Prices and Auctions in Markets with Complex Constraints

Paul Milgrom
Stanford University & Auctionomics
August 2016
Market Design Perspective

• Market design – implementing practical market rules – forces economists to deal explicitly with some issues that are usually de-emphasized.

• One such issue cluster, emphasized here, is product definition, product heterogeneity, and constraints on market allocations...
Three Market-Design Questions

1. What is a product? Debreu described commodities by their physical characteristics and their time & place of availability.
   • Electricity in Palo Alto, CA at 2:05pm is not the same commodity as electricity in Palo Alto at 2:06pm. ...nor the same as electricity in San Jose.

2. If heterogeneous products are lumped together (as they must be!), then there may be more constraints than just total resource constraints. How can we handle those?
   • My return flight to SFO will use “time-space” resources.

3. When can economic models that are drastically simpler than “realistic” engineering models be useful for market design?
   • “SFO airport has a capacity of X passengers/hour”
Examples

COMPLEX CONSTRAINTS AND THE ROLES OF PRICES & ORGANIZATION
Real-Estate Holdout Examples

- Seattle, WA
- San Antonio, TX
Co-Channel Interference Around One Station
Reallocating Radio Spectrum

About 130,000 co-channel interference constraints, and about 2.7 million constraints in the full model.

The graph-coloring is NP-complete. The FCC may sometimes be unable to determine, in reasonable time, whether a certain set of stations can be assigned to a given set of channels.
An “In-Between” Model of O’Hare Airport?!?

PRICES AND INCENTIVES IN THE “KNAPSACK PROBLEM”
Knapsack Problem

• Notation
  • Knapsack size: $\tilde{S}$.
  • Items $n = 1, \ldots, N$
  • Each item has a value $v_n > 0$ and a size $s_n > 0$.
  • Inclusion decision: $x_n \in \{0, 1\}$.

• Knapsack problem:
  $$V^* = \max_{x \in \{0,1\}^N} \sum_{n=1}^{N} v_n x_n \text{ subject to } \sum_{n=1}^{N} s_n x_n \leq \tilde{S}.$$  

• The class of knapsack problems (verification) is NP-complete, but there are fast algorithms for “approximate optimization.”
Dantzig’s “Greedy Algorithm”

• Order the items so that $\frac{v_1}{s_1} > \cdots > \frac{v_N}{s_N}$.

• Algorithm:
  1. $S_1 \leftarrow \bar{S}$. (Initialize “available space”)
  2. For $n = 1, \ldots, N$
  3. If $s_n \leq S_n$, set $x_n = 1$, else set $x_n = 0$.
  4. Set $S_{n+1} = S_n - x_n s_n$.
  5. Next $n$.
  6. End

• The items selected are $\alpha^{Greedy}(v; s) \overset{\text{def}}{=} \{ n | x_n = 1 \}$.
  • This function is “monotonic.”
“Nearly the Same” Algorithm
Can be Formulated as an Auction

- Let $p(0) > \frac{v_1}{s_1}; S(0) = \bar{S}$; and label all $n$ “Out.”

**Discrete greedy algorithm**

- For $t = 1, 2, ... p(0)/\varepsilon$.
  - Mark “Rejected” any $n$ for whom $s_n + S(t - 1) > \bar{S}$
  - Set $p(t) = p(t - 1) - \varepsilon$ (where $\varepsilon > 0$ is the “bid decrement”)
  - If some $n$ who is marked “Out” has $v_n > p(t)s_n$, mark it “Accepted” and set $S(t) = S(t - 1) - s_n$.

- Next $t$

**Generalizable Insight:**

- Close equivalence between various clock auctions and related greedy algorithms (Milgrom & Segal).
The LOS Auction

- **Model** (*Lehman, O’Callaghan and Shoham (2002))*
  - Each item $n$ is owned by a separate bidder.
  - Sizes $s_n$ are observable to the auctioneer.

- **“LOS” Direct Mechanism**
  - Each bidder $n$ reports its value $v_n$.
  - Allocate space to the set of bidders $\alpha^{Greedy}(v; s)$.
  - Charge each bidder its “threshold price”, defined by:
    $$p_n(v_n; s) \overset{\text{def}}{=} \inf\{v' \mid n \in \alpha^{Greedy}(v'_n, v_n; s)\}.$$

- **Theorem.** The LOS auction is truthful.
LOS Mechanism Can Lead to Excessive Investments

• Consider a game in which, before the auction, each bidder $n$ can, by investing $c$, reduce the size of its item to $s_n - \Delta$.

• **Examples:** Suppose there are $N$ items, each of size 1 and the knapsack has size $N - 1$, so $\alpha^{Greedy}(v, s) = \{1, \ldots, N - 1\}$.

• **Losses from excessive investment:** If $\Delta = \frac{1}{N}$ and $c = v_N - \varepsilon$, then there is a Nash equilibrium in which all invest, even though the cost is nearly $N$ times the benefit.

• *Can a uniform price mechanism perform similarly well?*
Uniform-Price Greedy Mechanism

• Determine the allocation
  1. Order the items so that $\frac{v_1}{s_1} > \cdots > \frac{v_N}{s_N}$.
  2. Initialize $n \leftarrow 1$ and set $S_1 \leftarrow S$.
  3. If $s_n > S_n$, go to step 5
  4. Increment $n$ and go to step 3
  5. Set $\alpha^{Alt}(v, s) \leftarrow \{1, \ldots, n - 1\}$.

• Set a supporting price (a “uniform” price of space)
  6. Define $\hat{p} \overset{\text{def}}{=} \frac{v_n}{s_n}$ (the “pseudo-equilibrium” price of space)
  7. Set $p_j \leftarrow \hat{p}s_j$ for $j = 1, \ldots, n - 1$, and $p_j \leftarrow 0$ for $j = n, \ldots, N$.
  8. End

• Define $V^{Alt}(v, s) \overset{\text{def}}{=} \sum_{n \in \alpha^{Alt}(v, s)} v_n$. 
Packing Efficiency and Truthfulness

- **Theorem.** The “Alt” mechanism is truthful. Moreover,

\[ V^{\text{Greedy}}(v, s) \geq V^{\text{Alt}}(v, s) \geq V^*(v, s) - (S - S^{\text{Alt}})\hat{p}. \]

- The bound on efficiency loss is observable: it is equal to the “value” of the unused space in the knapsack.
Investment Efficiency

• Suppose that each bidder $n$ can, by expending $c_n \in C$, determine the size $s_n(c_n)$ of its item, where $C$ is a finite set. Let $n^*$ denote the index of the first bidder not packed by the Alt mechanism.

• Notation:

$$V^{**} \overset{\text{def}}{=} \max_{\{x, c|c_{n^*} = \cdots = c_N = 0\}} \sum_n (x_n v_n - c_n) \text{ s.t. } \sum_n x_n s_n(c_n) \leq S$$

$$c_n^* \overset{\text{def}}{=} \arg\max_{c_n \in C} \max_{x_n} v_n x_n - \hat{p} s_n(c_n) - c_n$$

• **Theorem.** The combined loss from packing and investment in $Alt$ is (“observably”) bounded as follows:

$$V^{**} - V^{Alt}(s(c^*)) \leq \left( S - S^{Alt}(c^*) \right) \hat{p}$$
Greedy Algorithms and Auctions

• In an auction with single-minded bidders, a greedy algorithm can be used to sort the bidders into two sets, winners and losers.
  • LOS Algorithm and Related: Select winners by a greedy algorithm; other bidders are losers.
  • DAA Algorithm and Related: Select losers by a greedy algorithm; other bidders are winners.

• The two categories are *economically* distinct because
  • winners collect payments
  • losers do not

• The FCC descending clock auction is a DAA algorithm.
  • One could specify an ascending clock for an LOS-style algorithm.
More Greedy Algorithms

GREEDY ON MATROIDS
Matroid Terms Defined

• Given a fine “ground set” $X$, let $\varnothing(X)$ denote its power set.
  • Example: $X$ is the set of rows of a finite matrix
• Let $\mathcal{R} \subseteq \varnothing(X)$ be non-empty and all its elements “independent sets.”
  • Example: all linearly independent sets of rows in $X$.
• A “basis” is a maximal independent set.
  • Example: a maximal linearly independent set of rows of $X$.

• The pair $(X, \mathcal{R})$ (or just the set $\mathcal{R}$) is a matroid if
  1. [Free disposal] If $S' \subseteq S \in \mathcal{R}$, then $S' \in \mathcal{R}$.
  2. [Augmentation Property] Given $S, S' \in \mathcal{R}$, if $|S| > |S'|$, then there exists $n \in S - S'$ such that $S' \cup \{n\} \in \mathcal{R}$.
Greedy Algorithm

• Given any collection of independent sets $\mathcal{R}$.
• Order the items so that $v_1 > \cdots > v_N$. (No volumes)

• Algorithm:
  1. Initialize $S_0 \leftarrow \emptyset$.
  2. For $n = 1, \ldots, N$
  3. $S_n \leftarrow \begin{cases} S_{n-1} \cup \{n\} & \text{if } S_{n-1} \cup \{n\} \in \mathcal{R} \\ S_{n-1} & \text{otherwise} \end{cases}$
  4. Next $n$
  5. Output $S_N$. 
Optimization on Matroids

- For simplicity, assume a unique optimum.
- **Theorem.** If $\mathcal{R}$ is a matroid and $S_N$ is the greedy solution, then
  \[ S_N = \arg\max_{S \in \mathcal{R}} \sum_{n \in S} v_n \]

- **Intuition.** Suppose that $S^* = \{i_1, \ldots, i_k\} \in \mathcal{R}$ does not include the most valuable item, which is item 1.
  - Then $S^*$ is not optimal, because we can *augment* the set $\{1\}$ using items from $S^*$ to create a $k$ item set that is strictly more valuable.
Full Proof is by Induction

- Suppose that the set selected by the greedy algorithm is \( \{g_1, \ldots, g_k\} \) and that the \( g_n \) is the element with the lowest index that such that for the optimal set \( S \), \( g_n \notin S \). So, \( S = \{g_1, \ldots, g_{n-1}\} \cup S' \) and for each element \( s \in S' \), \( v_g > v_s \).
- By the augmentation property, it is possible to augment \( \{g_1, \ldots, g_n\} \) to a basis \( B \) set by iteratively adding elements from \( S' \), while omitting just one element, say \( \hat{s} \).
- By then \( S \) was not optimal, because \( B \) is better:
  \[
  \sum_{j \in B} v_j - \sum_{j \in S} v_j = v_{g_n} - v_{\hat{s}} > 0. \]

Matroids and Substitutes

• Let $\mathcal{R}$ be a non-empty collection of independent sets satisfying free disposal.

• For each good in $x \in X$, there is a buyer $v(x)$ and a price $p(x)$. The buyer’s demand is described by:

$$V^*(\mathcal{R}, v) \overset{\text{def}}{=} \max_{S \in \mathcal{R}} \sum_{x \in S} (v(x) - p(x))$$

$$d^*(p|\mathcal{R}, v) \overset{\text{def}}{=} \arg\max_{S \in \mathcal{R}} \sum_{x \in S} (v(x) - p(x))$$

• **Theorem.** The items in $X$ are substitutes if and only if $\mathcal{R}$ is a matroid.

• **Intuition:** pretty much the same as optimality of greedy algorithm!
Proof Sketch

• Suppose that $n \in d^*(p|\mathcal{R}, v)$ and consider a price $p'(n) > p(n)$ such that $n \notin d^*(p \setminus p'(n)|\mathcal{R}, v)$. Let $n' \notin d^*(p|\mathcal{R}, v)$ be the first new item chosen instead during the greedy algorithm with prices $p \setminus p'(n)$. Let the state of the greedy algorithm when it is chosen be $S'$ and let $S = S' \cup \{n\} - \{n'\}$.

• By the augmentation property, the feasible next choices to augment $S'$ and $S$ are identical. Hence, $d(p \setminus p'(n)|\mathcal{R}, v) = (d(p|\mathcal{R}, v) - \{n\}) \cup \{n'\}$, as required.

• Conversely, if $\mathcal{R}$ is not a matroid, then...
Necessity of Matroids

• **Theorem.** If \( \mathcal{R} \) is a non-empty family that satisfies free disposal but not the augmentation property, then there is some vector of values \( \nu \) such that (the greedy algorithm “fails”) \( S_N \notin \arg\max_{S \in \mathcal{R}} \sum_{n \in S} \nu_n \).

• **Proof.** \( \mathcal{R} \) does not have the augmentation property, so there is some \( S, S' \in \mathcal{R} \) such that \( |S| > |S'| \) and there is no \( n \in S - S' \) such that \( S' \cup \{n\} \in \mathcal{R} \).

• Let \( \epsilon > 0 \) be small and take:

\[
\nu_n = \begin{cases} 
1 & \text{if } n \in S' \\
1 - \epsilon & \text{if } n \in S - S' \\
0 & \text{otherwise}
\end{cases}
\]

• Then the greedy algorithm selects \( S' \) and no elements of \( S - S' \), so its value is \( |S'| \), but \( S \) achieves at least \( (1 - \epsilon)|S| > |S'| \).
Approximate Substitutes and the Incentive Auction

WHY SHOULD WE CARE?
How Well Does The Incentive Auction Work?

- Can’t run VCG on national-scale problems: can’t find an optimal packing
  - Restrict attention to all stations within **two constraints of New York City**
    - a very densely connected region
    - 218 stations met this criterion
- Reverse auction simulator (UHF only)
- Simulation **assumptions**:
  - 100% participation
  - 126 MHz clearing target
  - valuations generated by sampling from a prominent model due FCC chief economist (before her FCC appointment)
  - 1 min timeout given to SATFC
“Greedy”: check whether existing solution can be directly augmented with new station
The Substitution Index

• Why does the DA algorithm perform so well?
• Two conjectured reasons:
  • Special constraints: the independent sets $C$?
  • Special values: a set $\mathcal{O} \subseteq C$ where the optimum may lie?
    • “Zero knowledge case”: $\mathcal{O} = C$.

• Definitions. Given the ground set $\mathcal{X}$ and the constraints $C$ and possible optimizers $\mathcal{O}$ that both satisfy free disposal,

\[
R^*(C, \mathcal{O}) \overset{\text{def}}{=} \arg\max_{R \text{ a matroid}} \min_{X \in \mathcal{O}} \max_{X' \in R} \frac{|X'|}{|X|}
\]

\[
\rho(C, \mathcal{O}) \overset{\text{def}}{=} \max_{R \text{ a matroid}} \min_{X \in \mathcal{O}} \max_{X' \in R} \frac{|X'|}{|X|}
\]
Approximation Theorem

• Given the ground set $\mathcal{X}$, any $\mathcal{S} \subseteq \mathcal{P}(\mathcal{X})$ and any $v \in \mathbb{R}^{\mathcal{X}}_+$, define notation as follows:

$$V^* (\mathcal{S}; v) \overset{\text{def}}{=} \max_{X \in \mathcal{S}} \sum_{n \in X} v_n$$

• **Theorem.** The greedy solution on $\mathcal{R}^*$ approximates the optimum in worst case as follows:

$$\min_{v > 0} \frac{V^*(\mathcal{R}^*; v)}{V^*(\mathcal{O}; v)} = \rho(C, \mathcal{O}).$$
Proof Sketch, 1

• Let

\[ \nu^* \in \arg\min_{\nu > 0} \frac{V^*(R^*; \nu)}{V^*(O; \nu)}, \rho^* = \frac{V^*(R^*; \nu^*)}{V^*(O; \nu^*)} \]

• Among optimal solutions, choose \( \nu^* \) to be one with the smallest number of strictly positive components.

• Without loss of optimality, we rescale \( \nu^* \) so that the smallest strictly positive component is 1.

• The next step will show that every component of \( \nu^* \) is zero or one, so that the values of the two minimization problems must exactly coincide.
Proof Sketch, 2

• Consider the family of potential minimizers $\hat{v}(\alpha)$, where

$$\hat{v}_n(\alpha) \overset{\text{def}}{=} \begin{cases} \alpha & \text{if } v^*_n = 1 \\ v^*_n & \text{otherwise} \end{cases}$$

• Then, $v^* = \hat{v}(1)$. The value of the objective for $\hat{v}(\alpha)$ is

$$\hat{\rho}(\alpha) = \frac{V^*(R^*; \hat{v}(\alpha))}{V^*(\emptyset; \hat{v}(\alpha))} = \frac{\alpha |\hat{X} \cap X_{R^*}| + \sum_{n \in (X-\hat{X}) \cap X_{R^*}} v^*_n}{\alpha |\hat{X} \cap X_{\emptyset}| + \sum_{n \in (X-\hat{X}) \cap X_{\emptyset}} v^*_n}$$

where

$$\hat{X} \overset{\text{def}}{=} \{ n | v^*_n = 1 \}$$

$$X_{\emptyset} \in \arg\max_{S \in \emptyset} \sum_{n \in S} v^*_n$$

$$X_{R^*} \in \arg\max_{S \in R^*} \sum_{n \in S} v^*_n$$
Proof Sketch, 3

\[ \hat{\rho}(\alpha) = \frac{\alpha |\hat{X} \cap X_R^*| + \sum_{n \in (X-\hat{X}) \cap X_R^*} v_n^*}{\alpha |\hat{X} \cap X_P| + \sum_{n \in (X-\hat{X}) \cap X_P} v_n^*} \]

For \( \hat{\rho}(\cdot) \) to achieve its minimum of \( \rho^* \) when \( \alpha = 1 \), it must be a constant function, which requires \( \frac{|\hat{X} \cap X_R^*|}{|\hat{X} \cap X_P|} = \rho^* \). Then, since \( v^* \) is the minimizer with the fewest strictly positive elements, \( \{ n | v_n^* > 1 \} = \emptyset \). □
WHAT IS GOING ON NOW?

The US Incentive Auction
$86.422 Billion? What does it mean?

## Incentive Auction Dashboard - Stage 1

- **Clearing Target**: 126 MHz
- **Licensed Spectrum**: 100 MHz

### Reverse Auction
- **Current Round**: Bidding concluded
- **Clearing Cost as of Stage 1**: $86,422,558,704

### Forward Auction
- **Current Round**: Bidding not started
- **Auction Proceeds**: N/A
The Incentive Auction “Stages”: A Conceptual Illustration

- Reverse Price
- Forward Price

TV spectrum supply

Net Revenue > Target

Quantity Traded
Thank you!

END