Some simple Bitcoin Economics

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Outline

1. Introduction.
2. The Model
3. Analysis
4. Bitcoins and Monetary Policy
5. Examples
6. Conclusions
Introduction.

Questions

- Bitcoin, cybercurrencies: increasingly hard to ignore.
- Increasing number of cybercurrencies. Regulatory concerns.
- Blockchain technology. (Not a topic today)
- Imagine a world, where Bitcoin (or cybercurrencies) are important.
- Key questions:
  - How do Bitcoin prices evolve?
  - What are the consequences for monetary policy?
Bitcoin Price, 2011-09-13 to 2018-02-07

Data: quandl.com
Bitcoin Price, 2017-01-01 to 2018-02-07

Data: quandl.com
This paper

Approach: a simple model, with money as a medium of exchange.

- A novel, yet simple endowment economy: two types of agents keep trading.
- Two types of money: Bitcoins and Dollars.
- A central bank keeps real value of Dollars constant...
- ... while Bitcoin production is private and decentralized.

Results:

- “Fundamental condition”: a version of Kareken-Wallace (1981)
- “Speculative condition”.
- Under some conditions: no speculation.
- Under some conditions: Bitcoin price converges.
- Implications for monetary policy: two scenarios.
- Construction of equilibria.
Literature

Bitcoin Pricing
- Athey et al
- Garratt and Wallace (2017)
- Huberman, Leshno, Moallemi (2017)

Currency Competition
- Kareken and Wallace (1981)

(Monetary) Theory
- Bewley (1977)
- Townsend (1980)
- Kyotaki and Wright (1989)
- Lagos and Wright (2005)
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The model

- $t = 0, 1, 2, \ldots \text{ Randomness: } \theta_t$, at beg. of per.. History: $\theta^t$.
- Two types of money: Bitcoins $B_t$ and Dollars $D_t$ (aggregates).
- Assume: Central Bank keeps Dollar price constant, $P_t \equiv 1$.
- Goods ( = Dollar) price of Bitcoins: $Q_t = Q(\theta_t)$.
- Two types of infinitely lived agents: green and red.
- Green agent $j$ in even periods $t$:
  - receives lump sum Dollar transfer ( “tax”, if $< 0$ ) from Central Bank.
  - purchases goods from red agents, with Bitcoins or Dollars.
  - enjoys consumption $c_{t,j}$, utility $\beta^t u(c_{t,j})$.
- Green agents in odd periods $t$:
  - mines new Bitcoins $A_{t,j} = f(e_{t,j}; B_t)$ at effort $e_{t,j} \geq 0$, disutil. $-\beta^t e_{t,j}$.
  - receives goods endowment $y_{t,j}$. Not storable.
  - can sell goods to red agents, against Bitcoins or Dollars.
- Red agents: flip even and odd periods.
- Assume: whoever consumes first has all the money.
Timeline

The Model

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Timeline

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Optimization problem of green agents: (drop “\(j\")

Maximize \[ U = E \left[ \sum_{t=0}^{\infty} \beta^t (\xi_{t,g} u(c_t) - e_t) \right] \]

where \(\xi_{t,g} = 1\) in even periods, \(\xi_{t,g} = 0\) in odd periods, s.t.

in even periods \(t\):
- \(0 \leq b_t \leq Q_t B_{t,g}\) \hspace{1cm} (1)
- \(0 \leq P_t d_t \leq D_{t,g}\) \hspace{1cm} (2)
- \(0 \leq c_t = b_t + d_t\) \hspace{1cm} (3)
- \(0 \leq B_{t+1,g} = B_{t,g} - b_t/Q_t\) \hspace{1cm} (4)
- \(0 \leq D_{t+1,g} = D_{t,g} - P_t d_t\) \hspace{1cm} (5)

in odd periods \(t\):
- \(A_t = f(e_t; B_t)\), with \(e_t \geq 0\) \hspace{1cm} (6)
- \(y_t = x_t + z_t\), with \(x_t \geq 0\), \(z_t \geq 0\) \hspace{1cm} (7)
- \(0 \leq B_{t+1,g} = A_t + B_{t,g} + x_t/Q_t\) \hspace{1cm} (8)
- \(0 \leq D_{t+1,g} = D_{t,g} + P_t z_t + \tau_{t+1}\) \hspace{1cm} (9)
Monetary Policy and Market clearing

- The **Central Bank** achieves \( P_t \equiv 1 \), per suitable transfers \( \tau_t \).
- Markets clear:

  - Bitcoin market: \( B_t = B_{t,r} + B_{t,g} \) \hspace{1cm} (10)
  - Dollar market: \( D_t = D_{t,r} + D_{t,g} \) \hspace{1cm} (11)
  - Bitcoin denom. cons. market: \( b_t = x_t \) \hspace{1cm} (12)
  - Dollar denom. cons. market: \( d_t = z_t \) \hspace{1cm} (13)
Equilibrium

An equilibrium is a stochastic sequence

\((A_t, [B_t, B_{t,g}, B_{t,r}], [D_t, D_{t,g}, D_{t,r}], \tau_t, (P_t, z_t, d_t), (Q_t, x_t, b_t), e_t)_{t \geq 0}\)

- Given prices, choices maximize utility for green and red agents.
- Budget constraints
  - \(0 \leq b_{t,j} \leq B_{t,j}Q_t\)
  - \(0 \leq P_t d_{t,j} \leq D_{t,j}\)
- Evolution money stock
  - \(B_{t+1,j} = B_{t,j} - b_{t,j}/Q_t \geq 0\)
  - \(D_{t+1,j} = D_{t,j} - P_t d_{t,j} \geq 0\)
  - \(B_{t+1,j} = B_{t,j} + x_{t,j}/Q_t + A_{t,j}(e_{t,j})\)
  - \(B_{t+1,j} = B_{t,j} + x_{t,j}/Q_t + A_{t,j}(e_{t,j})\)
- Markets clear (for goods, Bitcoin, Dollars):
  - \(y_t = \int_0^2 c_{t,j} \, dj\)
  - \(\int_0^2 z_{t,j} \, dj = \int_0^2 d_{t,j} \, dj\)
  - \(\int_0^2 x_{t,j} \, dj = \int_0^2 b_{t,j} \, dj\)
  - \(D_t = D_{t,g} + D_{t,r}\)
  - \(B_t = B_{t,g} + B_{t,r}\)
- Dollar monetary policy: \(P_t = 1\)
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Consolidate:

\[
\begin{align*}
B_{t+1} &= B_t + f(e_t; B_t) \\
D_t &= D_{t-1} + \tau_t \\
c_t &= y_t
\end{align*}
\]
Avoid speculation with Dollars

Assumption A.

Assume throughout: for all $t$,

$$u'(y_t) - \beta^2 \mathbb{E}_t[u'(y_{t+2})] > 0$$

(14)

Proposition

(All Dollars are spent:) Agents will always spend all Dollars. Thus, $D_t = D_{t,g}$ and $D_{t,r} = 0$ in even periods and $D_t = D_{t,r}$ and $D_{t,g} = 0$ in odd periods.

This is a consequence of assumption 14 and $P_t \equiv 1$.

Proposition

(Dollar Injections:) In equilibrium,

$$D_t = z_t \text{ and } \tau_t = z_t - z_{t-1}$$
Bitcoin Production

Proposition

(Bitcoin Production Condition:) Suppose that Dollar sales are nonzero, \( z_t > 0 \) in period \( t \). Then

\[
1 \geq \beta E_t \left[ u'(c_{t+1}) \frac{\partial f(e_t; B_t)}{\partial e_t} Q_{t+1} \right]
\]  

(15)

This inequality is an equality, if there is positive production \( A_t > 0 \) of Bitcoins and associated positive effort \( e_t > 0 \) at time \( t \) as well as positive spending of Bitcoins \( b_{t+1} > 0 \) in \( t + 1 \).
The Fundamental Condition

The following is a version of Kareken-Wallace (1981).

Proposition

(Fundamental Condition:)

Suppose that sales happen both in the Bitcoin-denom. cons. market as well as the Dollar-denom. cons. market at time $t$ as well as at time $t + 1$, i.e. suppose that $x_t > 0$, $z_t > 0$, $x_{t+1} > 0$ and $z_{t+1} > 0$. Then

$$E_t [u'(c_{t+1})] = E_t [u'(c_{t+1}) \frac{Q_{t+1}}{Q_t}]$$  \hspace{1cm} (16)

In particular, if consumption and production is constant at $t + 1$, $c_{t+1} = y_{t+1} \equiv \bar{y}$, then

$$Q_t = E_t [Q_{t+1}]$$  \hspace{1cm} (17)

i.e., the price of a Bitcoin in Dollar is a martingale.
The Speculative Condition

Proposition

(Speculative Condition:)
Suppose that $B_t > 0$, $Q_t > 0$, $z_t > 0$ and that $b_t < Q_t B_t$. Then,

$$u'(c_t) \leq \beta^2 \mathbb{E}_t \left[ u'(c_{t+2}) \frac{Q_{t+2}}{Q_t} \right]$$

(18)

where this equation furthermore holds with equality, if $x_t > 0$ and $x_{t+2} > 0$. 
**Seller Participation Condition**

**Proposition**

*(Seller Participation Condition:)*

*Suppose that $B_t > 0$, $Q_t > 0$, $z_t > 0$. Then*

$$
\mathbb{E}_t \left[ u'(c_{t+1}) \right] \geq \mathbb{E}_t \left[ u'(c_{t+1}) \frac{Q_{t+1}}{Q_t} \right]
$$

(19)
The Sharpened No-Speculation Assumption

Assumption A.

For all $t$,

$$u'(y_t) - \beta \mathbb{E}_t[u'(y_{t+1})] > 0$$

(20)

This is a slightly sharper version of assumption 1, which only required

$$u'(y_t) - \beta^2 \mathbb{E}_t[u'(y_{t+2})] > 0$$
The No-Bitcoin-Speculation Theorem

**Theorem**

*(No-Bitcoin-Speculation Theorem.)* Suppose that $B_t > 0$ and $Q_t > 0$ for all $t$. Impose assumption 2. Then in every period, all Bitcoins are spent.

**Proof.**

\[
\beta^2 E_t[u'(c_{t+2})Q_{t+2}] = \beta^2 E_t[E_{t+1}[u'(c_{t+2})Q_{t+2}]] \\
\leq \beta^2 E_t[E_{t+1}[u'(c_{t+2})] \cdot Q_{t+1}] \\
< \beta E_t[u'(c_{t+1})Q_{t+1}] \\
\leq \beta E_t[u'(c_{t+1})]Q_t \\
< u'(c_t)Q_t
\]

(law of iter. expect.)

(equ. (19) at $t + 1$)

(ass. 2 at $t + 1$)

(equ. (19) at $t$)

(ass. 2 at $t$)

Thus, the specul. cond. (18) cannot hold in $t$. Hence $b_t = Q_tB_t$. □
A (very high) bound for Bitcoin Prices

Corollary

(Bitcoin price bound) Suppose that $B_t > 0$ and $Q_t > 0$ for all $t$. The Bitcoin price is bounded by

$$0 \leq Q_t \leq \bar{Q}$$

where

$$\bar{Q} = \frac{\bar{y}}{B_0}$$

(21)
Bitcoin Correlation-Pricing

Rewrite (16) as

$$Q_t = \frac{\text{cov}_t(u'(c_{t+1}), Q_{t+1})}{\mathbb{E}_t[u'(c_{t+1})]} + \mathbb{E}_t[Q_{t+1}]$$  \hspace{1cm} (22)$$

Corollary

(Bitcoin Correlation Pricing Formula:)

Suppose that $B_t > 0$ and $Q_t > 0$ for all $t$. Impose assumption 2. In equilibrium,

$$Q_t = \kappa_t \cdot \text{corr}_t(u'(c_{t+1}), Q_{t+1}) + \mathbb{E}_t[Q_{t+1}]$$  \hspace{1cm} (23)$$

where

$$\kappa_t = \frac{\sigma_{u'(c)|t} \sigma_{Q_{t+1}|t}}{\mathbb{E}_t[u'(c_{t+1})]} > 0$$  \hspace{1cm} (24)$$

where $\sigma_{u'(c)|t}$ is the standard deviation of marginal utility of consumption, conditional on date-t information, etc..
Martingale Properties

**Corollary**

(Martingale Properties of Equilibrium Bitcoin Prices:) Suppose $B_t > 0$ and $Q_t > 0$ for all $t$. Impose ass. 2. If and only if for all $t$, marg. util. of cons. and Bitcoin price are positively correlated at $t + 1$, given $t$ info, the Bitcoin price is a supermartingale and strictly falls in expectation,

$$Q_t > \mathbb{E}_t[Q_{t+1}]$$

(25)

If and only if marginal utility and the Bitcoin price are always neg. corr.,

$$Q_t < \mathbb{E}_t[Q_{t+1}]$$

(26)

If and only if marginal utility and the Bitcoin price are always uncorr., the Bitcoin price is a martingale,

$$Q_t = \mathbb{E}_t[Q_{t+1}]$$

(27)
Bitcoin Price Convergence

Theorem

(Bitcoin Price Convergence Theorem.) Suppose that $B_t > 0$ and $Q_t > 0$ for all $t$. Impose assumption 2. For all $t$ and conditional on information at date $t$, suppose that marginal utility $u'(c_{t+1})$ and the Bitcoin price $Q_{t+1}$ are either always nonnegatively correlated or always non-positively correlated. Then the Bitcoin price $Q_t$ converges almost surely pointwise as well as in $L^1$ norm to a (random) limit $Q_\infty$,

$$Q_t \to Q_\infty \ a.s. \quad \text{and} \quad E[|Q_t - Q_\infty|] \to 0$$

(28)

Proof.

$Q_t$ or $-Q_t$ is a bounded supermartingale. Apply Doob's martingale convergence theorem.
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Scenario 1 - Conventional approach

Assume that Bitcoin prices move independently of central bank policies. Impose assumption 2. Then

Proposition

(Conventional Monetary Policy:)

The equilibrium Dollar quantity is given as

\[ D_t = y_t - Q_t B_t \]  \hspace{1cm} (29)

The central bank’s transfers are

\[ \tau_t = y_t - Q_t B_t - z_{t-1} \] \hspace{1cm} (30)
Scenarios 1 - Conventional approach

Proposition

(Dollar Stock Evolution:)
Tomorrow’s expected Dollar quantity equals today’s Dollar quantity corrected for deviation from expected production, purchasing power of newly produced Bitcoin and correlation

\[ \mathbb{E}_t[D_{t+1}] = D_t - (y_t - \mathbb{E}_t[y_{t+1}]) - A_t Q_t + \kappa_t B_{t+1} \cdot \text{corr}_t(u'(c_{t+1}), Q_{t+1}) \]

Likewise, the central bank’s expected transfers satisfy

\[ \mathbb{E}_t[\tau_{t+1}] = - (y_t - \mathbb{E}_t[y_{t+1}]) - A_t Q_t + \kappa_t B_{t+1} \cdot \text{corr}_t(u'(c_{t+1}), Q_{t+1}) \]

If the Bitcoin price is a martingale, then

\[ \mathbb{E}_t[D_{t+1}] = D_t - (y_t - \mathbb{E}_t[y_{t+1}]) - A_t Q_t \]
\[ \mathbb{E}_t[\tau_{t+1}] = - (y_t - \mathbb{E}_t[y_{t+1}]) - A_t Q_t \]
Scenario 2 - Unconventional approach

- Unconventional view, but compatible with equilibrium: the Central Bank can maintain the price level $P_t \equiv 1$ independently of the transfers she sets.
- Further, assume that she sets transfers independently of production.
- Note that

\[
Q_t = \frac{y_t - D_t}{B_t}
\]  

(31)

- Intuitively, the causality is in reverse compared to scenario 1: now central bank policy drives Bitcoin prices.
- However, the process for the Dollar stock cannot be arbitrary.
  - To see this, suppose that $y_t \equiv \bar{y}$ is constant. We already know that $Q_t$ must then be a martingale. Suppose $B_t$ is constant as well. Equation (31) now implies that $D_t$ must be a martingale too.
Scenario 2 - Unconventional approach

Proposition

(Submartingale Implication:)

If the Dollar quantity is set independently of production, the Bitcoin price process is a submartingale, $\mathbb{E}_t[Q_{t+1}] \geq Q_t$. 
Scenario 2 - Unconventional approach

Suppose that production $y_t$ is iid. Let $F$ denote the distribution of $y_t$, $y_t \sim F$. The distribution $G_t$ of the Bitcoin price is then given by

$$G_t(s) = \mathbb{P}(Q_t \leq s) = F(B_t s + D_t).$$

Proposition

(Bitcoin Price Distribution:)

In “scenario 2”, if Bitcoin quantity or Dollar quantity is higher, high Bitcoin price realizations are less likely in the sense of first order stochastic dominance.
Scenario 2 - Unconventional approach

Compare two economies with \( y_t \sim F_1 \) vs \( y_t \sim F_2 \), iid.

**Definition**
- Economy 2 is **more productive** than economy 1, if \( F_2 \) first order stochastically dominates \( F_1 \).
- Economy 2 has **more predictable production** than economy 1, if \( F_2 \) second order stochastically dominates \( F_1 \).

**Proposition**
(Bitcoins and Productivity)
Assume “scenario 2”. In more productive economies or economies with higher predictability of production, the Bitcoin price is higher in expectation.
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Constructing an equilibrium: an example.

- Suppose $\theta_t \in \{L, H\}$, each with probability $1/2$.
- Let $m_t$ be iid, $m_t = m(\theta_t)$, with $m(L) \leq m(H)$ and $\mathbb{E}[m_t] = (m_L + m_H)/2 = 1$. Pick $0 < \beta < 1$ such that $m(L) > \beta$.
- At date $t$ and for $\epsilon(\theta^t) = \epsilon_t(\theta_t)$, consider two cases
  - **Case A:** $\epsilon_t(H) = 2^{-t}$, $\epsilon_t(L) = -2^{-t}$
  - **Case B:** $\epsilon_t(H) = -2^{-t}$, $\epsilon_t(L) = 2^{-t}$.
- Pick $Q_0 > \xi + (m(H) - m(L))/2$. Set

  $$Q_{t+1} = Q_t + \epsilon_{t+1} - \frac{\text{cov}_t(m_{t+1}, \epsilon_{t+1})}{E_t[m_{t+1}]}$$

- Fix some strictly concave $u(\cdot)$. Let $y_t = (u')^{-1}(m_t)$.
- Start with some initial $B_0$. With $B_t$ and $Q_t$, equation (15) delivers new Bitcoin mining $A_t$ and thus $B_{t+1}$.
- The No-Bitcoin-Speculation Theorem now implies the purchases $x_t = b_t = Q_t/B_t$ and $z_t = d_t = y_t - b_t$.
- Be careful with $B_0$, so that $b_t \leq y_t$ for all $t$. Or: fix “ex post”.

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Super-, sub-, non-martingale examples

Consider three constructions,

**Always A:** Always impose case A, i.e. $\epsilon_t(H) = 2^{-t}$, $\epsilon_t(L) = -2^{-t}$. “Always A” results in supermartingale $Q_t > E_t[Q_{t+1}]$.

**Always B:** Always impose case B, i.e. $\epsilon_t(H) = -2^{-t}$, $\epsilon_t(L) = 2^{-t}$. “Always B” results in submartingale $Q_t < E_t[Q_{t+1}]$.

**Alternate:**
- In even periods, impose case A, i.e. $\epsilon_t(H) = 2^{-t}$, $\epsilon_t(L) = -2^{-t}$.
- In odd periods, impose case B, i.e. $\epsilon_t(H) = -2^{-t}$, $\epsilon_t(L) = 2^{-t}$.

This results in a price process that is neither a supermartingale nor a submartingale, but which one still can show to converge almost surely and in $L_1$ norm.
Bitcoin Price, 2017-01-01 to 2018-02-07

Data: quandl.com
“Bubble and bust” examples

- \( \theta_t \in \{L, H\} \), but now \( \mathbb{P}(\theta_t = L) = p < 0.5 \).
- Suppose that \( m(L) = m(H) = 1 \).
- Pick some \( Q > 0 \) as well as some \( Q^* > Q \).
- Pick some \( Q_0 \in [Q, Q^*] \). If \( Q_t < Q^* \), let

\[
Q_{t+1} = \begin{cases} 
\frac{Q_t - pQ}{1-p} & \text{if } \theta_t = H \\
\frac{Q}{Q^*} & \text{if } \theta_t = L
\end{cases}
\]

If \( Q_t \geq Q^* \), let \( Q_{t+1} = Q_t \).
- Therefore \( Q_t \) will be a martingale and satisfies (22).
- If \( Q_0 \) is sufficiently far above \( \bar{Q} \) and if \( p \) is reasonably small, then typical sample paths will feature a reasonably quickly rising Bitcoin price \( Q_t \), which crashes eventually to \( Q \) and stays there, unless it reaches the upper bound \( Q^* \) first.
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Recap and Conclusions.

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- Two types of money: Bitcoins and Dollars.
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- ... while Bitcoin production is private and decentralized.

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