Comparative Valuation Dynamics in Models with Financial Frictions

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June 20, 2018
Research Objective

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  - Financial intermediaries
  - Heterogeneous productivity, market access and preferences
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  - Macroeconomic quantity implications
  - Asset pricing implications
  - Macro- and micro-prudential policy
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- **Approach**: Nesting model
“Nesting” Model

- **Starting Point:** Brunnermeier & Sannikov (2016)
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  - A-K production function with $a_e \geq a_h$
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  - Idiosyncratic shocks
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  - Complete financial markets for households
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- **Preferences**
  - Recursive utility
  - Households and experts potentially different
  - OLG for technical reasons
“Nesting” Model

“Experts”

Assets
- Physical Capital
- Net Worth

Liabilities
- Risk Free
- External Equity

“Households”

Assets
- Physical Capital
- Risk Free
- Equities

Liabilities
- Net Worth
- Short Term
- Bonds

Dividends

Interest
Models Nested

- Complete markets with long run risk
  - Bansal & Yaron (2004)
  - Hansen, Heaton & Li (2008)
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  - Di Tella (2017)
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- **Incomplete market/capital misallocation models**
  - Brunnermeier & Sannikov (2014, 2016)
Overview of Solution Method

- Markov equilibrium – aggregate state vector $X_t$:
  - Exogenous states $g_t$ (growth), $s_t$ (agg. stochastic vol.), and $\varsigma_t$ (idio. stochastic vol.)
  - Endogenous state $w_t := \frac{N_{e,t}}{N_{e,t} + N_{h,t}}$ (wealth share)

"Value function" approach: $V_i(n_t, X_t) = n_{i+1} - \gamma_i \xi^i(X_t)$ (solutions to second order non-linear PDEs – implicit FD scheme with artificial time derivative ("false transient")

Each time-step: compute aggregate state dynamics and prices using the value functions from the previous time-step

Endogenous state partition due to occasionally-binding constraints

Implementation in C++ allowing for HPC

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Quantities

- Consumption/wealth ratio \((c_i/n_i)(X)\)
- Investment rate \(\iota(X)\)
- Output growth \(\mu_y(X)\)
Diagnostic Tools I

- **Quantities**
  - Consumption/wealth ratio $(c_i/n_i)(X)$
  - Investment rate $\iota(X)$
  - Output growth $\mu_y(X)$

- **Prices**
  - Risk-free rate $r(X)$
  - Risk-price vectors $\pi_i(X)$ (one per agent)
  - Capital price $q$
Quantities
- Consumption/wealth ratio \((c_i/n_i)(X)\)
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Aggregate state dynamics
- Drift \(\mu_X(X)\) and diffusion \(\sigma_X(X)\) of aggregate state vector
- Ergodic density \(f(X)\)
Diagnostic Tools II

- Transition dynamics and valuation through altering cash flow exposure to shocks

Formal Definition

\[ \epsilon_C(X, t; \vec{\nu}) \] gives us:

\[
\% \text{ ∆ in expected future cash-flow } C_t \text{ given a unit increase in exposure of that cash-flow to a shock (in direction } \vec{\nu} ) \text{ today}
\]

counterpart to IRF for models with non-linear state dynamics

\[ \epsilon_C(X, t; \vec{\nu}) - \epsilon_S \]

\[ \% \text{ ∆ in expected return (per unit of risk) of cash-flow } C_t \text{ perceived by investor } i \text{ given a unit increase in exposure of that cash-flow to a shock (in direction } \vec{\nu} ) \text{ today}
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counterpart to (investor-specific) Sharpe ratio for dividend strips

Those shock exposure and price elasticities depend on the current state \( X \), depends on the horizon \( t \), depends on the marginal investor \( i \) (for price elasticities).

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Diagnostic Tools II

- Transition dynamics and valuation through altering cash flow exposure to shocks
- Shock exposure elasticity $\epsilon_C(X, t; \vec{v})$ gives us:
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DSGE Models with Financial Frictions
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- Shock price elasticity $\epsilon_C (X, t; \vec{\nu}) - \epsilon_{S_iC} (X, t; \vec{\nu})$
  - % $\Delta$ in expected return (per unit of risk) of cash-flow $C_t$ perceived by investor $i$ given a unit increase in exposure of that cash-flow to a shock (in direction $\vec{\nu}$) today
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  - depends on the current state $X$;
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What shocks do investors care about as measured by expected return compensation?

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→ vary the initial state $X$ and the time-horizon $t$
Diagnostic Tools II – Why do we care?

- What shocks do investors care about as measured by expected return compensation?
  → vary the shock direction $\tilde{\nu}$

- How do these compensations vary across states and over horizons?
  → vary the initial state $X$ and the time-horizon $t$

- How severe are financial frictions?
  → how do shadow compensations differ across agents
Always vs. Occasionally Binding Constraints

- When is an always-binding-constraint assumption legitimate?
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- Economic setting of focus
  - Experts are the only producers (i.e. $a_h = -\infty$)
  - Skin-in-the-game constraint $\chi \geq \underline{\chi}$
  - TFP shocks only
  - EIS $\psi^{-1} = 1$
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- Compare
  1. homogeneous RRA ($\gamma_e = \gamma_h$) vs.
  2. heterogeneous RRA ($\gamma_e < \gamma_h$)
Always vs. Occasionally Binding Constraints

Expert’s risk-retention $\chi$ in the two models.

\[ \chi (\gamma_c = \gamma_h = 3) \]

\[ \chi (\gamma_c = 1, \gamma_h = 3) \]
Wealth share diffusion $\sigma_w$ in the two models.

$\sigma_w(\gamma_c = \gamma_h = 3)$

$\sigma_w(\gamma_c = 1, \gamma_h = 3)$
Expert’s shadow risk prices $\pi_e$ in the two models.
How do financial frictions affect agents’ attitudes about other shocks?
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- RRA $\gamma = 3$, EIS $\psi^{-1} = 1$
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Compare
1. model with frictions ($\chi = 0.5$) vs.
2. model without frictions ($\chi = 0$)
Expert’s and Household’s TFP risk prices $\pi_e^{(1)}, \pi_h^{(1)}$. 

\[ \pi_e : \text{TFP Shock} \]

\[ \pi_h : \text{TFP Shock} \]
Other Shocks and Financial Frictions

Expert’s and Household’s volatility risk prices $\pi_e^{(3)}, \pi_h^{(3)}$.

$\pi_e$ : Volatility Shock

$\pi_h$ : Volatility Shock
Who are the “expert” agents in the economy – productive or risk-tolerant?
Productivity vs. Risk-Tolerance

- Who are the “expert” agents in the economy – productive or risk-tolerant?

- Economic setting of focus
  - Experts and households can both produce (i.e. \( a_e \geq a_h > -\infty \))
  - No equity-issuance \( \chi \equiv \chi = 1 \)
  - Shocks to TFP level, growth rate, and volatility
  - EIS \( \psi^{-1} = 1 \)
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Compare
1. experts more productive ($a_e > a_h$ but $\gamma_e = \gamma_h$) vs.
2. experts more risk-tolerant ($\gamma_e < \gamma_h$ but $a_e = a_h$)
Capital distribution $\kappa$ in the two models.
Productivity vs. Risk-Tolerance

Expert’s TFP risk price $\pi_e^{(1)}$ in the two models.

Note: only $a_h$ and $\gamma_h$ differ between the two models.
Productivity vs. Risk-Tolerance

- Previous approach akin to “opening the black-box”
  - Keep all parameters fixed except $a_h$ and $\gamma_h$
  - Clean comparison of productivity versus risk-aversion
  - Obvious differences emerged
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Now, impose observational constraints to compare models
- Vary parameters across models to match desired empirical moments
- E.g., engineer two models with similar wealth distributions
- What distinguishes the models under this approach?
Expert’s TFP risk price $\pi_e^{(1)}$ in the two models.
TFP shock-exposure elasticities in the two models.

Perturbed variable: expert wealth share $w_t$
Conclusion / Next Steps

- Numerical approach: computation using GPU (instead of CPU)
- Consider additional types of financial constraints
- Analyze link between heterogenous preference models, heterogenous belief models, financial frictions’ models
- User-friendly web application to compare and contrast models
Efficiency units of capital $k_t$ follow

$$ dk_t = k_t \left[ (g_t + \iota_t - \delta) \, dt + \sqrt{s_t} \sigma \cdot dZ_t \right], \quad (1) $$

Exogenous state variables $(s_t, g_t)$ follow

$$ dg_t = \lambda_g (\bar{g} - g_t) \, dt + \sqrt{s_t} \sigma_g \cdot dZ_t \quad (2) $$

$$ ds_t = \lambda_s (\bar{s} - s_t) \, dt + \sqrt{s_t} \sigma_s \cdot dZ_t \quad (3) $$

Adjustment costs: investment $\iota_t k_t \, dt$ costs $\Phi(\iota_t) k_t \, dt$ in output
Markets

- Capital is freely traded, at price $q_t$
  
  $$dq_t = q_t[\mu_{q,t} dt + \sigma_{q,t} \cdot dZ_t]$$  (4)

- Households facing dynamically complete markets, leading to SDF
  
  $$dS_{h,t} = -S_{h,t}[r_t dt + \pi_{h,t} \cdot dZ_t]$$  (5)

- Experts face skin-in-the-game constraint via minimum risk retention:
  
  $$\chi_t \geq \chi$$  (6)

  - $\chi_t$ is fraction of capital held by experts that is “retained”

- Experts SDF
  
  $$dS_{e,t} = -S_{e,t}[r_t dt + \pi_{e,t} \cdot dZ_t]$$  (7)
Agent $i$ will solve the following problem:

$$U_{i,t} = \max_{\{k_{i} \geq 0, c_{i}, \theta_{i}, \iota_{i}\}} \mathbb{E} \left[ \int_{t}^{+\infty} \varphi (c_{i,s}, U_{i,s}) \, ds \right]$$

s.t. $$\frac{dn_{i,t}}{n_{i,t}} = \left[ \mu_{n,i,t} - \frac{c_{i,t}}{n_{i,t}} \right] dt + \sigma_{n,i,t} \cdot dZ_{t}$$

$$\mu_{n,i,t} = r_{t} + \frac{q_{t} k_{i,t}}{n_{i,t}} \left( \mu_{R,i,t} - r_{t} \right) + \theta_{i,t} \cdot \pi_{t}$$

$$\sigma_{n,i,t} = \frac{q_{t} k_{i,t}}{n_{i,t}} \sigma_{R,t} + \theta_{i,t}$$

$$\theta_{i,t} \in \Theta_{i,t}$$

Financial constraint set $\Theta_{i,t}$:

- $\Theta_{i,t} = \{0\}$: agent cannot issue “equity” securities
- $\Theta_{i,t} = \{(\chi_{t} - 1) \frac{q_{t} k_{i,t}}{n_{i,t}} \sigma_{R,t}, \chi_{t} \geq \chi\}$: “skin-in-the-game” constraint
- $\Theta_{i,t} = \mathbb{R}^{d}$: unconstrained agent
Numerical Implementation: Value Functions

Statement of the problem. Scaled value functions $\xi_i$ solve PDEs like

$$0 = K_i + A_i \xi_i + B_i \cdot \partial_x \xi_i + \text{trace}[C_i C_i' \partial_{xx'} \xi_i], \quad x = (w, g, s, \varsigma),$$

where the coefficients are:

$$K_i = K_i(x, \xi_e, \xi_h, \partial_x \xi_e, \partial_x \xi_h)$$
$$A_i = A_i(x, \xi_e, \xi_h, \partial_x \xi_e, \partial_x \xi_h)$$
$$B_i = B_i(x, \xi_e, \xi_h, \partial_x \xi_e, \partial_x \xi_h)$$
$$C_i = C_i(x, \xi_e, \xi_h, \partial_x \xi_e, \partial_x \xi_h)$$

The dependence of $A, B, C$ on $(\xi_e, \xi_h)$ arises due to general equilibrium.

We solve this PDE system with a 2-step iterative approach:

- given coefficients, we solve the linear PDE and obtain $\{\xi_i\}_{i=e,h}$
- given PDE solution $\{\xi_i\}_{i=e,h}$, we update coefficients
Step 1. Augment the PDE with a “false transient,” which is an artificial time-derivative $\partial_t \xi_i$:

$$\partial_t \xi_i = K_i + A_i \xi_i + B_i \cdot \partial_x \xi_i + \text{trace}[C_i C'_i \partial_{xx'} \xi_i],$$

where

$$K_i = K_i(x, \xi_e, \xi_h, \partial_x \xi_e, \partial_x \xi_h)$$
$$A_i = A_i(x, \xi_e, \xi_h, \partial_x \xi_e, \partial_x \xi_h)$$
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$$C_i = C_i(x, \xi_e, \xi_h, \partial_x \xi_e, \partial_x \xi_h)$$
Step 2. Given an iterant or guess \((\xi^{(t)}_e, \xi^{(t)}_h)\), we substitute the coefficients \( (K^{(t)}_i, A^{(t)}_i, B^{(t)}_i, C^{(t)}_i) \).

\[ \partial_t \xi_i = K^{(t)}_i + A^{(t)}_i \xi_i + B^{(t)}_i \cdot \partial_x \xi_i + \text{trace}[C^{(t)}_i C^{(t)}_i', \partial_{xx'} \xi_i], \]

where

\[
\begin{align*}
K^{(t)}_i &= K_i(x, \xi^{(t)}_e, \xi^{(t)}_h, \partial_x \xi^{(t)}_e, \partial_x \xi^{(t)}_h) \\
A^{(t)}_i &= A_i(x, \xi^{(t)}_e, \xi^{(t)}_h, \partial_x \xi^{(t)}_e, \partial_x \xi^{(t)}_h) \\
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C^{(t)}_i &= C_i(x, \xi^{(t)}_e, \xi^{(t)}_h, \partial_x \xi^{(t)}_e, \partial_x \xi^{(t)}_h)
\end{align*}
\]
Numerical Implementation: Value Functions

**Step 3.** Discretize the time derivatives and write all spatial derivatives in terms of $\xi_i^{(t+\Delta)}$ (“implicit”, as opposed to “explicit” scheme), i.e.,

$$
\frac{\xi_i^{(t+\Delta)} - \xi_i^{(t)}}{\Delta} = K_i^{(t)} + A_i^{(t)} \xi_i^{(t+\Delta)} + B_i^{(t)} \cdot \partial_x \xi_i^{(t+\Delta)} + \text{tr}[C_i^{(t)} C_i^{(t)'} \partial_{xx} \xi_i^{(t+\Delta)}],
$$

where

$$
K_i^{(t)} = K_i(x, \xi_e^{(t)}, \xi_h^{(t)}, \partial_x \xi_e^{(t)}, \partial_x \xi_h^{(t)})
$$

$$
A_i^{(t)} = A_i(x, \xi_e^{(t)}, \xi_h^{(t)}, \partial_x \xi_e^{(t)}, \partial_x \xi_h^{(t)})
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$$
C_i^{(t)} = C_i(x, \xi_e^{(t)}, \xi_h^{(t)}, \partial_x \xi_e^{(t)}, \partial_x \xi_h^{(t)})
$$

To insure scheme “monotonicity”,

- “Upwinding” for discretization of $\partial_x \xi_i^{(t+\Delta)}$;
- Cross-partial derivatives computed using $\xi_i^{(t)}$ and added to previous iterant $K_i^{(t)}$. 
**Numerical Implementation: Value Functions**

**Step 4.** By discretizing the spatial derivatives $\partial_x \xi_i^{(t+\Delta)}$ and $\partial_{xx'} \xi_i^{(t+\Delta)}$, the PDE becomes a system of linear equations in the unknown value function at the discretization points:

$$\left[ I - \Delta L_i^{(t)} \right] \xi_i^{(t+\Delta)} = \xi_i^{(t)} + \Delta K_i^{(t)}$$

Solve this system for $(\xi_e^{(t+\Delta)}, \xi_h^{(t+\Delta)})$. 
Numerical Implementation: Value Functions

**Computational Considerations.**

- Brownian information structure implies $L_i^{(t)}$ is a highly sparse matrix, with $I - \Delta L_i^{(t)}$ diagonally dominant for $\Delta$ sufficiently small.

- Solving $\left[ I - \Delta L_i^{(t)} \right] \xi_i^{(t+\Delta)} = \xi_i^{(t)} + \Delta K_i^{(t)}$:
  - direct approach: Pardiso (efficient LU decomposition, parallel computing, exact linear system solution)
  - iterative approach: conjugate gradient with different preconditioners, utilize initial guess from previous time iteration

- Explore numerical performance across examples of choice of $\Delta$, choice of grid, number of chores, etc.
Numerical Implementation: Constraints

**Statement of the problem.** Capital distribution $\kappa \in [0, 1]$ and expert equity issuance $\chi \in [\underline{\chi}, 1]$ determine occasionally-binding constraints

$$0 = \min(1 - \kappa, -\alpha_h)$$

$$0 = \min(\chi - \underline{\chi}, \alpha_e),$$

where $\alpha_i := \mu_{R,i} - r - \pi \cdot \sigma_R$ is agent $i$’s endogenous premium on capital.

**Economic intuition.**

- Experts hold all capital ($\kappa = 1$) if and only if households obtain no premium for holding it ($\alpha_h < 0$)
- Experts issue as much equity as possible ($\chi = \underline{\chi}$) if and only if their inside equity premium exceeds the outside equity premium ($\alpha_e > 0$)
Numerical Implementation: Constraints

**Variational inequalities.** Algebraic equations on part of the state space (when constraints bind) and first-order non-linear elliptic PDEs on the complement (when constraints are slack).

\[
0 = \min(1 - \kappa, -\alpha_h)
\]

\[
0 = \min(\chi - \underline{\chi}, \alpha_e),
\]

where

\[
\alpha_h = F_h(x, \kappa, \partial_x \kappa, \chi, \partial_x \chi)
\]

\[
\alpha_e = F_e(x, \kappa, \partial_x \kappa, \chi, \partial_x \chi).
\]

**Solution method.**

- Explicit FD scheme with false transient and “CFL” condition

\[
\frac{\kappa^{t+\Delta} - \kappa^t}{\Delta} = \min \left(1 - \kappa^t, F_h(x, \kappa^t, \partial_x \kappa^t, \chi^t, \partial_x \chi^t)\right)
\]

- See Oberman (2006)
Consider a martingale perturbation $H^s_t$ in direction $\nu$

$$d \log H^s_t = -\frac{\|\nu(X_t)\|^2}{2} dt + \nu(X_t) \cdot dZ_t \quad 0 \leq t \leq s$$

$$d \log M_t = \mu_M(X_t) dt + \sigma_M(X_t) \cdot dZ_t$$

$$\epsilon_M(x, t) := \lim_{s \to 0} \frac{1}{s} \log \mathbb{E} \left[ \frac{M_t H^s_t}{M_0} | X_0 = x \right]$$
Consider a martingale perturbation $H^s_t$ in direction $\nu$

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Applications for a cash-flow $C_t$ received at time $t$

- Shock exposure elasticity $\epsilon_C(x, t)$;
- Shock cost elasticity $\epsilon_{SC}(x, t)$;
- Shock price elasticity $\epsilon_C(x, t) - \epsilon_{SC}(x, t)$
Diagnostic Tools II

- Consider a martingale perturbation $H^s_t$ in direction $\nu$

\[
d \log H^s_t = -\frac{||\nu(X_t)||^2}{2} dt + \nu(X_t) \cdot dZ_t \quad 0 \leq t \leq s
\]

\[
d \log M_t = \mu_M(X_t) dt + \sigma_M(X_t) \cdot dZ_t
\]

\[
\epsilon_M(x, t) : = \lim_{s \to 0} \frac{1}{s} \log \mathbb{E} \left[ \frac{M_t}{M_0} H^s_t | X_0 = x \right]
\]

- Applications for a cash-flow $C_t$ received at time $t$
  - Shock exposure elasticity $\epsilon_C(x, t)$;
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- Two interpretations
  - Altering the exposure of cashflow
  - Altering the probability distribution of cashflow