Random Risk Tolerance: a Model of Asset Pricing and Trade Volume

Fernando Alvarez   Andrew Atkeson

U Chicago   UCLA

in Honor of Bob Lucas

When I met Bob

- Trading Volumes in Asset Markets

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- Preference Shocks and Risk Sharing

When I met Bob

- Trading Volumes in Asset Markets
- Preference Shocks and Risk Sharing
- 25 years later put them in the same paper

Empirical Literature

- Large empirical literature which studies trade volume and asset prices:
  - Expected returns and trade volume.
  - Expected returns and trade volume $\times$ autocorrelation of returns.
  - Trade volume as a pricing factor
- Campbell, Grossman & Wang 93, Amihud 02, Llorente et al 02, Pastor Stambaugh 03, Lo & Wang 02, 06, . . .

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- Campbell, Grossman & Wang 93, Amihud 02, Llorente et al 02, Pastor Stambaugh 03, Lo & Wang 02, 06, . . .
- We develop a general equilibrium model of the pricing of the risk that one will want to trade.

Basic Idea

- Dispersion in idiosyncratic shocks to desired portfolios generates trade volumes
- Aggregate shocks to desired portfolios drive changes in asset prices
- Interaction of idiosyncratic and aggregate risks are priced
  - similar to idiosyncratic endowment shocks
  - Mankiw 86 and Constantinides and Duffie 96

Model Features

- General equilibrium model: prices and quantities.
- Three periods.
  - $t = 0$ Everyone ex-ante identical
  - $t = 1$ “Identity" risk: idiosyncratic and aggregate shocks to risk tolerance
  - $t = 2$ “Outcome" risk: endowments realized and consumed
- Dispersion of risk-tolerance: trade volumes at $t = 1$
- Aggregate shocks: asset prices at $t = 1$
  - How are these two risks priced at $t = 0$?
  - impact of frictions on prices and welfare — a transactions tax

Main Results

- Develop a tractable GE asset pricing framework:
  - solve trading volumes and asset prices
  - preferences imply a seller of risky securities has suffered a negative idiosyncratic shock
  - precise mathematical analogy to idiosyncratic endowment shocks Mankiw 86
  - idiosyncratic risk measured from interaction of trading volumes and aggregate risk premia
- First order welfare loss from a tax on asset transactions

Plan for Talk

1. 3 period model set up.
2. preferences.
3. equilibrium w/complete and incomplete market.
4. key properties of equicautious HARA preferences
5. asset pricing and trade volumes
6. Tobin tax on asset trade.

Three period model: time line and shocks

ex-ante identical investors at $t = 0$.

time $t = 0$	time $t = 1$	time $t = 2$
aggregate shocks:	$z \sim \pi(\cdot)$	$y \sim \rho(\cdot \mid z)$

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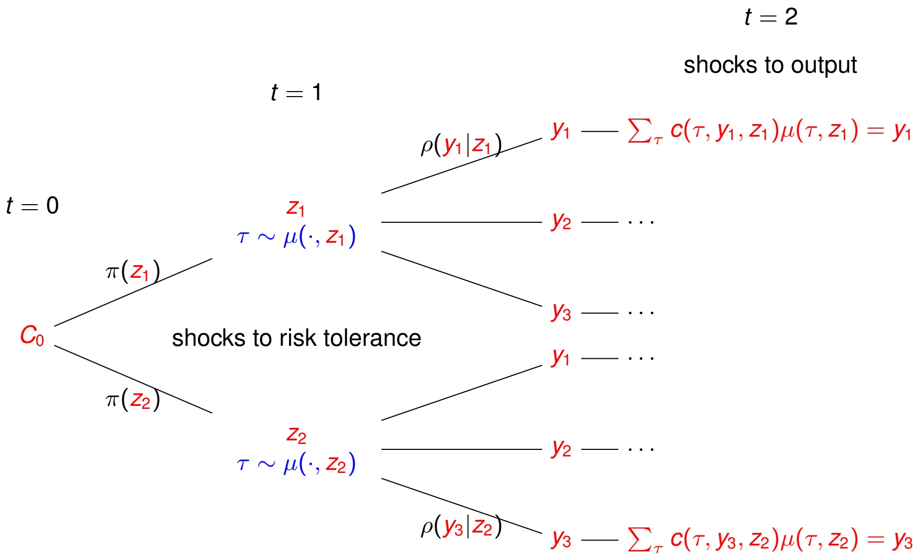
C_0 shocks to risk-tolerance $\frac{-U'_\tau}{U''_\tau}$ $c(\tau, y, z)$

Investors: *ex-ante identical*, ex-post difference in τ



Investor's preferences

- From time $t = 1$ to $t = 2$, equicautious HARA utility w/risk tolerance τ.
- Distribution of risk tolerance $\tau \sim \mu(\cdot, z)$ at each z in time $t = 1$.
- At $t = 0$ investor use expected utility on time $t = 1$ Certainty Equivalent (C.E.)
- Using C.E. isolates other attitudes of investor's preferences

$$V(C_0) + \beta \sum_z \sum_\tau V \left(\underbrace{U_\tau^{-1} \left(\sum_y U_\tau \left(c(\tau, y, z) \right) \rho \left(y | z \right) \right)}_{C_1(\tau, z)\,:\,\text{Certainty Equivalence for } \tau \text{ at } z} \right) \mu(\tau, z) \pi(z)$$

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$$V(C_0) + \beta \sum_z \sum_\tau V\left(C_1(\tau, z) \right) \mu(\tau, z) \, \pi(z)$$

$$C_1(\tau, z) \equiv U_\tau^{-1} \left(\sum_y U_\tau \left(c(\tau, y, z) \right) \rho\left(y|z\right) \right)$$

- Arrow-Pratt Theorem — lower risk tolerance $\tau \implies$ lower $C_1(\tau, z)$ from same allocation $c(\cdot, y, z)$ at $t = 2$

U_τ equicautious HARA preferences

$$U_\tau(c) = \left(\frac{\gamma}{1-\gamma}\right)\left(\frac{c}{\gamma} + \tau\right)^{1-\gamma} \quad \gamma \neq 1$$

$$U_\tau(c) = \log(c+\tau) \text{ for } \{c : \tau + c > 0\} \text{ for } \gamma = 1$$

$$U_\tau(c) = -\tau \exp(-c/\tau) \text{ as } \gamma \to \infty,$$

$$U_\tau'(c) = \left(\frac{c}{\gamma} + \tau\right)^{-\gamma} > 0\,,\ U_\tau''(c) = -\left(\frac{c}{\gamma} + \tau\right)^{-\gamma-1} < 0$$

τ shifts risk tolerance

$$\mathcal{R}_\tau(c) \equiv -\frac{U_\tau'(c)}{U_\tau''(c)} = \frac{c}{\gamma} + \tau$$

Optimum and Equilibrium with Complete Markets

- From $t = 1$ onwards, given τ, maximize

$$\mathbb{E}U_\tau(c(\tau, y, z)) = \sum_y U_\tau(c(\tau, y; z))\rho(y|z)$$

$$\sum_y p(y; z)c(\tau, y; z)\rho(y|z) \leq \sum_y p(y; z)\left[y + B(\tau; z)\right]\rho(y|z)$$

- implied $C_1(\tau; z)$

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- implied $C_1(\tau; z)$

- Time $t = 0$ choose initial consumption and τ - contingent bonds s.t.

$$C_0 + \sum_{\tau, z} Q(\tau; z)B(\tau; z)\mu(\tau; z)\pi(z) = \bar{C}_0$$

- bond market clearing

$$\sum_\tau B(\tau; z)\mu(\tau; z) = 0$$

Equilibrium with Incomplete Markets

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$$B(z) = 0$$

Asset Prices

- Time $t = 1$ risk free bond price is numeraire

$$\sum_y p(y;z)\rho(y|z)dy \equiv 1$$

- Time $t = 1$ share price

$$D_1(z) \equiv \sum_y p(y;z)y\rho(y|z)$$

- Time $t = 1$ asset $d(y;z)$

$$P_1(z;d) \equiv \sum_y p(y;z)d(y;z)\rho(y|z)$$

- Time $t = 0$ asset $d(y;z)$

$$P_0(d) = \sum_z Q(z)P_1(z;d)\pi(z)$$

Two Stage Budgeting

- At $t = 1$, cost of C.E. consumption C_1: given prices $p(y; z)$ and τ,

$$H_\tau(C_1; z) = \min_{c(y;z)} \sum_y p(y; z)c(y; z)\rho(y|z)$$

subject to $c(y; z)$ delivers C.E. consumption C_1 for investor τ

- Budget constraints in C.E. consumption

$$H_\tau(C_1(\tau; z); z) = D_1(z) + B(z)$$

$$C_0 + \sum_z Q(z)B(z)\pi(z) = \bar{C}_0$$

- Date $t = 0$ asset prices with risk to τ

$$Q(z) = \beta \sum_\tau \left[\frac{V'(C_1(\tau; z))}{V'(C_0)} \Big/ \frac{\partial}{\partial C_1} H_\tau(C_1(\tau; z); z) \right] \mu(\tau; z)$$

Conditionally Efficient Allocations of C.E. Consumption

- Psuedo-resource constraint for C.E. consumption
- Average Risk Tolerance:

$$\bar{\tau}(z) = \sum_{z} \tau \mu(\tau; z)$$

- C.E. consumption for average investor from consuming endowment y

$$\bar{C}_1(z) \equiv U^{-1}_{\bar{\tau}(z)} \left(\sum_{y} U_{\bar{\tau}(z)}(y) \rho(y|z) \right)$$

- All conditionally optimal allocations at $t = 1$ satisfy

$$\sum_{\tau} C_1(\tau; z) \mu(\tau; z) = \bar{C}_1(z)$$

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- Implies optimal allocation as of $t = 0$ has $C_1(\tau; z) = \bar{C}_1(z)$ for all τ
- Identity risk not priced with complete asset markets

Gorman Aggregation

- At $t = 1$ endowment risk priced by investor with average risk tolerance $\bar{\tau}(z)$
- $p(y; z) = \bar{p}(y; z)$ independent of bondholdings and dispersion in τ
- same with share price $D_1(z) = \bar{D}_1(z) = \sum_y \bar{p}(y; z) y \rho(y|z)$
- C.E. cost functions $H_\tau(C_1; z)$ pinned down and common marginal cost

$$J(z) \equiv 1 \bigg/ \frac{\partial}{\partial C_1} H_\tau(C_1(\tau; z); z)$$

- Equilibrium allocation ($B(z) = 0$)

$$C_1^e(\tau; z) = \bar{C}_1(z) + \left(\frac{\tau - \bar{\tau}(z)}{\frac{\bar{D}_1(z)}{\gamma} + \bar{\tau}(z)} \right) \left[\bar{C}_1(z) - \bar{D}_1(z) \right]$$

Two Fund Separation and Trade Volumes

- Equilibrium allocation of C.E. consumption at $t = 1$

$$C_1^e(\tau; z) = \bar{C}_1(z) + \left(\frac{\tau - \bar{\tau}(z)}{\frac{\bar{D}_1(z)}{\gamma} + \bar{\tau}(z)} \right) \left[\bar{C}_1(z) - \bar{D}_1(z) \right]$$

- aggregate risk premium $\left[\bar{C}_1(z) - \bar{D}_1(z) \right]$
  - $\bar{C}_1(z)$ cost of aggregate C.E. consumption in bonds
  - $\bar{D}_1(z)$ cost of aggregate C.E. consumption in shares
- equilibrium share trade volume

$$\phi^e(\tau; z) - 1 = \left(\frac{\tau - \bar{\tau}(z)}{\frac{\bar{D}_1(z)}{\gamma} + \bar{\tau}(z)} \right)$$

- C.E. consumption risk seen in trade volumes and aggregate risk premia

Asset Pricing at $t = 0$

- Date $t = 0$ bond prices

$$Q^e(z) = \beta \frac{V'(\bar{C}_1(z))}{V'(\bar{C}_0)} J(z) L(z)$$

- $L(z)$ reflects dispersion in C.E. consumption

$$L(z) \equiv \sum_{\tau} \frac{V'(C_1^e(\tau; z))}{V'(\bar{C}_1(z))} \mu(\tau; z)$$

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$$L(z) \equiv \sum_{\tau} \frac{V'(C_1^e(\tau; z))}{V'(\bar{C}_1(z))} \mu(\tau; z)$$

- and thus aggregate risk premia and trade volumes

$$L(z) \approx 1 + \frac{V'''(\bar{C}_1(z))}{V'(\bar{C}_1(z))} \left(\bar{C}_1(z) - \bar{D}_1(z)\right)^2 \sum_{\tau} \left(\phi^e(\tau; z) - 1\right)^2 \mu(\tau; z)$$

- and precautionary motives $V'''(\cdot) > 0$

- aggregate trade volume

$$TV^e(z) = \frac{1}{2} \sum_{\tau} |\phi^e(\tau; z) - 1| \mu(\tau; z)$$

Trade Volume as a Pricing Factor

- ex-ante expected excess returns in the complete markets economy

$$\mathcal{E}^*(d) - 1 = -\,Cov\,(Q^*(z), R_1(z;d))$$

- and in the incomplete markets economy

$$\mathcal{E}_1^e(d) - 1 = (\mathcal{E}^*(d) - 1) - \frac{\beta}{V'(\bar{C}_0)} Cov\,(J(z)\Delta(z), R_1(z;d))$$

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- $\Delta(z)$ reflects dispersion in C.E. consumption

$$\Delta(z) \equiv \sum_{\tau} \left[V'(C_1^e(\tau; z)) - V'(\bar{C}_1(z))\right] \mu(\tau; z)$$

- and thus aggregate risk premia and trade volumes

$$\Delta(z) \approx \frac{\beta}{2} \frac{V'''(\bar{C}_1(z))}{V'(\bar{C}_0)} \left(\bar{C}_1(z) - \bar{D}_1(z)\right)^2 \sum_{\tau} \left(\phi^e(\tau; z) - 1\right)^2 \mu(\tau; z)$$

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Impact of a transactions tax

- We have examined an environment with no trading frictions
- What is the impact of trading frictions on asset prices and welfare?
- Example: Tobin taxes on trading shares
  - Transaction tax ω on rebalancing trade of shares vs bonds at time $t = 1$.
  - Proceeds rebated equally to all investors at time $t = 1$.
  - Tax ω: wedge between the buying and selling price of shares to dividend y.
- $W(\omega)$ time $t = 0$ ex-ante welfare with tax ω:

$$W(\omega) = V(C_0) + \sum_z \sum_\tau V\left(C_1(\tau, z; \omega)\right) \mu(\tau, z)\, \pi(z)$$

First order effect of transaction tax

- $W(\omega)$ time $t = 0$ ex-ante welfare with tax ω:

$$\frac{dW}{d\omega} = \beta \sum_{z} \pi(z) \sum_{\tau} \mu(\tau; z) V'(C_1(\tau; z)) \frac{d}{d\omega} C_1(\tau; z)$$

- initial equilibrium marginal utility of C.E. consumption

$$V'(C_1(\tau; z))$$

- incidence of tax on sellers ($\tau < \bar{\tau}(z)$) and buyers ($\tau > \bar{\tau}(z)$) of shares

$$\frac{d}{d\omega} C_1(\tau; z)$$

Complete Asset Markets

- Complete Mkts: "standard" Ramsey-Harberger results

$$\frac{\partial W(\omega)}{\partial \omega}|_{\omega=0} = 0$$

- initial equilibrium marginal utility of C.E. consumption all equal

$$V'(C_1(\tau; z)) = V'(\bar{C}_1(z))$$

- incidence of tax averages to zero

$$\sum_{\tau} \frac{d}{d\omega} C_1(\tau; z)\mu(\tau; z) = 0$$

Incomplete Asset Markets

- incomplete Mkts:

$$\frac{\partial W(\omega)}{\partial \omega}\Big|_{\omega=0} < 0$$

- initial equilibrium marginal utility of C.E. consumption higher for low τ

$$V'(C_1(\tau_{Low}; z)) > V'(C_1(\tau_{High}; z))$$

- incidence of tax averages falls on low τ

$$\frac{d}{d\omega} C_1(\tau_{Low}; z) < 0$$

- low risk tolerant investors have relatively inelastic desire to sell shares

Conclusion

- General Equilibrium model of the risk that one will want to trade
- Wanting to sell risky assets is a negative shock
- analogous to a negative endowment shock
- risk manifest in data on trade volumes and aggregate risk premia
- seen in pricing if distribution of trade volumes is correlated with aggregate shocks
- Tobin taxes exacerbate this risk