sorting mechanism in a purely competitive market. This is potentially important because although the labor market is certainly characterized by long search times and a high degree of decentralization, it is less true of financial markets. Even when over the counter, transactions occur among a small set of rather well identified dealer banks, and finding a counterparty is often for dealers only a matter of one or a few phone calls. Understanding the rationale for over-the-counter markets is important because they apply to a very large set of financial assets and they received significant attention during the recent financial crisis. For example, the model could provide a framework to assess the ongoing regulatory efforts trying to push banks’ bilateral trades onto exchanges. A particularly important question in this context is to understand why banks do not use Central (Clearing) Counterparties when they engage in repurchase agreements. Again, the model says

¹There is no mathematical contradiction here: when agents are ex-ante identical, an equally valid equilibrium has one agent invest on behalf of everyone else, or intermediating everyone’s savings. It is not the one we usually consider, as we invoke symmetry. In the light of this observation, the large volumes observed, for example on currency markets, are perhaps not so puzzling.

²For instance, Duffie (2014) writes: "Certain liquid OTC derivatives (such as simple interest rate swaps and credit derivative index products), seem like natural candidates for exchange-based trade but are normally traded over the counter. At this point, we lack convincing theories that explain why such simple and heavily traded instruments are traded over the counter."
that there is a natural tendency for borrowers to look for the counterparty that exactly suits their specific needs in terms of margining, and likewise that lenders also gain a few basis points (at least in expectation) when they use margins that they exactly want to set. Neither borrowers nor lenders necessarily want to be restricted by the kind of margin requirements that exchanges offer them. Finally, the model generates a dispersion in credit spreads, for the same type of collateral, that do not come from search frictions. The dispersion in repo rates is not necessarily a sign of non-competitive behavior, but in the model results from a competitive market characterized by anonymity.

Fourth, the framework allows to understand and rationalize the existence of pyramiding lending arrangements (or, equivalently, the tranching of securities). They emerge very naturally in the model: both result from the desire of lenders to leverage the excess premia on collateralized bonds they earn in equilibrium. I am therefore able to model the "double leverage cycle" conjectured by Geanakoplos (2011), which was at the forefront of the last financial crisis: leveraging on securities in the repo market, and on homes through the mortgage market. This extended model shows in particular that the possibility of using mortgage backed securities as collateral in repos is not redundant to the use of leverage by homeowners, but leads to a substantial increase in homeowners’ leverage as well as their expected returns. The qualitative intuition is that it relaxes the budget constraint of relatively optimistic lenders: these lenders, who agree to lend with very low margins, can now fund part of their loans by repo-ing their Mortgage-Backed Securities, which leads to higher leverage ratios for final borrowers. Quantitatively, pyramiding lending arrangements are shown to shift the leverage ratio distribution to the right on a log survivor-log leverage scale, as well as to a lowering of the Pareto coefficient. The theory as well as the evidence therefore suggest that monitoring ultimate borrowers’ leverage ratio distribution could provide some information on unmonitored bilateral transactions between banks, and on the overall stability of the financial system. Monitoring aggregate leverage is shown to be only an imperfect substitute to monitoring the cross-section of leverage ratios, as rising leverage can also signal that prices are more in line with fundamentals.

The disagreement model developed in this paper has other potential applications. For example, since the leverage ratio distribution of borrowers is Pareto, the model can potentially be used to understand why the returns to speculating are very skewed, and also why many top income earners can be found among investment bankers (Bell and Van Reenen (2013)). More generally, it suggests a novel mechanism through which a market economy can very naturally generate a large amount of inequality, coming from borrowers’ endogenous unequal equilibrium credit. Moreover, because the Pareto distribution for top incomes generated in this model relies on a microfounded mechanism, the model can be potentially used for policy and counterfactual analysis. For example, the model suggests a tight connection between margin requirements, or banking regulation, and top income inequality.

---

3I will in fact show that the tranching of loans into six different levels of riskiness (A, AA, AAA, ...) actually is equivalent to six layers of pyramiding lending arrangements. So it was more a "septuple leverage cycle".
The results also have implications for other borrower and lender relationships more generally, for example in a setting where the entrepreneur is more optimistic than his financier. In this interpretation, one should expect entrepreneurs to be endogenously overoptimistic on average: this is what makes them borrow and invest in the first place. And indeed, that entrepreneurship yields negative excess returns on average is a well-known puzzling feature of the data (see, for example, Moskowitz and Vissing-Jørgensen (2002)). In contrast, models based on information asymmetries have the opposite, counterfactual prediction: returns to entrepreneurship should be too high rather than too low. In the same line of idea, the model could also shed a new light on the key determinants of heterogenous returns to entrepreneurship. In the model, high returns to entrepreneurship obtain under three necessary and sufficient conditions: that the agent has the correct expectations about the asset he is investing in, that he is not competing for funds with too many similarly optimistic entrepreneurs, and that beliefs are on the extreme end of the belief distribution. Existing models of entrepreneurship, in contrast, fail to generate such a high heterogeneity (see, for example Cagetti and Nardi (2006)).

The rest of the paper proceeds as follows. Section 1 presents the simplest version of the model, where financial contracts are exogenously restricted to the set of Borrowing Contracts. Section 2 discusses at length the properties of this model. Section 3 extends the model to the case of pyramiding lending arrangements or tranching. Section 4 extends the model to a case where short-selling the asset is unrestricted. Section 5 shows three empirical applications of the framework, with three different types of borrowers: homeowners, entrepreneurs, and hedge funds. Section 6 concludes.

Literature. This paper is part of a large literature investigating the consequences of belief disagreement in financial markets. Miller (1977) was perhaps the first to highlight that because of short-sales constraints, asset prices can be higher than mean beliefs because pessimists cannot express their negative opinions. Dynamic versions of this model, such as Scheinkman and Xiong (2003), embed a resale value option, where asset prices can be higher than the value anyone would derive from holding the asset forever, following Harrison and Kreps (1978).

This paper follows the pioneering work of Geanakoplos (1997), Geanakoplos and Zame (2002), and Geanakoplos (2003) in embedding these insights in a model with endogenous collateral constraints, where optimists must borrow from lenders who have different beliefs and are therefore reluctant to lend. In Geanakoplos (1997)’ equilibrium as well as subsequent models, there is however only one leverage ratio for collateralized loans, because all agents agree on the value of the asset conditional upon default. This is a modeling simplification, which allows Geanakoplos (2010) to speak conveniently of a leverage factor, and to study its dynamics over the cycle. This merely technical assumption is however at odds with empirical evidence and even casual empiricism, as homebuyers purchase houses using collateralized loans with different loan-to-value ratios, for example. In contrast, the general model allows for dissent on the default state also, so that contracts with different endogenous margins are traded in equilibrium. Fostel and
Geanakoplos (2012) study the fall in the bubble following the appearance of Credit Default Swaps; and Simsek (2013) extends the model to different types of disagreement, showing that the optimism about the probability of upside states has more effect on asset prices than that on downside states. A common feature of these models is that they all need a fair amount of disagreement between agents in order to matter quantitatively. (and therefore, are usually thought of as pertaining to the behavioral finance literature) Such is not the case in this paper, as Pareto distributions for leverage ratios, hedonic interest rates, customization of financial markets, and pyramiding lending arrangements arise regardless of the underlying level of disagreement, however small.

This paper offers a new application of the standard assignment model to the study of leveraged investing, with several new insights. The possibility of hedonic interest rates relates the paper to an important literature in empirical finance documenting anomalies in the pricing of fixed income securities. Among many other example, it can explain why Bartolini et al. (2010) find that repurchase agreements’ rates on government-sponsored agencies Mortgage-Backed Securities (MBS) as well as private label MBS are not meaningfully different from rates on interbank unsecured loans. When interest rates are viewed as hedonic prices, this observation is not puzzling: indeed, repo rates are connected to underlying disagreement about MBS, while rates on unsecured interbank loans are connected to banks’ creditworthiness. The order of magnitudes for the repo spreads I obtain are similar to those reported in Gorton and Metrick (2012).

More broadly, this paper is part of a larger literature investigating the effect of borrowing constraints on asset prices and investment. The pioneering works on collateral are Bester (1985) and Shleifer and Vishny (1992) for example. Kiyotaki and Moore (1997) is another important paper emphasizing the feedback from the fall in collateral prices to that in debt capacity. In particular, a key element of my paper is that in order to benefit from one’s beliefs, one needs capital as in Shleifer and Vishny (1997). This is because other investors in the market do not share the same beliefs, and hence want to be protected if there is default. There are of course many other theories of borrowing limits, that stem from information asymmetries, lack of commitment, or exogenously imposed margin requirements: Holmstrom and Tirole (1997), Holmstrom and Tirole (1998), Bernanke et al. (1999), Gromb and Vayanos (2002), Brunnermeier and Pedersen (2009), among many others. None of these theories however has predictions about the Power law distribution of leverage ratios, which I document both theoretically and empirically in this paper.

The fact that the model is able to generate Pareto distributions for leverage ratios, expected returns and realized returns connects this paper to a large literature rationalizing power laws through a random growth mechanism (a survey of which is given in Gabaix (2009)). However, in the context of initial leverage ratio distributions for homeowners or entrepreneurs, it is hard to see what the underlying idiosyncratic shocks would be, and why these would have to be multiplicative.

Finally, the model casts some new light on the entrepreneurship literature. The model can rationalize the fact that returns are very heterogenous across entrepreneurs, as shown in Moskowitz
and Vissing-Jorgensen (2002). Quantitatively, it is capable of producing Pareto distributions for
the returns to entrepreneurship, again from only epsilon ex-ante differences in beliefs across en-
trepreneurs. In the model, entrepreneurship results from an occupational choice that is belief
dependant, and entrepreneurs are way too optimistic on average, also a key empirical insight
from Moskowitz and Vissing-Jorgensen (2002). Note that again, borrowing constraints cannot
explain these two facts. Average returns on entrepreneurship should on the contrary be higher
than average, under information asymmetries (Hurst and Lusardi (2004) provide more direct
evidence against information asymmetries playing a big role for determining entrepreneurship).

1 Borrowing Economy

This section presents the main results of the paper in the simplest possible economic environment.
I consider an Economy \( E_B \) where the only available instruments are equity and debt contracts.
In particular, I assume that agents cannot sell the real or financial assets short; and that loans
cannot be used as collateral. I allow for pyramiding lending arrangements in Section 3, and for
short-sales in Section 4. I first present the set-up, then discuss the assumptions, and finally solve
for the equilibrium.

1.1 Setup

There are two periods 0 and 1. To simplify notations, I omit the time zero subscript: for example,
the price at time 0 is denoted by \( p \) instead of \( p_0 \). There is a continuum of agents \( i \in [0, 1] \) of
measure one born in period 0 with initial wealth normalized to 1. Agents care only about period
1’s consumption, and therefore need assets to store their wealth.

**Assets.** To transfer wealth into period 1, agents can invest in a *Storage Technology* with
return normalized to \( R = 1 \). For concreteness, I refer to this asset as *Cash*. They can also
invest in an asset in finite supply normalized to 1, with exogenous resale value \( p_1 \) in period 1,
and endogenous price \( p \) in period 0, which I will refer to as the *Real Asset* in the following.
In addition, they can agree to collateralized Borrowing Contracts with each other. Formally, I
define a Borrowing Contract in Economy \( E_B \) as follows.

**Definition 1** (Borrowing Contract, Economy \( E_B \)). A Borrowing Contract \( (\phi) \) in economy \( E_B \) is
a promise of \( \phi \geq 0 \) units of Cash in period 1, the face value, collateralized by one unit of Real
Asset.

Without loss of generality, the set of contracts is restricted to a set containing contracts using
exactly one unit of Real Asset as collateral, and that this allows to name contract by their face
value \( (\phi) \). Contracts are traded in an anonymous market at competitive price \( q(\phi) \), and payment
is only enforced by the collateral: agents default as long as the value of the collateral is lower
than the face value of the loan they have to repay. The payoff of contract \( (\phi) \) in period 1 is
therefore:

\[
\min \{ \phi, p_1 \}
\]

for a contract with face value \( \phi \). In period 0, this contract is sold by the borrower, who gets \( q(\phi) \) units of Cash in exchange for the contract. The interest on this borrowing contract is:

\[
r(\phi) = \frac{\phi}{q(\phi)}.
\]

**Beliefs.** Agents have heterogeneous point expectations about the price of the asset in period 1. Namely, agent \( i \in [0,1] \) believes that the asset price will be \( p_1^i \) with probability 1. This assumption of point expectations may seem extreme. However the model generalizes very straightforwardly to a case where agents are risk neutral and agree about a probability distribution for \( p_1 \) around this mean. In particular, in that case, all Borrowing Contracts will in equilibrium be indexed by these states, in which the payoff of the asset differs from its expected mean by an amount that everybody agrees on. The key is therefore that I focus on disagreement about means of future asset payoffs rather than about probabilities of certain events, as in the previous heterogenous beliefs and endogenous margins literature. (which also has risk neutrality) Finally, note that I do not need risk neutrality in the case where agents have point expectations, as for those agents, speculation entails no risk.

More precisely, the cumulative distribution function representing the number of agents with beliefs \( p_1^i \) for future prices is denoted by \( F(.) \), with corresponding density \( f(.) \). The upper bound on agents’ beliefs is assumed to be 1 without loss of generality.\(^4\) The most pessimistic agents have beliefs \( 1 - \Delta \), with \( \Delta > 0 \). \( f(.) \) has full support on \([1 - \Delta, 1]\), so \( \Delta \) is a natural measure of belief heterogeneity. \( \Delta \) will be referred to as the belief heterogeneity parameter.

Note that the assumption made earlier of equal endowment for all agents also is without loss of generality. If initial endowments were heterogenous, then the density function would subsume both how many agents have those beliefs and how wealthy they are.

**Equilibrium.** All units of the Real Asset are initially endowed to unmodeled agents who sell their asset holdings in period 0 and then consume. Agent \( i \) chooses his positions in the Real Asset \( n_A^i \), a menu of financial Borrowing Contracts (\( \phi \)) denoted by \( dN_B^i(\phi) \) (where \( N_B^i(\phi) \) is the cumulative measure of contracts with face value less than \( \phi \)), and Cash \( n_C^i \), in order to maximize his expected wealth (\( W \)) in period 1 according to his subjective beliefs \( p_1^i \) about the Real Asset, subject to his budget constraint (\( BC \)), and the collateral constraint (\( CC \)):

\[
\max_{(n_A^i, dN_B^i(), n_C^i)} n_A^ip_1^i + \int_{\phi} \min\{\phi, p_1^i\} dN_B^i(\phi) + n_C^i \tag{W}
\]

\[
\text{s.t. } n_A^i p + \int_{\phi} q(\phi)dN_B^i(\phi) + n_C^i \leq 1 \tag{BC}
\]

\(^4\)The model being linear, all quantities in the model are multiplied by \( M \) in the case where this maximum belief is \( M \).
\[ \text{s.t. } \int_{\phi} \max\{0, -dN^i_B(\phi)\} \leq n^i_A \]  

\[ \text{s.t. } n^i_A \geq 0, \quad n^i_C \geq 0 \]

Note this portfolio problem is subject to the additional restriction that \( n^i_A \geq 0, n^i_C \geq 0 \): agents have to choose positive amount of Real Asset and Cash holdings (again, this is relaxed later in the paper). When \( dN^i_B(\phi) > 0 \), agent \( i \) buys Borrowing Contract \((\phi)\), and therefore lends. When \( dN^i_B(\phi) < 0 \), agent \( i \) sells Borrowing Contract \((\phi)\), and therefore borrows. Each time a borrower sells of unit of Borrowing Contract, he needs to own unit of asset hence equation (CC). The equilibrium concept is that of a Collateral Equilibrium, as defined in Geanakoplos (1997), with contracts being treated as commodities. Formally,

**Definition 2** (Competitive Equilibrium of \( \mathcal{E}_B \)). A Competitive Equilibrium for Economy \( \mathcal{E}_B \) is a price \( p \) for the Real Asset and a distribution of prices \( q(.) \) for all traded Borrowing Contracts \((\phi)\), and portfolios \((n^i_A, dN^i_B(\phi), n^i_C)\) for all agents \( i \) in the Real Asset, Borrowing Contracts and Cash, such that all agents \( i \) maximize expected wealth in period 1 \((W)\) according to their subjective beliefs, subject to their budget constraint (BC), their collateral constraint (CC), and markets for the Real Asset and Borrowing Contracts clear:

\[ \int n^i_A di = 1, \]  

\[ \text{and } \forall \phi, \quad \int dN^i_B(\phi) di = 0. \]

**1.2 Discussion**

Before studying the equilibrium of this model, it is worth discussing some of the assumptions I have so far made. This also leads to go through the potential applications of the theoretical model presented above.

**Agreeing to disagree.** A key assumption in the model is that agents agree to disagree. This disagreement does not come from information, as information alone cannot generate trade (no-trade theorems), and so agents do not learn from the fact that other disagree with them. The assumption of belief disagreement may seem problematic to some readers. It is certainly very likely that agents would have different priors about the returns of a yet unobserved technological advance (such as the internet in the 1990s). As argued in Morris (1995) and Morris (1996), it is very hard to conceive any criteria - rational or otherwise - that would require traders to have the same prior over the dividends of a yet unobserved asset. This perhaps also would apply to the impact of new subprime arrangements in the years 2001-2006. Moreover, Acemoglu et al. (2015), Borovicka (2015) and Cao (2014) provide theoretical mechanisms through which heterogenous beliefs can survive market selection. Furthermore, it is well known that disagreement models are isomorphic to models with noisy information aggregation - that is, to models where agents have heterogenous information, but do not learn perfectly from each other, because noise traders are also trading for random reasons.
The most salient argument in support of assuming agents agree to disagree is certainly that the main insights of the model rely on only an epsilon of disagreement between agents, in particular the Pareto results as well as the positive sorting results. As long as there is even an epsilon of noise on financial markets, then agents will never agree perfectly, even in the long run.

**Disagreement on means.** This model departs from previous papers in the literature on endogenous leverage, in that I assume disagreement on means of future asset returns. (and a continuum of agents) The assumption that agents disagree on means actually arises naturally in a number of different existing models. (which do not have endogenous leverage, though) For example, models where agents overestimate the precision of their signal, as Scheinkman and Xiong (2003), also lead to disagreement on mean values of asset prices.

In contrast, in Geanakoplos (1997), there is only one leverage ratio: all agents agree on the value of collateral conditional upon default, and disagree either on the value of collateral conditional on other states occurring or on probabilities of different states. In the case of a bond, this means all agents agree on the recovery rate. Such is also the case in Simsek (2013), where there is only two types of traders, optimistic and pessimistic. Therefore there must also be only one type of lender, who uses one type of contract. Interest rates compensate for expected default probabilities. This assumption of disagreement on means combined with that of a continuum of traders are the ones leading to a distribution of equilibrium leverage ratios, as well as to hedonic interest rates. All I need in fact is more than two belief types, and the continuum is an especially tractable way to do it.

Another advantage with working with more than two belief types, is that the model can allow for pyramiding lending arrangements, as well as short-selling at the same time as lending. The treatment of pyramiding lending arrangements with endogenous margins is new. Simsek (2013) has a treatment of short-sales, but the model cannot explain short-sales together with lending and borrowing in equilibrium, as it features only two types of agents. In Simsek (2013), one assumption is that some agents (those who short in equilibrium) are inhibited from lending, which is obviously counterfactual. In contrast, the tractability of the present model allows to have all these types of investment strategies coexisting in equilibrium, while everyone has a possibility to short and lend ex-ante. For the same reason, Simsek (2013) cannot model pyramiding, which was central during the last financial crisis.

**No short-sales, no pyramiding.** The assumption that short-sales is not possible is arguably a good approximation for the housing market, as well as for the financing of newly created firms, whose value is not quoted on financial markets: there is then no publicly observ-

---

5An exception of course, was the US housing market in 2006, during which derivative securities on mortgage-backed securities made possible to short housing. However, this crucially relied on loans being made to subprime borrowers, the loan being non recourse, and other conditions which are only rarely present simultaneously. Moreover, note that housing was not shorted directly, but loans to homeowners were. Lewis (2010) is a narrative account of how some investors were able to short the housing market in 2007.
able price which would allow traders to bet against the performance of this firm. But again, most of the insights obtained in Economy $E_R$ are maintained, if not strengthened, when short-sales are possible, as shown in Section 4. An exception concerns the price of the real asset, which is naturally lower than it is in Economy $E_R$ since pessimistic agents can now express their views.

I also assume that "pyramiding lending arrangements", or the use of loans as collateral are impossible. Equivalently, the rehypothecation of collateral, or the tranching of loans, which are all equivalent in this model, are not not feasible either. Why all these are equivalent and an investigation of what happens when they are feasible is the subject of Section 3.

**The real asset.** There are many different possible interpretations as to what the real asset, borrowers and lenders refer to. A homeowner could be investing in a house, borrowing from a bank. An entrepreneur could be investing in a project, also borrowing from a bank. Or a hedge fund could be financing the purchase of mortgage-backed securities, borrowing from his broker, for example through repo. The seller of the repurchase agreement is the borrower and sells the security used in collateral, agreeing to buy it back at a later date. In contrast, the lender buys the repurchase agreement as well as the security. (reverse-repo)

The risky asset should not be thought of too narrowly. In principle, it could represent some combination of trades, which together amount to some idiosyncratic risk exposure. At an abstract level, the "risky asset" of the paper may indeed very well be Shleifer and Vishny (1997)'s introductory example: the difference between the price of two Bund futures contracts. The belief that two Bund futures contracts, delivering the same exact value at time $T$, will converge before that, leads some traders to potentially take extreme positions in that direction. But they are able to maintain this position only if they have enough capital until time $T$, when it will converge for sure. Long-Term Capital Management was actually doing these types of trades, acting as the infinitely leveraged, very optimistic trader of the model.

**No recourse.** Note also that it is assumed that agents cannot impose penalties upon each other other than collateral seizure. In this model, agents do not have any income in period 1. In a model where they do have such income, or, in the case of financial intermediaries, who may have other assets, this assumption is tantamount to a no-recourse assumption.

Whether no recourse is a good assumption or not naturally depends on the nature of the risky asset, and on the identity of borrowers and lenders. In the case of a homeowner, some states in the United States are no recourse states, and the model applies very well to them. Even in recourse states, lenders do not usually go after a homeowners’ personal funds after default; so the model also applies quite well to them too. Entrepreneurs getting loans through banks usually pledge their homes or other personal assets as collateral, but these would then be included in the "equity" of the entrepreneurs, as they have the same function as personal funds. Again, because forced servitude is not allowed in modern societies, the banker will mostly want to assess the value of the enterprise, lending on the basis of a business plan for example. Finally, things are more subtle on financial markets, as most repurchase agreements legally allow recourse over
the balance sheet of banks. However market practice suggest that lenders want their collateral to protect them fully from the risks of the investment, and do not want to worry about the creditworthiness of their counterparty. This could be because of delays, uncertainties over the deficiency judgement, or because they have no information, or do not want to know, which other trades the counterparty is engaging into. They therefore set margins as if they had no recourse on the balance sheet of their borrower.

1.3 Equilibrium

The collateral equilibrium defined in Definition 2 is characterized by two thresholds dividing the set of agents into cash investors, lenders and borrowers, a matching function $\Gamma(.)$ linking borrowers to lenders, and a bond price function $Q(.)$. The price of the asset, the two thresholds, $\Gamma(.)$, and $Q(.)$ are linked by two first-order ordinary differential equations and five algebraic equations, which can be found in the next Proposition.

Proposition 1 (Equilibrium of Economy $E_B$). A competitive equilibrium of Economy $E_B$ is described by a price $p$ for the Asset, two thresholds $\xi \leq \tau$, a strictly increasing matching function $\Gamma(.)$ mapping $[\tau,1]$ onto $[\xi,\tau]$, and a loan pricing function $Q(.)$, such that:

1. The space of agents’ beliefs $[1 - \Delta, 1]$ is partitioned into three intervals:
   - Agents with beliefs $p_i^1 \in [1 - \Delta, \xi]$ (cash investors) invest in Cash.
   - Agents with beliefs $p_i^1 \in [\xi, \tau]$ (lenders) buy Borrowing Contracts.
   - Agents with beliefs $p_i^1 \in [\tau, 1]$ (borrowers) buy the Asset and sell Borrowing Contracts.

2. Lenders with beliefs $y$ sell Borrowing Contracts with face value $y$. Borrowers with beliefs $y$ buy Borrowing Contracts with face value $\Gamma(y)$, at price $q(\Gamma(y)) = Q(y)$.

3. $p$, $\xi$, $\tau$, $\Gamma(.)$, and $Q(.)$ are such that:
   
   \begin{align}
   (1a) & \quad \forall y \in [\tau, 1], \quad (p - Q(y)) \Gamma(y) = Q(y) \Gamma'(y), \\
   (1b) & \quad \forall y \in [\tau, 1], \quad (y - \Gamma(y))Q'(y) = (p - Q(y))\Gamma'(y).
   \end{align}

   \begin{align}
   (2a) & \quad 1 - F(\xi) = p, \quad (2b) \quad \Gamma(\tau) = \xi, \quad (2c) \quad \Gamma(1) = \tau, \\
   (2d) & \quad Q(\tau) = \xi, \quad (2e) \quad Q(1) = \frac{\tau(p - \xi)}{\tau - \xi}.
   \end{align}

Borrowers and lenders are thus matched in an assortative way through function $\Gamma(.)$: lender with beliefs $x$ are effectively lending to borrower with beliefs $\Gamma^{-1}(x)$, through their respective choice of contracts. I refer the reader to Appendix D for a full proof of this result. In the main text, I give an intuition when possible or derive the most important elements of this proposition.

The first part of Proposition 1 establishes an intuitive result. There exists a cutoff $\xi$ for $p_i^1$ such that agent $i$ with $p_i^1 < \xi$ is too pessimistic to invest in the asset or lend using this asset as
collateral. Agents with intermediary levels of pessimism such that \( p_1^i \in [\xi, \tau] \) do not invest in the asset but lend with the asset in collateral. Finally, more optimistic agents with beliefs \( p_1^i \in [\tau, 1] \) invest in the asset using it as collateral.

The second and third part of the proposition all come from lenders’ and borrowers’ portfolio problems once these occupational choices are given. Cash investors’ problem is trivial, they just invest everything they have in cash. The proof then proceeds in four steps, corresponding Lemmas 1, 2, 3 and 4. I first state borrowers problem, then the lenders’ problem, then the positive sorting results, and I finally derive the two differential equations and five algebraic equations for the model.

**Lemma 1** (Borrowers’ problem). A borrower with beliefs \( p_1^i \) chooses the face value of the Borrowing Contract \( \phi \) to solve:

\[
\max_{\phi} \frac{p_1^i - \phi}{p - q(\phi)}
\]

The intuition for the proof is as follows. Because of the linearity of the problem, whenever he finds it optimal to invest in the asset, he also wants to use this asset as collateral to the maximum, in order to get more funds. Each time he buys a unit of the real asset, he will also sell one unit of borrowing contract (recall from Definition 1 that one borrowing contract is collateralized by one unit of the asset). The chart below illustrates the financing of a purchase of one unit of asset using a Borrowing Contract with face value \( \phi \), with the balance sheet in period \( t = 0 \) on the left-hand side and the expected balance sheet in \( t = 1 \) on the right-hand side:

|\( L \)| |\( A \)||\( L \)| |\( A \)||\( L \)| |\( A \) ||\( L \)| |\( A \)| |
|---|---|---|---|---|---|---|---|---|
|\( p - q(\phi) \)| |\( p \)| |\( p_1^i - \phi \)| |\( p_1 \)| |\( \phi \)| |\( p \)| |\( \phi \)| |
|\( q(\phi) \)| |\( p \)| |\( \phi \)| |\( p_1^i \)| |\( \phi \)| |\( p \)| |\( \phi \)| |

When he sells a contract \( (\phi) \) with face value \( \phi \), he borrows \( q(\phi) \) for each borrowing contract (the price of the contract). When using contract \( (\phi) \), the borrower therefore needs to finance

\[
\text{Return on Asset} = \frac{p_1^i}{p},
\]

\[
\text{Return on Debt} = \frac{\phi}{q(\phi)} = r(\phi),
\]

\[
\text{Return on Equity} = \frac{p_1^i - \phi}{p - q(\phi)}.
\]
\[ p - q(\phi) \] from his own funds. The number of assets he can then buy with one unit of endowment is \( 1/(p - q(\phi)) \) in period 0. From the expected balance sheet in period 1, his expected wealth in period 1 is then \((p_1^i - \phi)/(p - q(\phi))\). The problem of the borrower is thus to choose the contract he uses to lend to solve:

\[
\max_{\phi} \frac{p_1^i - \phi}{p - q(\phi)}.
\]

If the problem is interior, then Lemma 1 implies:

\[
- (p - q(\phi)) + q'(\phi)(p_1^i - \phi) = 0 \Rightarrow \frac{p_1^i - \phi}{p - q(\phi)} = \frac{1}{q'(\phi)}.
\]

This equation has a straightforward intuition: at optimum, the benefits from choosing a higher face value for the Borrowing Contract \( \phi \) needs to be equal to the costs. The benefit is that the borrower then can relax his borrowing constraint, and increase his equity by the amount borrowed equal to the number of Borrowing Contracts times \( dq(\phi) \). The cost is that he then has to repay more in period 1, by an amount \( d\phi \) times the number of Borrowing Contracts:

\[
\frac{1}{p - q(\phi)} \frac{dq(\phi)}{\# \text{ of Borrowing Contracts}} = \frac{p_1^i - \phi}{p - q(\phi)} = \frac{1}{\# \text{ of Borrowing Contracts}} \frac{d\phi}{\text{Extra Repayment Per Unit}},
\]

which leads to the previous equation. For traded contracts, it must be that \( q(\phi) \) increases in \( \phi \): no borrower would ever promise to repay more in period 1 and get less in period 0. Therefore \( q(\phi) \) is increasing in \( \phi \). Consequently, leverage as well as \( q(\phi)/(p - q(\phi)) \) are increasing with \( \phi \).

The previous expected return on equity can be written as:

\[
\frac{p_1^i - \phi}{p - q(\phi)} = \left( 1 + \frac{q(\phi)}{p - q(\phi)} \right) \frac{p_1^i}{p} - \frac{q(\phi)}{p - q(\phi)} r(\phi)
\]

\[
\frac{p_1^i - \phi}{p - q(\phi)} = \frac{p_1^i}{p} + \left( \frac{p_1^i}{p} - r(\phi) \right) \frac{q(\phi)}{p - q(\phi)}.
\]

Note that different concepts are used in the financial markets to compare the amount borrowed to the value of collateral, which are all related to the price of the asset and to the amount borrowed \( q(\phi) \). The chart below shows the balance sheet of a borrower for each unit of asset bought in period 0, and the definitions of leverage, loan-to-value, haircut and margin in that case.

<table>
<thead>
<tr>
<th>L</th>
<th>A</th>
</tr>
</thead>
<tbody>
<tr>
<td>p-q(\phi)</td>
<td>p</td>
</tr>
<tr>
<td>q(\phi)</td>
<td></td>
</tr>
</tbody>
</table>

Leverage = \( \frac{p}{p - q(\phi)} \)  \quad Loan-To-Value = \( \frac{q(\phi)}{p} \)

Haircut = \( \frac{p - q(\phi)}{p} \)  \quad Margin = \( \frac{p - q(\phi)}{q(\phi)} \).
For the solution to this problem to be interior, it must be that there is a trade-off between leverage and return, thus \( r(\cdot) \) must also be increasing in \( \phi \). This brings us to lenders’ problem.

**Lemma 2** (Lenders’ problem). *A lender with beliefs \( p_i \) chooses the face value of the Borrowing Contract equal to \( p_i \).*

If he buys an overcollateralized contract with \( \phi < p_i \), the lender’s expected payoff is:

\[
\min\left\{ \phi, p_i \right\} \frac{q(\phi)}{q(\phi)} = \phi \frac{q(\phi)}{q(\phi)} = r(\phi),
\]

which is increasing in \( \phi \). Hence, the lender will find it optimal to choose \( \phi \geq p_i \). The lender does not choose an undercollateralized contract either, such that the promise exceeds the expected value of the collateral \( \phi > p_i \). The expected payoff on an under collateralized contract is:

\[
\min\left\{ \phi, p_i \right\} \frac{q(\phi)}{q(\phi)} = p_i \frac{q(\phi)}{q(\phi)} = \frac{q(\phi)}{q(\phi)},
\]

which is decreasing in \( \phi \). Therefore, it has to be that \( \phi \leq p_i \), so \( \phi = p_i \).

This result is intuitive. The lender does not choose an overcollateralized contract, as interest rates on loans are increasing in their face value: he is always better off lending up to the point where he thinks that the collateral will exactly cover the promised payment. He does not choose an undercollateralized contract either, as these default for sure and he needs to lend more in period 0 to get the same collateral in period 1.

**Lemma 3** (Positive sorting). *More optimistic borrowers borrow with higher leverage ratio loans. They therefore effectively borrow from more optimistic lenders, through their choice of Borrowing Contracts.*

There is supermodularity of the expected wealth (and return, both being equivalent) with respect to his beliefs \( p_i \) and the face value of the Borrowing Contract he uses:

\[
\frac{p_i - \phi}{p - q(\phi)} = \frac{p_i}{p} + \left( \frac{p_i}{p} - r(\phi) \right) \frac{q(\phi)}{p - q(\phi)}.
\]

Therefore, the cross derivative of wealth with respect to \( \phi \) and \( p_i \) is strictly positive:

\[
\frac{\partial^2}{\partial p_i \partial \phi} \left( \frac{p_i - \phi}{p - q(\phi)} \right) = \frac{q'(\phi)}{(1 - q(\phi))^2} > 0.
\]

We therefore have that:

\[
\frac{\partial}{\partial \phi} \left( \frac{p_i - \phi}{p - q(\phi)} \right) = 0 \Rightarrow \frac{\partial^2}{\partial \phi^2} \left( \frac{p_i - \phi}{p - q(\phi)} \right) = \frac{\partial^2}{\partial p_i \partial \phi} \left( \frac{p_i - \phi}{p - q(\phi)} \right) dp_i = 0
\]

\[
\Rightarrow \frac{d\phi}{dp_i} = -\frac{\partial^2}{\partial \phi^2} \left( \frac{p_i - \phi}{p - q(\phi)} \right) > 0,
\]

16
from the fact that φ maximizes the expected return of the borrower:

\[
\frac{\partial^2}{\partial^2 \phi} \left( \frac{p_i^1 - \phi}{p - q(\phi)} \right) < 0
\]

Therefore, the face value of the Borrowing Contract chosen by a borrower increase when he is relatively more optimistic. Since lenders choose contracts whose face value is equal to their beliefs, this means that he is also effectively borrowing from a more optimistic lender.

This result has an economic intuition. If a borrower is relatively more optimistic, he likes more buying an extra unit of the asset. Hence in the competitive equilibrium, he will be using the high leverage ratio loans, as his willingness to pay from them is higher than less optimistic borrowers. Intuitively, there is only a limited supply of optimistic lenders, and all borrowers would ideally like to borrow with high leverage ratios. But this cannot be an equilibrium, since markets need to clear. Interest rates on Borrowing Contracts thus discourage not so optimistic borrowers from using high leverage ratio loans. Denoting by \( \Gamma(y) \) the face value of the Borrowing Contract chosen by a borrower with beliefs \( y \), the positive sorting result means that \( \Gamma'(\cdot) > 0 \).

**Lemma 4** (Equations). The equilibrium quantities of the model \( p, \xi, \) and \( \tau \) and \( \Gamma(\cdot) \) and \( Q(\cdot) \) for \( y \in [\tau, 1] \) are given as a function of the belief cumulative distribution function \( F(\cdot) \) and its associated density \( f(\cdot) \) by two first-order ordinary differential equations as well as five algebraic equations:

\[
\forall y \in [\tau, 1], \quad (p - Q(y)) f(\Gamma(y)) \Gamma'(y) = Q(y) f(y) \quad (1a)
\]

\[
\forall y \in [\tau, 1], \quad (y - \Gamma(y)) Q'(y) = (p - Q(y)) \Gamma'(y) \quad (1b)
\]

\[
(2a) \quad 1 - F(\xi) = p, \quad (2b) \quad \Gamma(\tau) = \xi, \quad (2c) \quad \Gamma(1) = \tau,
\]

\[
(2d) \quad Q(\tau) = \xi, \quad (2e) \quad Q(1) = \frac{\tau(p - \xi)}{\tau - \xi}. \quad (2)
\]

The first differential equation comes from market clearing for Borrowing Contracts with face value \( \Gamma(y) \). Borrowers in an infinitesimal interval \([y, y + dy]\) of measure \( dy \) around \( y \) will sell \( f(y)dy/(p - Q(y)) \) financial contracts with face value \( \Gamma(y) \). Again, this is because for each unit asset they buy, they pay \( p \) and finance \( Q(y) \) through borrowing, so they contribute \( p - Q(y) \) of personal funds to the purchase of one asset. The corresponding measure of lenders \( dx \) with beliefs around \( x = \Gamma(y) \) have to buy the same quantity of Borrowing Contracts, which costs \( Q(y) \) each, which gives the following equation for lenders in interval \([x, x + dx]\) with wealth \( f(x)dx\):

\[
\frac{f(x)dx}{Q(y)} = \frac{f(y)dy}{p - Q(y)}
\]

\([^\# \text{ of Borrowing Contracts \text{\tiny bought by lenders in } } [x, x + dx] \text{\tiny sold by borrowers in } [y, y + dy] \^\# \text{ of Borrowing Contracts} \]

\( \Rightarrow \forall y \in [\tau, 1], \quad (p - Q(y)) f(\Gamma(y)) \Gamma'(y) = Q(y) f(y). \)

The second differential equation results from the previously derived optimality condition for borrowers with beliefs \( y \):

\[
\frac{y - \phi}{p - q(\phi)} = \frac{1}{q'(\phi)}
\]
The optimal choice of the face value for borrowers being given by \( \phi = \Gamma(y) \), this writes:

\[
\frac{y - \Gamma(y)}{p - q(\Gamma(y))} = \frac{1}{q'(\Gamma(y))}.
\]

Since \( Q(y) \) is the price at which a borrower with beliefs \( y \) sells a unit Borrowing Contract with face value \( \Gamma(y) \), we have:

\[
Q(y) = q(\Gamma(y)) \quad \Rightarrow \quad Q'(y) = q'(\Gamma(y))\Gamma'(y).
\]

Therefore, replacing in the optimality condition:

\[
\forall y \in [\tau, 1], \quad (y - \Gamma(y))Q'(y) = (p - Q(y))\Gamma'(y).
\]

Intuitively, this second differential equation expresses the fact that interest rates rise just enough so that more pessimistic borrowers are excluded from contracts with a higher leverage ratio. Let us now look at the properties of this equilibrium.

Positive sorting implied that the most optimistic borrowers sell the contracts bought by the most optimistic lenders, and symmetrically for the most pessimistic borrowers and lenders:

\[
\Gamma(1) = \tau \quad \Gamma(\tau) = \xi.
\]

It must be that agents with belief \( p_i^1 = \xi \) are indifferent between investing in cash and lending, which pins down the return to Borrowing Contracts with the lowest leverage ratio:

\[
r(\xi) = 1 \quad \Rightarrow \quad Q(\tau) = q(\Gamma(\tau)) = q(\xi) = \xi.
\]

Finally, agents with belief \( p_i^1 = \tau \) must be indifferent between lending with the highest leverage ratio Borrowing Contracts, and investing in the asset using the lowest leverage ratio Borrowing Contracts, whose return is one:

\[
r(\tau) = \frac{\tau - \xi}{p - \xi} \quad \Rightarrow \quad Q(1) = q(\Gamma(1)) = q(\tau) = \frac{\tau}{r(\tau)} = \frac{\tau(p - \xi)}{\tau - \xi}.
\]

Finally, there is market clearing for the Real Asset:

\[
p = 1 - F(\xi).
\]

The supply of the Real Asset and wealth of each agent are both normalized to one. The left hand side is the total value of the Asset, and the right hand side is the total wealth that purchases these assets.

2 Equilibrium Properties

In this section, I study the properties of the equilibrium described above. I show that it depends on very few characteristics of the distribution of beliefs represented by density \( f(.) \). Assuming that \( f(.) \) is sufficiently smooth, the equilibrium properties of this model depend on only two parameters: the heterogeneity parameter \( \Delta \) (defining the support \([1 - \Delta, 1]\)) and the optimism
scarcity parameter $\rho$, which represents the first non zero term in the Taylor expansion of $f(.)$ near 1, that is $\rho$ is such that:

$$f ( p^1_i ) \sim_{p^1_i \to 1} (1 - p^1_i)^\rho .$$

One can then form equivalent classes of density functions, depending on their Taylor expansions near $\max_i p^1_i = 1$, and study only the behavior of a representative of these equivalence classes, which I choose to be the following polynomial density functions:

$$f ( p^1_i ) = \begin{cases} \frac{\rho + 1}{\Delta^\rho + 1} (1 - p^1_i)^\rho & \text{if } p^1_i \in [1 - \Delta, 1] \\ 0 & \text{if } p^1_i > 1 \text{ or } p^1_i < 1 - \Delta. \end{cases}$$

For example, when the density function is bounded away from zero near 1, the equilibrium is studied through the lens of the uniform distribution for beliefs, which corresponds to $\rho = 0$ in the formula above. In that case, the optimism scarcity parameter is zero, as there are many agents with the maximum beliefs. The upper tail of the leverage ratio distribution then always is a truncated Pareto distribution, with a tail coefficient equal to two, regardless of the distribution of beliefs in this very large equivalence class.

**Uniform Distribution, $\rho = 0$.** The model can be solved completely in closed form when the distribution is uniform, as a function of $\Delta$. This helps build intuition on the properties of the model, for all density functions which are bounded away from zero.

**Corollary 1.** If the density of beliefs $f(.)$ is uniform on $[1 - \Delta, 1]$, $Q(.)$, $\Gamma(.)$, $p$, $\xi$, $\tau$, are obtained in closed form as a function of $\Delta$:

$$Q(y) = p - p \sqrt{ \left( 1 - \frac{Q(1)}{p} \right)^2 + 2 \frac{Q(1)}{p} \left( 1 - \frac{Q(1)}{p} \right) \frac{1 - y}{1 - \Gamma(1)} },$$

$$\Gamma(y) = y - \frac{Q(y)}{Q'(y)}$$

$$\tau = \frac{1}{2} + \frac{1}{2\sqrt{1 + 2\Delta^2}}$$

$$p = \frac{2(1 + \Delta^2)}{1 + \Delta + 2\Delta^2 + 2\Delta^3 + (1 - \Delta)\sqrt{1 + 2\Delta^2} - \Gamma(1)}$$

$$\xi = 1 - \Delta p.$$

where $Q(1)$ and $\Gamma(1)$ are given as a function of $p$, $\xi$, $\tau$ in Proposition 1.

In particular, the leverage ratio function $p/(p - Q(.))$, which is the focus the paper, can in the same way be expressed explicitly as a function of the cutoffs:

$$\frac{p}{p - Q(y)} = \frac{1}{\sqrt{ \left( 1 - \frac{Q(1)}{p} \right)^2 + 2 \frac{Q(1)}{p} \left( 1 - \frac{Q(1)}{p} \right) \frac{1 - y}{1 - \Gamma(1)} } .$$

One can perhaps already see that it is a truncated Pareto distribution with coefficient 2. I come back to this in Section 2.4. The proof for deriving $Q(.)$ (and therefore $\Gamma(.)$) follows, the derivation of the cutoffs is in Appendix E.

**Proof.** Equation (1a) implies when $f$ is uniform.\(^6\)

$$(p - Q)\Gamma' = Q \Rightarrow \Gamma' = \frac{Q}{p - Q} \Rightarrow (1b) \quad y - \Gamma = \frac{Q}{Q'}$$

\(^6\)I sometimes omit the dependance of functions on $y$ in the following. For example, $Q(y)$ is denoted $Q$. 

19
\[ 1 - \Gamma' = \frac{Q'^2 - Q''Q}{Q'^2} \Rightarrow (1a) \quad 1 - \frac{Q}{p - Q} = 1 - \frac{Q''Q}{Q'^2} \Rightarrow Q \neq 0 \quad Q''(p - Q) - Q'^2 = 0. \]

This is a non-linear second order differential equation in \( Q(.) \), which together with initial conditions \( Q(1) \) and \( Q'(1) \) forms a well-defined initial value problem. Somewhat unexpectedly, this has a closed form solution as:

\[
Q''(p - Q) - Q'^2 = 0 \quad \Rightarrow \quad (Q'(p - Q))' = 0 \quad \Rightarrow \quad \left( -\frac{(p - Q)^2}{2} \right)'' = 0.
\]

This together with \( Q'(1) = \frac{Q(1)}{1 - \Gamma(1)} \) gives \( Q(y) \) in the corollary:

\[
-\frac{(p - Q(y))^2}{2} = -\frac{(p - Q(1))^2}{2} + \frac{Q'(1)(p - Q(1))}{1 - \Gamma(1)}(y - 1)
\]

\[
\Rightarrow \quad Q(y) = p - p\sqrt{\left( 1 - \frac{Q(1)}{p} \right)^2 + 2\frac{Q(1)}{p} \left( 1 - \frac{Q(1)}{p} \right) \frac{1 - y}{1 - \Gamma(1)}}
\]

\[
\square
\]

### 2.1 High Amount of Financial Intermediation

When disagreement goes to zero, very few borrower buy the real asset and effectively intermediate all the economy’s funds. This is shown on Figure 2, for the case of a uniform distribution of beliefs.

[INSERT FIGURE 2 ABOUT HERE]

Note that the price of the real asset is naturally higher than it would be if no Borrowing Contracts would be available (in which case, the price would simply be given by \( 1 - F(p) = p \)). This is the leverage effect described by Geanakoplos (2003): leverage allows optimists to express themselves more, which raises asset prices.

When the density of beliefs is bounded away from zero, the value for the cutoffs naturally changes but the cutoffs Taylor expansion do not. We have that:

\[
p = 1 - O(\Delta^2) \quad \tau = 1 - O(\Delta^2) \quad \tau - p = O(\Delta^3) \quad \xi = 1 - O(\Delta).
\]

This result comes from writing a Taylor expansion of the density function for optimistic beliefs near one. Intuitively, when disagreement tends to 0, the density function can be approximated by a uniform distribution. (the invariance of the upper tail of the leverage ratio distribution also comes from the same reasoning)

Average leverage and maximum leverage are given by:

\[
\text{Max Leverage} = \frac{p}{p - Q(1)} = \frac{p\tau - \xi}{\xi \tau - p} \quad \text{Avg Leverage} = \frac{F(\tau) - F(\xi)}{1 - F(\tau)}.
\]

Therefore, using the previous Taylor expansions one gets that maximum leverage is a \( O(1/\Delta^2) \) and average leverage is a \( O(1/\Delta) \) (again, because the density function is bounded away from zero so one can calculate the order of \( F(\tau) - F(\xi) \) and of \( 1 - F(\tau) \)).
When disagreement $\Delta$ goes to zero, Economy $\mathcal{E}_B$ therefore does not converge to the commonly used benchmark where all agents invest their unit wealth in one unit of asset. Instead, there is complete intermediation: all agents lend to one buyer of measure zero. (one could perhaps think of these buyers as hedge funds, wealth managers, etc.) Note that there is no mathematical contradiction here: the equilibrium where one agent invests for everyone else is one of many equilibria of the setup with common beliefs. One can see the introduction of an epsilon disagreement as a refinement which selects an equilibrium, albeit one we do not usually favor.

2.2 Hedonic Interest Rates

**Corollary 2 (Hedonic interest rates).** The implicit interest rate on Borrowing Contracts with face value $(\phi)$ is strictly higher than the returns to cash for all $\phi > \xi$:

$$r(\phi) = \frac{\phi}{q(\phi)} = \frac{\Gamma(y)}{Q(y)} > R = 1.$$  

This occurs while Borrowing Contracts are fully secured according to lenders buying them, as well as borrowers selling them.

Proof. The maximization problem of a consumer with beliefs $p_i^1 = y$ yields a differential equation over $q(.)$:

$$\max_{\phi} \frac{y - \phi}{p - q(\phi)} \Rightarrow \frac{y - \phi}{p - q(\phi)} = \frac{1}{q'(\phi)}.$$  

Rewriting this in terms of the interest rate on the Borrowing Contract $r(\phi) = \phi/q(\phi)$, one gets:

$$\frac{r(\phi)^2}{r(\phi) - r'(\phi)\phi} = \frac{y - \phi}{pr(\phi) - \phi}r(\phi) \Rightarrow \frac{r'(\phi)\phi}{r(\phi)} = \frac{y - pr(\phi)}{y - \phi}.$$  

Borrower leverage only leverage themselves if the return on the asset is higher than the one they pay on the Borrowing Contract, so that $y/p > r(\phi)$. Moreover $y > \phi$. So the interest rate function is strictly increasing. With the initial condition $r(\xi)$ this proves the corollary. □

Note that because of hedonic interest rates the marginal buyer of the real asset does not have beliefs equal to the price of the Real Asset, even though the reference asset has an interest rate equal to $R = 1$. Because his outside option is to lend and not to invest in Cash, and that lending with a high leverage ratio brings an excess return relative to cash, the marginal buyer has beliefs $p_i^1 = \tau > p$.

Given the importance of fixed income securities in modern financial markets, this new determinant of credit spreads for Borrowing Contracts I uncover is not a theoretical curiosity. Depending on the level of disagreement and on the underlying distribution of beliefs, the credit spreads can be of substantial magnitude.

[INSERT FIGURE 3 ABOUT HERE]

[INSERT FIGURE 4 ABOUT HERE]
This result on hedonic prices can potentially explain several puzzles in finance theory. For example, the credit spread puzzle states that risk premia on bonds are too high to be explained by default probabilities and losses upon default. More directly connected to the paper, as it concerns the pricing of collateralized fixed income securities: Bartolini et al. (2010) show that repurchase agreements’ rates on Mortgage Backed Securities secured by government-sponsored agencies, and private-label MBS are not meaningfully different from rates on unsecured interbank loans. In the same line, Collin-Dufresne et al. (2001) show that variables which should in principle determine credit spread changes have little if any explanatory power. Huang and Huang (2012) is a good survey of this literature. More work is of course needed to connect the model to these empirical facts.

2.3 Demand for Customization

The market features a high degree of customization and fragmentation. Each lender is effectively lending to a particular borrower, using one type of borrowing contract, with a specific leverage ratio (or haircut), and a specific interest rate.

This results directly from Proposition 1. Borrowers with beliefs $p_i^1 = y$ sell contracts $\Gamma(y)$, which are bought by lenders with beliefs $p_i^1 = \Gamma(y)$. $\Gamma(.)$ is strictly increasing. The measure of agents buying contract $\Gamma(y)$ is zero. The market is therefore highly customized and fragmented.

At the competitive equilibrium of this economy, there is consequently assortative matching between borrowers and lenders, as in classic assignment models (Sattinger (1975)), with interest rates playing the role of hedonic prices. The assignment function is shown in the case of the uniform distribution on Figure 5.

An interesting feature of the model is therefore that starting from anonymous markets as in Geanakoplos (1997)’ collateral equilibrium, the market reveals to have a high degree of customization and fragmentation, as in real world repos markets which are organized Over-The-Counter (OTC): in this model, each borrower is effectively borrowing from a different lender with a different interest rate, and a different leverage ratio, both of which are higher when borrowers and lenders are relatively more optimistic. Recent policy proposals suggest to migrate OTC markets to exchanges in which investors would trade standardized contracts. My model suggests that investors may be discouraged from trading such contracts because they demand a high degree of customization.

Moreover, to the extent that the model also applies to loans made from banks to firms, with more or less optimistic banks potentially lending to more or less optimistic firms, the sorting result can explain why matching between banks and firms appears to be non-random, as evidence from Spanish loan data in Jiménez et al. (2014) suggest (pp 468, 493): "we find that controlling for unobserved time-varying firm heterogeneity (time*firm fixed effects) alters the main coefficients of interest, suggesting nonrandom matching between banks and firms."
2.4 Pareto Distributions for Leverage Ratios

The leverage \( l = L(y) \) of borrower with beliefs \( p^*_i = y \) is given by:

\[
l = L(y) = \frac{p}{p - Q(y)}
\]

We already saw that when the belief distribution is uniform, this leverage can be written as:

\[
\frac{p}{p - Q(y)} = \frac{p}{\sqrt{2\xi} \sqrt{\frac{p - \xi}{\tau - \xi} \frac{1}{2\xi}} - y}
\]

If one defines \( \bar{y} \) as the following belief cutoff:

\[
\bar{y} = \frac{(p + \xi)\tau - \xi(p - \xi)}{2\xi}
\]

then \( \bar{y} \) is the belief that the most optimistic borrower would need to have so that the leverage ratio goes to infinity for the most optimistic borrowers \( (y = 1) \). This cutoff therefore measures what the level of truncation is.

The leverage ratio of borrowers in the case of a uniform beliefs distribution is plotted on Figure 7. Figure 7 shows that leverage ratios become very high for the most optimistic borrowers, all the more so that disagreement is low. This comes from the level of truncation \( \bar{y} \) going to 1 as the level of disagreement decreases. On Figure 7 it is also apparent that the leverage ratio is obtained as the derivative of the assignment function on Figure 5, through the formula \( L(y) = 1 + \Gamma'(y) \).

\[\text{[INSERT FIGURE 7 ABOUT HERE]}\]

Denote by \( L^{-1}(l) \) the inverse of this function. The cumulative distribution function for leverage of optimists can be expressed as a function of \( L^{-1}(\cdot) \):

\[
G(l) = P(L(y) \leq l | y \geq \tau) = P(y \leq L^{-1}(l) | y \geq \tau)
\]

\[
G(l) = \frac{(F \circ L^{-1})(l) - F(\tau)}{1 - F(\tau)}
\]

The leverage ratio distribution does not only become more unbounded as disagreement goes to zero, it also becomes closer and closer to a Pareto distribution. On Figure 10, I plot this leverage ratio distribution in the uniform case on a log-log scale, with the log in base 10 of the survivor function on the \( y \)-axis, and the log in base 10 of the leverage ratio on the \( x \)-axis. That is, the log of the probability that the leverage of a borrower is higher than a certain number is plotted against the log of this number. The distribution of leverage ratios is then a truncated Pareto distribution with a coefficient equal to 2.

The distribution is not exactly Pareto. In particular, it is truncated, while the Pareto distribution has an infinite support. Again, this is because the formula in the uniform case is given by a truncated Pareto:

\[
\frac{p}{p - Q(y)} = \frac{p}{\sqrt{2\xi} \sqrt{\frac{p - \xi}{\tau - \xi} \frac{1}{\sqrt{\bar{y} - y}}}}
\]
In practice, it is in fact well known and intuitive that empirical Pareto distributions are always truncated, as Figure 24 for the distribution of leverage ratios of hedge funds shows (see Gabaix (2009)). This also happens in this model, as long as disagreement is strictly positive (but does not in random growth models, for example). Moreover, it should be noted that as disagreement goes to zero, one approaches the benchmark of common prior. Taking disagreement as being low amounts to studying the economy’s behavior not too far from the benchmark.

An important result feature of the model is that the prediction for the Pareto coefficient is obtained irrespective of the precise shape of the function of beliefs, as long as its density function is bounded away from zero near the most optimistic beliefs. For example, I show on 10 the example of an increasing distribution over \([1 - \Delta, 1]\), according to \(f(x) = \frac{2}{\Delta} (x - (1 - \Delta))\) for \(x \in [1 - \Delta, 1]\) (and zero elsewhere).

There is actually a clear intuition for why the distribution of leverage ratios converges to that of an asymptotic Pareto distribution of tail coefficient equal to two, it is worth giving first an intuition for it. The leverage ratio, in the space of lenders’ beliefs, is proportional to a ratio of one over the difference between the collateral price and the price of the Borrowing Contract, which goes to zero when more optimistic agents are lending. The reason for why margins go to zero is that the most optimistic lenders would almost be willing to buy the asset at the going price. When the distribution of beliefs is sufficiently regular, this difference can be approximated by a uniform distribution in the limit, because it is close to a difference between lenders’ beliefs and a constant (up to an interest rate term though, which is negligible in the limit). The reason why the exponent of this Pareto is two, and not one, is that what is measured is the leverage ratio in borrowers’ space, not in lenders’ space. Because of the properties of the matching function \(\Gamma(.)\), many lenders are matched with very few borrowers asking for close to zero margins, and therefore the measure of the corresponding lenders is higher than the measure of borrowers. Going from lenders’ space to borrowers’ space then involves a square transformation. Therefore, the distribution has a shape exponent of exactly two as long as the density function of beliefs is bounded away from zero.

In section 5, I will compare this distribution to the empirical counterpart for Hedge Funds in the TASS database on Figure 24.

2.5 Expected Returns of Borrowers

As shown previously, the expected return of a borrower with beliefs \(y\) is given by:

\[
\frac{y - \Gamma(y)}{p - Q(y)} = \frac{y}{p} + \left(\frac{p}{p - Q(y)} - 1\right) \left(\frac{y - \Gamma(y)}{p Q(y)}\right).
\]

\(\text{Returns from Own Equity} \quad \text{Leverage} - 1 \quad \text{Returns from Leveraging for One Unit}\)
In other words, there are two key component of this expected return. On the one hand, optimistic borrowers tend to have a very high leverage, as shown before. On the other, for markets to clear, borrowers give some part of the surplus they expect to gain on each unit of the asset they buy to the lender they borrow from. It is not clear a priori which of these two forces should dominate. Again, it is useful to investigate the case of a uniform distribution, which is representative of density functions which are bounded away from zero, and then take a look at polynomial function.

The return from leveraging in one unit is shown on Figure 11 for the case of the uniform distribution. Because borrowers compete a lot for the highest leverage ratio loans, they effectively give all their return to lenders, so that the unit return is going to zero for the most optimistic borrowers. As shown on Figure 13, this undoes the leverage effect. The expected returns of borrowers are of the order of magnitude of the underlying disagreement on the real asset: there is no amplification through leverage.

Things are different when $\rho > 0$. The case of $\rho = 1$ is illustrated on Figure 12 and Figure 14. Because the most optimistic borrowers are in relatively scarce supply, then compete less for the most optimistic lenders. While the magnitude of the leverage ratio they achieve is still very high, they now do not need to transfer all their expected surplus from buying assets to lenders through hedonic interest rates. (see Figure 12) Multiplied by leverage, their expected returns are very large and heterogeneous (Figure 14).

### 2.6 Realized Returns

Conditional on a price realization in period $1$ for the risky asset, denoted by $p_1$, the agent making the highest return is intuitively the one who was entertaining that belief at time $0$. Therefore, there is a non-monotone relationship between leverage and ex-post realized returns. Of course, the cash investors’ realized returns are always equal to one.

**Borrowers’ Realized Returns.** In the case of borrowers, we have that the realized return of borrowers is non monotonic as a function of his beliefs, and therefore his leverage, when the price of realization $p_1$ of the risky asset is interior ($p_1 < 1$):

$$\frac{\partial}{\partial y} \left( \frac{p_1 - \Gamma(y)}{p - Q(y)} \right) = \frac{-\Gamma'(y)(p - Q(y)) + Q'(y)(p_1 - \Gamma(y))}{(p - Q(y))^2}.$$
Using borrowers’ optimality condition \( (y - \Gamma)Q' = (p - Q)\Gamma' \), this gives:

\[
\frac{\partial}{\partial y} \left( \frac{p_1 - \Gamma(y)}{p - Q(y)} \right) = \frac{Q'(y)(p_1 - y)}{(p - Q(y))^2}.
\]

Therefore the derivative of borrowers’ has the sign of \( p_1 - y \), which shows a non-monotone relationship between leverage and returns. Borrowers’ realized return increases with leverage if they were too pessimistic relative to the truth but it decreases with leverage if they were too optimistic. The intuition is straightforward and comes once again from the hedonic interest rates. In practical terms, it is not sufficient for an entrepreneur to know whether a technology will yield positive excess returns or not, but by how much.

**Lenders’ Realized Returns.** Assuming that investors are right on average, and that the true distribution of returns for the risky asset is given by the distribution of point expectations that lender entertain, one obtains that the expected return on the Borrowing Contract is given by:

\[
\frac{1}{q(\phi)} \int_{1-\Delta}^{1} \min\{p_1, \phi\} f(p_1) dp_1 = \frac{1}{q(\phi)} \int_{1-\Delta}^{\phi} p_1 f(p_1) dp_1 + \frac{\phi (1 - F(\phi))}{q(\phi)}.
\]

This quantity is shown on Figure 15 in the case of a uniform distribution function. As can be seen, interest rates on Borrowing Contracts are too low to compensate the losses they incur on average. However, again, this is assuming that there is no inherent bias in agents’ expectations.

3 Pyramiding / Tranching Economies

In Borrowing Economy \( E_B \), I exogenously imposed that agents could only use the real asset as collateral. From a theoretical perspective, there is no reason why that should be. A Borrowing Contract is also an asset, albeit a financial asset, available in zero net supply. In principle, they could also be used to back promises.

3.1 Setup

In this section, I work in the same setting as in Economy \( E_B \) (see Section 1.1), except I allow agents to agree to collateralized Borrowing Contracts not only using the Real Asset as collateral, but also using Borrowing Contracts collateralized by the Real Asset as collateral, which I refer to as Borrowing Contracts Squared. I call this Economy the Pyramiding or Tranching Economy \( E_B^2 \). Formally,

**Definition 3** (Borrowing Contract Squared, Economy \( E_B^2 \)). A Borrowing Contract Squared \( (\phi')^{(2)} \) in Economy \( E_B^2 \) is a promise of \( \phi' \geq 0 \) units of Cash in period 1, the face value, collateralized by one unit of Borrowing Contract \( (\phi) \).
Note that restricting ourselves to $\phi' \leq \phi$ is without loss of generality, as Borrowing Contracts Squared with $\phi < \phi'$ have the same payoffs as Borrowing Contracts ($\phi'$), no matter what $p_1$ is. The reason is that the holder of the Borrowing Contract always wants to default for $\phi < p_1 < \phi'$. Note also that the face value $\phi$ of the Borrowing Contract backing the Borrowing Contract Squared ($\phi'$) is irrelevant, as long as $\phi' \leq \phi$. The reason is that the collateral matters only for price realization such that $p_1 \leq \phi'$ and that for those price realizations, the corresponding Borrowing Contract also is in default, such that only the collateral matters. This is why the Borrowing Contract Squared ($\phi'$) can be referred to by its face value only. The price of this Borrowing Contract Squared will be denoted by $q_2(\phi')$. On the other hand, regular Borrowing Contracts are now traded at competitive price $q_1(\phi)$ when of face value $\phi$ (to distinguish their price from that of Borrowing Contracts Squared).

**Equilibrium.** Again, all units of the real asset are initially endowed to unmodeled agents who sell their asset holdings and then consume. Agent $i$ chooses his position in the Real Asset $n_A^i$, a menu of financial Borrowing Contracts ($\phi$) denoted by $dN_B^i(\cdot), a menu of Borrowing Contracts Squared denoted by $dN_B^2(\cdot)$, and Cash $n_C^i$, in order to maximize his expected wealth in period 1 according to his subjective beliefs $p_1^i$ about the Real Asset ($W$), subject to his budget constraint (BC), collateral constraint (CC) and collateral constraint for Borrowing Contracts Squared (CC2):

$$\max_{(n_A^i,dN_B^i(\cdot),dN_B^2(\cdot),n_C^i)} n_A^i p_1^i + \int_\phi \min\{\phi, p_1^i\} dN_B^i(\phi) + n_C^i + \int_{\phi'} \min\{\phi', p_1^i\} dN_B^2(\phi')$$  \hspace{1cm} (W)

subject to

$$n_A^i p + \int_\phi q_1(\phi) dN_B^i(\phi) + n_C^i + \int_{\phi'} q_2(\phi') dN_B^2(\phi') \leq 1$$ \hspace{1cm} (BC)

$$\int_\phi \max\{-dN_B^i(\phi),0\} d\phi \leq n_A^i$$ \hspace{1cm} (CC)

$$\int_{\phi'} \max\{dN_B^2(\phi'),0\} \leq \int_\phi dN_B^i(\phi)$$ \hspace{1cm} (CC2)

$$n_A^i \geq 0, \quad n_C^i \geq 0$$

Note that each time a borrower/lender sells of unit of Borrowing Contract Squared, he needs to own unit of Borrowing Contract hence equation (CC2). A **Competitive Equilibrium** for Economy $E_B^2$ is then a price $p$ for the Real Asset, a distribution of prices $q_1(\cdot)$ for all traded Borrowing Contracts ($\phi$), of prices $q_2(\cdot)$ for all traded Borrowing Contracts Squared ($\phi'$) and portfolios $(n_A^i, dN_B^i(\cdot), dN_B^2(\cdot), n_C^i)$ for all agents $i$ in the Real Asset, Borrowing Contracts, Borrowing Contracts Squared and Cash, such that all agents $i$ maximize expected wealth ($W$) according to their subjective beliefs, subject to budget constraint (BC), collateral constraints (CC) and (CC2), and markets for the Real Asset Borrowing Contracts and Borrowing Contracts Squared clear:

$$\int_i n_A^i di = S,$$ \hspace{1cm} (MC_A)
\begin{align*}
\forall \phi, \quad & \int dN^i_B(\phi) di = 0. \quad (MC_B) \\
\text{and} \quad & \forall \phi', \quad \int dN^i_{B2}(\phi') di = 0. \quad (MC_{B2})
\end{align*}

3.2 Equilibrium

I now characterize the equilibrium of the pyramiding economy as three scalars dividing the set of agents into cash investors, lenders of type-2 (lending to lenders), lenders, and borrowers, one scalar for the price of the real asset, a function for the price all traded Borrowing Contracts, another for the price of all traded Borrowing Contracts Squared, a function determining the optimal choice of borrowers’ contracts, and another determining the optimal choice of lenders’. These equilibrium objects are linked by four first-order ordinary differential equations and eight algebraic equations, given in the next proposition.

**Proposition 2** (Equilibrium of Economy \( \mathcal{E}_{B^2} \)). A competitive equilibrium of Economy \( \mathcal{E}_{B^2} \) is described by a price \( p \) for the Asset, three thresholds \( \nu \leq \xi \leq \tau \), two strictly increasing matching functions \( \Gamma_1(.) \) mapping \( [\tau, 1] \) onto \([\xi, \tau]\) and \( \Gamma_2(.) \) mapping \([\xi, \tau]\) onto \([\nu, \xi]\), and two loan pricing functions \( Q_1(.) \) and \( Q_2(.) \), such that:

1. The space of agents’ beliefs \( [1 - \Delta, 1] \) is partitioned into four intervals:
   - Agents with beliefs \( p_i^1 \in [1 - \Delta, \nu] \) (cash investors) invest in Cash.
   - Agents \( i \) with beliefs \( p_i^1 \in [\nu, \xi] \) (lenders of Type-2) buy Borrowing Contracts Squared.
   - Agents \( i \) with \( p_i^1 \in [\xi, \tau] \) (lenders) buy Borrowing Contracts and sell Borrowing Contracts Squared.
   - Agents \( i \) with \( p_i^1 \in [\tau, 1] \) (borrowers) buy the Asset and sell Borrowing Contracts.

2. Lenders with beliefs \( x \) buy Borrowing Contracts with face value \( x \). Borrowers with beliefs \( y \) sell Borrowing Contracts with face value \( \Gamma_1(y) \), at price \( Q_1(y) \). Lenders of type 2 with beliefs \( z \) buy Borrowing Contracts Squared with face value \( z \). Lenders with beliefs \( \Gamma_1(y) \) sell Borrowing Contracts Squared with face value \( \Gamma_2(y) \), at price \( Q_2(y) \).

3. \( p, \nu, \xi, \tau, \Gamma_1(.) \), \( \Gamma_2(.) \), \( Q_1(.) \), and \( Q_2(.) \) are such that:

\begin{align*}
(3a) \quad & (p - Q_1)f (\Gamma_2) \Gamma'_2 = Q_2f, \\
(3c) \quad & (y - \Gamma_1)Q'_1 = (p - Q_1)\Gamma'_1, \\
(4a) \quad & 1 - F(\nu) = p, \\
(4b) \quad & \Gamma_1(\tau) = \xi, \\
(4c) \quad & \Gamma_1(1) = \tau, \\
(4d) \quad & \Gamma_2(\tau) = \nu \\
(4e) \quad & \Gamma_2(1) = \xi \\
(4f) \quad & Q_2(\tau) = \nu, \\
(4g) \quad & \frac{\xi}{Q_2(1)} = \frac{\xi - \nu}{Q_1(\tau) - \nu} \\
& \frac{\tau - \xi}{Q_1(1) - Q_2(1)} = \frac{\tau - \xi}{p - Q_1(\tau)}. 
\end{align*}
Figure 1: Pyramiding Arrangement, Tranching of Collateral

\[
\begin{align*}
\text{Cash Investors} & \quad \text{Lenders of Type 2} \quad \text{Lenders of Type 1} \quad \text{Borrowers} \\
1-\Delta & \quad \nu & \quad \xi & \quad p & \quad \tau & \quad 1 \\
\end{align*}
\]

Note: The borrowers’ balance sheet is shown on the right hand side: he finances the purchase of the asset using this asset as collateral with a lender of type 1, financing \(q_1(\phi)\) towards the purchase. This lender himself gets financing from lenders of type 2, who contribute \(q_2(\phi')\) towards the purchase of the collateral. For example, an investor buying a Mortgage-Backed Security lends to a homeowner; this investor finances part of his position through using the Mortgage-Backed Security in repo. Another example is a hedge fund who posts collateral with his broker dealer. The broker dealer rehypothecates the collateral to finance part of the position, with a money market fund. A final example is the tranching of loans. Lender of type-2 would have a senior tranche, and be repaid in full whenever the value of the collateral exceeds \(\phi'\). Lender of type 1 would have a junior tranche, and will start being repaid only if the value of the collateral exceeds \(\phi''\), and be repaid in full if it exceeds \(\phi\).

The fact that \(\Gamma_2(.)\) does not match the beliefs of a lender to a that of a lender of type-2 might be a bit confusing at first, but it actually allows more simple expressions. The reason is that borrowers are of particular interest here, and so it is simpler to calculate all functions taking their beliefs as a reference.

A full proof of this proposition would exactly mirror that of Proposition 1. I shall only focus on the minor difficulties brought by the presence of pyramiding arrangements. Obviously, the main difference is that Borrowing Contracts can now be used by Lenders to finance themselves from even less optimistic agents. The intuition for why they want to do that in equilibrium is that
borrowers give them part of their returns because of the allocative argument, and that lenders therefore want to purchase as many of these contracts as possible, possibly using Borrowing Contracts themselves as collateral. In equilibrium, this is what they do. This is illustrated on Figure 1.

As for the Borrowing Economy, a first equation results from the market clearing for Borrowing Contracts Squared with face value $\Gamma_2(y) = z$. These contracts are bought by lenders of type 2 in a small interval $[z, z + dz]$, with wealth $f(z)dz$, and sold by lenders of type 1 in a small interval $[x, x + dx]$. Lenders of type 2 therefore contribute $Q_2(y)$ to the total loan amount that lenders of type 1 make to borrowers, and lenders of type 1 contribute $Q_1(y) - Q_2(y)$ from their own funds. (see the balance sheet on Figure 1)

\[
\frac{f(z)dz}{Q_2(y)} = \frac{f(x)dx}{Q_1(y) - Q_2(y)}
\]

\# of Borrowing Contracts Squared bought by lenders of Type 2 in $[z, z + dz]$
\# of Borrowing Contracts Squared sold by lenders of type 1 in $[x, x + dx]$

\[\Rightarrow \forall y \in [\tau, 1], \quad (Q_1(y) - Q_2(y)) f(\Gamma_2(y)) \Gamma_2'(y) = Q_2(y) f(\Gamma_1(y)) \Gamma_1'(y).\]

Moreover, one can write the market clearing for Borrowing Contracts with face value $\Gamma_1(y) = x$. These contracts are bought by lenders of type 1 (using their own funds and lenders of type 2’s funds, as seen above) in a small interval $[x, x + dx]$, so that:

\[
\frac{f(x)dx}{Q_1(y) - Q_2(y)} = \frac{f(y)dy}{p - Q_1(y)}
\]

\# of Borrowing Contracts bought by lenders of Type 1 in $[x, x + dx]$
\# of Borrowing Contracts sold by borrowers in $[y, y + dy]$

\[\Rightarrow \forall y \in [\tau, 1], \quad (p - Q_1(y)) f(\Gamma_1(y)) \Gamma_1'(y) = (Q_1(y) - Q_2(y)) f(y).\]

Multiplying the two sides of the above equation, and keeping the second equation, allows to conclude that the allocation functions are given by the following differential equations:

\[(p - Q_1(y)) f(\Gamma_2(y)) \Gamma_2'(y) = Q_2(y) f(y)\]
\[(p - Q_1(y)) f(\Gamma_1(y)) \Gamma_1'(y) = (Q_1(y) - Q_2(y)) f(y)\]

The choice of Borrowers with beliefs $y$, who choose the face value of Borrowing Contract they choose, gives a first differential equation:

\[\max_{\phi} \frac{y - \phi}{p - q_1(\phi)} \Rightarrow - (p - q_1(\phi)) + q_1(\phi)(y - \phi) = 0\]

\[\Rightarrow (p - Q_1(y)) \Gamma_1'(y) = (y - \Gamma_1(y)) Q_1'(y).\]

The choice of Lenders of Type 1, with beliefs $x_1$, and who choose Borrowing Contracts Squared to leverage themselves into the spreads given by traditional Borrowing Contracts, gives a second differential equation:

\[\max_{\phi} \frac{x_1 - \phi}{q_1(x_1) - q_2(\phi)} \Rightarrow - (q_1(x_1) - q_2(\phi)) + q_2(\phi)(x_1 - \phi) = 0\]
\[ (Q_1(y) - Q_2(y)) \Gamma'_2(y) = (\Gamma_1(y) - \Gamma_2(y)) Q'_2(y). \]

Just as previously, the market clearing equation for the real asset writes:

\[ 1 - F(\nu) = p. \]

This is because now lenders of type 2 funds are also invested in the asset. Positive sorting at the boundaries brings:

\[ \Gamma_1(\tau) = \xi, \quad \Gamma_1(1) = \tau, \quad \Gamma_2(\tau) = \nu, \quad \Gamma_2(1) = \xi. \]

Finally, indifference for agents with beliefs \( \nu, \xi \), and \( \tau \) respectively imply:

\[
\begin{align*}
1 = \frac{\nu}{q_2(\nu)} & \Rightarrow Q_2(\tau) = \nu \\
\frac{\xi}{q_2(\xi)} = \frac{\xi - \nu}{q_1(\xi) - q_2(\nu)} & \Rightarrow \frac{\xi}{Q_2(1)} = \frac{\xi - \nu}{Q_1(1) - Q_2(\tau)} \Rightarrow \frac{\xi}{Q_2(1)} = \frac{\xi - \nu}{Q_1(\tau) - \nu} \\
\frac{\tau - \xi}{q_1(\tau) - q_2(\xi)} = \frac{\tau - \xi}{p - q_1(\xi)} & \Rightarrow \frac{\tau - \xi}{Q_1(1) - Q_2(1)} = \frac{\tau - \xi}{p - Q_1(\tau)}.
\end{align*}
\]

### 3.3 Discussion

**Equivalence to tranching.** Note that the above decomposition of claims between Borrowing Contracts and Borrowing Contracts Squared has an equivalent representation in terms of a junior and a senior tranche from a securitized mortgage. On Figure 1, Lender of Type-2 gets repaid whenever \( p_1 \geq \phi' \), and lender of Type-1 is junior as he gets repaid in full only if \( p_1 \geq \phi \).

**Importance of pyramiding.** Borrowing Contracts Squared can have several real world interpretations. For example, prime brokers to hedge funds are often allowed to use their clients’ collateral for financing from money market funds (money market funds usually are not allowed to lend to hedge funds directly by regulation). Another round of lending occurs when Mortgage-Backed Securities are financed through repurchase agreements. Greenwood and Scharfstein (2013) calculate that fixed income securities grew from totaling 57% of GDP in 1980 to 182% of GDP in 2007, and 58% of the growth of fixed income securities came from securitization.

**Limits to pyramiding.** The assumption made in Section 1 is still a good description for some markets. One reason is regulation. In the United States for example, Regulation T and SEC’s Rule 15c3 limit Prime Brokers’ use of rehypothecated collateral from a client. Similarly, securitization in the housing market is a fairly recent phenomenon, the extent of which was broadened in the nineties in the United States. However, it is still not possible in many countries around the world for regulatory reasons. Apart from institutional and regulatory obstacles, securitization entails some cost, which as we will see later, have to be weighted against its expected benefits (again, see Greenwood and Scharfstein (2013)).
More layers of pyramiding? For simplicity, I have investigated only the case where Borrowing Contracts can be used as collateral to get more funds. Naturally, there is no reason why there should only be two layers of lending. The model straightforwardly generalizes to a case with an arbitrary number of pyramids of lending. The number of these layers can be pinned down by adding a small cost of adding a layer of pyramiding, as the gains from adding another layer are decreasing in the number of layers. A real world example would be the tranching of a mortgage into six different levels of riskiness (AAA, AA,...). Formally, this would correspond to Economy $E_B^6$.

Limitations of the analysis. A key assumption of the model is that investors are investing in only one security (the only other options are safe cash, and financial assets, whose value is contingent on that of this asset). The consequence is that the borrower never worries that the lender will be able to give the collateral back, even in the event of rehypothecation. Of course, this is too simplistic if one wants to think about banks. In particular, some hedge funds with Lehman Brothers as their prime broker could not recoup their collateral during the 2008 financial crisis, even though this collateral was not the cause of Lehman’s failure.

3.4 Results

In this section, I present the results obtained from the equilibrium of the Pyramiding Economy (Proposition 2), and compare them to that of equilibrium of the Borrowing Economy (Proposition 1).

Importantly, I show that pyramiding lending arrangements are not redundant to the use of leverage by borrowers. The reason is that it allows lenders to also obtain more funds from less optimistic lenders. Just as leverage increases asset prices and allows optimists to express their opinions more, pyramiding, that is lending using Borrowing Contracts as collateral (or tranching), also increase the price of the real asset. Leverage ratios of borrowers increase dramatically in the Pyramiding Economy, as can be seen in the case where $\rho = 1$ and $\Delta = 10\%$ on Figure 16. Leverage ratios of borrowers are defined as follows in the Borrowing and the Pyramiding Economy respectively.

**Borrowing:** $\frac{p}{p - Q(y)}$

**Pyramiding:** $\frac{p}{p - Q_1(y)}$

[INSERT FIGURE 16 ABOUT HERE]

Expected returns of borrowers also increase a lot, as can be seen on Figure 18. Expected returns by borrowers in the Borrowing and Pyramiding economies are defined as follows.

**Borrowing:** $\frac{y - \Gamma(y)}{p - Q(y)}$

**Pyramiding:** $\frac{y - \Gamma_1(y)}{p - Q_1(y)}$

[INSERT FIGURE 18 ABOUT HERE]
Finally, expected returns of lenders obviously rise in the pyramiding economy, as lenders are able to leverage themselves in hedonic interest rates. Figure 19 shows the results: lenders can achieve returns of 12% in the pyramiding economy, while they get at most 1.5% in the Borrowing Economy. Fixed income thus can generate revenue. Note also on this graph that direct lenders to borrowers are also relatively more optimistic in the pyramiding economy, which helps explain why the leverage of ultimate borrowers is so high. Expected returns in the Borrowing and Pyramiding economies are defined as follows.

\[
\text{Borrowing: } \frac{\Gamma(y)}{Q(y)} \quad \text{Pyramiding: } \frac{\Gamma_1(y) - \Gamma_2(y)}{Q_1(y) - Q_2(y)}
\]

[INSERT FIGURE 19 ABOUT HERE]

Importantly, it is not that pyramiding completes market in the sense that new contingent assets are introduced. The effect of pyramiding is to allow the same collateral to back two different promises at the same time, and in a credible way. Lenders of type-2 are in equilibrium using Borrowing Contracts Squared, which are equivalent (in terms of payoffs) to regular Borrowing Contracts with the same face value. However, they were not using them in equilibrium of Economy \( E_B \) for lack of available collateral. To the best of my knowledge, it is the first time that what Geanakoplos (1997) calls pyramiding lending arrangements or double leverage cycle are modeled.

Moreover, rehypothecation has sometimes been used as a proof for search frictions. Why else would promises pass on from lenders to lenders, with the same asset used as collateral repeatedly. This paper gives another candidate explanation, one that is consistent with competitive behavior.

4 Extension: Borrowing Economy with Short-Sales

The above analysis was carried out with a maintained assumption of no short-sales constraints, in the tradition of Miller (1977). This is a good approximation for some markets, for example if it concerns housing or a new entrepreneurial investment. However in many markets today (large stocks are an example), going short is almost as easy as going long. The framework I have developed above extends very naturally to the case where such short-selling is possible. I study such an Economy \( E_S \), where both borrowing and short-selling are limited by the endogenous availability of collateral.

Simsek (2013) was, to the best of my knowledge, the first to introduce endogenous short-sales constraints in a disagreement model. However the scope of his analysis is limited by his assumption that there are only two agents in the economy, which leads him to rule out borrowing in the main section of the paper when he considers short-selling, and to rule out borrowing (and short-selling) at least for some agents in the Appendix.

In contrast, the very tractable setup of this paper allows a very natural treatment of short-selling, lending, securities lending, and borrowing at the same time. A number of new results come out of this analysis. Endogenous rebate rates arise, which do not represent short-selling
constraints, but instead equilibrium prices coming from the assignment mechanism. All assets are potentially available for loans, but only a fraction of them are in equilibrium: short interest is endogenous. Quantitatively, endogenous rebate rates are of the order of magnitude reported in the finance literature, that is a few tens of basis points (D’Avolio (2002)). In the same way, short interest, is of similar magnitude as the level of disagreement, that is a few percentage points, as in the data.

Finally, short-selling reinforces most of the conclusions from the preceding analysis. Interest rates on bonds are higher than the returns to cash, not only because of the assignment mechanism, but also to incentivize natural short-sellers of the asset to lend rather than short. Power law distributions for the leverage ratios of borrowers arise in Economy $\mathcal{E}_S$ in the same way as in Economy $\mathcal{E}_B$. Again, this contrasts with existing common wisdom, according to which conclusions from disagreement models are for the most part invalidated when short-selling is introduced.

4.1 Setup and Equilibrium

Economy $\mathcal{E}_S$ has Borrowing Contracts like Economy $\mathcal{E}_B$ (see Definition 1). Agents can also agree to collateralized Short-Sales Contracts with each other. A Short-Sales Contract is formally defined as follows.

**Definition 4** (Short-Sales Contract, Economy $\mathcal{E}_S$). A Short-Sales Contract $(\gamma)^s$ in economy $\mathcal{E}_S$ is a promise of 1 unit of asset in period 1, collateralized by $\gamma$ units of Cash, the cash-collateral.

Note that the normalization is this time done with respect to the loan amount, normalized to one unit of asset, not the collateral amount. This is because the loan is in terms of units of assets, and beliefs are expressed in these terms. Note that again, the normalization is without loss of generality. A Short-Sales Contract replicates the payoff of a Real Asset, aside from default risk. Short-Sales Contracts are traded in an anonymous market at competitive price $q_s(\gamma)$, and payment is again only enforced by collateral, this time in the form of cash. The payoff of contract $(\gamma)$ is given by:

$$\min\{\gamma, p_1\},$$

for a contract with cash-collateral amount $\gamma$.

**Endogenous rebate rates.** Note that Short-Sales Contracts replicate the payoff of a real asset, if the collateral amount is sufficient. (in equilibrium, it will be) The price at which they are sold by short-sellers is $q_s(\gamma)$, which in equilibrium will be lower than $p$: this represents represents the cost of shorting. This cost of shorting is in market practice expressed in terms of a rebate rate on the cash collateral kept with the lender of the security, whose return is lower than the return to cash. Because the return to cash is normalized to one, the rebate rate is here given by:

$$r_s(\gamma) = 1 - \frac{p - q_s(\gamma)}{\gamma}.$$
Equilibrium. In Economy $E_S$, agents choose their positions in Assets $n_A^i$, Borrowing and Short-Sales Contracts $n_B^i(\cdot)$, $n_S^i(\cdot)$, and Cash $n_C^i$, so as to maximize their expected wealth in period 1 according to their subjective beliefs $p^i_1$ about the Real Asset (W), subject to their budget constraint (BC), their collateral constraint (CC), and their cash-collateral constraint for Short-Sales Contracts (CCC):

$$\max_{(n_A^i, n_B^i(\cdot), n_C^i, n_S^i(\cdot))} n_A^i p^i_1 + \int \phi n_B^i(\phi) d\phi + n_C^i + \int \gamma n_S^i(\gamma) d\gamma \quad \text{(W)}$$

subject to:

$$n_A^i p^i_1 + \int \phi n_B^i(\phi) d\phi + n_C^i + \int \gamma n_S^i(\gamma) q_s(\gamma) d\gamma \leq 1 \quad \text{(BC)}$$

$$\int \max\{-n_B^i(\phi), 0\} d\phi \leq n_A^i \quad \text{(CC)}$$

$$\int \gamma \max\{-\gamma n_S^i(\gamma), 0\} d\gamma \leq n_C^i \quad \text{(CCC)}$$

and

$$n_A^i \geq 0, \quad n_C^i \geq 0 \quad \text{(MC)}$$

As in Economy $E_B$, a Competitive Equilibrium for Economy $E_S$ is a price $p$ for the Real Asset and a distribution of prices $q(\cdot)$ for all traded Borrowing Contracts (B) and a distribution of prices $q_s(\cdot)$ for all traded Short-Sales Contracts (S), and portfolios $(n_A^i, n_B^i(\cdot), n_C^i, n_S^i(\cdot))$ for all agents $i$ in the Real Asset, Borrowing Contracts, Cash, and Short-Sales Contracts such that all agents $i$ maximize expected wealth according to their subjective beliefs (W), subject to budget constraint (BC), collateral constraint (CC), cash-collateral constraint (CCC) and markets for the Real Asset, Borrowing Contracts, and Short-Sales Contracts clear:

$$\int n_A^i di = S, \quad \int n_B^i(\phi) di = 0. \quad \int n_S^i(\gamma) di = 0.$$

The next proposition shows that Economy $E_S$ can be characterized in a very similar way as the two preceding economies.

Proposition 3 (Equilibrium of Economy $E_S$). A competitive equilibrium of Economy $E_S$ is described by a price $p$ for the Asset, three thresholds $\xi \leq \tau \leq \sigma$, two strictly increasing matching functions $\Gamma(\cdot)$ mapping $[\sigma, 1]$ onto $[\xi, \tau]$, and $\Gamma_s(\cdot)$ mapping $[1 - \Delta, \xi]$ onto $[\tau, \sigma]$, and two loan and short-sales pricing functions $Q(\cdot)$ and $Q_s(\cdot)$ such that:

1. The space of agents’ beliefs $[1 - \Delta, 1]$ is partitioned through $\xi$, $\tau$, and $\sigma$ into four intervals:
   - Agents with beliefs $p^i_1 \in [1 - \Delta, \xi]$ (short-sellers) sell Short-Sales Contracts and invest in Cash.
   - Agents with beliefs $p^i_1 \in [\xi, \tau]$ (lenders) buy Borrowing Contracts.
   - Agents with beliefs $p^i_1 \in [\tau, \sigma]$ (securities lenders) buy Short-Sales Contracts.
• Agents with beliefs $p_i^1 \in [\sigma, 1]$ (borrowers) buy the Asset and sell Borrowing Contracts.

2. Lenders with beliefs $x$ buy Borrowing Contracts with face value $x$. Borrowers with beliefs $y$ sell Borrowing Contracts with face value $\Gamma(y)$, at price $Q(y)$. Securities lenders with beliefs $z$ buy Short-Sales Contracts with cash collateral $z$. Short-sellers with beliefs $y$ sell Short-Sales Contracts with cash collateral $\Gamma_s(y)$, at price $Q_s(y)$.

3. $p, \xi, \tau, \sigma, \Gamma(\cdot), \Gamma_s(\cdot), Q(\cdot)$ and $Q_s(\cdot)$ are such that:

\begin{align*}
(p - Q) f(\Gamma) \Gamma' = Qf, \\
(\Gamma_s - Q_s) f(\Gamma_s) \Gamma_s' = Qsf \\
(y - \Gamma) Q' = (p - Q) \Gamma', \\
(\Gamma_s - y) Q_s' = (Q_s - y) \Gamma_s'.
\end{align*}

\begin{align*}
1 - F(\sigma) + F(\tau) - F(\xi) &= p, \\
\Gamma(\sigma) &= \xi, \\
\Gamma(1) &= \tau, \\
\Gamma_s(1 - \Delta) &= \tau \\
\Gamma_s(\xi) &= \sigma \\
\frac{\xi}{Q(\sigma)} &= \frac{\sigma - \xi}{\sigma - Q_s(\xi)}, \\
\frac{\tau}{Q_s(1 - \Delta)} &= \frac{\tau}{Q_s(1)} \\
\frac{\sigma}{Q_s(\xi)} &= \frac{\sigma - \xi}{p - Q(\sigma)}.
\end{align*}

Proof. See Appendix F.

Although the proof of this proposition is in Appendix F, it is useful to look at the balance sheets of short-sellers to understand the intuition for these Short-Sales Contracts. Below these balance sheets are represented in period 0 and in period 1.

<table>
<thead>
<tr>
<th>L</th>
<th>A</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\gamma - q_s(\gamma)$</td>
<td>$\gamma$</td>
</tr>
<tr>
<td>$q_s(\gamma)$</td>
<td>$p_1$</td>
</tr>
</tbody>
</table>

Return of Asset Lender = $\frac{p_1^1}{q_s(p_1^1)} = \frac{p_1^1}{p} \frac{p}{q_s(p_1^1)}$.

Return of Short-Seller = $\frac{\gamma - p_1^1}{\gamma - q_s(\gamma)}$.

Short-sellers sell Short-Sales Contracts $(\gamma)^s$ at price $q_s(\gamma)$, by which they promise to give back one unit of the real asset in period 1 to the buyer of the Short-Sales Contracts (the securities

\[ p_1 \]
lender). Because the short-seller must provide $\gamma$ in total in terms of collateral, and can only raise $q_s(\gamma)$ through the sale of the security (less than $p$, because of the anticipation of a cost of short-selling, the rebate rate), he must finance $\gamma - q_s(\gamma)$ from his own funds. On the other hand, the return of the securities lender is composed of the normal return from the ownership of the real asset, plus an amount corresponding to the lower rebate rate on the collateral as the benchmark rate, which is a return $p/q_s(p_1^i)$.

### 4.2 Results

Proposition 3 shows several new insights about short-selling in disagreement economies: there can be coexistence of short-selling and lending, interest rates even on the safest (lowest haircuts) bonds are strictly higher than the benchmark rate, and short interest and rebate rates are determined in equilibrium, even though no agent is constrained from short-selling, and short-selling entails no physical cost. Let us examine each in turn.

**Coexistence of Short-selling and Lending.** A key insight from Proposition 3 is that lending and short-selling can very well coexist in equilibrium. A key shortcoming of existing endogenous leverage models, from Geanakoplos (1997) to Simsek (2013) is indeed that in these models, lenders would a priori rather short-sell than lend if they were given this possibility. The equilibrium of Economy this conclusion is in fact incorrect.

Figure 20 shows for the case of a uniform distribution of beliefs how the population of investors split into borrowers, securities lenders, cash lenders, and short-sellers, as a function of disagreement $\Delta$. One notes that as disagreement goes to zero, the proportion of investors selling the asset short also goes to zero. Limited short-selling on financial markets is thus no evidence that short-selling is limited by constraints, as previously thought.

**Corollary 3** (Coexistence of Short-Selling and Lending). Lending arises in disagreement economies with unrestricted short-sales. The conclusions from economies with impossible short-sales remain: hedonic interest rates, customization and fragmentation, Pareto distributions for leverage ratios, etc. The only exception is that the price represents now the opinion of pessimists when disagreement goes to zero, rather than that of optimists.

The equilibrium works as follows. Optimistic investors have a high expected return from borrowing, and are therefore willing to give lenders high interest rates on bonds, which compensate them for not short-selling. On the other hand, lenders to short-sellers, who are relatively optimistic, must be willing to give up a levered bet on the risky asset, which is the source of rebate rates in equilibrium.

**Equilibrium short interest.** The fact that only a limited number of shares are sold short at any time has often been cited as evidence of short-sales constraints, or of the fact that there was
no disagreement on financial markets. In this model, short-interest is of very small magnitude, even though no agent is constrained from selling-short, as long as he satisfies the collateral constraints. The short-interest is the percentage of Real Assets which is in period 0 on loan. It is therefore given by the following expression:

\[ \text{Short Interest} = \frac{F(\sigma) - F(\tau)}{p}. \]

Figure 21 shows that the short interest is of comparable magnitude as the amount of disagreement, when the belief density function is uniform. Short interest can therefore be used to recover the amount of disagreement in an economy.

**Interest rates on the safest securities.** The conclusions from Section 2.2 are in fact reinforced in the presence of short-sales. The reason is that lenders now earn larger than benchmark returns, not just because of the assignment process, but because lenders are natural short-sellers of the asset, and need to be compensated for not short-selling in equilibrium.

Figure 22 displays the equilibrium interest rates on bonds in Economy \( \mathcal{E}_S \). They comprise two components: the hedonic component, as in Economy \( \mathcal{E}_B \), together with a component which encourages lenders to lend rather than to short.

For the uniform density function, the second component is much greater than the first. This could potentially explain the findings in Gorton and Metrick (2012), showing that interest rates on several collateralized loans started to increase in 2007, just when shorting the US housing market was made possible.

**Equilibrium rebate rates.** D’Avolio (2002) studies the market for borrowing securities in detail and finds evidence for non zero rebate rates. This study has however been interpreted as evidence that short-sales constraints were somewhat present, but not commensurate with the ex-post potential gains from selling short in periods of irrational exuberance. I show here that non-zero rebate rates are perfectly compatible with a model of unconstrained short-selling. Figure 23 shows that the order of magnitude of rebate rates are have an order of magnitude of some tens of basis points, again in line with available empirical evidence.

**5 Empirical Applications**

I now illustrate three potential applications of the model. The first one concerns homeowners’ initial leverage ratios, as measured in the microdata from DataQuick. A second takes a look at entrepreneurs’ leverage, using the Survey of Consumer Finances, and the skewed returns
to entrepreneurship. Finally, I apply the model to hedge funds’ leverage, noting that a key prediction of the model is verified in the TASS Hedge Fund Database.

5.1 Homeowners’ Leverage (DataQuick microdata)

The first application concerns homeowners’ leverage. Dataquick microdata lends support to a main prediction of the model, mainly the Pareto coefficient diminishes as asset prices reflect more and more the optimism of a marginal buyer for a given level of disagreement. It shows a stunning decline in the Pareto coefficient from 2000 leading to 2006, and an increase thereafter. It suggests that monitoring ultimate borrowers’ leverage ratio distribution could provide some insights on monitoring the buildup of risk.

Data construction. DataQuick collects and digitized public records from county register of deeds and assessor offices and provides a detailed transaction history of each property sold in the United States from 1988 to 2013, which includes transaction prices and first, second, third mortgage loan amount. From this dataset, I sum the amounts for the first, second, and third mortgages and construct a variable I call Loan Amount in the formula below. If Price denotes the price of the transaction, then the leverage ratio as defined in the model \( p/(p - Q(y)) \) or \( p/(p - Q_1(y)) \) in the model with pyramiding lending arrangements), is given by:

\[
\text{Leverage Ratio} = \frac{\text{Price}}{\text{Price} - \text{Loan Amount}}.
\]

Results. In Tables 1 and 2, I plot the leverage ratio distribution of homeowners on a log survivor-log leverage ratio scale, at different points in time, using a month as a unit of observation. A first observation is that the resulting distribution of leverage ratios takes the form of an approximate power law in the upper tail. A second is that an important feature of the US mortgage system is that it is heavily regulated, and that strong incentives push homeowners to bunch around a 80% loan-to-value threshold (or a leverage ratio of 5).

[INSERT TABLE 1 ABOUT HERE]

[INSERT TABLE 2 ABOUT HERE]

The most interesting observation concerns the evolution of this leverage ratio distribution, and in particular the decrease in the Pareto coefficient in the years leading up the the financial crisis, from 2000-2006, and the subsequent rise in this Pareto coefficient, when mortgage-backed securities issuance came to a halt. According to the model in Section 3, this can happen when pyramiding lending arrangements, or tranching, enable optimists to express themselves more, and to borrow from more and more pessimistic investors through complex chains of lending. As Section 3 has shown, the price represents then more and more the opinion of a very optimistic investor. Table 1 and Table 2 thus suggest that monitoring the leverage ratio distribution could have alerted policymakers that prices were diverging a lot from fundamentals in the runup to the financial crisis.
5.2 Entrepreneurs’ Leverage and Returns (Survey of Consumer Finances)

A second application of the model concerns entrepreneurial leverage and returns. By entrepreneurs I mean owners of enterprises who are not simply managers, but who also have risked at least some of their wealth in the business they have invested in.

**Model insights.** The seminal paper of Moskowitz and Vissing-Jorgensen (2002) emphasizes that the returns to entrepreneurship are low on average ("private equity premium puzzle"), and discusses several potential explanations for this fact, such as large nonpecuniary benefits of working in one own’s company. Another potential candidate explanation from this model would be that entrepreneurs are optimistic (for example about a specific new technology) because this is why they became entrepreneurs in the first place. As emphasized in section 1, if investors are right on average, then entrepreneurs must be too optimistic on average. Note that the no short-sales assumption is a natural one in the case of entrepreneurship. There usually are no markets to bet on the failure of a specific entrepreneur. Allowing banks to securitize their loans to small and medium enterprises, as in Section 3 would only reinforce this observation that entrepreneurs tend to be overoptimistic on average.

**Evidence in the Survey of Consumer Finances.** Empirically, the Survey of Consumer Finances provides a measure of this leverage. More precisely, I compute using the Publicly Available Data, the following ratio:

\[
\text{Leverage} = \frac{X_{3130}}{X_{3130} - X_{3121}},
\]

the variable \(X_{3130}\) representing the solution to the following question: "If you sold the business now, what would be the cost basis for tax purposes (of your share of this business)?". The tax basis is the amount of the original investment (or the value when it was received) plus additional investments minus depreciation. The variable \(X_{3121}\) represents in contrast how much is still owed on the loan. Unlike what I obtain for housing in DataQuick microdata, the measure is here more indirect, as I do not measure initial leverage. One can however restrict the set of entrepreneurs who have started their business less than three years ago, and obtain a roughly similar picture.

[INSERT FIGURE 25 ABOUT HERE]

5.3 Hedge Funds’ Leverage (TASS Hedge Fund Database)

A final illustration of the model is given using the TASS Hedge Fund Database, which contains 50% of the universe of hedge funds. Hedge Funds report their average leverage to this database, on a voluntary basis. Hedge funds are asked to report average leverage as well as maximum leverage. The leverage ratio distribution of these hedge funds is shown on the figure below (Figure 24). The point estimate for the regression of the log survivor function on log leverage is \(-2.02\), with an \(R^2\) in the range of 98%.
Models generating Pareto distributions usually revolve around a random growth assumption, where skewness is generated through a multiplicative stochastic process with statistical frictions. The case of hedge funds perhaps provides one of the best case where the random growth model would not fit very well. It would be hard to see which shocks would hit leverage ratios multiplicatively, in a mechanic and idiosyncratic way. The model presented in this paper fits the upper part of the leverage distribution extremely well, and the point estimate for the tail coefficient is 2.03, with a standard error of 0.2, computed according to Gabaix and Ibragimov (2011)’s method, using $\text{std} = b \sqrt{\frac{2}{N}}$, with $N$ the number of observations. Note that leverage ratios are self-reported here, which may explain accumulation points towards round numbers, like 200 or 300 for example. Disagreement $\Delta$ can then be calibrated to the data, for example through the maximum or the minimum leverage in the upper tail and in August 2006, is estimated to be around $h \approx 1.9\%$ through both methods. Note however that the exact value for disagreement depends on the structure of beliefs. The calibration above is therefore conditional on the structure of beliefs being uniform.

6 Conclusion

The model presented in this paper has a number of novel properties. Hedonic interest rates can arise, which do not at all compensate lenders for expected default probability, but instead allow the market for high leverage ratio loans to clear. Given the importance of fixed income securities in modern financial markets, this insight could potentially be very important. This fact could be further developed into many potential promising directions. For example, a natural extension of the model would be one where savers-depositors do not entertain any beliefs about asset prices, and invest in money market funds promising the highest returns (perhaps because of lack of financial sophistication, or otherwise). Hedonic interest rates then would explain why money market funds with more risky investment strategies would tend to intermediate more money. This is not the usual risk-shifting hypothesis, even though money market managers taking more risks would earn higher returns: the corresponding lenders would be willing to put their own wealth at stake, as they would not expect to lose money. Skin in the game would therefore not solve the problem. This could also explain why financial institutions which are run as partnerships also run in trouble during typical financial crises.

The model also has implications for finance. For example, average leverage is increasing when disagreement decreases. When the wealth distribution becomes more skewed towards optimists, leverage rises as optimists with the same beliefs become more numerous; and conversely when the fat tail of optimistic investors is wiped out because of a crisis. In other words, the model could generate an increase in margins during crises without an assumption of "scary bad news", an important stylized fact that models with a single leverage ratio do not generate (Geanakoplos (2003)). The model also allows to investigate the consequences of introducing margin require-
ments on the price of risky assets. An interesting result which would come out of the analysis is that margin requirements interact with short-selling regulation: when leverage caps are introduced in a trading environment where short-selling is permitted, the margins both prevent optimists from leveraging and short-sellers from expressing their negative opinions about the asset, so that margin requirements can lead to exacerbate episodes of "irrational exuberance" about asset prices instead of dampening them. Methodologically, addressing this question requires having a model where agents can both short and lend in equilibrium, something that previous disagreement models did not allow, because all pessimists wanted to short rather than lend if they were given the possibility.

A model of this type could also be useful to study the heterogenous returns to entrepreneurship as well as top income inequality coming from the banking industry more generally. This is interesting because since Pareto (1897), economists have sought to explain why earnings distributions were always skewed to the right and had a fat tail: that is, the top percentile of earners always account for a disproportionate share of total earnings. This paper gives a new intuition for why returns to entrepreneurship or to specific trading strategies are skewed to the right in a Pareto manner: some agents are more optimistic (about a new technology, or a trading strategy), and they are able to take large bets because their optimistic beliefs allow them to leverage with agents who ask for very low margins. When they turn out to be right, they can enjoy very high returns. The model could therefore perhaps provide a link between the deregulation of finance (such as a decrease in margin requirements), in particular in Anglo-saxon countries, and increased top incomes in the finance industry. In the case of traders, deregulation can for example allow proprietary trading desks at investment banks to use the banks’ regulatory capital for leveraging into sometimes very large speculative bets. According to Bell and Van Reenen (2013), three fifths of the gains for the top 1% income share between 1997 to 2007 in the United Kingdom went to workers in "Financial Intermediation". Kaplan and Rauh (2009) also suggests this effect could be at work. However, according to the model, margin requirements should impact entrepreneurs, and more generally borrowers, broadly defined. They could therefore be detrimental to Main Street as well as Wall Street. These important questions are left for future research.

Note that the returns to speculating are very high only conditional on being exactly right ex-post about the value of the asset, so that successful speculation could as well be interpreted as skill or luck.
References


Borovicka, J. (2015), ‘Survival and Long-Run Dynamics with Heterogeneous Beliefs under Recursive Preferences’, *NYU, mimeo*.


Figure 2: **Investment Type, Uniform Distribution**

![Figure 2: Investment Type, Uniform Distribution](image)

A Figures
Figure 3: **Hedonic Spreads as a Function of Haircuts, Uniform Case**

Disagreement $\Delta=10\%$

Disagreement $\Delta=5\%$

Disagreement $\Delta=2\%$

Figure 4: **Hedonic Spreads as a Function of Haircuts, $\rho = 1$**

Disagreement $\Delta=10\%$

Disagreement $\Delta=5\%$

Disagreement $\Delta=2\%$
Figure 5: **Assignment of Borrowers to Lenders through Borrowing Contracts**

Figure 6: **Leverage Ratio as a function of Borrowers' Beliefs**
Figure 7: **Leverage Ratio** as a function of Borrowers’ Beliefs, $\rho = 1$

Figure 8: **Leverage Ratio Distribution**, Uniform Density
Figure 9: **Leverage Ratio Distribution, Increasing Density**

![Leverage Ratio Distribution, Increasing Density](image1)

Figure 10: **Leverage Ratio Distribution, $\rho = 1$**

![Leverage Ratio Distribution, $\rho = 1$](image2)
Figure 11: **Unit Value from Leveraging (Expected), \( \rho = 0 \)**

Figure 12: **Unit Value from Leveraging (Expected), \( \rho = 1 \)**
Figure 13: Expected Return of Borrowers, $\rho = 0$

Figure 14: Expected Return of Borrowers, $\rho = 1$
Figure 15: **Hedonic Spreads and True Average Returns on Borrowing Contracts**
Figure 16: **Leverage Ratios in Economy $E_B$ and $E_B^2$.** $\rho = 1$, $\Delta = 10\%$.

![Leverage Ratios Graph]

Figure 17: **Leverage Ratio Distributions in Economy $E_B$ and $E_B^2$.** $\rho = 1$, $\Delta = 10\%$.

![Leverage Ratio Distributions Graph]
Figure 18: **Expected Returns of Borrowers** in $\mathcal{E}_B$ and $\mathcal{E}_B^2$. $\rho = 1$, $\Delta = 10\%$.

![Figure 18](image)

Figure 19: **Expected Returns of Lenders** in Economy $\mathcal{E}_B$ and $\mathcal{E}_B^2$. $\rho = 1$, $\Delta = 10\%$.

![Figure 19](image)
Figure 20: **Investment Type**, Uniform Distribution, Economy $\mathcal{E}_S$

![Diagram](Figure 20)

Figure 21: **Short Interest**, Uniform Distribution, Economy $\mathcal{E}_S$

![Diagram](Figure 21)
Figure 22: **Hedonic Spreads and Haircuts, Uniform Case, Economy $\mathcal{E}_S$**

Figure 23: **Rebate Rates and Cash Collateral, Uniform Case, Economy $\mathcal{E}_S$**
B Evidence

Below is evidence on the distribution of leverage ratio distribution for Hedge Funds, from the TASS Lipper Database in August 2006, comprising about 50% of the universe of hedge funds. (the point estimate of the Pareto tail coefficient is -1.95, with a standard error of 0.2, and an $R^2 = 98\%$).

Figure 24: Distribution of Hedge Funds’ Leverage Ratios.

Figure 25: Initial Leverage Ratios for Entrepreneurs (Source: SCF 2013)
Proposition 1 fully characterizes prices at the competitive equilibrium in Definition 2: the price for the real asset $p$, and prices for each borrowing contract indexed by his face value $Q(\cdot)$. Portfolios of agents are however only defined implicitly. To matched the level of formalism used in Definitions 1 and 2, one needs to use mathematical distributions because portfolio problem consists in choosing between a continuum of commodities. In the following, $\delta_x(\cdot)$ denotes the Dirac measure with mass point at $x$. The following statements complete Proposition 1 in characterizing equilibrium in this economy.

- Portfolios of cash investors with beliefs such that $p^i_1 \in [1-\Delta, \xi]$ are:
  
  $$n^i_A = 0, \quad n^i_B(\cdot) = 0, \quad n^i_C = 1.$$  

- Portfolios of lenders with beliefs such that $p^i_1 \in [\xi, \tau]$ are:
  
  $$n^i_A = 0, \quad n^i_B(\cdot) = \frac{1}{q(p^i_1)} \delta_{p^i_1}(\cdot), \quad n^i_C = 0.$$  

- Portfolios of borrowers with beliefs such that $p^i_1 \in [\tau, 1]$ are:
  
  $$n^i_A = \frac{1}{p - q(\phi')}, \quad n^i_B(\cdot) = -\frac{1}{p - q(\phi')} \delta_{\phi}(\cdot), \quad n^i_C = 0,$$
  
  with $\phi'$ given by: $\phi' = \text{arg max}_\phi \frac{p^i_1 - \phi}{p - q(\phi)}$.

Note that equation (2a) obtains from aggregating all consumers’ budget constraint (at equality), using market clearing for the real asset ($MC_A$) and all Borrowing Contracts ($MC_B$):

$$\int_i n^i_A p_i \, di + \int_i \int \phi n^i_B(\phi) q(\phi) \, d \phi \, di + \int_i n^i_C \, di = \int_i \, di \quad \Rightarrow \quad p + \int_i n^i_C \, di = 1.$$  

Because only agents $i$ with $p^i_1 \in [1-\Delta, \xi]$ buy cash in quantity $n^i_C = 1$, we get equation (2a) as:

$$\int_i n^i_C \, di = F(\xi) \quad \Rightarrow \quad p = 1 - F(\xi). \quad (2a)$$

Finally, note that the first differential equation (1a) obtains with these notations from ($MC_B$):

$$\int_i n^i_B(\phi) \, di = 0 \quad \Rightarrow \quad -\frac{1}{p - q(x)} f(y) \, dy + \frac{1}{q(x)} f(\Gamma(y)) \, d\Gamma(y) = 0$$

$$\Rightarrow \quad -\frac{1}{p - Q(y)} f(y) \, dy + \frac{1}{Q(y)} f(\Gamma(y)) \, d\Gamma(y) = 0$$

$$\int_i n^i_B(\phi) \, di = 0 \quad \Rightarrow \quad (p - Q) f(\Gamma) \Gamma' = Q f. \quad (1a)$$

D Completing the Proof of Proposition 1

There is also a way of getting to the results in Proposition 1 in a completely formal way. If only contract $\phi$ is used, the collateral constraint (CC) can be used to replace $n_B(\cdot)$ in the budget
Table 1: **Homeowners’ Leverage Ratio Distribution, 2001-2008** (Source: Dataquick)

See [Online Video for More Pictures](#)

Leverage Ratio Distribution of US Homeowners (Leverage Ratio on New Loans)

Date: 1/2001

Date: 1/2002

Date: 1/2003

Date: 1/2004

Date: 1/2005

Date: 1/2006

Date: 1/2007

Date: 1/2008

60
Table 2: Homeowners’ Leverage Ratio Distribution, 1993-2000 (Source: Dataquick) 
See Online Video for More Pictures
constraint (BC), and expressed the fact that each unit of real asset that the borrower owns is going to be used in equilibrium to back a Borrowing Contract:

\[-\int_\phi n_B^i(\phi) d\phi = n_A^i \quad \text{and} \quad n_C^i = 0 \quad \Rightarrow \quad n_A^i(p - q(\phi)) = 1.\]

Replacing the equilibrium quantity of real asset, Borrowing Contracts, and cash used in equilibrium in the agent’s maximization of wealth (W) gives:

\[\phi = \arg \max_\phi \frac{p_A^i - \phi}{p - q(\phi)}.\]

**Partition of Agents.** Note that the sets of beliefs corresponding to borrowers, lenders and cash investors are necessarily convex because of the form of each agents’ optimization program. One comes easily to a contradiction by assuming initially that there exist \((x, y)\) and \(z \in [x, y]\) with \((x, y)\) being invested in the Real Asset and \(z\) buying financial contract, for example, or through a revealed preference argument. Moreover, note that because of the assumed bound on agents’ beliefs, that it be lower than one, the three sets of agents are necessarily present in equilibrium, from the market clearing equation (2a) which is proved in the main text \(1 - F(\xi) = p\). Note first that some agents need to be buying the Real Asset in equilibrium, for the market for the Real Asset to clear.

Assume that there are no lenders in this economy. Then we would have \(1 - F(p) = p\) from market clearing. If there was no cash investors, we would have \(p = 1 - \Delta\) so \(p = 1\) and hence \(1 - \Delta = 1\), which contradicts the heterogenous beliefs assumption. This configuration with cash investors and investors in the Real Asset would not be an equilibrium, because asset investors with \(p_A^i > p\) would have an incentive to sell Borrowing Contracts to cash investors, giving them the return of cash, to make a higher return. Hence, there are lenders in this economy.

Assume now that there are only lenders and borrowers in this economy. Then from market clearing and \(\xi = 1 - \Delta\) agents’ funds must all be invested in the Real Asset, and so \(1 - F(1 - \Delta) = 1 = p\). Therefore \(p = 1\). Once again, there would be no Real Asset investors in this economy, who are in the segment \([p, 1]\) reduced to a set of measure zero, a contradiction as lenders would have no one to buy their Borrowing Contracts from. We shall now verify that no lender would like to be a borrower and vice versa.

**Final check.** Finally, it is important to check whether given the above defined prices, borrowers would not rather be lenders or cash-investors given their beliefs, and the same for lenders and cash-investors.

Cash investors with beliefs \(p_A^i < \xi < p\) do not want to buy the asset which gives them a lower expected return than cash, let alone do leveraged investing into this asset. They do not want to lend either since if they did they would choose the lowest leverage ratio loans from the above reasoning (all of them give them a unit of collateral in period 1 in expectation, and the lowest leverage loans are the cheapest), and such lowest leverage ratio loans have face value \(\xi\) with \(q(\xi) = \xi\) so they give \(p_A^i/\xi < 1\) to cash investors.
Lenders with beliefs \( p_1^i \in [\xi, \tau] \) do not want to invest in cash because they expect to make \( r(p_1^i) > 1 \) from lending. The fact that they do not want to borrow results from the fact that borrowing with the optimum face value for them, that is \( \phi = \xi \) from the above remarks, would give them a return \( \frac{p_1^i - \xi}{p - \xi} \), which is linear in \( p_1^i \). It therefore suffices to check that the return from borrowing and investing with these loans with face values equal to \( \xi \) always brings a lower return than lending. Therefore, to show that \( r(p_1^i) \) is always higher for a lender than the return he would get from borrowing \( \frac{p_1^i - \xi}{p - \xi} \), it remains to prove that:

\[
r'(\tau) < \frac{1}{p - \xi}. 
\]

This inequality obtains from noting that:

\[
r(\tau) = \frac{\tau - \xi}{p - \xi} > \frac{\tau}{p} \quad \Rightarrow \quad 1 - pr(\tau) < 1 - \tau.
\]

Using the assignment equation (1a), together with the last inequality:

\[
r'(\tau) = \frac{r(\tau) - pr(\tau)^2}{\tau(1 - \tau)} = \frac{r(\tau)}{\tau} - \frac{pr(\tau)}{1 - \tau} = \frac{\tau - \xi}{\tau} - \frac{1}{p - \xi} < \frac{1}{p - \xi}. 
\]

Finally, it is much easier to see why borrowers with beliefs \( p_1^i \in [\tau, 1] \) would never invest in cash nor lend: their expected return is higher than both that of cash investors and that of lenders, from the indifference equation \( r(\tau) = \frac{\tau - \xi}{p - \xi} \) together with the envelope theorem on borrowers’ optimization program with respect to \( p_1^i \).

**Second-order differential equation for \( \Gamma(.) \).** Equation (1a) implies:

\[
(p - Q)f(\Gamma)\Gamma' = Qf \quad \Rightarrow \quad Q = \frac{f(\Gamma)\Gamma'}{f + f(\Gamma)\Gamma p} \quad \Rightarrow \quad p - Q = \frac{f}{f + f(\Gamma)\Gamma p}
\]

\[
\Rightarrow \quad \log(p - Q) = \log(p) - \log \left(1 + \frac{f(\Gamma)\Gamma'}{f} \right) \quad \Rightarrow \quad \frac{Q'}{p - Q} = \frac{f(\Gamma)\Gamma'}{1 + f(\Gamma)\Gamma'}
\]

Equation (1b) is:

\[
(y - \Gamma)Q' = (p - Q)\Gamma' \quad \Rightarrow \quad \frac{Q'}{p - Q} = \frac{\Gamma'}{y - \Gamma}.
\]

Together these imply that \( \Gamma(.) \) satisfies:

\[
(y - \Gamma) \left( \frac{f(\Gamma)\Gamma'}{f} \right)' - \Gamma' - \frac{f(\Gamma)}{f} \Gamma'^2 = 0.
\]

In algebraic form:

\[
\forall y \in [\tau, 1], \quad (y - \Gamma(y)) \left( \frac{f(\Gamma(y))\Gamma'(y)}{f(y)} \right)' - \Gamma'(y) - \frac{f(\Gamma(y))}{f(y)} \Gamma'(y)^2 = 0.
\]
E  Completing the Proof of Corollary 1 - Closed Form Expressions for the cutoffs

A closed form expression for \( p, \xi \) and \( \tau \) in the Borrowing Economy and when the density is uniform with heterogeneity parameter \( \Delta \) obtains as follows. If \( f \) is uniform, then \( F(.) \) is linear with slope \( 1/\Delta \) so that the market clearing equation (2a) writes \( \xi = 1 - p\Delta \). Because \( Q'(p-Q) \) is constant one can then write:

\[
Q'(\tau) (p - Q(\tau)) = Q'(1) (p - Q(1)) \Rightarrow \frac{Q(\tau)}{1 - \Gamma(\tau)} (p - Q(\tau)) = \frac{Q(1)}{1 - \Gamma(1)} (p - Q(1)) \Rightarrow \frac{\xi}{\tau - \xi} (p - \xi) = \frac{\tau(p - \xi)}{(\tau - \xi)((1 - \tau)} (p - \frac{\tau(p - \xi)}{\tau - \xi}) \Rightarrow \frac{\tau - \xi}{\tau - p} = \frac{\tau}{1 - \tau}.
\]

Using \( \xi = 1 - p\Delta \), one can express \( p \) as a function of \( \tau \):

\[
(1 - \tau)(\tau - 1 + p\Delta) = \tau(\tau - p) \Rightarrow p = \frac{2\tau^2 - 2\tau + 1}{\Delta + (1 - \Delta)\tau}.
\]

Using the integration of \((p - Q)^2\)' = 0 in the main proof of Corollary 1 we also have:

\[
(p - Q(\tau))^2 = (p - Q(1))^2 + 2Q(1) (p - Q(1)) \Rightarrow (p - Q(\tau))^2 = p^2 - Q(1)^2.
\]

Then, we have:

\[
p - \xi = p - (1 - \Delta p) = \frac{(1 + \Delta)^2 + (1 + \Delta)(1 - \tau)^2 - \tau - (1 - \tau)\Delta}{\Delta + (1 - \Delta)\tau} = \frac{(2\tau - 1)[(1 + \Delta)\tau - 1]}{\Delta + (1 - \Delta)\tau}
\]

\[
\tau - \xi = \tau - 1 + \Delta p = \frac{(\tau - 1)\Delta + (1 - \Delta)\tau^2 - (1 - \Delta)\tau + 2(2\tau^2 - 2\tau + 1)}{\Delta + (1 - \Delta)\tau} = \frac{(1 + \Delta)\tau - 1}{\Delta + (1 - \Delta)\tau}.
\]

Therefore, using equation (2e):

\[
Q(1) = \frac{\tau p - \xi}{\tau - \xi} = 2\tau - 1.
\]

Equation \((p - \xi)^2 = p^2 - Q(1)^2\) thus writes:

\[
(2\tau - 1)^2 [(1 + \Delta)\tau - 1] = (2\tau^2 - 2\tau + 1)^2 - (2\tau - 1)^2 [\Delta + (1 - \Delta)\tau]^2
\]

\[
\Rightarrow (2\tau - 1)^2(1 + \Delta^2)(2\tau^2 - 2\tau + 1) = (2\tau^2 - 2\tau + 1)^2.
\]

Because \(2\tau^2 - 2\tau + 1 = \tau^2 + (1 - \tau)^2 \neq 0\) this implies:

\[
(2\tau - 1)^2(1 + \Delta^2) = 2\tau^2 - 2\tau + 1 \Rightarrow \tau^2 - \tau + \frac{2\Delta^2}{4(1 + 2\Delta^2)} = 0 \Rightarrow \tau = \frac{1}{2} \left( 1 \pm \sqrt{\frac{1}{1 + 2\Delta^2}} \right).
\]

When \( \Delta \to 0 \), \( \tau \) must go to one and so the highest solution is the equilibrium one. One then gets a closed form expression for \( p \) and \( \xi \) as a function of \( \Delta \) as well, so that:

\[
\tau = \frac{1}{2} + \frac{1}{2\sqrt{1 + 2\Delta^2}} \quad \quad p = \frac{2(1 + \Delta^2)}{1 + \Delta + 2\Delta^2 + 2\Delta^3 + (1 - \Delta)\sqrt{1 + 2\Delta^2}} \quad \xi = 1 - \Delta p.
\]
F Short-Sales Economy - Proof of Proposition 3

Because of the linearity of the problem, whenever a short-seller finds it optimal to sell short, he will use all his cash as collateral to short as much as he can. Therefore, when using Short-Sales Contract \((\gamma)\) with cash collateral \(\gamma\), the short-seller gets \(q_{s}(\gamma)\) out of selling the contract, and so needs to contribute \(\gamma - q_{s}(\gamma)\) of his personal funds to the purchase. He will be able to purchase a number \(1/(\gamma - q_{s}(\gamma))\) of these contracts. On the asset side of his balance sheet, he gets the return on cash, which is \(\gamma\) per unit, and on the liability side buys back one unit of the contract to give it to the securities lender. Thus the Short-Seller chooses \((\gamma)\) to maximize:

\[
\max_{\gamma} \frac{\gamma - p_{1}}{\gamma - q_{s}(\gamma)}.
\]

The optimality condition thus gives equation (5d):

\[
1 - q'_{s}(\gamma) = \frac{\gamma - q_{s}(\gamma)}{\gamma - p_{1}} \Rightarrow \forall y \in [1 - \Delta, \xi], \quad Q'_{s}(y) (\Gamma_{s}(y) - y) = \Gamma'_{s}(y) (Q_{s}(y) - y).
\]

The market clearing equation for Short-Sales Contracts leads to equation (5b):

\[
\frac{f(x)dx}{Q_{s}(y)} = \frac{f(y)dy}{x - Q_{s}(y)} \Rightarrow \forall y \in [1 - \Delta, \xi], \quad (\Gamma_{s}(y) - Q_{s}(y)) f(\Gamma_{s}(y)) \Gamma'_{s}(y) = Q_{s}(y) f(y).
\]

The expressions concerning Borrowing Contracts, and leading to equations (5a) and (5c) are similar as those in Proposition 1. The reason why the cash collateral value is equal to the beliefs of securities lenders also arises in an exact symmetric way as for lenders. Finally, we need to prove the eight algebraic equations given by equation (6) to complete the proof.

Market clearing for the Real Asset implies that lenders’ and borrowers’ fund end up invested in the real asset, which is equation (6a):

\[
1 - F(\sigma) + F(\tau) - F(\xi) = p.
\]

Positive sorting of borrowers and lenders on the one hand, and securities lenders and short-sellers on the other, gives equations (6b), (6c), (6d) and (6e):

\[
\Gamma(\sigma) = \xi, \quad \Gamma(1) = \tau, \quad \Gamma_{s}(1 - \Delta) = \tau, \quad \Gamma_{s}(\xi) = \sigma.
\]

Finally, indifference for agents with beliefs \(\xi, \tau\) and \(\sigma\) respectively imply equations (6f), (6g), and (6h):

\[
\frac{\xi}{q(\xi)} = \frac{\sigma - \xi}{\sigma - Q_{s}(\xi)} \Rightarrow \frac{\xi}{Q(\sigma)} = \frac{\sigma - \xi}{\sigma - Q_{s}(\xi)}
\]

\[
\frac{\tau}{p q_{s}(\tau)} = \frac{\tau}{q(\tau)} \Rightarrow \frac{\tau}{Q_{s}(1 - \Delta)} = \frac{\tau}{Q(1)}
\]

\[
\frac{\sigma}{p q_{s}(\sigma)} = \frac{\sigma - \xi}{p - Q(\sigma)} \Rightarrow \frac{\sigma}{Q_{s}(\xi)} = \frac{\sigma - \xi}{p - Q(\sigma)}.
\]