Random Risk Tolerance: 
a Model of Asset Pricing and Trade Volume

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in Honor of Bob Lucas
When I met Bob

- Trading Volumes in Asset Markets
When I met Bob

- Trading Volumes in Asset Markets
- Preference Shocks and Risk Sharing
When I met Bob

- Trading Volumes in Asset Markets
- Preference Shocks and Risk Sharing
- 25 years later put them in the same paper
Large empirical literature which studies trade volume and asset prices:

- Expected returns and trade volume.
- Expected returns and trade volume \( \times \) autocorrelation of returns.
- Trade volume as a pricing factor

Campbell, Grossman & Wang 93, Amihud 02, Llorente et al 02, Pastor Stambaugh 03, Lo & Wang 02, 06, \ldots
Empirical Literature

- Large empirical literature which studies trade volume and asset prices:
  - Expected returns and trade volume.
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- Campbell, Grossman & Wang 93, Amihud 02, Llorente et al 02, Pastor Stambaugh 03, Lo & Wang 02, 06, ...

- We develop a general equilibrium model of the pricing of the risk that one will want to trade.
Dispersion in idiosyncratic shocks to desired portfolios generates trade volumes.

Aggregate shocks to desired portfolios drive changes in asset prices.

Interaction of idiosyncratic and aggregate risks are priced.
- similar to idiosyncratic endowment shocks
- Mankiw 86 and Constantinides and Duffie 96
General equilibrium model: prices and quantities.

Three periods.

- $t = 0$ Everyone ex-ante identical
- $t = 1$ “Identity" risk: idiosyncratic and aggregate shocks to risk tolerance
- $t = 2$ “Outcome" risk: endowments realized and consumed

Dispersion of risk-tolerance: trade volumes at $t = 1$

Aggregate shocks: asset prices at $t = 1$

- How are these two risks priced at $t = 0$?
- impact of frictions on prices and welfare — a transactions tax
Develop a tractable GE asset pricing framework:

- solve trading volumes and asset prices
- preferences imply a seller of risky securities has suffered a negative idiosyncratic shock
- precise mathematical analogy to idiosyncratic endowment shocks
  Mankiw 86
- idiosyncratic risk measured from interaction of trading volumes and aggregate risk premia

First order welfare loss from a tax on asset transactions
Plan for Talk

1. 3 period model set up.
2. preferences.
3. equilibrium w/complete and incomplete market.
4. key properties of equicautious HARA preferences
5. asset pricing and trade volumes
6. Tobin tax on asset trade.
Three period model: time line and shocks

**ex-ante identical investors at** $t = 0$.

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aggregate shocks:  

$z \sim \pi(\cdot)$  

$y \sim \rho(\cdot | z)$
Three period model: time line and shocks

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### Three period model: time line and shocks

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<td>$C_0$</td>
<td>shocks to risk-tolerance $\frac{-U'<em>\tau}{U''</em>\tau}$</td>
<td>$c(\tau, y, z)$</td>
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Investors: *ex-ante identical*, *ex-post difference in* $\tau$

\[ t = 0 \]
\[ C_0 \]
\[ \pi(z_1) \]
\[ \pi(z_2) \]
\[ \tau \sim \mu(\cdot, z_1) \]
\[ \tau \sim \mu(\cdot, z_2) \]

\[ t = 1 \]
\[ \rho(y_1 | z_1) \]
\[ y_1 \sim \sum_{\tau} c(\tau, y_1, z_1) \mu(\tau, z_1) = y_1 \]
\[ y_2 \sim \cdots \]
\[ y_3 \sim y_1 \sim \cdots \]

\[ t = 2 \]
\[ \text{shocks to output} \]
\[ \rho(y_3 | z_2) \]
\[ y_3 \sim \sum_{\tau} c(\tau, y_3, z_2) \mu(\tau, z_2) = y_3 \]

\[ t = 0 \]
\[ \text{shocks to risk tolerance} \]
Investor’s preferences

- From time \( t = 1 \) to \( t = 2 \), equicautious HARA utility w/risk tolerance \( \tau \).

- Distribution of risk tolerance \( \tau \sim \mu(\cdot, z) \) at each \( z \) in time \( t = 1 \).

- At \( t = 0 \) investor use expected utility on time \( t = 1 \) Certainty Equivalent (C.E.)

- Using C.E. isolates other attitudes of investor’s preferences

\[
V(C_0) + \beta \sum_z \sum_\tau V \left( U^{-1}_\tau \left( \sum_y U_\tau(c(\tau, y, z)) \rho(y|z) \right) \right) \mu(\tau, z) \pi(z)
\]

\( C_1(\tau, z) : \text{Certainty Equivalence for} \ \tau \ \text{at} \ z \)
Investor’s preferences

- From time $t = 1$ to $t = 2$, equicautious HARA utility w/risk tolerance $\tau$.

- At $t = 0$ investors use expected utility on time $t = 1$ Certainty Equivalent (C.E.)

\[
V(C_0) + \beta \sum_z \sum_{\tau} V(C_1(\tau, z)) \mu(\tau, z) \pi(z)
\]

\[
C_1(\tau, z) \equiv U_{\tau}^{-1} \left( \sum_y U_{\tau} (c(\tau, y, z)) \rho(y|z) \right)
\]

- Arrow-Pratt Theorem — lower risk tolerance $\tau \implies$ lower $C_1(\tau, z)$ from same allocation $c(\cdot, y, z)$ at $t = 2$
$U_\tau(c) = \left( \frac{\gamma}{1 - \gamma} \right) \left( \frac{c}{\gamma} + \tau \right)^{1-\gamma}$ for $\gamma \neq 1$

$U_\tau(c) = \log(c + \tau)$ for $\{ c : \tau + c > 0 \}$ for $\gamma = 1$

$U_\tau(c) = -\tau \exp(-c/\tau)$ as $\gamma \to \infty$,

$U'_\tau(c) = \left( \frac{c}{\gamma} + \tau \right)^{-\gamma} > 0$, $U''_\tau(c) = - \left( \frac{c}{\gamma} + \tau \right)^{-\gamma-1} < 0$

$\tau$ shifts risk tolerance

$R_\tau(c) \equiv - \frac{U'_\tau(c)}{U''_\tau(c)} = \frac{c}{\gamma} + \tau$
From $t = 1$ onwards, given $\tau$, maximize

$$\mathbb{E} U_\tau(c(\tau, y, z)) = \sum_y U_\tau(c(\tau, y; z))\rho(y|z)$$

$$\sum_y p(y; z)c(\tau, y; z)\rho(y|z) \leq \sum_y p(y; z)[y + B(\tau; z)]\rho(y|z)$$

implied $C_1(\tau; z)$
From $t = 1$ onwards, given $\tau$, maximize

$$\mathbb{E} U_\tau (c(\tau, y, z)) = \sum_y U_\tau (c(\tau, y; z)) \rho(y|z)$$

$$\sum_y p(y; z)c(\tau, y; z)\rho(y|z) \leq \sum_y p(y; z)[y + B(\tau; z)]\rho(y|z)$$

implied $C_1(\tau; z)$

Time $t = 0$ choose initial consumption and $\tau$ - contingent bonds s.t.

$$C_0 + \sum_{\tau, z} Q(\tau; z)B(\tau; z)\mu(\tau; z)\pi(z) = \bar{C}_0$$

bond market clearing

$$\sum_\tau B(\tau; z)\mu(\tau; z) = 0$$
From \( t = 1 \) onwards, given \( \tau \), maximize

\[
\mathbb{E} U_\tau (c(\tau, y, z)) = \sum_y U_\tau (c(\tau, y; z)) \rho(y|z)
\]

\[
\sum_y p(y; z) c(\tau, y; z) \rho(y|z) \leq \sum_y p(y; z) [y + B(z)] \rho(y|z)
\]

implied \( C_1(\tau; z) \)
Equilibrium with Incomplete Markets

- From $t = 1$ onwards, given $\tau$, maximize

$$\mathbb{E} U_\tau(c(\tau, y, z)) = \sum_y U_\tau(c(\tau, y; z)) \rho(y|z)$$

$$\sum_y p(y; z) c(\tau, y; z) \rho(y|z) \leq \sum_y p(y; z) [y + B(z)] \rho(y|z)$$

- implied $C_1(\tau; z)$

- Time $t = 0$ choose initial consumption and bonds s.t.

$$C_0 + \sum_z Q(z) B(z) \pi(z) = \bar{C}_0$$

- bond market clearing

$$B(z) = 0$$
Asset Prices

- Time $t = 1$ risk free bond price is numeraire

$$\sum_y p(y; z)\rho(y|z)dy \equiv 1$$

- Time $t = 1$ share price

$$D_1(z) \equiv \sum_y p(y; z)y\rho(y|z)$$

- Time $t = 1$ asset $d(y; z)$

$$P_1(z; d) \equiv \sum_y p(y; z)d(y; z)\rho(y|z)$$

- Time $t = 0$ asset $d(y; z)$

$$P_0(d) = \sum_z Q(z)P_1(z; d)\pi(z)$$
Two Stage Budgeting

- At $t = 1$, cost of C.E. consumption $C_1$: given prices $p(y; z)$ and $\tau$,

$$H_\tau(C_1; z) = \min_{c(y; z)} \sum_y p(y; z)c(y; z)\rho(y|z)$$

subject to $c(y; z)$ delivers C.E. consumption $C_1$ for investor $\tau$

- Budget constraints in C.E. consumption

$$H_\tau(C_1(\tau; z); z) = D_1(z) + B(z)$$

$$C_0 + \sum_z Q(z)B(z)\pi(z) = \bar{C}_0$$

- Date $t = 0$ asset prices with risk to $\tau$

$$Q(z) = \beta \sum_\tau \left[ \frac{V'(C_1(\tau; z))}{V'(C_0)} \frac{\partial}{\partial C_1} H_\tau(C_1(\tau; z); z) \right] \mu(\tau; z)$$
Psuedo-resource constraint for C.E. consumption

Average Risk Tolerance:

$$\bar{\tau}(z) = \sum_{\tau} \tau \mu(\tau; z)$$

C.E. consumption for average investor from consuming endowment $y$

$$\tilde{C}_1(z) \equiv U_{\bar{\tau}(z)}^{-1} \left( \sum_{y} U_{\tilde{\tau}(z)}(y) \rho(y|z) \right)$$

All conditionally optimal allocations at $t = 1$ satisfy

$$\sum_{\tau} C_1(\tau; z) \mu(\tau; z) = \tilde{C}_1(z)$$
Conditionally Efficient Allocations of C.E. Consumption

- Psuedo-resource constraint for C.E. consumption

- Average Risk Tolerance:
  \[ \bar{\tau}(z) = \sum_z \tau \mu(\tau; z) \]

- C.E. consumption for average investor from consuming endowment \( y \)
  \[ \tilde{C}_1(z) \equiv U^{-1}_{\bar{\tau}(z)} \left( \sum_y U_{\bar{\tau}(z)}(y) \rho(y|z) \right) \]

- All conditionally optimal allocations at \( t = 1 \) satisfy
  \[ \sum_\tau C_1(\tau; z) \mu(\tau; z) = \tilde{C}_1(z) \]

- Implies optimal allocation as of \( t = 0 \) has \( C_1(\tau; z) = \tilde{C}_1(z) \) for all \( \tau \)

- Identity risk not priced with complete asset markets
Gorman Aggregation

- At $t = 1$ endowment risk priced by investor with average risk tolerance $\bar{\tau}(z)$

- $p(y; z) = \bar{p}(y; z)$ independent of bondholdings and dispersion in $\tau$

- same with share price $D_1(z) = \bar{D}_1(z) = \sum_y \bar{p}(y; z)y \rho(y|z)$

- C.E. cost functions $H_{\tau}(C_1; z)$ pinned down and common marginal cost

\[
J(z) \equiv 1 / \frac{\partial}{\partial C_1} H_{\tau}(C_1(\tau; z); z)
\]

- Equilibrium allocation ($B(z) = 0$)

\[
C_1^c(\tau; z) = \bar{C}_1(z) + \left( \frac{\tau - \bar{\tau}(z)}{\bar{D}_1(z) + \bar{\tau}(z)} \right) \left[ \bar{C}_1(z) - \bar{D}_1(z) \right]
\]
Equilibrium allocation of C.E. consumption at $t = 1$

$$C^e_1(\tau; z) = \tilde{C}_1(z) + \left( \frac{\tau - \bar{\tau}(Z)}{\bar{D}_1(z) / \gamma + \bar{\tau}(Z)} \right) [\tilde{C}_1(z) - \tilde{D}_1(z)]$$

Aggregate risk premium $[\tilde{C}_1(z) - \tilde{D}_1(z)]$

- $\tilde{C}_1(z)$ cost of aggregate C.E. consumption in bonds
- $\tilde{D}_1(z)$ cost of aggregate C.E. consumption in shares

Equilibrium share trade volume

$$\phi^e(\tau; z) - 1 = \left( \frac{\tau - \bar{\tau}(Z)}{\bar{D}_1(z) / \gamma + \bar{\tau}(Z)} \right)$$

C.E. consumption risk seen in trade volumes and aggregate risk premia
Asset Pricing at $t = 0$

- **Date $t = 0$ bond prices**
  \[
  Q^e(z) = \beta \frac{V'(\tilde{C}_1(z))}{V'(\tilde{C}_0)} J(z) L(z)
  \]

- **$L(z)$ reflects dispersion in C.E. consumption**
  \[
  L(z) \equiv \sum_{\tau} \frac{V'(C^e_1(\tau; z))}{V'(\tilde{C}_1(z))} \mu(\tau; z)
  \]
Asset Pricing at $t = 0$

- Date $t = 0$ bond prices

$$Q^e(z) = \beta \frac{V'(\tilde{C}_1(z))}{V'({\bar{C}}_0)} J(z)L(z)$$

- $L(z)$ reflects dispersion in C.E. consumption

$$L(z) \equiv \sum_{\tau} \frac{V'(C^e_1(\tau; z))}{V'(\tilde{C}_1(z))} \mu(\tau; z)$$

and thus aggregate risk premia and trade volumes

$$L(z) \approx 1 + \frac{V'''(\tilde{C}_1(z))}{V'(\tilde{C}_1(z))} (\tilde{C}_1(z) - \bar{D}_1(z))^2 \sum_{\tau} (\phi^e(\tau; z) - 1)^2 \mu(\tau; z)$$

- and precautionary motives $V'''(\cdot) > 0$

- aggregate trade volume

$$TV^e(z) = \frac{1}{2} \sum_{\tau} |\phi^e(\tau; z) - 1| \mu(\tau; z)$$
Trade Volume as a Pricing Factor

- ex-ante expected excess returns in the complete markets economy
  \[ E^*(d) - 1 = - \text{Cov}(Q^*(z), R_1(z; d)) \]

- and in the incomplete markets economy
  \[ E^e_1(d) - 1 = (E^*(d) - 1) - \beta \frac{\text{Cov}(J(z)\Delta(z), R_1(z; d))}{V'(\bar{C}_0)} \]
Trade Volume as a Pricing Factor

- ex-ante expected excess returns in the complete markets economy
  \[ E^*(d) - 1 = -\text{Cov}(Q^*(z), R_1(z; d)) \]

- and in the incomplete markets economy
  \[ E^e_1(d) - 1 = (E^*(d) - 1) - \frac{\beta}{V'(\bar{C}_0)} \text{Cov}(J(z)\Delta(z), R_1(z; d)) \]

- \( \Delta(z) \) reflects dispersion in C.E. consumption
  \[ \Delta(z) \equiv \sum_{\tau} \left[ V'(C^e_1(\tau; z)) - V'(\bar{C}_1(z)) \right] \mu(\tau; z) \]

- and thus aggregate risk premia and trade volumes
  \[ \Delta(z) \approx \frac{\beta}{2} \frac{V'''(\bar{C}_1(z))}{V'(\bar{C}_0)} (\bar{C}_1(z) - \bar{D}_1(z))^2 \sum_{\tau} (\phi^e(\tau; z) - 1)^2 \mu(\tau; z) \]

- and precautionary motives \( V''''(\cdot) > 0 \)
Impact of a transactions tax

- We have examined an environment with no trading frictions.

- What is the impact of trading frictions on asset prices and welfare?

- Example: Tobin taxes on trading shares
  - Transaction tax $\omega$ on rebalancing trade of shares vs bonds at time $t = 1$.
  - Proceeds rebated equally to all investors at time $t = 1$.
  - Tax $\omega$: wedge between the buying and selling price of shares to dividend $y$.

- $W(\omega)$ time $t = 0$ ex-ante welfare with tax $\omega$:

$$W(\omega) = V(C_0) + \sum_z \sum_\tau V(C_1(\tau, z; \omega)) \mu(\tau, z) \pi(z)$$
First order effect of transaction tax

- $W(\omega)$ time $t = 0$ ex-ante welfare with tax $\omega$:

$$\frac{dW}{d\omega} = \beta \sum_z \pi(z) \sum_\tau \mu(\tau; z) V'(C_1(\tau; z)) \frac{d}{d\omega} C_1(\tau; z)$$

- Initial equilibrium marginal utility of C.E. consumption

$$V'(C_1(\tau; z))$$

- Incidence of tax on sellers ($\tau < \bar{\tau}(z)$) and buyers ($\tau > \bar{\tau}(z)$) of shares

$$\frac{d}{d\omega} C_1(\tau; z)$$
Complete Mkts: "standard" Ramsey-Harberger results

\[ \frac{\partial W(\omega)}{\partial \omega} \bigg|_{\omega=0} = 0 \]

initial equilibrium marginal utility of C.E. consumption all equal

\[ V'(C_1(\tau; z)) = V'(\bar{C}_1(z)) \]

incidence of tax averages to zero

\[ \sum_{\tau} \frac{d}{d\omega} C_1(\tau; z)\mu(\tau; z) = 0 \]
Incomplete Asset Markets

- Incomplete Mkts:
  \[ \frac{\partial W(\omega)}{\partial \omega} \bigg|_{\omega=0} < 0 \]

- Initial equilibrium marginal utility of C.E. consumption higher for low \( \tau \)
  \[ V'(C_1(\tau_{Low}; z)) > V'(C_1(\tau_{High}; z)) \]

- Incidence of tax averages falls on low \( \tau \)
  \[ \frac{d}{d\omega} C_1(\tau_{Low}; z) < 0 \]

- Low risk tolerant investors have relatively inelastic desire to sell shares
Conclusion

- General Equilibrium model of the risk that one will want to trade

- Wanting to sell risky assets is a negative shock

- Analogous to a negative endowment shock

- Risk manifest in data on trade volumes and aggregate risk premia

- Seen in pricing if distribution of trade volumes is correlated with aggregate shocks

- Tobin taxes exacerbate this risk