Leverage Dynamics without Commitment

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Leverage dynamics is at the heart of dynamic corporate finance, but hard to be analyzed in models.

- Equilibrium debt prices interact with future equilibrium leverage policies

Existing literature relies on some ad hoc “commitment” of future debt policy.

- Refinance to keep outstanding debt face value constant (Leland 1994, 1998)
- Whenever adjusting debt, the firm has to retire the existing debt first, with some transaction costs (Fischer, Heinkel, and Zechner 1989; Goldstein, Leland, Ju 2001)
  - Abrupt adjustment to “target” leverage

Empirically counterfactual: firms are actively managing their debt, and often incrementally.
Introduction (2)
The firm cannot commit to future debt policies
  - Otherwise, standard trade-off setting (tax shield vs bankruptcy cost) with stochastic asset growth; no transaction cost
  - Rather than exogenous frictions to adjust leverage, we consider no commitment at all (so a more endogenous “friction”)

Two cases
  - Zero recovery: seniority structure irrelevant
  - Positive recovery: pari-passu debt

Leverage may go down via asset growth and debt maturing, but equity never reduces debt voluntarily
  - Repurchase debt is never optimal—leverage ratchet effect (Admati DeMarzo Hellwig Pfleiderer, 2016)
  - Robust to the canonical setting and debt overhang (endogenous investment)
This Paper (2)

- A general method to solve this class of models
  - A result reminiscent of Coase conjecture

- Closed-form solutions for work-horse log-normal cash-flow setting

- History-dependent leverage dynamics: issue more (less) following good (bad) shocks
  - Leverage dynamics tend to be mean-reverting; no immediate adjustment to leverage “target”
General Model: Environment

Preferences
▶ Risk-neutral world, with common discount rate $r$

Assets
▶ Assets in place generate operating income (could allow for jumps):

$$dY_t = \mu(Y_t) \, dt + \sigma(Y_t) \, dZ_t$$

▶ Focus on zero recovery now (debt seniority irrelevant); relaxed later

Debt contract: aggregate face value $F_t$ (endogenous)
▶ Each debt with coupon rate $c$, face value 1
▶ Exponentially retiring (Poisson maturing) with rate $\xi$

Corporate tax: $\pi \cdot (Y_t - cF_t)$
General Model: Debt Issuance and Default

Evolution of debt
- Equity issues/repurchases debt $d\Gamma_t$, so aggregate debt face value evolves as

$$dF_t = -\zeta F_t dt + d\Gamma_t$$

contractual debt maturing \hspace{1cm} active debt management

Timing within $[t, t + dt]$ & lack of commitment
- Cash flow realizes; either default or pay coupon/principal; announce $d\Gamma_t$; debt price is set and transaction is implemented; next period
- Unable to commit on future $d\Gamma_{t+s}$ for $s > 0$

Smooth equilibrium: issuance/repurchase policy $d\Gamma_t = G_t dt$
- $G_t/F_t$: endogenous debt adjustment speed
- Equity could adjust debt discretely, but not optimal in equilibrium

Equity default at endogenous stopping time $\tau_b$
General Model: Equity Value

State variables (Markov Perfect Equilibrium)
- Exogenous cash-flows $Y_t$, and endogenous debt obligation $F_t$

Equity’s problem, taking debt prices $p$ as given
- Equity receives cash-flows (if negative, covered by issuing equity)

\[
\begin{aligned}
Y_t & - \pi (Y_t - cF_t) - (c + \zeta) F_t + p_t G_t \\
\text{cash-flows} & \text{ corporate taxes} \quad \text{interest & principal} \quad \text{issuance/repurchase}
\end{aligned}
\]

- Endogenous debt price $p_t$ determined later
- Given $Y_t = Y$ and $F_t = F$, equity is solving

\[
V (Y, F) \equiv \max_{\{G_s\}, \tau_b} \mathbb{E}_t \left\{ \int_{t}^{\tau_b} e^{-r(s-t)} \left[ Y_s - \pi (Y_s - cF_s) - (c + \zeta) F_s + p_s G_s \right] ds \right\}
\]

- Controlling 1) debt evolution $dF_t = F_t dt + G_t dt$; and 2) when to default
Debt price:

- Competitive risk neutral debt investors price debt rationally
- Given equity default decision $\tau_b$, equilibrium debt price

$$p(Y, F) \equiv \mathbb{E}_t \left\{ \int_t^{\tau_b} e^{-(r+\xi)(s-t)} (c + \xi) \, ds \mid Y_t = Y, F_t = F \right\}$$

Why does commitment matter?

- $p_t$ depends on equilibrium default time $\tau_b$
- $\tau_b$ depends on firm’s future debt policy—the more the future debt, the more likely the default
General Model: Value Equivalence of No-Issuance

- Hamilton-Jacobi-Bellman equation for equity

\[ rV(Y, F) = \max_G \left[ Gp(Y, F) + (G - \xi F) V_F(Y, F) \right] \]

\[ Y - \pi (Y - cF) - (c + \xi) F + \mu(Y) V_Y(Y, F) + \frac{\sigma^2(Y)}{2} V_{YY}(Y, F) \]

- Objective linear in \( G \). Optimal \( G \) ⇒ First-Order Condition

\[ p(Y, F) + V_F(Y, F) = 0 \]

MB of issuance  MC on future value

- Under FOC, equity indifferent at any \( G \) (given equilibrium \( p \))
  - Linear control with interior solution (smooth policy \( G_t dt \))

- Equity value can be solved by setting \( G = 0 \) always. No gain in equilibrium by issuance/repurchase
  - Any potential tax shield gain is dissipated by bankruptcy cost caused by future excessive leverage
  - Reminiscent of Coase conjecture; DeMarzo and Urosevic (2006)
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- Get equity value \( V(Y, F) \) without knowing debt price
Proposition 1: Global optimality of local FOC holds if and only if debt price \( p(Y, F) = -V_F(V, F) \) is non-increasing in debt \( F \)

**Proof:** \( V(Y, F + \Delta) - V(Y, F) + \Delta p(Y, F + \Delta) \) equals equity value change + debt proceeds

\[
\int_0^\Delta V_F(Y, F + \delta) \, d\delta + \int_0^\Delta p(Y, F + \Delta) \, d\delta \\
\leq \int_0^\Delta V_F(Y, F + \delta) \, d\delta + \int_0^\Delta p(Y, F + \delta) \, d\delta \\
= \int_0^\Delta \left[ V_F(Y, F + \delta) + p(Y, F + \delta) \right] \, d\delta = 0
\]

- Debt price decreasing in \( F \) \( \Rightarrow \) a bad idea to repurchase by paying a higher price
- Leverage ratchet effect (Admati et al 2015)
Equilibrium Policies

Basic idea

- Debt price $p(Y, F)$ must satisfy the valuation equation

$$p(Y, F) = \mathbb{E}_t \left\{ \int_t^\tau b e^{-(r+\xi)(s-t)}(c+\xi) \, ds \right\}$$

- $V(Y, F)$ gives $-V_F(V, F) = p(Y, F)$ using equity’s FOC

- How to make both match? Via debt management $G(Y, F)$
  - HJB for $p$, which depends on $G(Y, F)$
  - ODE for $V_F(V, F)$ (HJB for $V$), which does not depend on $G(Y, F)$
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Equilibrium debt issuance policy

$$G^*(Y, F) = \frac{c \cdot \pi'(Y - cF)}{-p_F(Y, F)}$$

- $\pi'(Y - cF) \geq 0$, tax benefit $\Rightarrow$ always issuing debt
- Recall $-p_F(Y, F) = V_{FF}(Y, F) > 0$, capturing the price impact
Summary of General Model

1. Solve for equity value $V(Y, F)$ by setting $G(Y, F) = 0$

2. Set the equilibrium debt price $p(Y, F) = -V_F(Y, F)$

3. Check the equity holders’ global optimality condition
   - Verifying $p(Y, F)$ is non-increasing in $F$ (or $V(Y, F)$ is convex in $F$)

4. Equilibrium debt issuance $G^*(Y, F) = \frac{\pi'(Y-cF)\cdot c}{-p_F(Y,F)} > 0$
Strict Optimality in Discrete Time

- Taking the value function at $t + h$ as given, consider equity’s problem at $t$, where $h > 0$

- Denote the debt issuance by $G \cdot h$. Equity is maximizing

$$\max_G \pi c h \cdot Gh + Gh \cdot p (F + Gh, Y) + V (F + Gh, Y) - V (F, Y)$$

- Taylor expansion of the objective up to $h^2$

$$\pi c G h^2 + Gh \cdot [p (F, Y) + p_F Gh] + V_F (F, Y) Gh + \frac{1}{2} V_{FF} (F, Y) G^2 h^2$$

$$= [p (F, Y) + V_F (F, Y)] G \cdot h + \left[ \pi c G + \left[ p_F + \frac{1}{2} V_{FF} (F, Y) \right] G^2 \right] \cdot h^2$$

- Since $p (F, Y) + V_F (F, Y) = 0 \Rightarrow p_F (F, Y) = -V_{FF} (F, Y)$, the optimal issuance is

$$G^* (Y, F) = -\frac{\pi c}{2 p_F (F, Y) + V_{FF} (F, Y)} = -\frac{\pi c}{-p_F (F, Y)}$$
Log-Normal Cash-flows Model

- Scale-invariance, cash-flows $dY_t / Y_t = \mu dt + \sigma dZ_t$
  - The work-horse model of dynamic corporate finance
- One-dimensional state variable: scaled cash-flow $y_t \equiv Y_t / F_t$
  - Equity value $V(Y, F) = F \cdot v(y)$, debt price $p(Y, F) = p(y)$
  - Total Enterprise Value (TEV) multiple $\frac{TEV}{Y} = \frac{v(y) + p(y)}{y}$
- Let $g^* (y_t) \equiv G^* (Y_t, F_t) / F_t$, then

$$\frac{dy_t}{y_t} = \left( \mu + \zeta - g^*_t \right) dt + \sigma dZ_t$$

- CF growth  
- debt maturing  
- debt issuance  
- CF shocks

- Debt grows at the rate of $g^*_t - \zeta = \frac{dF_t}{F_t dt}$, with endogenous $g^*_t$
Model Solution

- **Equity value** and default threshold (endogenous constant $\gamma$)

\[
v(y) = \left(1 - \pi\right) \frac{y}{r - \mu} + \frac{\pi c}{r + \xi} - \frac{c + \xi}{r + \xi} + \frac{c \left(1 - \pi\right) + \xi}{(1 + \gamma)(r + \xi)} \left(\frac{y}{y_b}\right)^{-\gamma}
\]

- **asset in place**
- **tax shield**
- **debt value**
- **default option**

\[
y_b = \frac{\gamma}{1 + \gamma} \frac{r - \mu}{r + \xi} \left[c + \frac{\xi}{1 - \pi}\right]
\]

- **Debt price**, increasing in $y$ hence decreasing in $F$

\[
p(y) = -V_F(V, F) = yv(v) - v'(y) = \frac{c \left(1 - \pi\right) + \xi}{r + \xi} \left(1 - \left(\frac{y}{y_b}\right)^{-\gamma}\right)
\]

- **TEV multiple** $\left(1 - \left(\frac{y}{y_b}\right)^{-\gamma-1}\right)$, lower than unlevered TEV multiple $\frac{1 - \pi}{r - \mu}$

  - **Default cost more than offsets tax shield**

- **Debt issuance**, increasing in $y$

\[
g^*(y) = \frac{(r + \xi) \pi c}{c \left(1 - \pi\right) + \xi} \frac{1}{\gamma} \left(\frac{y}{y_b}\right)^\gamma > 0
\]
Net Debt Issuance $g^*(y) - \xi$, Debt Maturity
Two Benchmarks with Commitment

No future debt issuance:
- The firm commits to set $g_t = 0$ always (superscript 0)
- Equity value is the same (so does $y_b$), debt price is higher (by the tax shield)

$$p^0(y) = p(y) + \frac{\pi c}{r + \zeta} \left(1 - \left(\frac{y}{y_b}\right)^{-\gamma}\right)$$

- Less debt $\Rightarrow$ less likely to default (same $y_b$ but $y$ has a higher drift)

Fixed future debt:
- The firm commits to set $g_t = \zeta$ always; Leland 1998
Model Comparisons: Debt Prices and Credit Spreads

Implication of credit spreads: $p(y)\rightarrow\infty$ i.e. zero current leverage $\Rightarrow p(\xi(y))\rightarrow c^+\xi + \xi r^+\xi$, with zero credit spread $\Rightarrow p(y)\rightarrow c(1-\pi)+\xi r^+\xi$, non-zero credit spreads (high future leverage!)

- **Baseline no commitment, $g^*$**
- **No future debt $g=0$**
- **Fixed face value Leland '98 $g=\xi$**
Implication of credit spreads: $y \to \infty$ i.e. zero current leverage

- $p^{\xi}(y)$ and $p^0(y) \to \frac{c+\xi}{r+\xi}$, with zero credit spread

- $p(y) \to \frac{c(1-\pi)+\xi}{r+\xi}$, non-zero credit spreads (high future excessive leverage!)
**Proposition.** Given cash-flow history \( \{ Y_s : 0 \leq s \leq t \} \), time-\( t \) debt is (\( \hat{y}_\zeta \) is a constant)

\[
F_t = \frac{1}{\hat{y}_\zeta} \left[ \int_0^t \gamma \zeta Y_s \gamma e^{-\gamma \zeta (s-t)} ds \right]^{1/\gamma}
\]

- Start from \( t = 0 \) debt grows at the order of \( t^{1/\gamma} \)
- Outstanding debt is average past earnings, with decaying weights \( \gamma \zeta \)
- High mean-reverting speed, or more aggressive in adding leverage given high cash flows, when
  - shorter debt maturity \( \Rightarrow \) higher \( \zeta \)
Leverage Ratchet Effect

- What is the impact of debt repurchase on equity value?
  - Often the intuition is through firm value...but may mislead as existing debt holders could gain from repurchase

"Leverage Ratchet Effect" Admati et al. (2016)

The same logic to debt overhang—equity is optimizing investment decisions ex post
Leverage Ratchet Effect

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- Reducing current debt will alleviate future default—but does equity benefit strictly from it? No
  - Yes less debt ⇒ default less often
  - But equity optimizes default decision ex post already ⇒ zero indirect impact on equity value now (envelope theorem)
Leverage Ratchet Effect

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  - The same logic to debt overhang—equity is optimizing investment decisions **ex post**
Positive Recovery

- Now suppose the recovery in bankruptcy is (baseline $\alpha = 1$)

$$
(1 - \alpha) \frac{1 - \pi}{r - \mu} Y_{\tau_b}, \text{ for } \alpha \in (0, 1)
$$

- For tractability, assume pari-passu debt
- Equity as if has the option to capture recovery value ($1 - \alpha$ fraction of TEV) by issuing infinite new debt right before default
  - Why not wait till the end?
- The equilibrium is as if the firm can only pledge out $\alpha$ fraction of cash-flow $Y$ (and grab $1 - \alpha$ at default)

$$
\nu^\alpha (y) = \nu (\alpha y), \ p^\alpha (y) = p (\alpha y), \text{ and } g^\alpha (y) = g (\alpha y)
$$

- Similar debt issuance policy, but with infinite issuance at (right before) $y_{\tau_b} = \frac{y_{b}}{\alpha}$
- Interestingly, $\ g^\alpha (y) > g^{\alpha = 1} (y)$, so the final diluting option $\Rightarrow$ less debt issuance before default
Optimal Debt Maturity Structure

- So far the debt maturity structure $\zeta$ is taken as a parameter.
- Say the firm gets a one-time chance to set $\zeta$ optimally together with leverage decision.
- At time zero, $F_0^* = 0$; but the choice of $\zeta$ affects the maturity of future debt issuances.

**Proposition**: Equity holders are **indifferent** at any $\zeta$.
  - Why? Because equity value is as if there is no future debt issuance...
- This indifference result holds even if the firm gets a chance of readjusting $\zeta$ after time-0.
Long-term vs. Short-term Debt

- Two firms start with zero debt, with different debt maturities (both being optimal)—but have different leverage dynamics/targets.

With flexibility of shorter-term debt, the firm borrows more for higher debt tax shield.

But tax shield is a transfer from social perspective—so long-term debt is preferred to minimize bankruptcy cost.
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Investment

- Special case of log-normal process. Capital $K_t$ evolves as

$$\frac{dK_t}{K_t} = (i_t - \delta) \, dt + \sigma dZ_t$$

with quadratic investment cost $\frac{\kappa i^2_t}{2} K_t$, and output $Y_t = A K_t$

- Prove $\nu (\cdot)$ is strictly convex, so always an equilibrium

- Leverage ratchet effect prevails despite debt overhang considerations

- Equity issues debt more aggressively when controlling investment endogenously, compared to exogenous investment
  - Endogenous investment offers equity more protection later
Conclusion and Future Work

What we have done

- A general methodology solving dynamic corporate finance model without commitment
- Leverage policy depending on the entire earnings history, new insight on debt maturity and investment
- Slow initial adoption of leverage, but leads ultimately to excess

Future extensions

- Covenant of no debt issuance once in distress (say for $y < \hat{y}$)
  - Discrete debt issuance (jump to $\hat{y}$) may occur in equilibrium
- What changes when one has internal cash (with some equity issuance cost)? How about liquidity-driven default?