Large Matching Markets
Discussion of Konrad Menzel

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Chicago—September 2014
Major Achievements

First tractable framework to analyze flexible NTU matching models empirically.

Identification of pseudo-surplus:

$$W = U + V$$

Nice formula to recover complementarities in preferences from assortative matching patterns:

$$D_{xz} \log f(x, z) = D_{xz} W(x, z)$$

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Identification of pseudo-surplus $W = U + V$

Nice formula to recover complementarities in preferences from assortative matching patterns:

$$D_{xz} \log f(x, z) = D_{xz} W(x, z).$$
Mysteries

This is NTU, so how does $U + V$ come about? and why not $D_{xz}(x, z) = \frac{1}{2} D_{xz}(x, z)$ as in Choo-Siow 2006? What assumptions really matter?

1. NTU vs TU
2. many $i | x$, many $j | z$
3. many $x$ and $z$
4. $\eta_{ij}, \zeta_{ji}$ are in MDA (Gumbel)
5. they have the same $\sigma$
6. they are iid.

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My answers

1. NTU vs TU: does not matter that much (Galichon–Kominers–Weber)

2. \(\text{many } i | x, \text{many } j | z:\) very important

3. \(\text{many } x \text{ and } z:\) almost irrelevant

4. \(\eta_{ij}, \zeta_{ji}\) are in MDA (Gumbel): \(??\) — e.g. trade people use Fréchet

5. \(\text{they have the same } \sigma:\) otherwise \(D_{xz}\) formula changes slightly

6. \(\eta_{ij}, \zeta_{ji}\) are iid: important but not so crucial (Galichon–Salanié).

Discussion of Menzel
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Intuitive and sloppy explanation

Drop singles for simplicity (everyone has to match)

Utility of woman

\[ u_i = \max \left\{ U(x_i, z_j) + \sigma \eta_{ij} \mid V(z_j, x_i) + \sigma \zeta_{ji} \geq v_j \right\} \]

Gumbel is in MDA(Gumbel), so assume \( \eta, \zeta \) are Gumbel ("logit")

the max of logits is logits:

\[ u_i = u_{x_i} + \bar{\eta}_i \]

with

\[ u_{x_i} = \log \sum_{j \in M_i} \mu \exp \left( U(x, z_j) \right) \]

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with

$$u_{x_i} = \log \sum_{j \in M_i[\mu]} \exp (U(x, z_j)).$$
Men of type $z$ available to $i$

$N_z(i) = m_z Pr \left( V(z, x_i) + \sigma \zeta_{ji} \geq v_j \right)$

But $v_j = v_z + \bar{\zeta}_j$ is a logit, so $N_z(i) = 1 + \exp(v_z - V(z, x_i))$.

With many available $j$'s of type $z$, $v_z \gg V(z, x_i)$; hence $N_z(i) \approx \exp(V(z, x_i) - v_z) \equiv N_z(x)$. 

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N_z(i) \approx \exp(V(z, x_i) - v_z) \equiv N_z(x).
\]
\[ \log \sum_N z(x) \exp(U(x, z)) \approx \log \sum_m z \exp(U(x, z) + V(z, x) - v z) \equiv \log \sum_m z \exp(W(x, z) - v z). \]

Coupled with

\[ v z = \log \sum_w x \exp(W(x, z) - u x). \]

Or, more symmetric: for all \( x \) and \( z \),

\[ \sum m z \exp(W(x, z) - u x - v z) = \sum w x \exp(W(x, z) - u x - v z) = 1. \]
\[ u_x = \log \sum_z N_z(x) \exp(U(x, z)) \]
\[ \simeq \log \sum_z m_z \exp(U(x, z) + V(z, x) - v_z) \]
\[ \equiv \log \sum_z m_z \exp(W(x, z) - v_z) \]
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coupled with \[ v_z = \log \sum_x w_x \exp(W(x, z) - u_x) \].
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coupled with \( v_z = \log \sum_x w_x \exp(W(x, z) - u_x) \).
Or, more symmetric: for all \( x \) and \( z \),
\[ \sum_z m_z \exp(W(x, z) - u_x - v_z) = \sum_x w_x \exp(W(x, z) - u_x - v_z) = 1. \]
How did $W$ come up?

$i \in x$ chooses $z$ with probability $\exp(U(x,z))$. A man $j \in z$ is available to her with probability $\exp(V(z,x))$. With e.g. heteroskedasticity $\sigma_x, \tau_z$ 'a la Chiappori–Salanié–Weiss:

$$\exp(U(x,z)/\sigma_x) \times \exp(V(z,x)/\tau_z).$$
How did $W$ come up?

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Matching patterns and pseudo-surplus

$\text{choose types } z \text{ with frequency } f(z|x) \approx N(z(x)) \exp U(x,z) \sum_t N_t(x) \exp(U(x,t)) \equiv \exp(W(x,z) - d(x))$

so that

$D_{xz} \log f(x,z) = D_{xz} \log f(z|x) = D_{xz} W(x,z)$.
Matching patterns and pseudo-surplus

Types $x$ choose types $z$ with frequency

$$f(z|x) \sim \frac{N_z(x) \exp U(x, z)}{\sum_t N_t(x) \exp(U(x, t))} \equiv \exp(W(x, z) - d(x))$$
Types $x$ choose types $z$ with frequency

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$$D_{xz} \log f(x, z) = D_{xz} \log f(z|x) = D_{xz} W(x, z).$$
Extension

Allowing for "fixed effects" a la Choo and Siow,

\[ U_{ij} = U(x_i, z_j) + f_{w,i}, z_j + \eta_{ij} \]

\[ V_{ji} = V(z_j, x_i) + f_{m,j}, x_i + \zeta_{ji} \]

with any distribution a la Galichon–Salanié:

\[ f_{w|x}, f_{m|z} \]

and iid \( \zeta_{ij} \) in MDA(Gumbel) as before.

Then we look for fixed points in functional spaces:

\[ u_{x}(f_{w}) \]

\[ v_{z}(f_{m}) \]

for all \( x \) and \( z \), and all \( f_{m} \) and \( f_{w} \),

\[ \sum_{z} \exp \left( W(x, z) + f_{w}z - u_x(f_{w}) \right) E(f_{m}x - v_z(f_{m})|z) = 1 \]

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Allowing for “fixed effects” à la Choo and Siow,

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\begin{align*}
U_{ij} &= U(x_i, z_j) + f_{i,z_j}^w + \eta_{ij} \\
V_{ji} &= V(z_j, x_i) + f_{j,x_i}^m + \zeta_{ji}
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Then we look for fixed points in functional spaces: \( u_x(f^w), v_z(f^m) \) for all \( x \) and \( z \), and all \( f^m \) and \( f^w \),

\[
\begin{align*}
\sum_z m_z \exp \left( W(x, z) + f^w_z - u_x(f^w) \right) E \left( f^m_x - v_z(f^m) \mid z \right) &= 1 \\
\sum_x w_x \exp \left( W(x, z) + f^m_x - v_z(f^m) \right) E \left( f^w_z - u_x(f^w) \mid x \right) &= 1.
\end{align*}
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What Changes

Still true: only $W = U + V$ is identified. But now

$$D_{xz} \log f(x, z) = D_{xz} W(x, z) + D_{xz} \log E(\exp (f_w z - u_x (f_w)))|x) + D_{xz} \log E(\exp (f_m x - v_z (f_m)))|z).$$
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$$+ D_{xz} \log E \left( \exp \left( f^m_x - v_z(f^m) \right) \mid z \right).$$
Special Case of Fixed Effects

Many values of $z$: then

$$\log E(\exp (f w z - u x (f w)))|x) \approx \log E(\exp (f w z)|x) + \log E(\exp(-u x (f w))|x).$$

Then the $D_{xz}$ of the second term is zero; If moreover the distribution of $f w$ does not depend on $x$, the $D_{xz}$ of the first term is zero and we get again $D_{xz} \log f(x, z) = D_{xz} W(x, z)$ even though we now allow for many forms of dependence.

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Special Case of Fixed Effects

Many values of $z$: then

$$\log E (\exp (f^w_z - u_x(f^w)) \mid x) \simeq \log E (\exp (f^w_z) \mid x)$$

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Special Case of Fixed Effects

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