Games of Incomplete Information
Played by Statisticians

Annie Liang
Microsoft Research / UPenn

Becker Friedman Institute
Modeling Beliefs in Games

Classic approach:

→ payoff-relevant parameter $\theta \in \Theta$.
→ players observe (a sequence of) signals - data
→ common prior over combinations of parameters and data

Common data → same posterior beliefs.
Private data, CK of posterior beliefs → again same beliefs.
Modeling Beliefs in Games

Classic approach:

→ payoff-relevant parameter $\theta \in \Theta$.
→ players observe (a sequence of) signals - data
→ common prior over combinations of parameters and data

Common data $\rightarrow$ same posterior beliefs.
Private data, CK of posterior beliefs $\rightarrow$ again same beliefs.

These implications are:

→ descriptively wrong (politics, financial markets, etc.)
→ problematic for predictions in settings in which disagreement is important for behavior (trading, etc.).

Goal: relax the CPA, retain discipline on beliefs.
Introduce Model Uncertainty

Generalize CPA to a set of ways to form beliefs given data.

→ set of priors over combinations of parameters and data

→ set of frequentist estimators for inferring parameter from data
Introduce Model Uncertainty

Generalize CPA to a set of ways to form beliefs given data.

→ set of priors over combinations of parameters and data

→ set of frequentist estimators for inferring parameter from data

example: observe common data \((x_1, \theta_1), \ldots, (x_n, \theta_n)\), use different (consistent) estimators to predict out-of-sample
Introduce Model Uncertainty

Generalize CPA to a set of ways to form beliefs given data.

→ set of priors over combinations of parameters and data

→ set of frequentist estimators for inferring parameter from data

example: observe common data \((x_1, \theta_1), \ldots, (x_n, \theta_n)\), use different (consistent) estimators to predict out-of-sample
Introduce Model Uncertainty

Generalize CPA to a set of ways to form beliefs given data.

→ set of priors over combinations of parameters and data

→ set of frequentist estimators for inferring parameter from data

example: observe common data \((x_1, \theta_1), \ldots, (x_n, \theta_n)\), use different (consistent) estimators to predict out-of-sample
Approach

Discipline imposed by assuming:

→ all learning rules recover $\theta$ given sufficient data (statistical consistency).

→ players have common certainty in the (first-order) beliefs induced by the set of learning rules.

Under assumptions, every dataset induces a set of “plausible” hierarchies of beliefs.
Q: What can an analyst predict based on the proposed refinement alone (i.e. if he doesn’t know exact beliefs)?

→ Strict solutions are good predictions if players observe a lot of data.

→ But how much data? Quantity required is increasing in
  → “complexity” of informational environment (prev. slide: “roughness” and dimensionality of $\phi$)
  → richness of set of interpretations

Can be impractically large.

→ Implausibility of equilibria/rationalizable actions in complex informational environments.
Preliminaries
Environment: Normal-Form Games

finite set of agents $\mathcal{I}$

finite action sets $(A_i)_{i \in \mathcal{I}}$, $A := \prod_{i \in \mathcal{I}} A_i$

payoffs $U = \mathbb{R}^{\mid A \mid \times \mid \mathcal{I} \mid}$
Strict Solution Concepts

Strict Nash Equilibrium. Action profile \( a \) is a strict NE if

\[
u_i(a_i, a_{-i}) > u_i(a'_i, a_{-i}) \quad \forall \ a_i \neq a'_i
\]

for every player \( i \).

Strict Rationalizability. Action \( a_i \) is strictly rationalizable for player \( i \) if there is a family of sets \( (R_j)_{j \in I} \) such that every \( a_j \in R_j \) is a strict best response to some mixed strategy with support in \( R_{-j} \), and \( a_i \in R_i \).
Strict Solution Concepts

\[ \delta \text{-Strict Nash Equilibrium. Action profile } a \text{ is a } \delta \text{-strict NE if } \]
\[ u_i(a_i, a_{-i}) > u_i(a'_i, a_{-i}) + \delta \quad \forall \ a_i \neq a'_i \]

for every player \( i \).

\[ \delta \text{-Strict Rationalizability. Action } a_i \text{ is } \delta \text{-strictly rationalizable for player } i \text{ if there is a family of sets } (R_j)_{j \in I} \text{ such that every } a_j \in R_j \text{ is a } \delta \text{-strict best response to some mixed strategy with support in } R_{-j}, \text{ and } a_i \in R_i. \]
Incomplete Information

parameter space \( \Theta \subset \mathbb{R}^k \)

compact

map

\[ g : \Theta \rightarrow U \]

parameters \(-\text{payoffs}\)

bounded and Lipschitz continuous

- first-order belief \( \Delta(\Theta) \)
- uncertainty over parameters
- second-order belief \( \Delta(\Theta \times \Delta(\Theta)) \)
- uncertainty over opponent's uncertainty
  - a "hierarchy" or "type" \( t \in \Delta(\Theta \times \Delta(\Theta \times \Delta(\Theta))) \)
  - ...
Incomplete Information

- **Parameter space**: \( \Theta \subseteq \mathbb{R}^k \) compact

- **Map**: \( g : \Theta \rightarrow U \)
  - parameters \( \rightarrow \) payoffs
  - bounded and Lipschitz continuous

- **First-order belief**: \( \Delta(\Theta) \)
  - uncertainty over parameter

- **Second-order belief**: \( \Delta(\Theta \times \Delta(\Theta)) \)
  - uncertainty over opponent's uncertainty

- **A “hierarchy” or “type”**: \( t \in \Delta(\Theta) \times \Delta(\Theta \times \Delta(\Theta)) \times \ldots \)
Which Type Space?

“Universal type space”
(all coherent belief hierarchies)

\[ t_\mu \]
Which Type Space?

“Universal type space”
(all coherent belief hierarchies)

$T$

$t_\mu$
Related Literature


Related Literature


Related Literature


Related Literature


Related Literature


Related Literature


Approach
Agents Form Beliefs by Learning From Data

**Data** is a random sequence of observations from a set $\mathcal{Z}$

$$z_n = (z^1, \ldots, z^n) \sim_{\text{i.i.d.}} P.$$ 

Notation: $Z_n$ denotes random sequence of length $n$.

A **learning rule** is any map

$$f : \bigcup_{n=1}^{\infty} \mathcal{Z}^n \rightarrow \Delta(\Theta)$$

fixes $\mathcal{F}$ to be a set of learning rules.
Example Learning Rules

Bayesian Updating:

→ $\mu \in \Delta(\Theta)$ is a prior
→ conditional on $\theta$, stochastic process $\xi^\theta$ generates a sequence of signals $z$
→ $f$ maps data $z$ to the posterior induced by $\mu$ and $(\xi_\theta)_{\theta \in \Theta}$.
→ $\mathcal{F}$ identified with different priors

Confidence Intervals

→ data is a sequence $z = (x_1, \theta_1), \ldots, (x_n, \theta_n)$
→ $\phi_z : \mathcal{X} \rightarrow \Theta$ is the best linear fit to the data, $\Theta_z$ is the 95% confidence interval for out-of-sample prediction at fixed $x^*$.
→ $f$ maps $z$ to a uniform distribution over $\Theta_z$
→ $\mathcal{F}$ identified with different distributions
Set of Plausible Hierarchies

For every dataset \( z \),

\[
\Delta_z = \{ f(z) : f \in \mathcal{F} \} \subseteq \Delta(\Theta)
\]

is the set of “plausible” first-order beliefs.

Let \( T_z \) := set of types with common certainty in \( \Delta_z \):

\[
\rightarrow \text{first-order belief is in } \Delta_z.
\]

\[
\rightarrow \text{believes with probability 1 that every other agent’s first-order belief is in } \Delta_z, \text{ etc.}
\]
Summary of Approach

\[ zk \xrightarrow{\mathcal{F}} \Delta_z \xrightarrow{\text{common certainty}} T_z \]

- data
- set of first-order beliefs
- set of hierarchies
Statistical Consistency

Let $\mu$ be a point mass on the true parameter $\theta^*$. Focus on $\theta^*$-uniform consistency families $\mathcal{F}$:

$$\sup_{f \in \mathcal{F}} d\left( f(Z_n) , \mu \right) \rightarrow 0 \text{ a.s.}$$

belief induced by $f$  "correct" belief

(Impies common learning - see Prop 1).

Limit as $n \rightarrow \infty$ is a complete information game.
Limit is Complete Information

\[ z_n \xrightarrow{\mathcal{F}} \Delta_{z_n} \xrightarrow{\text{common certainty}} T_{z_n} \]

- \( z_n \): data
- \( \Delta_{z_n} \): set of first-order beliefs
- \( T_{z_n} \): set of hierarchies

\[ n \to \infty \]

Prop 1

\[ \{ \delta_{\theta^*} \} \xrightarrow{\text{complete information}} \{ t_{\theta^*} \} \]
Asymptotic Behavior
Behavior Given $n$ Random Observations

Should behavior predicted at $n = \infty$ also be predicted for large $n$?

→ Let $p_n^{NE}(a)$ be the probability (over datasets $z_n$) that

$$(\sigma_i)_{i \in I}, \quad \text{with } \sigma_i(t_i) = a_i \quad \forall \ i, \forall t_i \in T_{z_n}$$

is an (interim Bayesian Nash) equilibrium.

→ Let $p_n^R(a_i, i)$ be the probability (over datasets $z_n$) that $a_i$ is

(interim correlated) rationalizable $\forall t_i \in T_{z_n}$.

Eq property of $a$ is robust to inference if $p_n^{NE}(a) \to 1$ as $n \to \infty$.

→ “Probability that $a$ is guaranteed to be an equilibrium is arbitrarily

close to 1 when players observe enough data.”
Behavior Given $n$ Random Observations

Should behavior predicted at $n = \infty$ also be predicted for large $n$?

→ Let $p_{n}^{NE}(a)$ be the probability (over datasets $z_n$) that
  
  $$(\sigma_i)_{i \in I}, \text{ with } \sigma_i(t_i) = a_i \forall i, \forall t_i \in T_{z_n}$$

  is an (interim Bayesian Nash) equilibrium.

→ Let $p_{n}^{R}(a_i, i)$ be the probability (over datasets $z_n$) that $a_i$ is
  (interim correlated) rationalizable $\forall t_i \in T_{z_n}$.

Rationalizability of $a_i$ is robust to inference if $p_{n}^{R}(a_i, i) \to 1$ as $n \to \infty$.

→ “Probability that $a_i$ is guaranteed to be rationalizable is arbitrarily
  close to 1 when players observe enough data.”
Examples: Robustness to Inference Trivially Met

\[
\begin{array}{ccc}
  & a_1 & a_2 \\
  a_1 & \theta, \theta & 0, 0 \quad & \text{true value } \theta^* > 0 \\
  a_2 & 0, 0 & 1, 1 & \frac{1}{2}, \frac{1}{2}
\end{array}
\]

(1) Take \( F = \{f\} \), where \( f(z) \) is a point mass on \( \theta^* \) for all \( z \).
Examples: Robustness to Inference Trivially Met

<table>
<thead>
<tr>
<th></th>
<th>$a_1$</th>
<th>$a_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_1$</td>
<td>$\theta, \theta$</td>
<td>0, 0</td>
</tr>
<tr>
<td>$a_2$</td>
<td>0, 0</td>
<td>$1, 1$</td>
</tr>
<tr>
<td></td>
<td>$2', 2$</td>
<td></td>
</tr>
</tbody>
</table>

true value $\theta^* > 0$

(1) Take $\mathcal{F} = \{f\}$, where $f(z)$ is a point mass on $\theta^*$ for all $z$.

Assumption (Nontrivial Inference): For every $n$ sufficiently large, $\delta_{\theta^*} \in \text{Interior}(\Delta_{z_n})$ for a $\gamma$-measure of datasets $z_n$. 

Examples: Robustness to Inference Trivially Met

\[
\begin{array}{ccc}
  a_1 & a_2 \\
  \theta, \theta & 0, 0 & \text{true value } \theta^* > 0 \\
  0, 0 & 1, 1 & 2, 2 \\
\end{array}
\]

(1) Take \( \mathcal{F} = \{ f \} \), where \( f(z) \) is a point mass on \( \theta^* \) for all \( z \).

Assumption (Nontrivial Inference): For every \( n \) sufficiently large, \( \delta_{\theta^*} \in \text{Interior}(\Delta_{z_n}) \) for a \( \gamma \)-measure of datasets \( z_n \).

(2) Set \( \Theta := [0, \infty) \).
Examples: Robustness to Inference Trivially Met

\[
\begin{array}{ccc}
  a_1 & a_2 \\
  a_1 & \theta, \theta & 0, 0 \\
  a_2 & 0, 0 & 1, 1 \\
  & & \frac{1}{2}, \frac{1}{2}
\end{array}
\]

true value $\theta^* > 0$

(1) Take $\mathcal{F} = \{f\}$, where $f(z)$ is a point mass on $\theta^*$ for all $z$.

Assumption (Nontrivial Inference): For every $n$ sufficiently large, $\delta_{\theta^*} \in \text{Interior}(\Delta_{z_n})$ for a $\gamma$-measure of datasets $z_n$.

(2) Set $\Theta := [0, \infty)$.

Assumption (Richness): For every $i$ and $a_i \in A_i$, $\exists \theta \in \Theta$ such that $a_i$ is strictly dominant in the game $g(\theta)$. (e.g. WY, CvD)
Strict Equilibrium are Robust

**Theorem 1.** Assume nontrivial inference and richness. Then, the equilibrium property of action profile $a^*$ is **robust to inference** if and only if $a^*$ is a strict Nash equilibrium in the complete information game with payoffs $u^* = g(\theta^*)$. 
Strict-Rationalizability

Conjecture: robustness to inference for rationalizability characterized by strict rationalizability.

Simple counterexample:

\[
\begin{array}{ccc}
  \ a_3 & a_4 \\
  a_1 & \theta,0 & \theta,0 \\
  a_2 & 0,0 & 0,0 \\
\end{array}
\]

true value \( \theta^* = 1 \)

Action \( a_1 \) is robust to inference (strictly dominant at all nearby payoffs), but not strictly rationalizable for player 1.
Strict-Rationalizability

Conjecture: robustness to inference for rationalizability characterized by strict rationalizability.

Simple counterexample:

<table>
<thead>
<tr>
<th></th>
<th>$a_3$</th>
<th>$a_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_1$</td>
<td>$\theta, 0$</td>
<td>$\theta, 0$</td>
</tr>
<tr>
<td>$a_2$</td>
<td>$0, 0$</td>
<td>$0, 0$</td>
</tr>
</tbody>
</table>

true value $\theta^* = 1$

Action $a_1$ is robust to inference (strictly dominant at all nearby payoffs), but not strictly rationalizable for player 1.
Strict-Rationalizability

Conjecture: robustness to inference for rationalizability characterized by strict rationalizability.

Simple counterexample:

\[
\begin{align*}
a_1 & \quad \text{true value } \theta^* = 1 \\
a_2 &
\end{align*}
\]

Action \( a_1 \) is robust to inference (strictly dominant at all nearby payoffs), but not strictly rationalizable for player 1.
Weak Strict-Rationalizability

Alternative definition of strict rationalizability:

Recursively eliminate (up to) one action \( a_i \) that is not a strict best reply to any surviving opponent action.

\[
\begin{array}{c|cc}
 a_3 & a_4 \\
 a_1 & 1,0 & 1,0 \\
 a_2 & 0,0 & 0,0 \\
\end{array}
\]

Action \( a_i \) is weakly strict-rationalizable if survives every possible order of one-at-a-time elimination.
Weak Strict-Rationalizability

Alternative definition of strict rationalizability:

Recursively eliminate (up to) one action $a_i$ that is not a strict best reply to any surviving opponent action.

\[
\begin{array}{c|cc}
  & a_3 & a_4 \\
  a_1 & 1, 0 & 1, 0 \\
  a_2 & 0, 0 & 0, 0 \\
\end{array}
\]

Action $a_i$ is \textit{weakly strict-rationalizable} if survives every possible order of one-at-a-time elimination.
Weak Strict-Rationalizability

Alternative definition of strict rationalizability:

Recursively eliminate (up to) one action $a_i$ that is not a strict best reply to any surviving opponent action.

<table>
<thead>
<tr>
<th>$a_3$</th>
<th>$a_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_1$</td>
<td>1, 0</td>
</tr>
<tr>
<td>$a_2$</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Action $a_i$ is weakly strict-rationalizable if survives every possible order of one-at-a-time elimination.
Theorem 2. Assume nontrivial inference and richness. Then, the rationalizability of $a_i^*$ is robust to inference if $a_i^*$ is strictly rationalizable in the complete information game with payoffs $u^*$, and only if it is weakly strict-rationalizable.
Recap

Theorems 1 and 2 show that strict solutions of the limit game are solutions with high probability when players observe sufficient data:

\[ p_n^{NE}(a) \rightarrow 1 \quad \text{and} \quad p_n^{R}(i, a_i) \rightarrow 1 \]

for strict equilibria \( a \) and strict rationalizable \( a_i \).

How large are \( p_n^{NE}(a) \) and \( p_n^{R}(i, a_i) \) for \( n \) far from the limit?
Finite Data Behavior
Lower Bound on $p_n^{NE}(a)$

$\exists$ constant $c > 0$ such that for every strict Nash Equilibrium $a$,

$$p_n^{NE}(a) \geq 1 - \frac{c}{\delta_{NE}^a} \mathbb{E}_{P^n} \left( \sup_{f \in F} d(f(Z_n), \mu) \right)$$
Lower Bound on $p_n^{NE}(a)$

$\exists$ constant $c > 0$ such that for every strict Nash Equilibrium $a$,

$$p_n^{NE}(a) \geq 1 - \frac{c}{\delta_a^{NE}} \mathbb{E}_{P^n} \left( \sup_{f \in F} d(f(Z_n), \mu) \right)$$

degree of "strictness" of equilibrium
Equilibrium:

\[ \delta^\text{NE}_a = \text{largest } \delta \text{ s.t. } a \text{ is a } \delta\text{-strict Nash equilibrium}. \]

Rationalizability:

\[ \delta^R_{a_i} = \text{largest } \delta \text{ s.t. } a_i \text{ is } \delta\text{-strictly rationalizable. } \]
∃ constant $c > 0$ such that for every strict Nash Equilibrium $a$,

$$p_{n}^{NE}(a) \geq 1 - \frac{c}{\delta_{a}^{NE}} \mathbb{E}_{p_{n}} \left( \sup_{f \in \mathcal{F}} d(f(Z_{n}), \mu) \right)$$
Lower Bound on $p_{n}^{NE}(a)$

\[ \exists \text{ constant } c > 0 \text{ such that for every strict Nash Equilibrium } a, \]

\[ p_{n}^{NE}(a) \geq 1 - \frac{c}{\delta_{a}^{NE}} \mathbb{E}_{P_{n}} \left( \sup_{f \in F} d(f(Z_{n}), \mu) \right) \]

rate at which rules in $\mathcal{F}$ jointly learn parameter
Rate of Joint Learning

Decompose into:

(1) Rate of individual learning: how fast does
\[ \mathbb{E}_{P^n} (d(f(Z_n), \mu)) \to 0 \]

for each \( f \in \mathcal{F} \)? Depends on “complexity” of learning problem.

e.g. OLS: number of covariates, kernel regression: “smoothness” of underlying function, classification: VC dimension
(2) Opinion diversity across $\mathcal{F}$

$\rightarrow$ nature of correlation across beliefs induced by rules in $\mathcal{F}$

**Proposition:** For every finite $\mathcal{F} = \{f_1, \ldots, f_K\}$,

$$1 - \sum_{k=1}^{K} p_{k}^{NE} \leq p_{n}^{NE}(a) \leq 1 - \min_{k\in\{1,\ldots,n\}} p_{k}^{NE}(a)$$

where $p_{k}^{NE}$ is defined for $\mathcal{F}_k = \{f_k\}$. (Frechet-Hoeffding bound.)
Lower Bound on $p_n^R(i, a_i)$

\[ \exists c > 0 \text{ such that for every strictly rationalizable action } a_i, \]

\[ p_n^R(i, a_i) \geq 1 - \frac{c}{\delta_{a_i}^R} E_{\mu} \left( \sup_{f \in \mathcal{F}} d(f(Z_n)), \mu \right). \]
Example: Obfuscation using Irrelevant Data
Example: Risky Joint Investment

→ Two banks decide whether or not to invest in a risky project
→ Payoffs are

<table>
<thead>
<tr>
<th></th>
<th>Invest</th>
<th>Don’t Invest</th>
</tr>
</thead>
<tbody>
<tr>
<td>Invest</td>
<td>(\theta, \theta)</td>
<td>(\theta - c, 0)</td>
</tr>
<tr>
<td>Don’t Invest</td>
<td>(0, \theta - c)</td>
<td>(0, 0)</td>
</tr>
</tbody>
</table>

where \(c > 0\) and \(\theta \in \mathbb{R}\) is the unknown return to the invest.

→ Is Invest rationalizable?
Central Bank Releases Data

→ Central bank observes \( n \) past projects and their returns.
→ Each project described by \((x^1_k, \ldots, x^p_k) \in \mathbb{R}^p\), and its return is

\[
\theta_k = \beta_0 + \beta_1 x^1_k + \cdots + \beta_{p^*} x^{p^*}_k + \epsilon_k, \quad \epsilon_k \sim \mathcal{N}(0, \sigma^2)
\]

where \( p^* < p \).
→ Bank reports \( \{(x^1_k, \ldots, x^{p'}_k, \theta_k)\}_{k=1}^n \), where \( p' \geq p^* \).
→ Players observe data, find best linear fit, predict return for current project (described by \((x^*_1, \ldots, x^*_{p'})\)).
→ \( \mathcal{F} \) consists of all maps from data into a belief over the 95% confidence interval for the prediction.
**Extraneous Data Decreases Investment**

**Corollary.** Suppose $\theta^* > 0$. Then for every $n \geq 1$,

$$p^R_n (i, \text{Invest}) \geq 1 - \frac{1}{|\theta^*|} \phi(p')$$

where $\phi$ is monotonically increasing in the number of reported features $p'$.

→ reporting extraneous variables reduces probability that investment is rationalizable.
Concluding Remarks
Conclusion

Proposed a learning-based refinement for the universal type space.

Weak solutions are not good predictions.

Strict solutions are good predictions if players observe a lot of data.

Non-equilibrium/rationalizable play may be expected in complex informational environments with small quantities of data.
Thank You
Discussion

Misspecification

- can relax uniform consistency to uniform convergence to $\delta^a_{NE}$-ball (or $\delta^R_{ai}$-ball) around point mass on $\theta^*$. 

Private Data

- unmodeled: how players form beliefs over other players’ data.
- without restriction, results need not hold (email-game style counterexamples).

Limit Uncertainty

- in slides, beliefs converge to point mass on $\theta^*$.
- can generalize to convergence to a “limit common prior” on $\Theta$. 