Leverage and Disagreement
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François Geerolf

UCLA

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- **Endogenous Leverage**
- Interest Rates on Collateralized Bonds

among competitive investors with **heterogenous beliefs**.
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- Only **one** leverage ratio (simplifying assumption on the structure of beliefs / or on the number of agents).
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- Counterfactual. **Many** leverage ratios, even for same asset: homebuyers, entrepreneurs, hedge funds, investment banks...
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Relaxing the hypotheses leading to one leverage ratio, the model yields two key predictions.
1) When disagreement goes to 0, the upper tail of the distribution of leverage ratios goes to a Pareto with endogenous tail coefficient 2, for any smooth and bounded away from zero density of beliefs.
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- Cross section of Hedge Funds (TASS Lipper, 2006)

**Pareto in the upper tail** ($l \in [150, 3000]$)
1) When disagreement goes to 0, the upper tail of the distribution of leverage ratios goes to a Pareto with endogenous tail coefficient 2, for any smooth and bounded away from zero density of beliefs.

- Cross section of Hedge Funds (TASS Lipper, 2006)

- Pareto in the upper tail (\( l \in [150, 3000] \))

- Point estimate for tail coefficient: \( \alpha = 1.95 \) (std: 0.2).
Cross-section of homowners’ initial leverage ratios (Dataquick, August 2010).
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Pareto of leverage ratios found also for:
- Entrepreneurs in the SCF.
- Firms in Compustat.
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Leverage Ratio Distribution of US Homeowners
(Leverage Ratio on New Loans)

Pareto of leverage ratios found also for:
- Entrepreneurs in the SCF.
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⇒ Pareto for borrowers’ expected / realized returns, however small belief heterogeneity:
- Pareto Returns to entrepreneurship.
- Pareto Returns to speculation in general.
2) Distribution of interest rates adjusts so that borrowers and lenders are matched assortatively: interest rates are assignment / hedonic prices, disconnected from expected and true default probability:
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- New determinant for pricing fixed income securities. (⇒ Credit Spread Puzzle? / CDS-Bond Basis)
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- New determinant for pricing fixed income securities. \((\Rightarrow \text{Credit Spread Puzzle? / CDS-Bond Basis})\)
- Investing in high yield not necessarily risk shifting.
2) Distribution of interest rates adjusts so that borrowers and lenders are matched assortatively: **interest rates are assignment / hedonic prices**, disconnected from expected and true default probability:

- New determinant for pricing fixed income securities. (⇒ Credit Spread Puzzle? / CDS-Bond Basis)
- Investing in high yield not necessarily risk shifting.
- High customization / fragmentation of the market = Endogenous OTC structure. ⇒ OTC versus exchanges debate.
Model Ingredients:

- Disagreement on mean rather than on default probabilities.

Key Results:

- Pareto distributions for leverage ratios / expected and realized returns. Also gives information on:
  - Representativeness of marginal buyer
  - Elements of the belief distribution. (⇒ monitoring systemic risk?)
  - Underlying financial structure.
- Credit spreads as hedonic interest rates.

Other Theoretical / Methodological contributions:

- Pyramiding Lending Arrangements.
- Endogenous Short-sales:
  - Endogenous rebate rates, without transactions costs / risk aversion.
  - Endogenous short interest.
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Literature

- **Heterogeneous Priors.** Miller (1977), Harrison, Kreps (1978), Ofek, Richardson (2003), Hong, Scheinkman, Xiong (2006), Hong, Stein (2007), Hong, Sraer (2012).


- **Credit Spread Puzzle.** Chen, Colling-Dufresne, Goldstein (2009), Buraschi, Trojani, Vedolin (2011), Huang and Huang (2012), Albagli, Hellwig, Tsyvinski (2012), McQuade (2013).

Model with Borrowing Contracts Only

Setup
Equilibrium Definition
Equilibrium Solution
Equilibrium Properties

Extension 1: "Pyramiding" Lending Arrangements

Extension 2: Short-Sales

Conclusion
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Conclusion
Set-up

- Two Periods: 0 and 1.
Set-up

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- Continuum of agents. Measure 1.
Set-up

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- Consume in period 1.
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Assets

- Storage’s Return $R = 1. \rightarrow \text{Cash}.$
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- **Borrowing Contracts** collateralized by the Real Asset.
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  - No-recourse.
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▶ **Borrowing Contracts** collateralized by the Real Asset.
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  ▶ Normalization: 1 unit of Real Asset in Collateral.
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  - $\phi$: **Face Value** - promised payment in period 1.
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  - $\phi$: **Face Value** - promised payment in period 1.
  - Notation for contract: $(\phi)$.
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  - Competitive Markets (Anonymous). Price: $q(\phi)$. ”Loan amount”. Implicit interest rate: $r(\phi) = \phi/q(\phi)$. 

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  - Notation for contract: $(\phi)$.
  - Competitive Markets (Anonymous). Price: $q(\phi)$. "Loan amount". Implicit interest rate: $r(\phi) = \phi/q(\phi)$.
  - Payoff: $\min\{\phi, p_1\}$. 
Beliefs

- Agents agree to disagree on $p_1$.
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- Agent $i$: point expectations $p_1^i \in [1 - \Delta, 1]$. 

Diagram:

```
1-\Delta          1
-------------------
p_1^i
```
Beliefs

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Key difference with Geanakoplos (1997), where agents agree on value upon default.
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- Generalization:
  - Agents agree on a probability distribution around mean.
  - Risk neutral.
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Density $f(.)$, c.d.f $F(.)$ on $[1 - \Delta, 1]$.
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$1 - \Delta \quad 1$ $\quad p^i_1$

- Key difference with Geanakoplos (1997), where agents agree on value upon default.
- Generalization:
  - Agents agree on a probability distribution around mean.
  - Risk neutral.
- Density $f(.)$, c.d.f $F(.)$ on $[1 - \Delta, 1]$.
- Exogenously given.
- No learning.
Agents’ Problem

Given \((p, q().)\), agent \(i\) chooses \((n_i^A, n_i^B(.), n_i^C)\) to max. expected wealth \((W)\) in period 1 under:

- Budget Constraint (BC).

\[
\text{max} \quad (n_i^A, n_i^B(.), n_i^C)
\]

\[
\text{s.t.} \quad n_i^A p_i + \int \phi n_i^B(\phi) q(\phi) d\phi + n_i^C \leq 1 \quad \text{(BC)}
\]

\[
\text{s.t.} \quad \int \phi \max\{-n_i^B(\phi), 0\} d\phi \leq n_i^A \quad \text{(CC)}
\]

\[
n_i^A \geq 0, \quad n_i^C \geq 0
\]
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Given \((p, q(.))\), agent \(i\) chooses \( (n^i_A, n^i_B(.), n^i_C) \) to max. expected wealth \((W)\) in period 1 under:

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- Collateral Constraint (CC).
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\[
\max_{(n^i_A, n^i_B(.), n^i_C)} n^i_A p^i_1 + \int n^i_B(\phi) \min\{\phi, p^i_1\} d\phi + n^i_C
\]

\((W)\)

\(\text{(BC)}\)

\(\text{(CC)}\)
Agents’ Problem

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\max_{(n^i_A, n^i_B(.), n^i_C)} n^i_A p^i + \int \phi n^i_B(\phi) \min\{\phi, p^i\} d\phi + n^i_C \quad \text{(W)}
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\[
s.t. \quad n^i_A p + \int \phi n^i_B(\phi) q(\phi) d\phi + n^i_C \leq 1 \quad \text{(BC)}
\]

\[
\text{(CC)}
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Agents’ Problem

Given \((p, q(\cdot))\), agent \(i\) chooses \((n^i_A, n^i_B(\cdot), n^i_C)\) to max. expected
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\max_{(n^i_A, n^i_B(\cdot), n^i_C)} n^i_A p^i + \int_{\phi} n^i_B(\phi) \min\{\phi, p^i\} d\phi + n^i_C \quad \text{(W)}
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- **Collateral Constraint (CC).**

\[
\begin{align*}
\max_{(n^i_A, n^i_B(.), n^i_C)} & \quad n^i_A p^i + \int n^i_B(\phi) \min\{\phi, p^i\} d\phi + n^i_C \\
\text{s.t.} & \quad n^i_A p + \int n^i_B(\phi) q(\phi) d\phi + n^i_C \leq 1 \\
\text{s.t.} & \quad \int n^i_B(\phi) d\phi \leq n^i_A \\
\text{s.t.} & \quad n^i_A \geq 0, \quad n^i_C \geq 0
\end{align*}
\]
Model with Borrowing Contracts Only

Setup
Equilibrium Definition
Equilibrium Solution
Equilibrium Properties

Extension 1: "Pyramiding" Lending Arrangements

Extension 2: Short-Sales

Conclusion
Definition (Competitive Equilibrium for Economy $\mathcal{E}^B$)

A competitive equilibrium is a price system $(p, q(.))$, and portfolios $(n^i_A, n^i_B(.), n^i_C)$ for all $i$ such that:

1. Given $(p, q(.))$, agent $i$ chooses $(n^i_A, n^i_B(.), n^i_C)$ maximizing $(W)$ under (BC) and (CC),

2. Markets clear:
   \[ \int_i n^i_A \, di = 1 \]
   and
   \[ \forall \phi, \int_i n^i_B(\phi) \, di = 0 \]
Equilibrium

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Agents’ Types

Agents split into three types depending on optimism:

- Cash Investors
- Lenders
- Borrowers

\[ p_i \in [\tau, 1] \rightarrow \text{Borrowers (“Homeowners”, “Hedge Funds”, “Entrepreneurs”).} \]

\[ p_i \in [0, \tau] \rightarrow \text{Lenders.} \]

\[ p_i \in [1 - \Delta, \tau] \rightarrow \text{Cash Investors.} \]
Agents’ Types

Agents split into three types depending on optimism:

- **1 - Δ** ∈ [τ, 1] → Borrowers ("Homeowners", "Hedge Funds", "Entrepreneurs").
- **ξ** ∈ [τ, 1] → Lenders.
- **1 - Δ** ∈ [1 - Δ, ξ] → Cash Investors.
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Agents split into three types depending on optimism:

- $1 - \Delta 
- \xi 
- p \tau 

$ \(p^i_1 \in [\tau, 1] \rightarrow \text{Borrowers ("Homeowners", "Hedge Funds", "Entrepreneurs")} \)
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  $n^i_A > 0$  $\exists \phi$,  $n^i_B(\phi) < 0$. 

- $\xi \rightarrow$ Lenders

- $1 \rightarrow$ Cash Investors
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2. $p^i_1 \in [\xi, \tau] \rightarrow$ Lenders (”Banks”, ”Money-Market Fund”).

\[ n^i_C = \frac{w_1}{36} \]
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Agents split into three types depending on optimism:

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- \( p^i_1 \in [1 - \Delta, \xi] \) → Cash Investors.
  \[ n^i_C = 1. \]
Borrowers’ Problem
Lemma

A borrower $p_i$ chooses ($\phi$) s.t.:
Borrowers’ Problem

Lemma

A borrower $p^i_1$ chooses $(\phi)$ s.t.: 

$$\phi = \arg \max_{\phi} \frac{p^i_1 - \phi}{p - q(\phi)}.$$
Borrowers’ Problem

Lemma

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- Coll. Const. binds: 1 Real asset $\Rightarrow$ 1 Borrowing Contract.
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Lemma

A borrower $p_1$ chooses ($\phi$) s.t.: $\phi = \arg\max_{\phi} \frac{p_1 - \phi}{p - q(\phi)}$.

- Coll. Const. binds: 1 Real asset $\Rightarrow$ 1 Borrowing Contract.
- Number: $1/(p - q(\phi))$ of Real assets / Borrowing Contracts.
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- Leverage ratio of $(\phi)$: $l(\phi) = p/(p - q(\phi))$. 
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\[
\begin{array}{|c|c|c|}
\hline
A & L & E \\
\hline
p & p - q(\phi) & q(\phi) \\
\hline
\end{array}
\]
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▶ Leverage ratio of $(\phi)$: $l(\phi) = p/(p - q(\phi))$.

\[
\frac{1}{p - q(\phi)}(p_i^1 - \phi) = \frac{p_i^1}{p}l(\phi) - \frac{\phi}{q(\phi)}(l(\phi) - 1)
\]
Borrowers’ Problem

Lemma

A borrower $p^i_1$ chooses $\phi$ s.t.: $\phi = \text{arg max}_\phi \frac{p^i_1 - \phi}{p - q(\phi)}$.

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- Leverage ratio of $(\phi)$: $l(\phi) = p/(p - q(\phi))$.

$$
\frac{1}{p - q(\phi)}(p^i_1 - \phi) = \frac{p^i_1}{p} l(\phi) - \frac{\phi}{q(\phi)} (l(\phi) - 1)
= \frac{p^i_1}{p} + \left( \frac{p^i_1}{p} - r(\phi) \right) (l(\phi) - 1).
$$
Borrowers’ Problem

Lemma

A borrower $p_1^i$ chooses $(\phi)$ s.t.: $\phi = \arg \max_{\phi} \frac{p_1^i - \phi}{p - q(\phi)}$.

- Coll. Const. binds: 1 Real asset ⇒ 1 Borrowing Contract.
- Number: $1/(p - q(\phi))$ of Real assets / Borrowing Contracts.

- Leverage ratio of $(\phi)$: $l(\phi) = p/(p - q(\phi))$.

$$
\frac{1}{p - q(\phi)}(p_1^i - \phi) = \frac{p_1^i}{p} l(\phi) - \frac{\phi}{q(\phi)} (l(\phi) - 1)
= \frac{p_1^i}{p} + \left( \frac{p_1^i}{p} - r(\phi) \right) (l(\phi) - 1).
$$

- Promise $\phi \uparrow$
Borrowers’ Problem

Lemma

A borrower $p_1^i$ chooses $\phi$ s.t.: $\phi = \arg\max_\phi \frac{p_1^i - \phi}{p - q(\phi)}$.

- Coll. Const. binds: 1 Real asset $\Rightarrow$ 1 Borrowing Contract.
- Number: $1/(p - q(\phi))$ of Real assets / Borrowing Contracts.

- Leverage ratio of $\phi$: $l(\phi) = p/(p - q(\phi))$.
  
  \[
  \frac{1}{p - q(\phi)}(p_1^i - \phi) = \frac{p_1^i}{p}l(\phi) - \frac{\phi}{q(\phi)}(l(\phi) - 1) \\
  = \frac{p_1^i}{p} + \left(\frac{p_1^i}{p} - r(\phi)\right)(l(\phi) - 1).
  \]

- Promise $\phi \uparrow \Rightarrow q(\phi) \uparrow$
Borrowers’ Problem

Lemma

A borrower \( p^i_1 \) chooses \( \phi \) s.t.: \( \phi = \arg \max_{\phi} \frac{p^i_1 - \phi}{p - q(\phi)} \).

- Coll. Const. binds: 1 Real asset \( \Rightarrow \) 1 Borrowing Contract.
- Number: \( 1/(p - q(\phi)) \) of Real assets / Borrowing Contracts.

- Leverage ratio of \( \phi \): \( l(\phi) = p/(p - q(\phi)) \).

\[
\frac{1}{p - q(\phi)} (p^i_1 - \phi) = \frac{p^i_1}{p} l(\phi) - \frac{\phi}{q(\phi)} (l(\phi) - 1)
\]
\[
= \frac{p^i_1}{p} + \left( \frac{p^i_1}{p} - r(\phi) \right) (l(\phi) - 1).
\]

- Promise \( \phi \uparrow \Rightarrow q(\phi) \uparrow \Rightarrow q'(\phi) > 0 \)
Borrowers’ Problem

Lemma

A borrower $p^i_1$ chooses ($\phi$) s.t.: $\phi = \arg \max_{\phi} \frac{p^i_1 - \phi}{p - q(\phi)}$.

- Coll. Const. binds: 1 Real asset $\Rightarrow$ 1 Borrowing Contract.
- Number: $1/(p - q(\phi))$ of Real assets / Borrowing Contracts.

- Leverage ratio of ($\phi$): $l(\phi) = p/(p - q(\phi))$.

\[
\frac{1}{p - q(\phi)}(p^i_1 - \phi) = \frac{p^i_1}{p} l(\phi) - \frac{\phi}{q(\phi)} (l(\phi) - 1)
\]

\[
= \frac{p^i_1}{p} + \left( \frac{p^i_1}{p} - r(\phi) \right) (l(\phi) - 1).
\]

- Promise $\phi \uparrow \Rightarrow q(\phi) \uparrow \Rightarrow q'(\phi) > 0 \Rightarrow l'(\phi) > 0$
Borrowers’ Problem

Lemma

A borrower \( p_i \) chooses \( \phi \) s.t.: \( \phi = \arg \max_{\phi} \frac{p_i - \phi}{p - q(\phi)} \).

- Coll. Const. binds: 1 Real asset \( \Rightarrow 1 \) Borrowing Contract.
- Number: \( \frac{1}{(p - q(\phi))} \) of Real assets / Borrowing Contracts.

- Leverage ratio of \( \phi \): \( l(\phi) = \frac{p}{(p - q(\phi))} \).

\[
\begin{align*}
1 \frac{1}{p - q(\phi)} (p_i - \phi) &= \frac{p_i}{p} l(\phi) - \frac{\phi}{q(\phi)} (l(\phi) - 1) \\
&= \frac{p_i}{p} + \left( \frac{p_i}{p} - r(\phi) \right) (l(\phi) - 1).
\end{align*}
\]

- Promise \( \phi \uparrow \Rightarrow q(\phi) \uparrow \Rightarrow q'(\phi) > 0 \Rightarrow l'(\phi) > 0 \)

\Rightarrow \text{Leverage rises with face value } \phi.
Borrowers’ Problem

Lemma

A borrower $p^*_i$ chooses $\phi$ s.t.: $\phi = \arg\max_{\phi} \frac{p^*_i - \phi}{p - q(\phi)}$.

- Coll. Const. binds: 1 Real asset $\Rightarrow 1$ Borrowing Contract.
- Number: $1/(p - q(\phi))$ of Real assets / Borrowing Contracts.

- Leverage ratio of $(\phi)$: $l(\phi) = p/(p - q(\phi))$.

$$\frac{1}{p - q(\phi)} (p^*_i - \phi) = \frac{p^*_i}{p} l(\phi) - \frac{\phi}{q(\phi)} (l(\phi) - 1)$$

$$= \frac{p^*_i}{p} + \left( \frac{p^*_i}{p} - r(\phi) \right) (l(\phi) - 1).$$

- Promise $\phi \uparrow \Rightarrow q(\phi) \uparrow \Rightarrow \boxed{q'(\phi) > 0} \Rightarrow \boxed{l'(\phi) > 0}$

$\Rightarrow$ Leverage rises with face value $\phi$.

- Trade-off between higher $\phi$ but higher $r(\phi)$
Borrowers’ Problem

Lemma

A borrower $p_1^i$ chooses ($\phi$) s.t.: $\phi = \arg \max \frac{p_1^i - \phi}{p - q(\phi)}$.

- Coll. Const. binds: 1 Real asset $\Rightarrow$ 1 Borrowing Contract.
- Number: $1/(p - q(\phi))$ of Real assets / Borrowing Contracts.

- Leverage ratio of ($\phi$): $l(\phi) = p/(p - q(\phi))$.

$$\frac{1}{p - q(\phi)}(p_1^i - \phi) = \frac{p_1^i}{p} l(\phi) - \frac{\phi}{q(\phi)} (l(\phi) - 1)$$

$$= \frac{p_1^i}{p} + \left( \frac{p_1^i}{p} - r(\phi) \right) (l(\phi) - 1).$$

- Promise $\phi \uparrow \Rightarrow q(\phi) \uparrow \Rightarrow q'(\phi) > 0 \Rightarrow l'(\phi) > 0$
  $\Rightarrow$ Leverage rises with face value $\phi$.
- Trade-off between higher $\phi$ but higher $r(\phi) \Rightarrow r'(\phi) > 0$. 

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Lemma

A lender with beliefs $p_i^*$ chooses contract $(p_i^*)$. 

A lender with beliefs $p^i_1$ chooses contract $(p^i_1)$.

- For lenders: Face value of the loan = Beliefs about the Real Asset.
Lemma

A lender with beliefs $p_i^1$ chooses contract $(p_i^1)$.

- For lenders: Face value of the loan = Beliefs about the Real Asset.
  - Why not a higher $\phi$?

- Leverage rises with $\phi$, and $\phi = p_i^1$ of lenders ⇒ Leverage rises with beliefs of lenders.

- Lenders think they trade perfectly safe contracts.
Lenders

Lemma

A lender with beliefs \( p^i_1 \) chooses contract \((p^i_1)\).

▶ For lenders: Face value of the loan = Beliefs about the Real Asset.
  ▶ Why not a higher \( \phi \)? Default for sure.
Lenders

Lemma

A lender with beliefs $p_1^i$ chooses contract $(p_1^i)$.

- For lenders: Face value of the loan = Beliefs about the Real Asset.
  - Why not a higher $\phi$? Default for sure.

  Return: $\min\{p_1^i, \phi\}$

  $q(\phi)$
Lenders

Lemma

A lender with beliefs $p^i_1$ chooses contract $(p^i_1)$.

- For lenders: Face value of the loan = Beliefs about the Real Asset.
  - Why not a higher $\phi$? Default for sure.

\[
\text{Return: } \frac{\min\{p^i_1, \phi\}}{q(\phi)} = \frac{p^i_1}{q(\phi)}
\]
Lemma

A lender with beliefs $p^i_1$ chooses contract $(p^i_1)$.

- For lenders: Face value of the loan $\equiv$ Beliefs about the Real Asset.
  - Why not a higher $\phi$? Default for sure.

$$\text{Return: } \frac{\min\{p^i_1, \phi\}}{q(\phi)} = \frac{p^i_1}{q(\phi)} \searrow \phi.$$
Lenders

Lemma

A lender with beliefs $p_i$ chooses contract $(p_i^i)$.

- For lenders: Face value of the loan $= \text{Beliefs about the Real Asset}$.
  - Why not a higher $\phi$? Default for sure.
    \[
    \text{Return: } \frac{\min\{p_i, \phi\}}{q(\phi)} = \frac{p_i}{q(\phi)} \downarrow \phi.
    \]
  - Why not a lower $\phi$?
Lenders

Lemma

A lender with beliefs $p^i_1$ chooses contract $(p^i_1)$.

- For lenders: Face value of the loan = Beliefs about the Real Asset.
  - Why not a higher $\phi$? Default for sure.
    
    \[
    \text{Return: } \frac{\min\{p^i_1, \phi\}}{q(\phi)} = \frac{p^i_1}{q(\phi)} \searrow \phi.
    \]

- Why not a lower $\phi$?
  
  \[
  \text{Return: } \frac{\min\{p^i_1, \phi\}}{q(\phi)}
  \]
A lender with beliefs \( p_1^i \) chooses contract \((p_1^i)\).

- For lenders: Face value of the loan = Beliefs about the Real Asset.
  - Why not a higher \( \phi \)? Default for sure.

\[
\text{Return: } \frac{\min\{p_1^i, \phi\}}{q(\phi)} = \frac{p_1^i}{q(\phi)} \downarrow \phi.
\]

- Why not a lower \( \phi \)?

\[
\text{Return: } \frac{\min\{p_1^i, \phi\}}{q(\phi)} = \frac{\phi}{q(\phi)} \uparrow \phi.
\]
Lenders

Lemma

A lender with beliefs $p_1^i$ chooses contract $(p_1^i)$.

- For lenders: Face value of the loan $\equiv$ Beliefs about the Real Asset.
  - Why not a higher $\phi$? Default for sure.
    \[
    \text{Return: } \frac{\min\{p_1^i, \phi\}}{q(\phi)} = \frac{p_1^i}{q(\phi)} \downarrow \phi.
    \]

- Why not a lower $\phi$?
  \[
  \text{Return: } \frac{\min\{p_1^i, \phi\}}{q(\phi)} = \frac{\phi}{q(\phi)} = r(\phi)
  \]
Lenders

Lemma

A lender with beliefs $p^i_1$ chooses contract $(p^i_1)$.

- For lenders: Face value of the loan $=$ Beliefs about the Real Asset.
  - Why not a higher $\phi$? Default for sure.
    
    \[
    \text{Return: } \frac{\min\{p^i_1, \phi\}}{q(\phi)} = \frac{p^i_1}{q(\phi)} \downarrow \phi.
    \]

- Why not a lower $\phi$?

    \[
    \text{Return: } \frac{\min\{p^i_1, \phi\}}{q(\phi)} = \frac{\phi}{q(\phi)} = r(\phi) \uparrow \phi.
    \]
Lenders

Lemma

A lender with beliefs $p^i_1$ chooses contract $(p^i_1)$.

▶ For lenders: Face value of the loan $= \text{Beliefs about the Real Asset.}$
  ▶ Why not a higher $\phi$? Default for sure.

  Return:
  \[
  \frac{\min\{p^i_1, \phi\}}{q(\phi)} = \frac{p^i_1}{q(\phi)} \downarrow \phi.
  \]

  ▶ Why not a lower $\phi$?

  Return:
  \[
  \frac{\min\{p^i_1, \phi\}}{q(\phi)} = \frac{\phi}{q(\phi)} = r(\phi) \uparrow \phi.
  \]

▶ Leverage rises with $\phi$,  

\[
\text{(Symbolic expression for leverage rise)}
\]
Lenders

Lemma

A lender with beliefs $p^i_1$ chooses contract $(p^i_1)$.

- For lenders: Face value of the loan $= \text{Beliefs about the Real Asset}$.
  - Why not a higher $\phi$? Default for sure.
    \[
    \text{Return: } \frac{\min\{p^i_1, \phi\}}{q(\phi)} = \frac{p^i_1}{q(\phi)} \downarrow \phi.
    \]
  - Why not a lower $\phi$?
    \[
    \text{Return: } \frac{\min\{p^i_1, \phi\}}{q(\phi)} = \frac{\phi}{q(\phi)} = r(\phi) \uparrow \phi.
    \]
- Leverage rises with $\phi$, and $\phi = p^i_1$ of lenders.
A lender with beliefs $p_1^i$ chooses contract $(p_1^i)$.

For lenders: Face value of the loan $= \text{Beliefs about the Real Asset}$.

- Why not a higher $\phi$? Default for sure.

\[
\text{Return: } \frac{\min\{p_1^i, \phi\}}{q(\phi)} = \frac{p_1^i}{q(\phi)} \quad \downarrow \phi.
\]

- Why not a lower $\phi$?

\[
\text{Return: } \frac{\min\{p_1^i, \phi\}}{q(\phi)} = \frac{\phi}{q(\phi)} = r(\phi) \quad \uparrow \phi.
\]

- Leverage rises with $\phi$, and $\phi = p_1^i$ of lenders $\Rightarrow$ Leverage rises with beliefs of lenders.
Lenders

Lemma

A lender with beliefs $p_i^1$ chooses contract $(p_i^1)$.

- For lenders: Face value of the loan $\equiv$ Beliefs about the Real Asset.
  - Why not a higher $\phi$? Default for sure.
    
    \[
    \text{Return: } \frac{\min\{p_1^i, \phi\}}{q(\phi)} = \frac{p_1^i}{q(\phi)} \downarrow \phi.
    \]
  
  - Why not a lower $\phi$?
    
    \[
    \text{Return: } \frac{\min\{p_1^i, \phi\}}{q(\phi)} = \frac{\phi}{q(\phi)} = r(\phi) \uparrow \phi.
    \]

- Leverage rises with $\phi$, and $\phi = p_1^i$ of lenders $\Rightarrow$ Leverage rises with beliefs of lenders.

- Lenders think they trade perfectly safe contracts.
Positive Sorting

- Supermodularity of Expected Wealth of a Borrower with respect to his Beliefs $p_i$ and the face value $\phi$:
Positive Sorting

- Supermodularity of Expected Wealth of a Borrower with respect to his Beliefs $p^i_1$ and the face value $\phi$:

$$\frac{p^i_1 - \phi}{p - q(\phi)} = \frac{p^i_1}{p} (1 + l(\phi)) - \frac{\phi}{q(\phi)} l(\phi)$$

$$\Rightarrow \frac{\partial^2}{\partial \phi \partial p^i_1} (. ) = \frac{1}{p} l'(\phi) > 0.$$
Positive Sorting

- Supermodularity of Expected Wealth of a Borrower with respect to his Beliefs \( p^i_1 \) and the face value \( \phi \):

\[
\frac{p^i_1 - \phi}{p - q(\phi)} = \frac{p^i_1}{p} (1 + l(\phi)) - \frac{\phi}{q(\phi)} l(\phi)
\]

\[
\Rightarrow \quad \frac{\partial^2}{\partial \phi \partial p^i_1} (.) = \frac{1}{p} l'(\phi) > 0.
\]

- Complementarity between leverage \((\phi)\) and expected return on each asset \((p^i_1)\).
Positive Sorting

- Supermodularity of Expected Wealth of a Borrower with respect to his Beliefs \( p_1^i \) and the face value \( \phi \):

\[
\frac{p_1^i - \phi}{p - q(\phi)} = \frac{p_1^i}{p} (1 + l(\phi)) - \frac{\phi}{q(\phi)} l(\phi)
\]

\[\Rightarrow \quad \frac{\partial^2}{\partial \phi \partial p_1^i}(.) = \frac{1}{p} l'(\phi) > 0.\]

- Complementarity between leverage \( \phi \) and expected return on each asset \( p_1^i \).

- \( \phi = p_1^i \) of lenders \( \Rightarrow \) Positive Sorting of borrowers and lenders.
Positive Sorting

- Supermodularity of Expected Wealth of a Borrower with respect to his Beliefs $p^i_1$ and the face value $\phi$:

\[
\frac{p^i_1 - \phi}{p - q(\phi)} = \frac{p^i_1}{p} (1 + l(\phi)) - \frac{\phi}{q(\phi)} l(\phi)
\]

\[
\Rightarrow \frac{\partial^2}{\partial \phi \partial p^i_1} (.) = \frac{1}{p} l'(\phi) > 0.
\]

- Complementarity between leverage ($\phi$) and expected return on each asset ($p^i_1$).

- $\phi = p^i_1$ of lenders $\Rightarrow$ Positive Sorting of borrowers and lenders. Empirically: Over-The-Counter (OTC) Markets.
Positive Sorting

- Supermodularity of Expected Wealth of a Borrower with respect to his Beliefs $p^i_1$ and the face value $\phi$:

$$\frac{p^i_1 - \phi}{p - q(\phi)} = \frac{p^i_1}{p} \left(1 + l(\phi)\right) - \frac{\phi}{q(\phi)} l(\phi)$$

$$\Rightarrow \frac{\partial^2}{\partial \phi \partial p^i_1}(.) = \frac{1}{p} l'(\phi) > 0.$$

- **Complementarity** between leverage ($\phi$) and expected return on each asset ($p^i_1$).

- $\phi = p^i_1$ of lenders $\Rightarrow$ **Positive Sorting** of borrowers and lenders. Empirically: Over-The-Counter (OTC) Markets.

- $\Gamma(.)$: Belief of borrower $\rightarrow$ Belief of lender.
Positive Sorting

- Supermodularity of Expected Wealth of a Borrower with respect to his Beliefs $p^i_1$ and the face value $\phi$:

$$\frac{p^i_1 - \phi}{p - q(\phi)} = \frac{p^i_1}{p} (1 + l(\phi)) - \frac{\phi}{q(\phi)} l(\phi)$$

$$\Rightarrow \frac{\partial^2}{\partial \phi \partial p^i_1} (.) = \frac{1}{p} l'(\phi) > 0.$$

- **Complementarity** between leverage ($\phi$) and expected return on each asset ($p^i_1$).

- $\phi = p^i_1$ of lenders $\Rightarrow$ **Positive Sorting** of borrowers and lenders. Empirically: Over-The-Counter (OTC) Markets.

- $\Gamma(.)$: Belief of borrower $\rightarrow$ Belief of lender. Sorting: $\Gamma'(.) > 0$. 
2 first-order ODE for $\Gamma(.)$ and $q(.)$

$\Gamma(y)$

$\xi$

$1-\Delta$

Cash Investors

Lenders

Borrowers

$p^i_1 = y$ chooses $\phi$ s.t. lender choosing same $\phi$ is $\Gamma(y)$:
2 first-order ODE for $\Gamma(\cdot)$ and $q(\cdot)$

$\mathbf{p^*_1} = y$ chooses $\phi$ s.t. lender choosing same $\phi$ is $\Gamma(y)$:

$$\Gamma(y) = \arg\max_{\phi} \frac{y - \phi}{p - q(\phi)} \Rightarrow q'(\phi) \frac{y - \phi}{p - q(\phi)} = 1$$

$$\Rightarrow (y - \Gamma(y)) q'(\Gamma(y)) = p - q(\Gamma(y)).$$
2 first-order ODE for $\Gamma(.)$ and $q(.)$

- $p_1^i = y$ chooses $\phi$ s.t. lender choosing same $\phi$ is $\Gamma(y)$:
  
  $$\Gamma(y) = \arg \max_{\phi} \frac{y - \phi}{p - q(\phi)} \Rightarrow q'(\phi) \frac{y - \phi}{p - q(\phi)} = 1$$

  $$\Rightarrow (y - \Gamma(y)) q'(\Gamma(y)) = p - q(\Gamma(y)).$$

- Market clearing for contract $(x)$:
  
  $$\int_{i} n_B^i(x) di = 0 \Rightarrow \frac{f(\Gamma(y))d\Gamma(y)}{q(\Gamma(y))} = \frac{f(y)dy}{p - q(\Gamma(y))}$$

  $$\Rightarrow (p - q(\Gamma(y))) f(\Gamma(y)) \Gamma'(y) = q(\Gamma(y))f(y).$$
- Ununknowns: $q(.) (\equiv r(.)$, $\Gamma(.), \xi, p, \tau$.

- 2 First-Order ODEs $\Rightarrow$ Need 5 algebraic equations.

- Indifference Cash / Lending: $r(\xi) = 1$.

- Indifference Lending / Investing: $r(\tau) = \tau - \xi$.

- Most pessimistic lenders & borrowers: $\Gamma(\xi) = \tau$.

- Most optimistic lenders & borrowers: $\Gamma(\tau) = M$.

- Market clearing for the real asset: $w(1 - F(\xi)) = Sp$. 

- Cash Investors \hspace{2cm} Lenders \hspace{2cm} Borrowers
- Unknowns: $q(.) (\equiv r(.)), \Gamma(.), \xi, p, \tau$.

- 2 First-Order ODEs $\Rightarrow$ Need 5 algebraic equations.

- Indifference Cash / Lending: $r(\xi) = 1.$
- Unknowns: $q(.) \equiv r(.)$, $\Gamma(.)$, $\xi$, $p$, $\tau$.

- 2 First-Order ODEs $\Rightarrow$ Need 5 algebraic equations.

\[ 1-\Delta \quad \xi \quad p \quad \tau \quad 1 \quad p_i \]

- Indifference Cash / Lending: $r(\xi) = 1$.

- Indifference Lending / Investing:
- Unkowns: $q(.) \equiv r(.)$, $\Gamma(.)$, $\xi$, $p$, $\tau$.

- 2 First-Order ODEs $\Rightarrow$ Need 5 algebraic equations.

\[ 1-\Delta \quad \xi \quad p \quad \tau \quad 1 \]

- Indifference Cash / Lending: $r(\xi) = 1$.

- Indifference Lending / Investing: $r(\tau) = \frac{\tau - \xi}{p - \xi}$. 
- Unknowns: $q(.) \equiv r(.)$, $\Gamma(.)$, $\xi$, $p$, $\tau$.

- 2 First-Order ODEs $\Rightarrow$ Need 5 algebraic equations.

- Indifference Cash / Lending: $r(\xi) = 1$.
- Indifference Lending / Investing: $r(\tau) = \frac{\tau - \xi}{p - \xi}$.
- Most pessimistic lenders & borrowers:
▶ Unknowns: \( q(.) \) \( \equiv r(.) \), \( \Gamma(.) \), \( \xi \), \( p \), \( \tau \).

▶ 2 First-Order ODEs \Rightarrow \text{Need 5 algebraic equations.}

\begin{align*}
1 - \Delta & \quad \xi & \quad p & \quad \tau & \quad 1 \\
\text{Cash Investors} & \quad \text{Lenders} & \quad \text{Borrowers} & \quad p^i_1
\end{align*}

▶ Indifference Cash / Lending:
\[ r(\xi) = 1. \]

▶ Indifference Lending / Investing:
\[ r(\tau) = \frac{\tau - \xi}{p - \xi}. \]

▶ Most pessimistic lenders & borrowers:
\[ \Gamma(\tau) \equiv \xi. \]
Unknwons: \( q(.) \) (\( \equiv r(.) \)), \( \Gamma(.) \), \( \xi \), \( p \), \( \tau \).

2 First-Order ODEs \( \Rightarrow \) Need 5 algebraic equations.

- Indifference Cash / Lending: \( r(\xi) = 1 \).
- Indifference Lending / Investing: \( r(\tau) = \frac{\tau - \xi}{p - \xi} \).
- Most pessimistic lenders & borrowers: \( \Gamma(\tau) = \xi \).
- Most optimistic lenders & borrowers:
Unown: $q(.) (\equiv r(.))$, $\Gamma(.)$, $\xi$, $p$, $\tau$.

2 First-Order ODEs $\Rightarrow$ Need 5 algebraic equations.

Indifference Cash / Lending: $r(\xi) = 1$.

Indifference Lending / Investing: $r(\tau) = \frac{\tau - \xi}{p - \xi}$.

Most pessimistic lenders & borrowers: $\Gamma(\tau) = \xi$.

Most optimistic lenders & borrowers: $\Gamma(1) = \tau$. 
Unknowns: $q(.) \equiv r(.), \Gamma(.), \xi, p, \tau$.

2 First-Order ODEs $\Rightarrow$ Need 5 algebraic equations.

Indifference Cash / Lending: $r(\xi) = 1$.

Indifference Lending / Investing: $r(\tau) = \frac{\tau - \xi}{p - \xi}$.

Most pessimistic lenders & borrowers: $\Gamma(\tau) = \xi$.

Most optimistic lenders & borrowers: $\Gamma(1) = \tau$.

Market clearing for the real asset:
- **Ununknowns**: \( q(.) \) \((\equiv r(.))\), \( \Gamma(.) \), \( \xi \), \( p \), \( \tau \).

- **2 First-Order ODEs ⇒ Need 5 algebraic equations.**

\[
\begin{align*}
1-\Delta & \quad \xi & \quad p & \quad \tau & \quad 1 \\
\text{Cash Investors} & \quad & \text{Lenders} & \quad & \text{Borrowers} \\
\end{align*}
\]

- **Indifference Cash / Lending:**
  \[ r(\xi) = 1. \]

- **Indifference Lending / Investing:**
  \[ r(\tau) = \frac{\tau - \xi}{p - \xi}. \]

- **Most pessimistic lenders & borrowers:**
  \[ \Gamma(\tau) = \xi. \]

- **Most optimistic lenders & borrowers:**
  \[ \Gamma(1) = \tau. \]

- **Market clearing for the real asset:**
  \[ 1 - F(\xi) = p. \]
Model with Borrowing Contracts Only
Setup
Equilibrium Definition
Equilibrium Solution
Equilibrium Properties

Extension 1: ”Pyramiding” Lending Arrangements

Extension 2: Short-Sales

Conclusion
Illustrating examples: $f$ uniform, $f$ increasing
Illustrating examples: $f$ uniform, $f$ increasing

- Uniform: 2 first-order ODE $\rightarrow$ second-order ODE:
Illustrating examples: $f$ uniform, $f$ increasing

Uniform: 2 first-order ODE $\rightarrow$ second-order ODE:

$\Gamma'' (\Gamma - x) + \Gamma' + \Gamma'^2 = 0$
Illustrating examples: $f$ uniform, $f$ increasing

Uniform: 2 first-order ODE $\rightarrow$ second-order ODE:

$$\Gamma'' (\Gamma - x) + \Gamma' + \Gamma'^2 = 0 \quad \Rightarrow \quad \Gamma(x) = -x - a + b\sqrt{x + c}.$$
Illustrating examples: $f$ uniform, $f$ increasing

Uniform: 2 first-order ODE $\rightarrow$ second-order ODE:

$$
\Gamma'' (\Gamma - x) + \Gamma' + \Gamma'^2 = 0 \quad \Rightarrow \quad \Gamma(x) = -x - a + b\sqrt{x} + c.
$$

Closed form: $p$, $\xi$, $\tau$, $r(.)$, $q(.)$, $L(.)$, $a$, $b$, $c$. 
Illustrating examples: $f$ uniform, $f$ increasing

\[ \Gamma'' (\Gamma - x) + \Gamma' + \Gamma'^2 = 0 \quad \Rightarrow \quad \Gamma(x) = -x - a + b\sqrt{x + c}. \]

Closed form: $p, \xi, \tau, r(.), q(.), L(.), a, b, c$. Example:
Illustrating examples: $f$ uniform, $f$ increasing

- Uniform: 2 first-order ODE $\rightarrow$ second-order ODE:

$$\Gamma'' (\Gamma - x) + \Gamma' + \Gamma'^2 = 0 \quad \Rightarrow \quad \Gamma(x) = -x - a + b\sqrt{x + c}.$$

- Closed form: $p, \xi, \tau, r(.), q(.), L(.), a, b, c$. Example:

$$p = \frac{1 + \Delta + 2\Delta^2 + 2\Delta^3 - \sqrt{(-1 + \Delta)^2 (1 + 2\Delta^2)}}{2\Delta + \Delta^2 + 4\Delta^3 + 2\Delta^4}.$$
Illustrating examples: $f$ uniform, $f$ increasing

- Uniform: 2 first-order ODE $\rightarrow$ second-order ODE:
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- Closed form: $p$, $\xi$, $\tau$, $r(.)$, $q(.)$, $L(.)$, $a$, $b$, $c$. Example:
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- **Closed form**: \( p, \xi, \tau, r(.), q(.), L(.), a, b, c \). Example:

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Cutoffs as a function of $\Delta$ ($f$ uniform)

True across bounded away from zero density function:

$$p = 1 - O(\Delta^2),$$
Cutoffs as a function of $\Delta$ ($f$ uniform)

True across bounded away from zero density function:

$$p = 1 - O(\Delta^2), \quad \tau = 1 - O(\Delta^2),$$
Cutoffs as a function of $\Delta$ ($f$ uniform)

True across bounded away from zero density function:

$$p = 1 - O(\Delta^2), \quad \tau = 1 - O(\Delta^2), \quad \text{and} \quad \xi = 1 - O(\Delta).$$
Limiting Pareto Tail of Endogenous Tail Coefficient 2

- In uniform case, truncated Pareto with coeff 2:

\[
\frac{p}{p - Q(y)} = \frac{p}{\sqrt{2\xi}} \sqrt{\frac{p - \xi}{\tau - \xi}} \frac{1}{\sqrt{\frac{(p + \xi)\tau - \xi(p - \xi)}{2\xi} - y}}.
\]

Proposition (Limiting Pareto Distribution for Leverage Ratios of Optimists for smooth \(f(.)\))
Limiting Pareto Tail of Endogenous Tail Coefficient 2

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Proposition (Limiting Pareto Distribution for Leverage Ratios of Optimists for smooth \( f(.) \))

Let \( f(.) \) differentiable, \( f' \) continuous, \( f(.) \) bounded away from 0. \( G_\Delta(.) \) distribution function for the leverage of borrowers for \( f_\Delta(.) \):

\[
\exists A_\Delta, \quad \| l^2(1 - G_\Delta(l)) - A_\Delta \|_{\infty}^{[L_\Delta(1)/2, L_\Delta(1)]} \xrightarrow{\Delta \to 0} 0,
\]

- Heuristically:

\[
1 - G_\Delta(l) \sim \frac{A_\Delta}{l^2}.
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- Heuristically:

\[ 1 - G_\Delta(l) \sim \frac{A_\Delta}{l^2}. \]

- Upper tail behavior: not dependent on \( f(.) \).
Pareto Distributions for Leverage Ratios, Uniform Distribution

Coefficient: 2.

Disagreement $\Delta = 10\%$
Disagreement $\Delta = 5\%$
Disagreement $\Delta = 2\%$
Pareto Distributions for Leverage Ratios, Increasing Distribution

Still Coefficient: 2.

Disagreement $\Delta = 10\%$
Disagreement $\Delta = 5\%$
Disagreement $\Delta = 2\%$

Log$_{10}$ Leverage Ratio
Log$_{10}$ Survivor Function
Empirical Counterpart

TASS Hedge Fund Database, August 2006.

Log_{10} Survivor

Log_{10} Leverage Ratio

Slope: -1.95

Calibration: disagreement ≈ 1.8%.
Empirical Counterpart

TASS Hedge Fund Database, August 2006.

Slope: -1.95

Calibration: disagreement \( \approx 1.8\% \).
Non Bounded away from 0.

- If the density is s.t. \( f(x) \sim (1 - x)^\rho \Rightarrow \text{Pareto with coefficient } 2 + \rho \).

- Scale Independence Remains.

- Returns are Pareto from envelope condition.

![Graph showing survivor function and leverage ratio with regression lines for different values of \( \Delta \) and \( \rho \).]
Hedonic Interest Rates

- **Hedonic Interest rates** $r(.)$ on safe bonds for lenders. Can be substantial. Example with $f(x) = 2(1 - x)/\Delta$

![Graph showing the relationship between Haircuts (%) and Spreads on Collateralized Bonds (Basis Points). The graph illustrates three scenarios: 10% disagreement (blue line), 5% disagreement (red dashed line), and 2% disagreement (black line). Each scenario demonstrates an increase in spreads as haircuts decrease.](image-url)
Hedonic Interest Rates

- **Hedonic Interest rates** $r(\cdot)$ on safe bonds for lenders. Can be substantial. Example with $f(x) = 2(1 - x)/\Delta$

- $Corr(r(\cdot), l(\cdot)) > 0$ observed by econometrician from disagreement. **But**: no risk shifting $\Rightarrow$ Different regulatory implications.

- Non monotonic relationship between leverage and realized returns of borrowers, because of spreads.
Model with Borrowing Contracts Only

Setup
Equilibrium Definition
Equilibrium Solution
Equilibrium Properties

Extension 1: "Pyramiding" Lending Arrangements

Extension 2: Short-Sales

Conclusion
Pyramiding Lending Arrangements

- Allow Borrowing Contracts to be used as collateral.
Pyramiding Lending Arrangements

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Hedonic interest rates ⇒ Lenders want to leverage into them!
Pyramiding Lending Arrangements

- Allow Borrowing Contracts to be used as collateral.

- Hedonic interest rates ⇒ Lenders want to **leverage into** them!

- Example for houses, loans to SMEs: securitization. Or rehypothecation of collateral, repos of mortgage-backed securities, etc.
Pyramiding Lending Arrangements

▶ Allow Borrowing Contracts to be used as collateral.

▶ Hedonic interest rates ⇒ Lenders want to leverage into them!

▶ Example for houses, loans to SMEs: securitization. Or rehypothecation of collateral, repos of mortgage-backed securities, etc.

▶ Price $p$ increases even more.
Akin to tranching. The lender of type 2 is repaid until $\phi'$, then lender of type 1 is repaid on $\phi - \phi'$, then the borrower gets $p_1 - \phi$. 
Pyramiding Lending Arrangements

- Pareto Coefficients decrease (leverage distributions are multiplied). In general, with $L$ levels the coefficient is $(L + 1)/L$ when the distribution is bounded away from 0.
- Leverage Ratio distribution shifted to the right.
- Price expresses the opinion of superoptimists.
Evidence from Dataquick

Leverage Ratio Distribution of US Homeowners
(Leverage Ratio on New Loans)

Date: 9/1988

Date: 4/1991

Date: 5/2007

Date: 8/2010

Video: the leverage ratio distribution from 1987 to 2012.
The model allows to recover the corresponding increase in borrowers’ expected returns.

In a model with a little bit of risk aversion: more risk taking?

Another Extension: Short-Sales
Model with Borrowing Contracts Only
  Setup
  Equilibrium Definition
  Equilibrium Solution
  Equilibrium Properties

Extension 1: "Pyramiding" Lending Arrangements

Extension 2: Short-Sales

Conclusion
Short-Sales

- Unlike existing disagreement models, the model allows the treatment of short-sales.

- Price = pessimists’ valuations ⇒ Systematic undervaluation - similar to noise trader risks in De Long et al. (1990), but risk neutrality. Equity premium, discount of closed-end funds, etc.

- **Endogenous rebate rates** - apparent short-selling costs not evidence of constraints: about 100 bps, larger with more disagreement.

- **Endogenous Short-interest** (a few percent).
Short-Sales

Disagreement $\Delta$

Percentile (Ranked by Degree of Optimism)

Cutoff $\sigma$
Cutoff $\tau$
Price of Real Asset $p$
Cutoff $\xi$

Lenders
Borrowers
Securities Lenders
Short-Sellers
Rebate Rates and Cash Collateral

- No short-selling costs or costs of default.
Short Interest

- Only a few percent of stocks are on loan in equilibrium, even though all are potentially available.
Larger Spreads on Bonds, even the safest (AAA)
Model with Borrowing Contracts Only
  Setup
  Equilibrium Definition
  Equilibrium Solution
  Equilibrium Properties

Extension 1: ”Pyramiding” Lending Arrangements

Extension 2: Short-Sales

Conclusion
Conclusion

- Homeowners / Entrepreneurs’ / Hedge Funds data lend support to a very stylized model.
Conclusion

- Homeowners / Entrepreneurs’ / Hedge Funds data lend support to a very stylized model.
- New (static) source of Pareto distributions in returns independent from Gibrat’s law/ random growth.

Potential for future work:
- Empirical work on short interest, rebate rates, distributions of leverage ratios to recover disagreement.
- Financial regulation:
  - Costs of moving OTC onto exchanges.
  - Monitoring financial system through ultimate borrowers’ leverage ratio distribution?
Conclusion

▶ Homeowners / Entrepreneurs’ / Hedge Funds data lend support to a very stylized model.
▶ New (static) source of Pareto distributions in returns independent from Gibrat’s law/ random growth.
▶ New intuitions on key financial prices / quantities:
  ▶ Returns on Bonds.
  ▶ Short-selling ”costs”.
  ▶ Short interest

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Thank you
Histogram Leverage Ratios

Source: TASS Hedge Fund Database, August 2006 [To be included]
Short-Sales

- Adding Short-Sales Contracts collateralized by cash, promising $p_{t+1}$ (one unit of asset), with cash-collateral $\gamma$.

- First-order Credit Spreads: $r(.) - 1 \simeq h \Rightarrow$ A few hundred Basis Points.

- Indifference at $\xi$: $r(\xi) = \frac{\tau' - \xi}{\tau' - p} \Rightarrow$ Credit Spread for Aaa.

- Coexistence of lending, short-selling. Conclusions on Pareto robust. Application: Leverage Caps have ambiguous effects.
Short-Sales

Price of the Real Asset, Cutoffs

- **Borrowers**
- **Securities Lenders**
- **Lenders**
- **Short-Sellers**

Disagreement $h$
Pareto Robust
Leverage Ratios of Entrepreneurs

Slope: -2.02

![Graph showing the relationship between Leverage Ratio and Survivor log on a logarithmic scale. The graph includes a fitted line with a slope of -2.02.]