Term Structure of Uncertainty: New Evidence from Survey Expectations*

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Abstract

We construct measures of individual forecasters' subjective uncertainty at horizons ranging from one to five years, incorporating a rich information set from the European Central Bank's Survey of Professional Forecasters. Our measures have two key advantages over traditional measures: (i) they reflect the subjective perceptions of market participants; (ii) they are ex ante measures that do not require the knowledge of realized outcomes. We find that the uncertainty curve is much more linear than the disagreement curve — uncertainty at the one-year and two-year horizons can almost perfectly predict uncertainty at the five-year horizon, but not so for disagreement. We develop a learning model that captures the two channels through which large shocks affect the term structure of uncertainty: wake-up effect — the visibly large shocks induce immediate and synchronized updating of information for economic agents, and wait-and-see effect — the occurrence of those shocks generates increased uncertainty among agents. Depending on the magnitude of the shock, and whether it is temporary or permanent, the response of uncertainty afterwards displays heterogeneous patterns over different time horizons.

Keywords: Density forecasts; Heterogeneity; Inflation; Surveys; Term structure; Uncertainty

*The views expressed herein are those of the authors and do not necessarily reflect the views of U.S. Census Bureau.

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1 Introduction

Uncertainty varies over time and with the business cycle (Bloom, 2009; Bachmann and Bayer, 2013). Heightened uncertainty in the Great Recession has prompted renewed efforts to investigate the expectations formation process, the nature of economic agents' information problems, and sources of uncertainty.[1] Survey forecasts can provide valuable information about the expectations formation process and information environment (Mankiw et al., 2004; Coibion and Gorodnichenko, 2012).

This paper studies the uncertainty in multi-step ahead forecasts of economics agents, in the presence of both private and public channels of information that are subject to shocks. We model forecasters' signal extraction process under this information structure, and derive new implications for uncertainty and dispersion across forecast horizons. Then we construct measures of individual forecasters' subjective uncertainty at horizons ranging from one to five years, using density forecasts from the European Central Bank's Survey of Professional Forecasters (ECB SPF). Uncertainty measures the spread of an individual agent's probability distribution about an outcome, i.e. the variance of the individual's density forecast (Lahiri and Sheng, 2010; Abel et al., 2016). We also construct measures of disagreement, or dispersion of expectations across forecasters, at each horizon. We explore the properties and drivers of uncertainty and disagreement over shorter and longer horizons.

A substantial literature has examined the cyclicality and drivers of uncertainty (Bloom, 2014; Baker et al., 2016) and of disagreement (Kandel and Pearson, 1995; Woodford, 2001; Lahiri and Sheng, 2008; Wieland and Wolters, 2011; Dovern et al., 2012; Acemoglu et al., 2016). Most of this literature focuses on a single horizon, but several papers have examined either uncertainty or disagreement at multiple horizons. Patton and Timmermann (2010) find that U.S. growth and inflation disagreement increase with forecast horizon, using Con-

[1] Some of the many papers in the recent uncertainty literature include Bloom et al. (2007); Bloom (2009); Baker and Bloom (2013); Bachmann et al. (2013); Leduc and Liu (2016). Fluctuations in disagreement and uncertainty affect the financial sector and the macroeconomy through a variety of channels (Varian, 1985; Dixit and Pindyck, 1994; Hellwig and Venkateswaran, 2009; Coibion and Gorodnichenko, 2012).

sensus Forecast data available at horizons up to two years. Lahiri and Sheng (2008) use the same data to estimate a Bayesian learning model with heterogeneity; they show that the term structure of disagreement can help distinguish whether forecasters disagree because of differences in priors or differences in interpretation of new information. Andrade et al. (2016) examine the term structure of disagreement over longer horizons using Blue Chip Financial Forecast data, and note that disagreement at the longest horizon likely reflects disagreement about fundamentals, such as potential output growth or the central bank's inflation target. Giacomini et al. (2016) formulate a theory of expectations updating that fits the dynamics of disagreement in a new Bloomberg survey dataset where agents can update at any time while observing each other's expectations. Wright (2015) highlights the lack of research on the term structure of uncertainty, and shows that both the level and slope of the term structure of uncertainty are negatively correlated with capital investment.[2]

Recently, Barrero et al. (2017) examine the drivers of firm and macro implied volatility at the 30-day and 1-year horizons. Economic policy uncertainty is a driver of their longer-run volatility measures, while oil price volatility is a driver of their shorter-run measures and currency volatility impacts short- and long-run uncertainty similarly. We differ from Barrero et al. in both the measures of uncertainty and the horizons that we examine. We measure forecasters' subjective uncertainty, rather than implied volatility, and use three horizons: 1-, 2-, and 5-year. A benefit of looking at these longer horizons is that they correspond to

[2] A few studies have examined the term structure of inflation uncertainty, since inflation uncertainty at different horizons is of particular interest to monetary policymakers. Ball and Cecchetti (1990) show that short-run inflation uncertainty depends mainly on the variance of temporary inflation shocks, while long-run inflation uncertainty depends mainly on the variance of permanent shocks. They also find that the level of inflation has a much stronger effect on the variance of permanent than of temporary shocks, and therefore has a greater effect on longer-horizon inflation uncertainty. These authors do not use a direct measure of inflation uncertainty, but rather use inflation variability as a proxy. Binder (2017b) constructs an index of consumer inflation uncertainty in the United States at the five- to ten-year horizon and at the one-year horizon based on consumers' tendency to round their forecasts on the Michigan Survey of Consumers. Consumer inflation uncertainty was higher for the longer horizon than for the shorter horizon until the mid 1990s. As the Federal Reserve worked to anchor longer-run inflation expectations, longer-run inflation uncertainty declined more than shorter-run uncertainty, and the inflation uncertainty term structure inverted. Binder (2017a) shows that for consumers, economic policy uncertainty is most strongly correlated with uncertainty about 1-year-ahead inflation, while monetary policy uncertainty is most correlated with uncertainty about 5-year-ahead inflation.

the horizons over which some macroeconomic policies may take effect, and also the horizons that firms may use when evaluating investment projects. Given a concern that uncertainty depresses investment, these longer horizons are especially relevant. We also differ from the existing literature by examining both the term structure of uncertainty *and* the term structure of disagreement simultaneously, and using a model that provides predictions for each.

An important new result is that the term structures of uncertainty and disagreement are qualitatively quite different. Average uncertainty nearly always increases with forecast horizon, while disagreement may be increasing, decreasing, or non-monotonic in forecast horizon. The uncertainty curve is much more linear than the disagreement curve— that is, uncertainty at the one-year and two-year horizons can almost perfectly predict uncertainty at the five-year horizon, but not so for disagreement. Volatility data is also characterized by this linear curve property (Barrero et al., 2017). We thus contribute to a literature on the empirical and theoretical distinctions between disagreement and uncertainty, which typically cautions against using disagreement as a proxy for uncertainty (Zarnowitz and Lambros, 1987; Boero et al., 2008; D'Amico and Orphanides, 2008; Lahiri and Sheng, 2010; Rich and Tracy, 2010; Abel et al., 2016).

We also find that the term structures of uncertainty and disagreement differ by variable; most notably, there is a much larger difference between five-year and shorter-horizon disagreement for unemployment forecasts than for inflation and growth. This is consistent with evidence that the unemployment rate in the euro area contains a significant nonstationary component (Gali, 2015). Similarly, Andrade et al. (2016) show that in U.S. data, the shape of the term structure of forecast disagreement differs qualitatively across variables.

Another key finding is that uncertainty and disagreement exhibit striking behavior in the crisis period. For inflation and growth forecasts, the term structure of disagreement inverts in early 2009. The term structure of unemployment disagreement narrows in 2008 and 2009, with disagreement about the two-year horizon temporarily surpassing disagreement about the five-year horizon. Uncertainty also typically increases with forecast horizon, but the term structure narrows and exhibits occasional non-monotonicities during the crisis. Moreover, for a sizeable minority of individual forecasters, uncertainty about shorter horizons exceeds uncertainty about longer horizons.

2 Uncertainty and Disagreement Term Structures

The European Central Bank (ECB) has conducted the Survey of Professional Forecasters (SPF) since 1999. Each quarter, approximately 60 forecasters provide point and density forecasts for inflation (π), GDP growth (g), and unemployment (u) over several horizons. Density forecasts have a bin width of 0.5 percentage points. We collect data on the variables being forecasted from Eurostat. More detailed information on these variables appears in Appendix Tables A.1 and A.2. We consider forecasts at three horizons that are included on most survey dates: one-year, two-year, and five-year.[3] Abel et al. (2016) document that changes in the panel of respondents over time do not distort empirical analysis of uncertainty and disagreement.

We estimate the mean and variance of each forecaster's density forecasts both parametrically and non-parametrically. Our parametric estimates come from fitting a generalized beta distribution to the density forecast; uncertainty is the variance of this distribution. Alternatively, following D'Amico and Orphanides (2008), we compute these variances non-parametrically by assuming that the probability is concentrated at the midpoint of each bin.[4] Our parametric and non-parametric uncertainty estimates are extremely similar, with

[3]The one- and two-year forecasts have rolling horizons, where the rolling one-year horizon refers to the month (for inflation and unemployment) or quarter (for growth) one year ahead of the latest available observation at the time of the survey. The rolling two-year horizon refers to the month or quarter two years ahead of the latest available observation. For surveys on the third and fourth quarter of the year, the five-year horizon refers to five calendar years ahead, while on the first and second quarter of the year, the five-year horizon refers to four calendar years ahead.

[4]D'Amico and Orphanides show that estimates of uncertainty computed with the assumption of midpoint probability concentration are very similar if probability is assumed to be uniformly distributed across each bin or if a normal distribution is fitted to the density forecast. Therefore, estimates of uncertainty are not sensitive to this assumption.

correlation coefficients around 0.99. The parametric and non-parametric density mean estimates are likewise very similar to each other and to respondents' point forecasts. We also estimate average *ex post* uncertainty as the squared error of the mean forecast made at time t for horizon $t+k$.[5]

Disagreement could refer to the heterogeneity of point forecasts across forecasters, or the heterogeneity of the mean of the density forecast across forecasters. Since both measures are very similar, we use the heterogeneity of the (parametric) mean of the density forecast.

Figure 1 displays estimates of mean uncertainty by horizon for inflation, growth, and unemployment. For all variables, uncertainty increases with forecast horizon and increases in the Great Recession, remaining elevated thereafter.

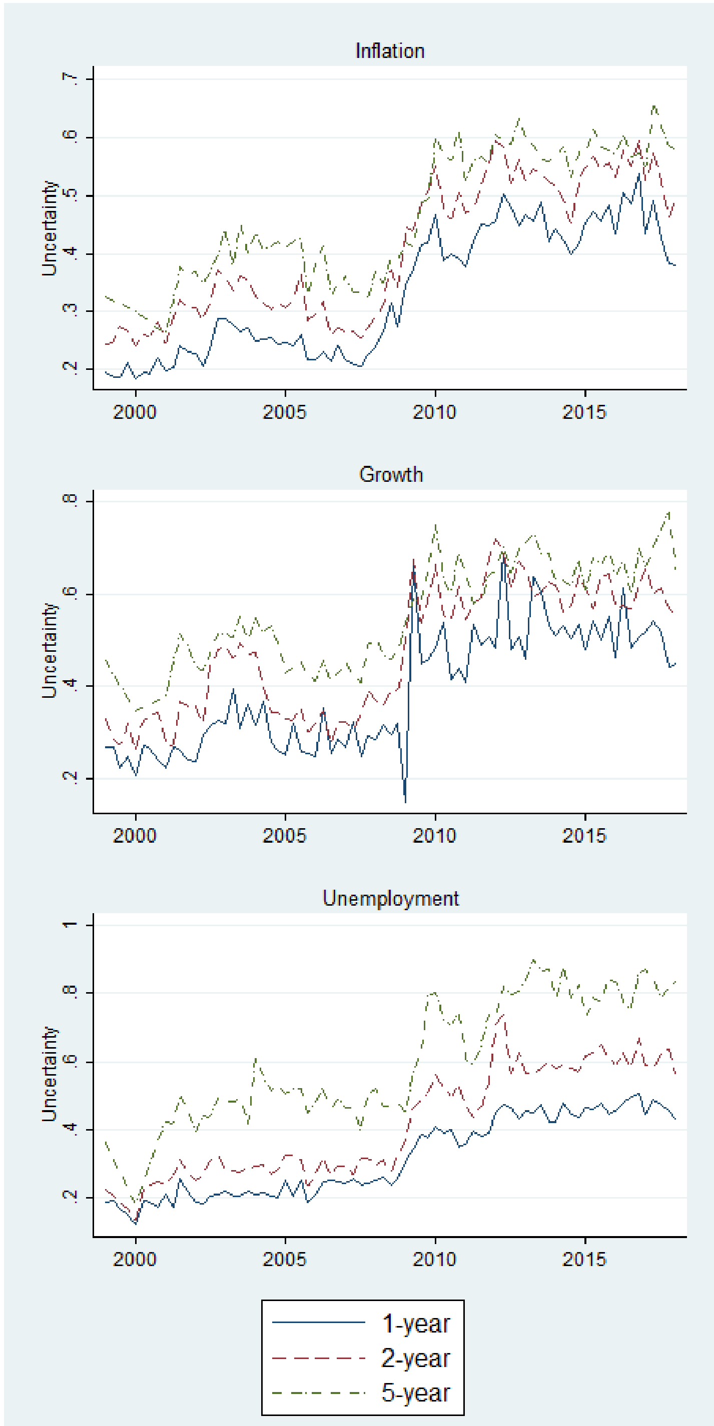
Figure 2 plots *ex post* uncertainty, the mean squared forecast error, for each variable and horizon. For growth, for example, the one-year forecast made in 2008Q4 had the largest ex post error. The mean forecast was 1.2%, and the realization -5.5%. For the longer horizon forecasts, of course, even earlier forecasts were associated with larger errors ex post due to the recession. But *ex ante*, forecasters were not particularly uncertain in 2008Q4. The mean variance of their density forecasts was 0.3. It was not until the recession was fully underway that ex ante uncertainty rose, and it remained elevated even after ex post uncertainty fell.

Figure 3 displays the term structure of disagreement for each variable. Disagreement also spikes in the Great Recession, but unlike uncertainty, does not remain elevated. For unemployment, the most persistent of the variables, disagreement increases monotonically with horizon, but the term structure is often inverted for the other variables. For growth, 1-year disagreement in 2009Q3 spiked to 0.61, while 5-year disagreement was just 0.07. Most forecasters expected growth to return to around 1.8% in the long-run, though they disagreed.

Individual forecasters may differ in their patterns of uncertainty across the term structure. Figure 4 displays the share of forecasters for whom longer-horizon uncertainty is greater than shorter-horizon uncertainty for each variable over time. For inflation, about 70% of

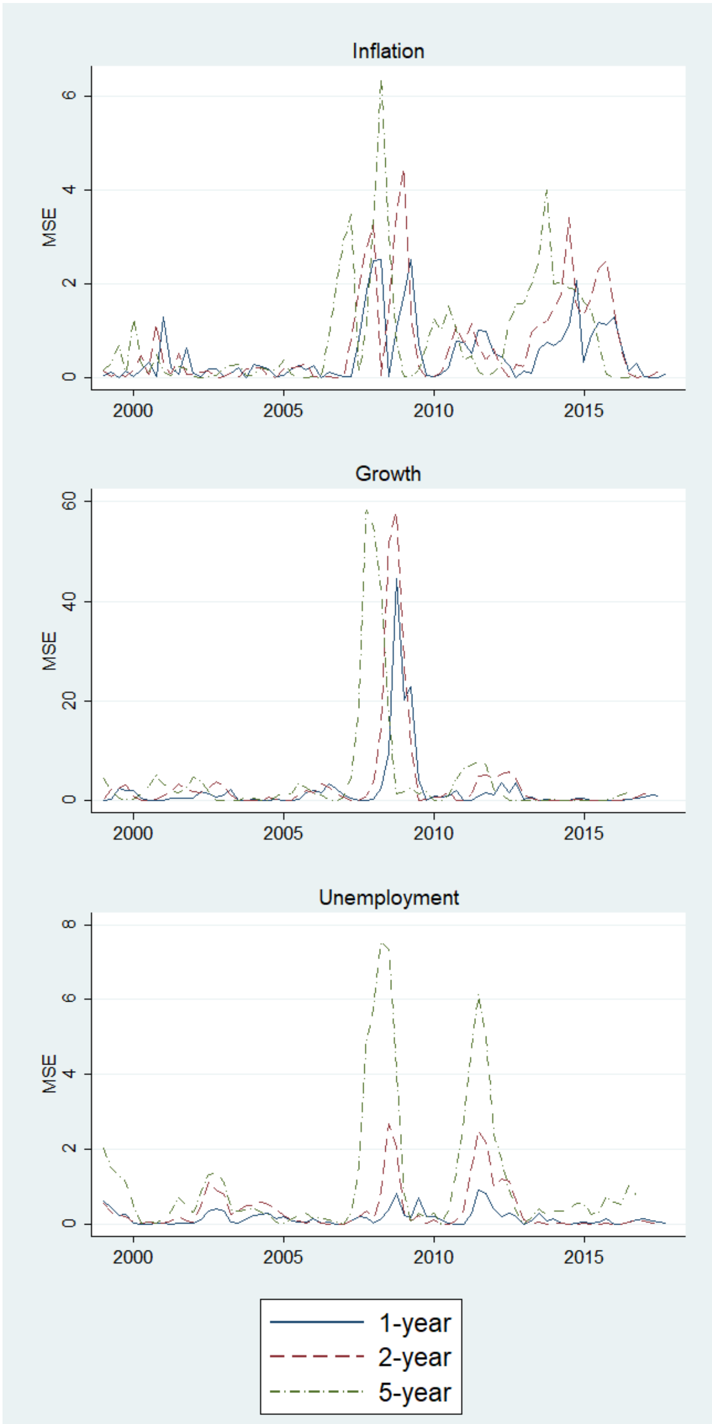
[5] We use the mean across forecasters of the parametric density mean estimate. Results are similar using the mean point forecast or non-parametric density mean.

Figure 1: Term Structure of Uncertainty



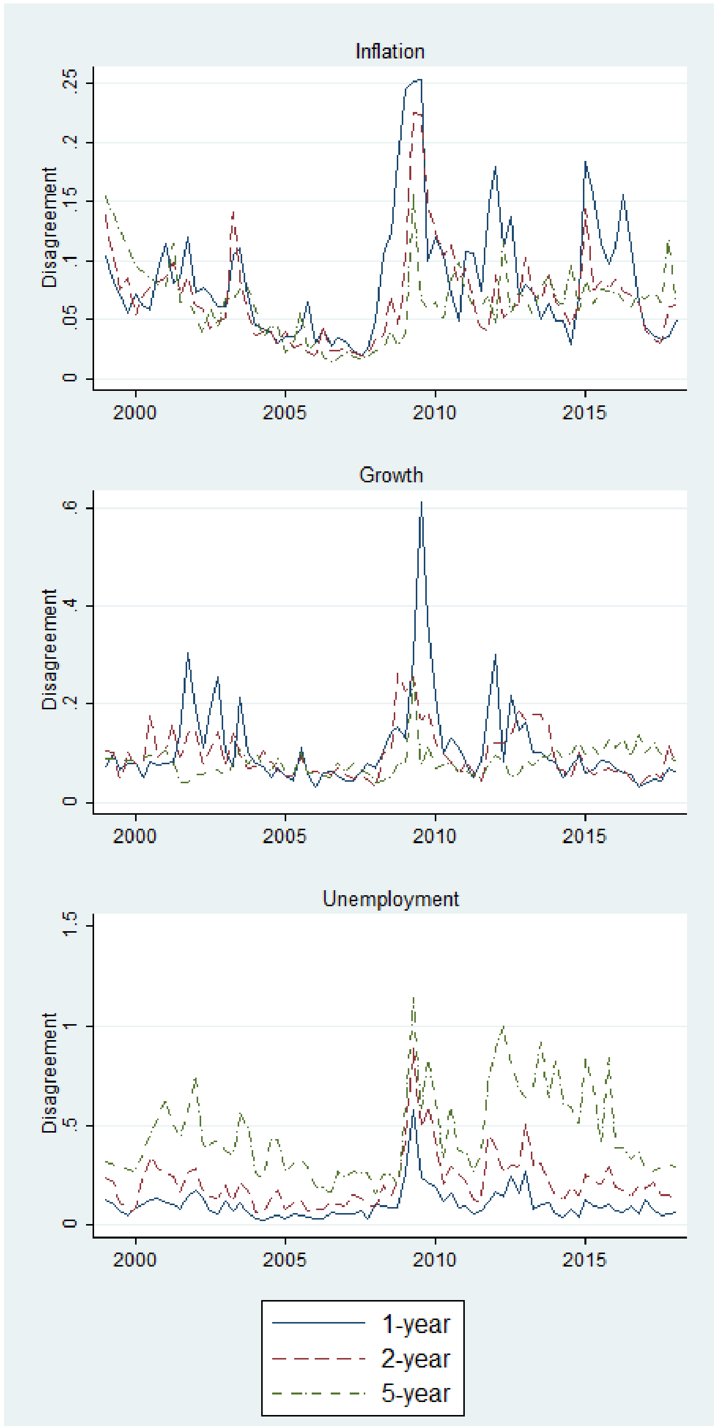
Notes: ECB SPF data. Uncertainty is the mean variance of a beta distribution fit to each individual's density forecast.

Figure 2: Term Structure of Mean Squared Error (Ex Post Uncertainty)



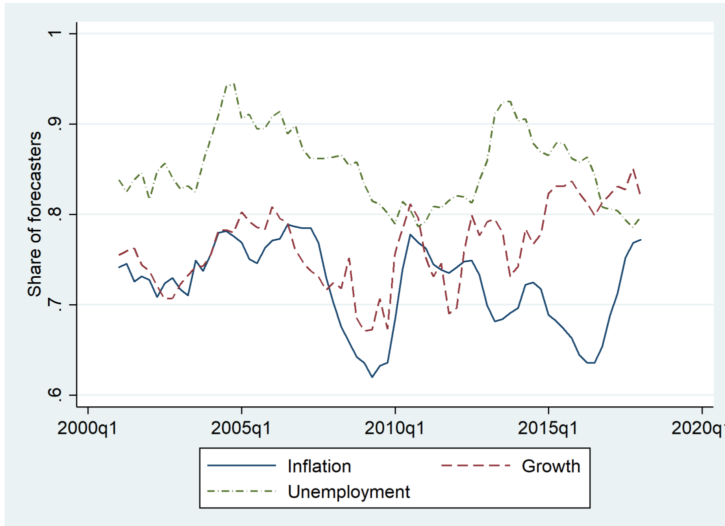
Notes: ECB SPF data. Squared difference between mean forecast made at time t and realization in $t+1$, $t+2$, or $t+5$..

Figure 3: Term Structure of Disagreement



Notes: ECB SPF data. Disagreement is the cross-sectional variance of the mean of each individual's density forecast.

Figure 4: Share of Forecasters with Higher Uncertainty at Longer Horizon



Notes: ECB SPF data. Centered 5-quarter moving average.

forecasters have higher uncertainty at the two-year than the one-year horizon, and 72% have higher uncertainty at the five-year than at the one-year horizon. For growth, these figures are 70% and 76%, and for unemployment, 75% and 85%.

2.1 Linearity of Uncertainty and Disagreement Curves

Barrero et al. (2017) document "the stylized fact that volatility curves are effectively linear, so that the entire curve is well characterized by the 30-day and 6-month (182-day) implied volatilities...It is useful conceptually and empirically to think of the curves as being characterized by a level (30-day implied vol.) and a slope statistic (difference between the 6-month and 30-day implied vol.), as is analogously done in the finance literature with the term structure of interest rates." They do so by "regressing quarterly measures of one- and two-year firm implied volatility against the corresponding level (30-day implied vol.) and slope (difference between the 6-month and 30-day volatilities) and observing that the R-squared from these regressions is .994 and .944, respectively."

Analogously, we regress 5-year uncertainty on 1-year uncertainty (the level) and the difference between 1- and 2-year uncertainty (the slope) in Table 1. Uncertainty at the 5-year horizon is largely explained by uncertainty at the 1-year horizon and the slope statistic, with R^2 around 0.9 for each variable.

We also regress 5-year disagreement on 1-year disagreement and the difference between 1- and 2-year disagreement (Table 2). For disagreement, however, the R^2 values are much lower: 0.35, 0.05, and 0.54 for inflation, growth, and unemployment, respectively. For growth, neither the level nor the slope has statistically significant predictive power for long-run disagreement. Table 3 shows analogous results for mean squared error (ex post uncertainty) and 4 for the realizations themselves (i.e. regression of X_{t+5} on X_{t+1} and $X_{t+2} - X_{t+2}$ where X is inflation, growth, or unemployment.)

Table 1: Regression of 5-Year Uncertainty on 1-Year Uncertainty and Difference between 2- and 1-Year Uncertainty

	(1) Inflation	(2) Growth	(3) Unemployment
1-year	0.882***	0.769***	1.223***
	(0.037)	(0.044)	(0.094)
Difference between 2-year and 1-year	0.985***	0.618***	0.612**
	(0.204)	(0.073)	(0.238)
Constant	0.098***	0.202***	0.158***
	(0.014)	(0.017)	(0.022)
Observations	71	71	71
r2	0.907	0.853	0.889

Standard errors in parentheses

* $p < 0.10$, ** $p < 0.05$, *** $p < 0.01$

Table 2: Regression of 5-Year Disagreement on 1-Year Disagreement and Difference betweent 2- and 1-Year Disagreement

	(1) Inflation	(2) Growth	(3) Unemployment
1-year	0.398***	0.130	0.954***
	(0.106)	(0.143)	(0.284)
Difference between 2-year and 1-year	0.533***	0.153	1.476***
	(0.136)	(0.140)	(0.315)
Constant	0.037***	0.070***	0.203***
	(0.007)	(0.013)	(0.025)
Observations	71	71	71
r2	0.356	0.048	0.539

Standard errors in parentheses

* $p < 0.10$, ** $p < 0.05$, *** $p < 0.01$

Table 3: Regression of 5-Year MSE on 1-Year MSE and Difference betweent 2- and 1-Year MSE

	(1) Inflation	(2) Growth	(3) Unemployment
1-year	0.565	-0.195	1.201*
	(0.387)	(0.132)	(0.653)
Difference between 2-year and 1-year	0.035	0.555**	2.528***
	(0.344)	(0.253)	(0.364)
Constant	0.552***	3.873***	0.423**
	(0.152)	(1.267)	(0.173)
Observations	72	71	72
r2	0.092	0.086	0.557

Standard errors in parentheses

* $p < 0.10$, ** $p < 0.05$, *** $p < 0.01$

Table 4: Regression of Realization at 5-Year Horizon on Realization at 1-Year Horizon and Difference betweent 2- and 1-Year Horizons

	(1) Inflation	(2) Growth	(3) Unemployment
1-year	0.443***	0.314***	0.851***
	(0.136)	(0.096)	(0.041)
Difference between 2-year and 1-year	1.147***	1.511***	2.949***
	(0.228)	(0.338)	(0.310)
Constant	0.960***	0.883***	1.412***
	(0.240)	(0.238)	(0.421)
Observations	72	71	72
r2	0.370	0.457	0.908

Standard errors in parentheses

* $p < 0.10$, ** $p < 0.05$, *** $p < 0.01$

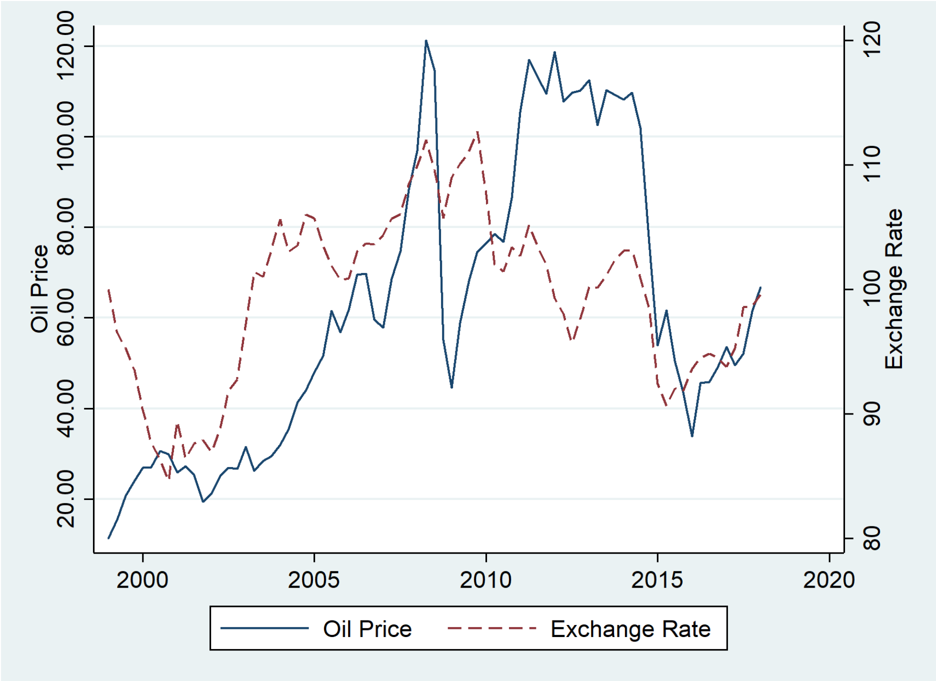
2.2 Drivers of Long- and Short-Run Uncertainty

We regress uncertainty or disagreement at each horizon h on several potential drivers of uncertainty and disagreement. Following Barrero et al. (2017), we consider the Economic Policy Uncertainty (EPU) index for Europe and measures of volatility of oil price returns and exchange rate returns. Abel et al. (2016, p. 534) find "evidence of an economically and statistically signicant relationship between uncertainty and aggregate point predictions—a negative association for GDP growth and a positive association for unemployment." Thus, we also include either the mean point forecast for inflation, growth, and unemployment or the realization of inflation, growth, and unemployment in the previous quarter in the regressions. (We do not include point forecasts and lagged realizations simultaneously because of high collinearity.) In some specifications, we also include the change or squared change in inflation, growth, and unemployment.

The EPU is based on newspaper articles regarding policy uncertainty.[6] Our oil price series is "Crude Oil Prices: Brent - Europe" from the U.S. Energy Information Administration. We compute the standard deviation of daily returns for the past 252 trading days, which is

[6]Data was downloaded in April 2018 from http://www.policyuncertainty.com/europe_monthly.html. The newspapers include Le Monde and Le Figaro for France, Handelsblatt and Frankfurter Allgemeine Zeitung for Germany, Corriere Della Sera and La Repubblica for Italy, El Mundo and El Pais for Spain, and The Times of London and Financial Times for the United Kingdom.

Figure 5: Oil Prices and Exchange Rate

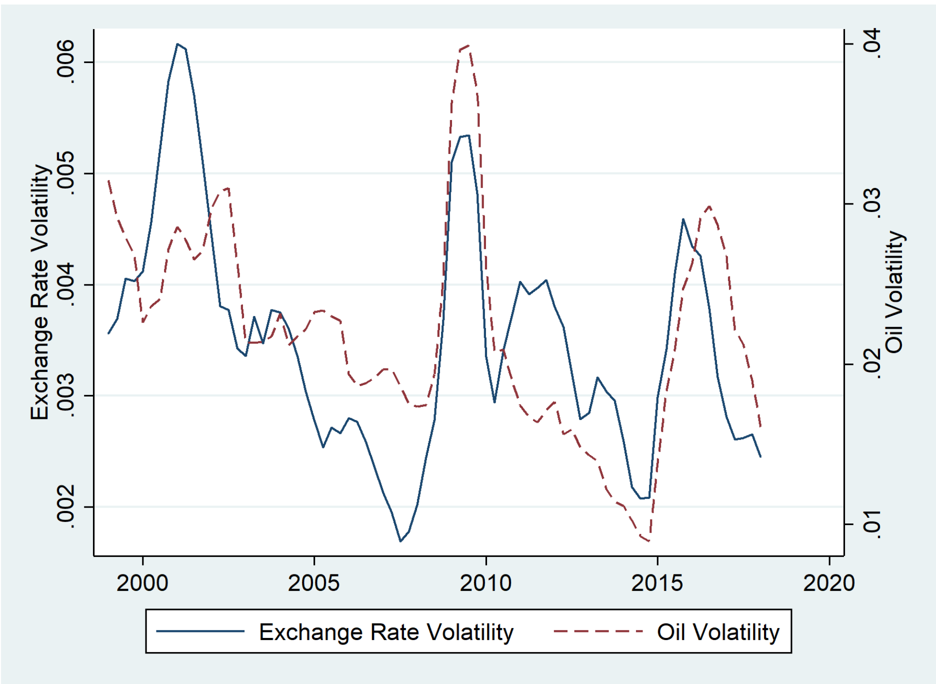


Notes:

the number in a typical year.

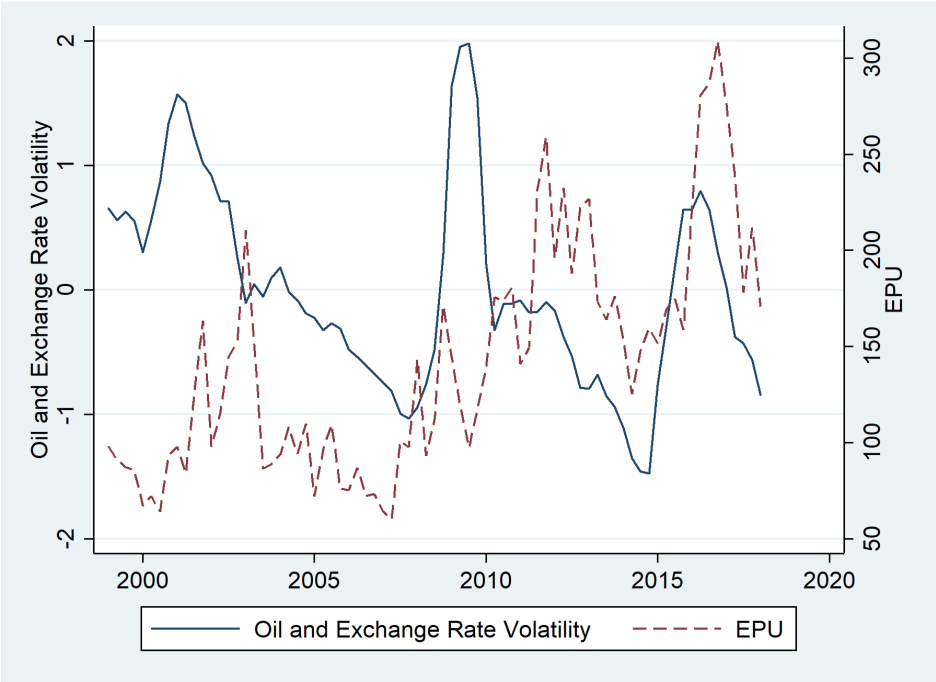
[TO BE DONE]

Figure 6: Volatility of Oil Price and Exchange Rate Returns



Notes: Volatility is the standard deviation of returns over past year (252 trading days).

Figure 7: Volatility and Economic Policy Uncertainty



Notes: Volatility is a weighted average of oil return volatility and exchange rate return volatility. EPU is European Economic Policy Uncertainty.

3 Model of Information Structure and Forecast Formation

Following Baker et al. (2017), we use a model in which N agents forecast a m-dimensional stationary Markov signal process $\{\pi_t\}$ using noisy public and private signals. The observation process for agent i is

$$y_t(i) = \begin{bmatrix} A^{(i)} \\ B \end{bmatrix} \pi_t + \begin{bmatrix} \nu_t(i) \\ \eta_t \end{bmatrix}. \tag{1}$$

where $\{\eta_t\}$ and $\{\nu_t(i)\}$ are the public and private noise and the matrix $A^{(i)}$ corresponds to the private manifestation of the signal. We assume that $\{\eta_t\}$ is serially uncorrelated and— to allow for time-varying uncertainty— has a stochastic covariance matrix Σ_t. The private noises $\{\nu_t(i)\}$ are serially uncorrelated with deterministic covariance matrix $\Sigma^{(i)}$.

3.1 State Space Representation

We first consider the forecasters' Kalman filtering process in the case that Σ_t, the parameters governing $\{\pi_t\}$, and the matrices B and $A^{(i)}$ are known. Later, to introduce first- and second-moment shocks, we will consider the unknown parameters case (Refer to Baker et al. (2017) for details).

Suppose there exists a matrix G such that $\pi_t = G\,x_t$, where x_t is a Markovian state vector x_t with transition equation:

$$x_t = \Phi\, x_{t-1} + \epsilon_t \tag{2}$$

for $t \geq 1$, with initial value x_0. We assume the transition matrix Φ has eigenvalues less than one in magnitude and that the signal innovations $\{\epsilon_t\}$ are uncorrelated with x_0, so that ϵ_t is uncorrelated with x_{t-1} for $t \geq 1$. The innovations' common covariance matrix is denoted

Σ^ϵ. Let

$$\delta_t(i) = \begin{bmatrix} \nu_t^{(i)} \\ \eta_t \end{bmatrix} \qquad H(i) = \begin{bmatrix} A^{(i)} \\ B \end{bmatrix} G,$$

so that combining (1) with $\pi_t = G\,x_t$ yields the observation equation

$$y_t(i) = H(i)\,x_t + \delta_t(i). \tag{3}$$

Evidently $\{\delta_t(i)\}$ is heteroscedastic white noise, with covariance matrix S_t given by

$$S_t = \mathrm{Var}[\delta_t(i)] = \begin{bmatrix} \Sigma^{(i)} & 0 \\ 0 & \Sigma_t \end{bmatrix}. \tag{4}$$

Note that the only data available to the ith agent is $\{y_t(i)\}$, and so estimates of the signal are to be constructed on this basis, without reference to the data available to some other agent j. Together, equations (3) and (2) describe the information structure in state space form (ssf). Then $x_t' = [\pi_t', \pi_{t-1}', \ldots, \pi_{t-p+1}']$ with $G = [I_m, 0, \ldots]$ (and I_m is the m-dimensional identity matrix) corresponds to the companion form, yielding a VAR(1) for the state vector, writing

$$\Phi = \begin{bmatrix} \Phi_1 & \Phi_2 & \ldots & \Phi_p \\ I_m & 0 & \ldots & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \ldots & I_m & 0 \end{bmatrix}.$$

We define the following quantities: the forecast of the state vector is

$$\widehat{x}_{t+1|t}(i) = \mathbb{E}[x_{t+1}|y_1(i), \ldots, y_t(i)],$$

and its mean square error matrix is $P_{t+1|t}(i) = \mathrm{Cov}[x_{t+1} - \widehat{x}_{t+1|t}(i)]$. The residual is the data minus its forecast, namely $e_t(i) = y_t(i) - \widehat{y}_{t|t-1}(i)$, and its mean square error matrix is denoted $V_t(i)$. The Kalman gain is by definition $K_t(i) = \mathrm{Cov}[x_{t+1}, e_t(i)]\,\mathrm{Var}[e_t(i)]^{-1}$, and plays a key role in updating a signal extraction estimate given new information. Initialization of the recursive Kalman filter algorithm is given by $\widehat{x}_{1|0}(i) = 0$ and $P_{1|0}(i) = \mathrm{Var}[x_1]$, which are the correct quantities given a stationary state vector. In the case of a VAR(p) signal process, this initial variance can be computed directly from the companion form. Then for $1 \leq t \leq T$, we compute

$$e_t(i) = y_t(i) - H(i)\,\widehat{x}_{t|t-1}(i) \tag{5}$$

$$V_t(i) = H(i)\,P_{t|t-1}(i)\,H(i)' + S_t \tag{6}$$

$$K_t(i) = \Phi\,P_{t|t-1}(i)\,H(i)'\,V_t(i)^{-1} \tag{7}$$

$$\widehat{x}_{t+1|t}(i) = \Phi\,\widehat{x}_{t|t-1}(i) + K_t(i)\,e_t(i) \tag{8}$$

$$P_{t+1|t}(i) = (\Phi - K_t(i)\,H(i))\;P_{t|t-1}(i)\,\Phi' + G'\,\Sigma^{\epsilon}\,G. \tag{9}$$

Equation (7) gives a recursive formula for the Kalman gain, and its dependence on the heteroscedastic noise is clearly given through $V_t(i)$ in (6). Moreover, equations (8) and (9) tell us how to update our one-step ahead prediction and forecast error variance for the state vector. Again, because the Kalman gain depends upon the heteroscedastic variance Σ_t, both the state vector forecast and its uncertainty will be impacted. In order to obtain $(k+1)$-step ahead forecasts ($k \geq 0$), we compute

$$\widehat{\pi}_{t+1|t-k}(i) = G\,\Phi^k\,\widehat{x}_{t+1-k|t-k}(i) \tag{10}$$

$$\mathrm{Var}[\widehat{\pi}_{t+1|t-k}(i) - \pi_{t+1}] = G\,P_{t+1|t-k}(i)\,G', \tag{11}$$

where $P_{t+1|t-k}(i) = \mathrm{Var}[\widehat{x}_{t+1|t-k}(i) - x_{t+1}]$ and is further described below. So (10) provides an optimal estimate of the forecasted signal, given the presence of noise with time-varying volatility.

We require a flexible and broad specification for the signal and noise processes, and propose the VAR(p) class for the signal, where p is taken sufficiently large to approximate a generic signal. We parametrize this process such that stationarity is guaranteed, as described in Roy, McElroy and Linton (2017); the private noise has variance matrix $\Sigma^{(i)}$, which can be parameterized as a member of the space of symmetric positive definite matrices. For the dynamics of the public signal, we describe Σ_t as a stochastic volatility process.[7] Following the work of Cogley and Sargent (2005), Primiceri (2005), and Neusser (2016), consider the Cholesky decomposition $\Sigma_t = B_t \, \Omega_t \, B_t'$, where B_t is unit lower triangular and Ω_t is diagonal. The diagonal entries of Ω_t each follow an exponential random walk, and the $B_t = \exp\{C_t\}$, where C_t is a lower triangular with zeroes on the diagonal. Each lower triangular element of C_t follows an independent random walk. This framework – of a VAR(p) signal with the exponential random walk model for volatility – can be tailored to the user's specification through the parameter settings, which determine the dispersion for the random walk increments in the volatility process.

A temporary shock at some time index τ can be generated by scaling the diagonal entries of a single Σ_t by some $a > 0$, but without altering B_t or Ω_t, so that the effect is transitory:

$$\Sigma_t = B_t \, \Omega_t \, B_t' \cdot (1 + a \, 1_{\{t=\tau\}}). \tag{12}$$

This ensures that Σ_τ has values multiplied by $1+a$, but the corresponding shock η_τ will not be large unless the random vector is generated from the right tail of the normal distribution. We proceed by generating m random variables Z independently from the marginal distribution $\mathbb{P}[Z > x | Z > 2] = (1 - \Phi(x))/(1 - \Phi(2))$, and multiplying the corresponding vector ζ by $\Sigma_\tau^{1/2}$ to obtain η_τ. This modification to η_t and Σ_t at time $t = \tau$ will be designated as temporary first-moment and second-moment shocks.

A permanent shock at some time index τ involves dilating Σ_t in the same manner as the

[7] Chiu, Leonard and Tsui (1996) modeled Σ_t as the matrix exponential of a symmetric matrix A_t (which can take negative values), whose vech was modeled as a VAR process. Uhlig (1997) modeled Σ_t via a generalized Cholesky decomposition, wherein the diagonal factor followed a positive random walk model. A Wishart autoregressive process was studied in Gouriéroux, Jasiak and Sufana (2009).

temporary shock, but with the effect lasting for all times $t \geq \tau$:

$$\Sigma_t = B_t \, \Omega_t \, B_t' \cdot (1 + a \, 1_{\{t \geq \tau\}}). \tag{13}$$

This second-moment shock is paired with a first-moment shock obtained by generating a temporary first-moment shock ζ at time $t = \tau$ in the manner described above, and for $t > \tau$ we draw an independent standard normal random vector υ_t, and set $\eta_t = \Sigma_t^{1/2} \, (\zeta + \upsilon_t)$. Note that in this construction, ζ corresponds to an initial up-swing at time τ, which is persistent at later times $t > \tau$.

4 A Framework for Expectation Formation

The key three ingredients of our framework are: (i) multi-step ahead forecasting, (ii) stochastic volatility, and (iii) Kalman filter updating. The ith agent ($1 \leq i \leq N$) observes both public and private signals and makes the forecast through a signal extraction process. Given the data $\{y_t(i)\}$ for $1 \leq t \leq T$, for each agent i, we obtain the $(k+1)$-step ahead forecast $\widehat{\pi}_{t+1|t-k}(i)$ from (10); calculation of the uncertainty relies upon $P_{t+1|t-k}(i)$ in (11), which is a special case of the multi-step ahead error covariance

$$R_{k,\ell}^{(ij)}(t) = \text{Cov}\left[\widehat{x}_{t+1|t-k}(i) - x_{t+1}, \widehat{x}_{t+1|t-\ell}(j) - x_{t+1}\right],$$

i.e., the covariance of forecast errors for the ith agent ($k+1$ steps ahead) and the jth agent ($\ell+1$ steps ahead). The following result provides a recursive algorithm for computing these covariances, based upon recursions for the one-step ahead error covariances

$$Q_{t+1|t}^{(ij)} = \text{Cov}[\widehat{x}_{t+1|t}(i) - x_{t+1}, \widehat{x}_{t+1|t}(j) - x_{t+1}].$$

Note that setting $j = i$ and $\ell = k$ we obtain $P_{t+1|t-k}(i) = R^{(ii)}_{k,k}(t)$, and furthermore $Q^{(ii)}_{t+1|t} = P_{t+1|t}(i)$.

Proposition 1 *The covariance of one-step ahead prediction errors across agents, $Q^{(ij)}_{t+1|t}$, can be computed recursively by*

$$Q^{(ij)}_{t+1|t} = [\Phi - K_t(i)\, H(i)]\, Q^{(ij)}_{t|t-1}[\Phi - K_t(j)\, H(j)]' + \Sigma^{\epsilon} - K_t(i) \begin{bmatrix} 0 & 0 \\ 0 & \Sigma_t \end{bmatrix} K_t(j)', \qquad (14)$$

with the initialization $Q^{(ij)}_{1|0} = Var[x_1]$ for all i and j. The covariance of prediction errors across forecast horizons and agents, $R^{(ij)}_{k,\ell}(t)$, can be computed if $k \leq \ell$ by

$$R^{(ij)}_{\ell,\ell}(t) = \Phi^{\ell}\, Q^{(ij)}_{t+1-\ell|t-\ell}\, \Phi'^{\ell} + \sum_{n=0}^{\ell-1} \Phi^n\, \Sigma^{\epsilon}\, \Phi'^n$$

$$R^{(ij)}_{\ell-1,\ell}(t) = \Phi^{\ell-1}\, (\Phi - K_{t+1-\ell}(i)\, H(i))\, Q^{(ij)}_{t+1-\ell|t-\ell}\, \Phi'^{\ell} + \sum_{n=0}^{\ell-1} \Phi^n\, \Sigma^{\epsilon}\, \Phi'^n$$

$$R^{(ij)}_{k,\ell}(t) = \Phi^k \prod_{n=k}^{\ell-1} (\Phi - K_{t-n}(i)\, H(i))\, Q^{(ij)}_{t+1-\ell|t-\ell}\, \Phi'^{\ell} + \sum_{m=2}^{\ell-k} \Phi^k \prod_{n=k}^{\ell-m} (\Phi - K_{t-n}(i)\, H(i))\, \Sigma^{\epsilon}\, \Phi'^{\ell-m+1} + \sum_{n=0}^{k} \Phi^n\, \Sigma^{\epsilon}\, \Phi'^n,$$

where $k \leq \ell - 2$ in the last case, and where the matrix products are computed with the lowest index matrix first, and multiplying on the right by matrices of higher index.

It is clear that Proposition 1 can be used to compute the mean squared error (MSE) of multi-step ahead forecast errors, and is a new result in the state space literature of interest in its own right. Beyond supplying the covariances needed to compute (11), the proposition also allows us to compute aggregate forecaster MSE. If we have interest in some linear composite of economic agents' results, say

$$\overline{\pi}_{t+k+1|t} = \sum_{i=1}^{N} w_i\, \widehat{\pi}_{t+k+1|t}(i) \qquad (15)$$

for given weights w_i, then the corresponding target is $\sum_{i=1}^{N} w_i \pi_{t+k+1}$, which equals π_{t+k+1} when the weights sum to one. The variance of the discrepancy between $\overline{\pi}_{t+k+1|t}$ and π_{t+k+1} is the forecast MSE, given by

$$\mathrm{MSE}_{t+k+1|t} = \sum_{i,j=1}^{N} w_i \, w_j \, G \, R_{k,k}^{(ij)}(t+k) \, G'. \tag{16}$$

To compute (16) recursively, for fixed k, we only need the first formula in Proposition 1, which in turn requires us to know $Q_{t+1|t}^{(ij)}$, and is calculated from (14). The disagreement across agents is defined as the sample variability of the forecasts across forecasters, i.e.,

$$D_{t+k+1|t} = \sum_{i=1}^{N} w_i \left(\widehat{\pi}_{t+k+1|t}(i) - \overline{\pi}_{t+k+1|t}\right) \left(\widehat{\pi}_{t+k+1|t}(i) - \overline{\pi}_{t+k+1|t}\right)'. \tag{17}$$

This is easily computed from (10) and (15). To see this, we can show that

$$\widehat{\pi}_{t+k+h+1|t}(i) - \overline{\pi}_{t+k+h+1|t} = G \, \Phi^h \, \left(\widehat{x}_{t+k+1|t}(i) - \overline{x}_{t+k+1}\right),$$

where $\overline{x}_{t+k+1} = \sum_{i=1}^{N} w_i \, \widehat{x}_{t+k+1|t}(i)$. Substituting into (17) yields

$$D_{t+k+h+1|t} = \sum_{i=1}^{N} w_i \, G \, \Phi^h \left(\widehat{x}_{t+k+1|t}(i) - \overline{x}_{t+k+1|t}\right) \left(\widehat{x}_{t+k+1|t}(i) - \overline{x}_{t+k+1|t}\right)' \Phi'^h \, G',$$

which expresses the $(h+k+1)$-step ahead dispersion in terms of $(k+1)$-step ahead dispersion (of the state vector).

Let the aggregate forecast uncertainty be defined as

$$U_{t+k+1|t} = \sum_{i=1}^{N} w_i \, P_{t+k+1|t}(i),$$

which represents an average (across agents) of the variability in k-step ahead forecasting of the state vector. Then we have the following result.

Proposition 2 *The covariance of prediction errors across one forecast horizon and multiple agents can be recursively computed via*

$$R^{(ij)}_{k+h,k+h}(t+k+h) = \Phi^h \, R^{(ij)}_{k,k}(t+k) \, \Phi'^h + \sum_{n=0}^{h-1} \Phi^n \, \Sigma^\epsilon \, \Phi'^n.$$

Hence the aggregate forecast uncertainty satisfies

$$U_{t+k+1|t} = \Phi^k \, U_{t+1|t} \, \Phi'^k + \sum_{n=0}^{k-1} \Phi^n \, \Sigma^\varepsilon \, \Phi'^n. \tag{18}$$

5 Simulation

[TO BE DONE]

6 Conclusion

In the Great Recession, professional forecasters in Europe not only became more uncertain about inflation, growth, and unemployment, but also expressed more disagreement in their forecasts. While disagreement has since fallen at most horizons, uncertainty remains elevated. Understanding the sources of disagreement and uncertainty is an important goal, with implications about the social value of public information and about the effects of policies that work partially through an expectations channel (such as monetary policy and tax policy) (Mankiw et al., 2004).

Patton and Timmermann (2010) point out that the term structure of forecaster disagreement is a way of deducing the relative importance of various causes of disagreement. They find that greater disagreement during recessions is not due to increased heterogeneity in information signals but rather to agents putting more weight on model-based forecasts during recessions. Similarly, analysis of the term structure of uncertainty can help reveal the causes of uncertainty and information about forecasters' models and beliefs.

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Appendix A Tables and Figures

Table A.1: Variables Forecasted by the European Central Bank Survey of Professional Forecasters

Name	Notation	Description
Inflation	π	Year on year percentage change of the Harmonised Index of Consumer Prices (HICP) published by Eurostat
Growth	g	Year on year percentage change of real GDP based on ESA definition
Unemployment	u	Unemployment as percentage of labor force based on Eurostat definition

Notes: For more information, see "ECB Survey of Professional Forecasters (SPF): Description of SPF Dataset."

Table A.2: Survey Return Dates and Available Information

Quarter	Return Date	Last π Obs.	Last g Obs.	Last u Obs.
1	Late Jan/early Feb	Dec	Q3	Nov
2	Late April/early May	March	Q4	Feb
3	Late July/early Aug	June	Q1	May
4	Late Oct/early Nov	Sep	Q2	Aug

Notes: For more information, see "ECB Survey of Professional Forecasters (SPF): Description of SPF Dataset."

Appendix B Proofs

Proof of Proposition 1 Equation (14) was proved in Proposition 1 of Baker et al. (2017). The multi-step ahead forecasting of the state vector is straightforward: $\widehat{x}_{t+1|t-k}(i) = \Phi^k\, \widehat{x}_{t+1-k|t-k}(i)$. Also, iteration of (2) yields $x_{t+1} = \Phi^k\, x_{t+1-k} + \sum_{n=0}^{k-1} \Phi^n\, \epsilon_{t+1-n}$. Therefore the multi-step ahead forecasting error can be expressed in terms of one-step ahead forecasting error, via

$$\widehat{x}_{t+1|t-k}(i) - x_{t+1} = \Phi^k\, \left(\widehat{x}_{t+1-k|t-k}(i) - x_{t+1-k}\right) - \sum_{n=0}^{k-1} \Phi^n\, \epsilon_{t+1-n}.$$

We can also express the one-step ahead forecasting error in terms of prior such errors. Suppose $k < \ell$:

$$\begin{aligned}\widehat{x}_{t+1-k|t-k}(i) - x_{t+1-k} &= \Phi\, \left(\widehat{x}_{t-k|t-k-1}(i) - x_{t-k}\right) + K_{t-k}(i)\, e_{t-k}(i) - \epsilon_{t-k+1} \\ \cdots &= \Phi^{\ell-k}\, \left(\widehat{x}_{t+1-\ell|t-\ell}(i) - x_{t+1-\ell}\right) + \sum_{n=0}^{\ell-k-1} \Phi^n\, \left(K_{t-k-n}(i)\, e_{t-k-n}(i) - \epsilon_{t+1-k-n}\right).\end{aligned}$$

This indicates that the multi-step ahead forecasting error is related to past one-step ahead forecasting errors as follows, when $k < \ell$:

$$\widehat{x}_{t+1|t-k}(i) - x_{t+1} = \Phi^\ell\, \left(\widehat{x}_{t+1-\ell|t-\ell}(i) - x_{t+1-\ell}\right) - \sum_{n=0}^{\ell-1} \Phi^n\, \epsilon_{t+1-n} + \sum_{n=0}^{\ell-k-1} \Phi^{n+k}\, K_{t-k-n}(i)\, e_{t-k-n}(i).$$

The errors $e_t(i)$ have the following form:

$$e_{t-k-n}(i) = y_{t-k-n}(i) - H(i)\, \widehat{x}_{t-k-n|t-k-n-1}(i) = -H(i)\, \left(\widehat{x}_{t-k-n|t-k-n-1}(i) - x_{t-k-n}\right) + \delta_{t-k-n}(i). \quad (19)$$

For $0 \leq n \leq \ell - k - 1$, $\delta_{t-k-n}(i)$ is uncorrelated with $\widehat{x}_{t+1-\ell|t-\ell}(i) - x_{t+1-\ell}$, and moreover for $0 \leq n \leq \ell - 1$, ϵ_{t+1-j} is uncorrelated with $\widehat{x}_{t+1-\ell|t-\ell}(i) - x_{t+1-\ell}$. Now using (19), we can re-express the one-step ahead forecasting errors as

$$\begin{aligned}\widehat{x}_{t+1-k|t-k}(i) - x_{t+1-k} &= (\Phi - K_{t-k}(i)\, H(i))\, \left(\widehat{x}_{t-k|t-k-1}(i) - x_{t-k}\right) + K_{t-k}(i)\, \delta_{t-k}(i) - \epsilon_{t-k+1} \\ \cdots &= \prod_{n=k}^{\ell-1} (\Phi - K_{t-n}(i)\, H(i))\, \left(\widehat{x}_{t-\ell+1|t-\ell}(i) - x_{t-\ell+1}\right) \\ &+ \prod_{n=k}^{\ell-2} (\Phi - K_{t-n}(i)\, H(i))\, \left(K_{t-\ell+1}(i)\, \delta_{t-\ell+1}(i) - \epsilon_{t-\ell+2}\right) \\ &\cdots + (\Phi - K_{t-k}(i)\, H(i))\, \left(K_{t-k-1}(i)\, \delta_{t-k-1}(i) - \epsilon_{t-k}\right) + K_{t-k}(i)\, \delta_{t-k}(i) - \epsilon_{t-k+1}.\end{aligned}$$

The convention regarding the product symbols is as discussed in Proposition 1. Hence the multi-step ahead forecasting errors can be re-expressed as

$$\widehat{x}_{t+1|t-k}(i) - x_{t+1} = \Phi^k \prod_{n=k}^{\ell-1}(\Phi - K_{t-n}(i)\,H(i))\,\left(\widehat{x}_{t-\ell+1|t-\ell}(i) - x_{t-\ell+1}\right) \tag{20}$$

$$+ \Phi^k \prod_{n=k}^{\ell-2}(\Phi - K_{t-n}(i)\,H(i))\,(K_{t-\ell+1}(i)\,\delta_{t-\ell+1}(i) - \epsilon_{t-\ell+2})$$

$$\cdots + \Phi^k\,(\Phi - K_{t-k}(i)\,H(i))\,(K_{t-k-1}(i)\,\delta_{t-k-1}(i) - \epsilon_{t-k}) + \Phi^k\,K_{t-k}(i)\,\delta_{t-k}(i) - \sum_{n=0}^{k}\Phi^n\,\epsilon_{t+1-n}$$

when $k \leq \ell - 2$; when $k = \ell - 1$ the simpler formula is

$$\widehat{x}_{t+1|t-k}(i) - x_{t+1} = \Phi^k\,(\Phi - K_{t-k}(i)\,H(i))\,\left(\widehat{x}_{t-k|t-k-1}(i) - x_{t-k}\right) + \Phi^k\,K_{t-k}(i)\,\delta_{t-k}(i) - \sum_{n=0}^{k}\Phi^n\,\epsilon_{t+1-n}.$$

From these expressions, the formulas for $R_{k,\ell}$ can now be deduced. The case $R_{\ell,\ell}$ is standard, whereas for $R_{\ell-1,\ell}$ indicates we should set $k = \ell - 1$, and together with

$$\widehat{x}_{t+1|t-\ell}(i) - x_{t+1} = \Phi^\ell\,\left(\widehat{x}_{t+1-\ell|t-\ell}(i) - x_{t+1-\ell}\right) - \sum_{n=0}^{\ell-1}\Phi^n\,\epsilon_{t+1-n} \tag{21}$$

we find $R_{\ell-1,\ell}$ has the stated expression. This uses the fact that $\widehat{x}_{t+1-\ell|t-\ell}(i) - x_{t+1-\ell}$ is uncorrelated with $\delta_{t-\ell+1}(i)$, as well as ϵ_{t+1-n} for $0 \leq n \leq \ell - 1$. Next, for $k \leq \ell - 2$ we compute $R_{k,\ell}$ using (21) together with (20). $\square$

Proof of Proposition 2 Forecasting the state vector $k+1$ steps ahead ($k \geq 0$) is given by

$$\widehat{x}_{t+k+1|t}(i) = \Phi^k\,\widehat{x}_{t+1|t}(i),$$

and the corresponding forecast error is

$$\widehat{x}_{t+k+1|t}(i) - x_{t+k+1} = \Phi^k\,\left(\widehat{x}_{t+1|t}(i) - x_{t+1}\right) - \sum_{n=0}^{k-1}\Phi^n\,\epsilon_{t+k+1-n}$$

(where the sum is omitted if $k = 0$). So for $h > 0$, we obtain

$$
\begin{aligned}
\widehat{x}_{t+k+h+1|t}(i) - x_{t+k+h+1} &= \Phi^{h+k} \left(\widehat{x}_{t+1|t}(i) - x_{t+1}\right) - \sum_{n=0}^{h+k-1} \Phi^n \, \epsilon_{t+k+1-n} \\
&= \Phi^h \left\{ \Phi^k \left(\widehat{x}_{t+1|t}(i) - x_{t+1}\right) - \sum_{n=0}^{k-1} \Phi^n \, \epsilon_{t+k+1-n} \right\} - \sum_{n=0}^{h-1} \Phi^n \, \epsilon_{t+h+k+1-n} \\
&= \Phi^h \left(\widehat{x}_{t+k+1|t}(i) - x_{t+k+1}\right) - \sum_{n=0}^{h-1} \Phi^n \, \epsilon_{t+h+k+1-n}.
\end{aligned}
$$

The two terms on the right hand side are uncorrelated with one another, and hence we obtain the relation (18). Next, setting $h = 0$ and using $P_{t+k+1|t}(i) = R^{(ii)}_{k,k}(t+k)$ and summing against w_i (which sum to one), we obtain the recursive relation for aggregate forecast uncertainty. $\square$