Large Matching Markets as Two-Sided Demand Systems

Konrad Menzel

New York University

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Matching models describe markets in which agents care about the identity of the party they transact or associate with. Examples include

- marriage markets,
- college or school admissions,
- hiring workers, interns, doctors, lawyers, or other professionals.

This paper focuses on **one-to-one** matchings in two-sided markets with **non-transferable utility**.

- transfers or side-payments may be illegal, unethical, or technically infeasible.
- wages, rates, or prices may be set by a centralized institution.

The aim of this paper is estimation of preference parameters from observed market outcomes.
Two-Sided Matching Markets

- A two-sided matching market consists of $n_w$ women and $n_m$ men, and we observe a random sample of “couples” $s = 1, \ldots, S$;
- our data set records observable characteristics $x_{w(s)}$ for the woman, and $z_{m(s)}$ for the man in the $s$th couple.
- The sample also includes individuals that are single, where e.g. for a single man $x_{w(s)}$ is coded as missing.
- We want to use data from one or several markets to learn about individual preferences over potential matches.
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- We want to use data from one or several markets to learn about individual preferences over potential matches.
- It is difficult to characterize the exact distribution of \( (x_w(s), z_m(s)) \) resulting from sufficiently rich models of random preferences.
- The main objective of the paper is to derive a tractable approximation to the joint distribution of \( (x_w(s), z_m(s)) \) for identification analysis and estimation.
- To that end, we propose an asymptotic experiment in which the number of individuals on both sides of the market goes to infinity.
Matching Markets - Preferences

We consider a matching model where

- woman $i$’s preferences over spouses $j = 1, \ldots, n_M$ are given by the latent random utility functions of the form $U_{ij}$, and
- $V_{ji}$ represents man $j$’s payoff from a match with woman $i = 1, \ldots, n_W$.
- Payoffs depend on both individuals’ characteristics, $x_i$ and $z_j$, respectively, through the **systematic parts** $U(x_i, z_j)$ and $V(z_j, x_i)$, respectively, and
- there is additional heterogeneity in the form of additive **idosyncratic taste shifters** $\eta_{ij}$ and $\zeta_{ji}$, i.e.

\[
U_{ij} = U(x_i, z_j) + \sigma \eta_{ij} \\
V_{ji} = V(z_j, x_i) + \sigma \zeta_{ji}
\]

- The random utilities for the outside option - i.e. of remaining single - are denoted by $U_{i0}$ and $V_{j0}$, respectively.
- Utilities are not transferable (NTU), i.e. if woman $i$ marries man $j$, her payoff from the match $U_{ij}$ does not depend on man $j$’s preferences, and there is no bargaining over shared resources after marriage.
Two-Sided Matchings

- Each individual can marry a person of the opposite sex or choose to remain single.
- We use $\mu_w(i)$ and $\mu_m(j)$ to denote woman $i$’s, and man $j$’s spouse, respectively, under the matching $\mu$.
- Given a matching $\mu$, we let the set $M_i \equiv M_i[\mu] \subset \{0, 1, \ldots, n_M\}$ denote the set of men $j$ preferring woman $i$ over their current match, $\mu_m(j)$, i.e.
  \[ j \in M_i[\mu] \text{ if and only if } V_{ji} \geq V_{j\mu_m(j)} \]
- We call $M_i[\mu]$ the set of men available to woman $i$ under the matching $\mu$, or woman $i$’s opportunity set, and similarly, we define man $j$’s opportunity set $W_j \equiv W_j[\mu]$.
- By default we assume that $0 \in M_i$ and $0 \in W_j$, i.e. the opportunity set always includes the outside option.
A matching $\mu$ is **pairwise stable** if given the preferences $(U_{ij}, V_{ji})$, every individual prefers her/his spouse under $\mu$ to any other available partner, i.e. for all $i, j = 1, \ldots, n$,

$$U_{i\mu_w(i)} \geq \max_{j \in M_i} U_{ij} \quad \text{and} \quad V_{j\mu_m(j)} \geq \max_{i \in W_j} V_{ji}$$

If preferences are strict, a stable matching always exists, and
A matching $\mu$ is **pairwise stable** if given the preferences $(U_{ij}, V_{ji})_{ij}$, every individual prefers her/his spouse under $\mu$ to any other available partner, i.e. for all $i, j = 1, \ldots, n$,

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If preferences are strict, a stable matching always exists, and

- the set of stable matchings has a minimal element $\mu^M$ and a maximal element $\mu^W$ with respect to the preferences of the female side (W-preferred and M-preferred stable matching).
- Specifically, for any stable matching $\mu^*$, the W-preferred stable matching satisfies $U_{i\mu^W_w(i)} \geq U_{i\mu^*_w(i)}$ and $V_{j\mu^W_m(j)} \leq V_{j\mu^*_m(j)}$, and for the M-preferred stable matching we always have $U_{i\mu^M_w(i)} \leq U_{i\mu^*_w(i)}$ and $V_{j\mu^M_m(j)} \geq V_{j\mu^*_m(j)}$. 

**Pairwise Stability**
A matching \( \mu \) is **pairwise stable** if given the preferences \((U_{ij}, V_{ji})_{ij}\), every individual prefers her/his spouse under \( \mu \) to any other available partner, i.e. for all \( i, j = 1, \ldots, n \),

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U_{i\mu_w(i)} \geq \max_{j \in M_i} U_{ij} \quad \text{and} \quad V_{j\mu_m(j)} \geq \max_{i \in W_j} V_{ji}
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- the set of stable matchings has a minimal element \( \mu^M \) and a maximal element \( \mu^W \) with respect to the preferences of the female side (W-preferred and M-preferred stable matching).

- Specifically, for any stable matching \( \mu^* \), the W-preferred stable matching satisfies \( U_{i\mu_w^W(i)} \geq U_{i\mu_w^*(i)} \) and \( V_{j\mu_m^W(j)} \leq V_{j\mu_m^*(j)} \), and for the M-preferred stable matching we always have \( U_{i\mu_m^M(i)} \leq U_{i\mu_m^*(i)} \) and \( V_{j\mu_m^M(j)} \geq V_{j\mu_m^*(j)} \).

- The M-preferred and W-preferred stable matching can be found in polynomial computing time using the well-known Gale-Shapley algorithm.

- The number of distinct stable matchings for a typical realization of preferences may increase exponentially in the number of individuals on each side of the market.
Pairwise Stability

When is pairwise stability a plausible solution concept for an empirical model?

- Centralized matching markets may employ a variant of the Gale-Shapley algorithm to assign matching partners.

- Decentralized matching markets: Roth and Van de Vate (1990) show that randomized myopic tâtonnement processes converge with probability 1 to pairwise stable matchings if allowed to continue indefinitely.
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- decentralized matching markets: Roth and Van de Vate (1990) show that randomized myopic tâtonnement processes converge with probability 1 to pairwise stable matchings if allowed to continue indefinitely.

- Pairwise stability does not require that agents have perfect knowledge of all participants’ preferences, but each agent only needs to know which matching partners are available to him or her.

- We also give a straightforward extension to the main model which allows for agents only to be aware of a random subset of potential matching partners.
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- We also give a straightforward extension to the main model which allows for agents only to be aware of a random subset of potential matching partners.

- Our model does allow for transfers that are deterministic given observable characteristics $x_i, z_j$. The only substantive difference to TU models is whether transfers are allowed to depend on $\eta_{ij}$ and $\zeta_{ji}$. 
Outline of the Talk

Introduction

Model

Asymptotic Approximation

Identification

Estimation

Simulation Results

Conclusion
We consider a matching model with nontransferable utilities (NTU), where preferences over spouses are given by the latent random utility functions of the form

\[ U_{ij} = U(x_i, z_j) + \sigma \eta_{ij} \]
\[ V_{ji} = V(z_j, x_i) + \sigma \zeta_{ji} \]

for \( i = 1, \ldots, n_W \) and \( j = 1, \ldots, n_M \). The random utility of remaining single is specified as

\[ U_{i0} = 0 + \sigma \max_{k=1,\ldots,J} \{\eta_{i0,k}\} \]
\[ V_{j0} = 0 + \sigma \max_{k=1,\ldots,J} \{\zeta_{j0,k}\} \]
Random Utility Model

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For the asymptotic sequence we allow \( J \) and \( \sigma \) change with market size \( n \) in order to keep the matching distribution from becoming degenerate.
Assumptions on Systematic Part

**Assumption 1 (Systematic Part of Payoffs)**

The functions $|U(x, z)| \leq \bar{U} < \infty$ and $|V(z, x)| \leq \bar{V} < \infty$ are uniformly bounded in absolute value and continuous in $\mathcal{X} \times \mathcal{Z}$. Furthermore, at all $(x', z') \in \mathcal{X} \times \mathcal{Z}$ the functions $U(x, z)$, and $V(z, x)$ are $p \geq 1$ times differentiable with uniformly bounded partial derivatives.
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- We also denote the p.d.f.s for the marginal distributions of types with $w(x)$ and $m(z)$, respectively, where characteristics can be discrete or continuous.
- For the approximation result, we do not need to distinguish which components of $x_i$ and $z_j$ are observed or unobserved.
- However, the assumption that the $U(x, z)$ and $V(z, x)$ are bounded is restrictive for some standard random coefficient models.
Assumptions on the Idiosyncratic Part

The upper tail of the distribution $G(\eta)$ is of type I if there exists an auxiliary function $a(s) \geq 0$ such that the c.d.f. satisfies

$$\lim_{s \to \infty} \frac{1 - G(s + a(s)v)}{1 - G(s)} = e^{-v} \quad \text{for all } v \in \mathbb{R}$$
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Assumption 2 (Idiosyncratic Part of Payoffs)

$\eta_{ij}$ and $\zeta_{ji}$ are i.i.d. draws from the distribution $G(s)$, and are independent of $x_i, z_j$, where (i) the c.d.f. $G(s)$ is absolutely continuous with density $g(s)$, and (ii) the upper tail of the distribution $G(s)$ is of type I with auxiliary function $a(s) := \frac{1 - G(s)}{g(s)}$. 
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Special cases:

- if $\eta_{ij} \sim \Lambda(\eta)$, the extreme-value type I (or Gumbel) distribution, then we choose $a(s) = 1$
- For $\eta_{ij} \sim N(0, \sigma^2)$, the auxiliary function can be chosen as $a(s) = \frac{\sigma}{s}$.
- if $\eta_{ij}$ is Gamma-distributed, we can choose $a(s) = 1$
Asymptotic Sequence

In general, there are many ways in which we can embed the $n$-player market in different sequences of markets. However, we want the approximation to retain the following features of the finite-agent market:

- the share of single individuals should not degenerate to one or zero.
- When the systematic part matters for outcomes in the finite-player market, it should also be predictive for match probabilities in the limit.
- However, the joint distribution of male/female match characteristics should not be degenerate in the limit.
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Assumption (Market Size)

The size of a given market is governed by $n = 1, 2, \ldots$, where

1. the number of men and women $n_m = \lceil n \exp\{\gamma_m\} \rceil$ and $n_w = \lceil n \exp\{\gamma_w\} \rceil$ grow proportionally with $n$, where $\gamma_w$ and $\gamma_m$ are bounded in absolute value across markets.

2. The size of the outside option is $J = \lceil n^{1/2} \rceil$.

3. The scale parameter for the taste shifters $\sigma \equiv \sigma_n = \frac{1}{a(b_n)}$, where $b_n = G^{-1} \left(1 - \frac{1}{\sqrt{n}}\right)$. 

Given a random matching \( \mu \), we define the *matching frequency distribution* as the expected number of matched pairs of men and women of observable types \( z \) and \( x \),

\[
F(x, z | \mu) := \frac{1}{n} \sum_{i=0}^{n_w} \sum_{j=0}^{n_m} P(x_i \leq x, z_j \leq z, j \in \mu_w(i) \text{ or } i \in \mu_m(j))
\]

where, as a convention, \( \mu_w(0) \) and \( \mu_m(0) \) denote the set of unmatched men and women, respectively.

The measure \( F(\cdot) \) is in general not a proper probability distribution but integrates to the mass of couples that form in equilibrium.
Matching Frequency Distribution

- Given a random matching $\mu$, we define the matching frequency distribution as the expected number of matched pairs of men and women of observable types $z$ and $x$,

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where, as a convention, $\mu_w(0)$ and $\mu_m(0)$ denote the set of unmatched men and women, respectively.

- The measure $F(\cdot)$ is in general not a proper probability distribution but integrates to the mass of couples that form in equilibrium.

- We also let $f(x, z)$ denote the joint density of observable characteristics of a matched pair, defined as the Radon Nikodym derivative of the measure $F$.

- We let $f(x, *)$ and $f(*, z)$ denote the density of characteristics among unmatched women and men, respectively, so that

$$\int_{Z} f(x, z)dz + f(x, *) = w(x) \exp\{\gamma_w\}$$
Main Result: Unique Limiting Distribution

We show that there is a unique limit for the matching frequency distributions resulting from any (random selection of) stable matchings. The density function of that limiting distribution is given by

\[
f(x, z) = \frac{\exp\{U(x, z) + V(z, x)\}w(x)m(z)}{(1 + \Gamma^*_w(x))(1 + \Gamma^*_m(z))} \quad x \in X, z \in Z
\]

\[
f(x, \ast) = \frac{w(x)}{1 + \Gamma^*_w(x)} \quad x \in X
\]

\[
f(\ast, z) = \frac{m(z)}{1 + \Gamma^*_m(z)} \quad z \in Z
\]

where \(\Gamma^*_w\) and \(\Gamma^*_m\) solve the fixed point problem

\[
\Gamma^*_w(x) := \int \frac{\exp\{U(x, s) + V(s, x)\}m(s)}{1 + \Gamma^*_m(s)} ds
\]

\[
\Gamma^*_m(z) := \int \frac{\exp\{U(s, z) + V(z, s)\}w(s)}{1 + \Gamma^*_w(s)} ds
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Conditional Logit Model

If the idiosyncratic components in the RUM were i.i.d. draws from an Extreme-Value Type I distribution, and the choice set $M$ of alternatives is exogenously given, then conditional choice probabilities (CCP) are of the form

$$P \left( U_{ij} \geq \max_{k \in M} U_{ik} \mid x_i, z_1, \ldots, z_J \right) = \frac{\exp \{ U(x_i, z_j) \}}{\sum_{k \in M} \exp \{ U(x_i, z_k) \}}$$

We’ll argue that this model is both convenient and appropriate when choice sets are large and possibly random:

- A new result in this paper is that Logit CCPs are limiting choice probabilities for a much broader class of RUMs when $J$ is large.

- Specifically, we only need to assume that the RUM is additively separable, and the idiosyncratic component is independent of the systematic part (plus regularity conditions).
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- Specifically, we only need to assume that the RUM is additively separable, and the idiosyncratic component is independent of the systematic part (plus regularity conditions).
- A sufficient statistic for an exogenously determined choice set $M$ is given by the **inclusive value**

$$I_{wi}[M] := \frac{1}{|M|} \sum_{k \in M} \exp \left\{ U(x_i, z_k) \right\}$$
Asymptotic Approximation

In order to derive the limit of the joint distribution of \((x_{w(s)}, z_{m(s)})\), we proceed by proving the following claims:

1. Dependence of taste shifters and opportunity sets is negligible for CCPs when \(n\) is large. Hence we can approximate choice probabilities using the inclusive values.

2. Convergence of conditional choice probabilities (CCP) to Logit probabilities, assuming that taste shifters \(\eta_{ij}\) are independent from the equilibrium opportunity sets \(W_i\) and \(M_j\).

3. Inclusive values \(I_{w}[M_i]\) and \(I_{m}[W_j]\) are approximated by their conditional means \(\hat{\Gamma}_w(x_i)\) and \(\hat{\Gamma}_m(z_j)\) which are solutions of an approximate fixed-point problem.

4. The fixed-point mapping characterizing the inclusive value functions is a contraction, so that a solution exists and is unique.

5. For stable matchings, \(\hat{\Gamma}_w\) and \(\hat{\Gamma}_m\) solve the sample analog of the fixed-point problem \(\Gamma = \Psi_0[\Gamma]\), and the solutions converge in probability to the limits \(\Gamma^*_w\) and \(\Gamma^*_m\).
Step 1: Multiplicity and Dependence

In order to analyze the joint dependence of $\eta_{ij}$ and the different opportunity sets, we consider the joint distribution of $\eta := (\eta_{i1}, \ldots, \eta_{in})'$ and woman $i$’s opportunity sets under the M- and W-preferred matching, $M_i^M$ and $M_i^W$, respectively.

- The main part of argument replaces $\eta_i$ with an independent copy $\tilde{\eta}_i$ from the marginal distribution for $\eta$, holding all other taste shifters in the market fixed.
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- Changing all of $i$’s taste-shifters from $\eta_i$ to $\tilde{\eta}_i$ affects only two men in $M_i$ since $i$ is available only to the most preferred option in $M_i$. Changing her preference over men $j \notin M_i$ has no effect on subsequent iterations.

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- Changing all of i’s taste-shifters from $\eta_i$ to $\tilde{\eta}_i$ affects only two men in $M_i$ since $i$ is available only to the most preferred option in $M_i$. Changing her preference over men $j \notin M_i$ has no effect on subsequent iterations.
- This starts two separate chains of subsequent adjustments where at each iteration at most one relevant availability indicator is switched.
- At each iteration, the process may reach the outside option with some probability, terminating the chain.
- On the other hand, the chain may alter one of the availability indicators $D_{i1}^W, \ldots, D_{inm}^W$. That probability is decreasing in $n$, so that

$$\lim_{n \to \infty} \left| \frac{P(D_{i1}^W, \ldots, D_{inm}^W | \eta_i)}{P(D_{i1}^W, \ldots, D_{inm}^W | \tilde{\eta}_i)} - 1 \right| = 0$$
Multiplicity and Dependence: Preference Cycles
Step 2: Approximation of CCPs

Lemma

Given our assumptions, suppose that \( z_1, \ldots, z_J \) are \( J \) i.i.d draws from a distribution \( M(z) \) with p.d.f. \( m(z) \). Then as \( J \to \infty \),

\[
P \left( U_{i0} \geq \max_{k=0,\ldots,J} U_{ik} \bigg| x_i \right) \to \frac{1}{1 + \int \exp\{U(x_i, s)\} m(s) ds}
\]

\[
JP \left( U_{ij} \geq \max_{k=0,\ldots,J} U_{ik} \bigg| x_i, z_j = z \right) \to \frac{\exp\{U(x_i, z)\}}{1 + \int \exp\{U(x_i, s)\} m(s) ds}
\]

almost surely for any fixed \( j = 1, 2, \ldots, J \).

- Since the number of alternatives grows with \( J \), we have to scale choice probabilities by \( J \) in order to obtain a non-degenerate limit.

- In the paper, we use a technical generalization to this result for the case in which the opportunity sets are not i.i.d. but result from a stable matching.

- Convergence of finite-sample inclusive values to population expectations follows from a law of large numbers.
Step 3: Inclusive Value Functions

For the third step, we show that woman $i$’s conditional choice probabilities given her opportunity set can be approximated using a state variable $\hat{\Gamma}_w(x_i)$ that depends only on her observable characteristics $x_i$.

Specifically, we consider the inclusive values associated with the extremal matchings, where for the M-preferred matching, we denote

$$I^M_{wi} := I_{wi}[M^M_i] = n^{-1/2} \sum_{j \in M^M_i} \exp \{ U(x_i, z_j) \}$$

$$I^M_{mj} := I_{mj}[W^M_j] = n^{-1/2} \sum_{i \in W^M_j} \exp \{ V(z_j, x_i) \}$$

We also define the average inclusive value function under the M-preferred matching as

$$\hat{\Gamma}^M_w(x) := \frac{1}{n} \sum_{j=1}^{n^m} \frac{\exp \{ U(x, z_j) + V(z_j, x) \}}{1 + I^M_{mj}}$$

$$\hat{\Gamma}^M_m(z) := \frac{1}{n} \sum_{i=1}^{n^w} \frac{\exp \{ U(x_i, z) + V(z, x_i) \}}{1 + I^M_{wi}}$$
Step 3: Law of Large Numbers

We can write the difference between $I_{wi}^M$ and the inclusive value function as

$$I_{wi}^M - \hat{\Gamma}_w^M(x_i) = \frac{1}{n^{1/2}} \sum_{j=1}^{n_m} \exp \{ U(x_i, z_j) \} \left[ \mathbf{1}\{ V_{ji} \geq V_j^M[W_j^M] \} - \frac{\exp \{ V(z_j, x) \}}{1 + I_{mj}^M} \right]$$
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By the previous step, the random variables

$$v_{ji}(I_{mj}) := 1 \{ V_{ji} \geq V_j^M[W_j^M] \} - \frac{\exp \{ V(z_j, x) \}}{1 + I_{mj}^M}$$

have zero mean and are approximately independent across $j = 1, \ldots, n_m$ conditional on $I_{mj}^M$.

We can show that the pairwise correlations vanish sufficiently fast as $n$ grows and obtain:

$$I_{wi}^M \geq \hat{\Gamma}_{wn}^M(x_i) + o_p(1) \text{ and } I_{mj}^M \leq \hat{\Gamma}_{mn}^M(z_j) + o_p(1)$$

for all $i = 1, \ldots, n_w$ and $j = 1, \ldots, n_m$. The analogous result holds for the $W$-preferred matching.

This result allows us to approximate inclusive values as a function of observable characteristics alone.
Step 3: Fixed Point Characterization

Therefore, by the continuous mapping theorem,

\[
\hat{\Gamma}_w^M(x_i) = \frac{1}{n} \sum_{j=1}^{n} \frac{\exp \{ U(x_i, z_j) + V(z_j, x_i) \}}{1 + I_{m_j}^M} 
\geq \frac{1}{n} \sum_{j=1}^{n} \frac{\exp \{ U(x_i, z_j) + V(z_j, x_i) \}}{1 + \hat{\Gamma}_m^M(z_j)} + o_p(1)
\]
Step 3: Fixed Point Characterization

Therefore, by the continuous mapping theorem,

\[ \hat{\Gamma}_w^M(x_i) = \frac{1}{n} \sum_{j=1}^{n^m} \exp \left\{ U(x_i, z_j) + V(z_j, x_i) \right\} \left( 1 + I_{m_j}^M \right) \]

\[ \geq \frac{1}{n} \sum_{j=1}^{n^m} \exp \left\{ U(x_i, z_j) + V(z_j, x_i) \right\} \frac{1}{1 + \hat{\Gamma}_m^M(z_j)} + o_p(1) \]

Since the opportunity sets \( M_i^* \) and \( W_j^* \) arising from any stable matching satisfy \( M_i^M \subset M_i^* \subset M_i^W \) and \( W_j^W \subset W_j^* \subset W_j^M \), we also have

\[ \hat{\Gamma}_w^M(x) \leq \hat{\Gamma}_w^*(x) \leq \hat{\Gamma}_w^W(x) \], and \( \hat{\Gamma}_m^M(z) \geq \hat{\Gamma}_m^*(z) \geq \hat{\Gamma}_m^W(z) \)
Step 3: Fixed Point Characterization

We define the fixed-point operators

\[
\hat{\psi}_w[\Gamma_m](x) = \frac{1}{n} \sum_{j=1}^{n_m} \frac{\exp \{U(x, z_j) + V(z_j, x)\}}{1 + \Gamma_m(z_j)}
\]

\[
\hat{\psi}_m[\Gamma_w](z) = \frac{1}{n} \sum_{i=1}^{n_w} \frac{\exp \{U(x_i, z) + V(z, x_i)\}}{1 + \Gamma_w(x_i)}
\]

Combining the inequalities from the previous step, it follows that the inclusive value functions \(\hat{\Gamma}_w^*, \hat{\Gamma}_m^*\) arising from any pairwise stable matching have to satisfy

\[
\hat{\Gamma}_m^* = \hat{\psi}_m[\hat{\Gamma}_w^*] + o_p(1) \quad \text{and} \quad \hat{\Gamma}_w^* = \hat{\psi}_w[\hat{\Gamma}_m^*] + o_p(1)
\]

Hence, we can rewrite the pairwise stability conditions as a fixed-point problem for the asymptotically sufficient parameters \(\hat{\Gamma}_w^*, \hat{\Gamma}_m^*\).
Step 4: Population Fixed-Point Mapping

In order to characterize the limit, we can state the population analogs of the equilibrium conditions for the inclusive value functions,

\[
\Gamma_w(x) = \int \frac{\exp\{U(x, s) + V(s, x)\}m(s)}{1 + \Gamma_m(s)} \, ds
\]

\[
\Gamma_m(z) = \int \frac{\exp\{U(s, z) + V(z, s)\}w(s)}{1 + \Gamma_w(s)} \, ds
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\[ \Gamma_m(z) = \int \frac{\exp\{U(s, z) + V(z, s)\} w(s)}{1 + \Gamma_w(s)} \, ds \]

We can define the operators \( \Psi_m : \Gamma_w \mapsto \Gamma_m \) and \( \Psi_w : \Gamma_m \mapsto \Gamma_w \) by

\[ \Psi_w[\Gamma_m](x) := \int \frac{\exp\{U(x, s) + V(s, x)\} m(s)}{1 + \Gamma_m(s)} \, ds \]

\[ \Psi_m[\Gamma_w](z) := \int \frac{\exp\{U(s, z) + V(z, s)\} w(s)}{1 + \Gamma_w(s)} \, ds \]

In that notation, we obtain the fixed-point problem for the population inclusive value functions

\[ \Gamma_w^* = \Psi_w[\Gamma_m^*] \quad \text{and} \quad \Gamma_m^* = \Psi_m[\Gamma_w^*] \]
Step 4: Existence and Uniqueness of Fixed Point

Theorem

Under our assumptions,

(i) the mapping \((\log \Gamma) \mapsto (\log \Psi[\Gamma])\) is a contraction mapping with

\[
\|\log \Psi[\Gamma] - \log \Psi[\tilde{\Gamma}]\|_{\infty} \leq \lambda \|\log \Gamma - \log \tilde{\Gamma}\|_{\infty}
\]

where \(\lambda := \frac{\exp\{\bar{U}+\bar{V}\}}{1+\exp\{U+V\}} < 1\) and does not depend on \(\theta\). Specifically, a solution to the fixed point problem exists and is unique.

(ii) Moreover the equilibrium distributions are characterized by functions \(\Gamma^*_w(x)\) and \(\Gamma^*_m(z)\) that are continuous and \(p\) times differentiable in \(x\) and \(z\), respectively, with bounded partial derivatives.
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(ii) Moreover the equilibrium distributions are characterized by functions \(\Gamma^*_w(x)\) and \(\Gamma^*_m(z)\) that are continuous and \(p\) times differentiable in \(x\) and \(z\), respectively, with bounded partial derivatives.

- Uniqueness of the inclusive values implies uniqueness of the distribution of matched characteristics.
- Hence, if the matching market converges to this limit, the difference in inclusive values between different stable matchings must converge to zero as \(n\) grows.
- The contraction property depends on the relevance of the outside option in the limiting market.
Step 5: Convergence of Inclusive Value Functions

- The fixed-point mapping \( \hat{\Psi}_n[\Gamma] := \left( \hat{\Psi}_w[\Gamma_m](x), \hat{\Psi}_m[\Gamma_w](x) \right) \) is a sample average over functions of \( x, z \) and \( \Gamma \).
- We can show that \( \hat{\Psi}_n \xrightarrow{p} \Psi \) uniformly in \( \Gamma_w \in \mathcal{T}_w \) and \( \Gamma_m \in \mathcal{T}_m \) and \( (x', z')' \in \mathcal{X} \times \mathcal{Z} \) as \( n \to \infty \).
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We can show that $\hat{\psi}_n \xrightarrow{p} \psi$ uniformly in $\Gamma_w \in \mathcal{T}_w$ and $\Gamma_m \in \mathcal{T}_m$ and $(x', z')' \in \mathcal{X} \times \mathcal{Z}$ as $n \to \infty$.

Furthermore, since $\psi[\cdot]$ was shown to be a contraction, the solution to the limiting problem is unique and well-separated in the sense that large perturbations of $\Gamma$ relative to the fixed point $\Gamma^*$ also lead to sufficiently large deviations from the fixed-point condition, $\Gamma - \psi[\Gamma]$. 
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- Furthermore, since \( \Psi[\cdot] \) was shown to be a contraction, the solution to the limiting problem is unique and well-separated in the sense that large perturbations of \( \Gamma \) relative to the fixed point \( \Gamma^* \) also lead to sufficiently large deviations from the fixed-point condition, \( \Gamma - \Psi[\Gamma] \).

Hence, the solutions to the fixed-point problem in a finite economy converge to the (unique) fixed point of the limiting problem:

**Theorem (Convergence of Inclusive Values)**

Given our assumptions, (a) for any stable matching, the inclusive values satisfy the fixed-point characterization in equation (3), and (b) we have convergence of the inclusive value functions \( \| \hat{\Gamma}_m - \Gamma^* \|_\infty \xrightarrow{p} 0 \) and \( \| \hat{\Gamma}_m - \Gamma^* \|_\infty \xrightarrow{p} 0 \).
The unique solution for the inclusive values implies a unique match level distribution of observed characteristics:

\[ f(x, z) = \frac{\exp\{U(x, z) + V(z, x)\}w(x)m(z)}{(1 + \Gamma_w^*(x))(1 + \Gamma_m^*(z))} \quad x \in \mathcal{X}, z \in \mathcal{Z} \]

\[ f(x, \ast) = \frac{w(x)}{1 + \Gamma_w^*(x)} \quad x \in \mathcal{X} \]

\[ f(\ast, z) = \frac{m(z)}{1 + \Gamma_m^*(z)} \quad z \in \mathcal{Z} \]

where \( \Gamma_w^* \) and \( \Gamma_m^* \) solve the fixed point problem

\[ \Gamma_w^*(x) := \int \frac{\exp\{U(x, s) + V(s, x)\}m(s)}{1 + \Gamma_m^*(s)} \, ds \]

\[ \Gamma_m^*(z) := \int \frac{\exp\{U(s, z) + V(z, s)\}w(s)}{1 + \Gamma_w^*(s)} \, ds \]
Convergence of Matching Frequencies

Finally, convergence of the inclusive value functions implies convergence of the matching frequencies from a stable matching.

Corollary (Convergence of Matching Frequencies)
For a given matching \( \mu \) define the empirical matching frequencies

\[
\hat{F}_n(x, z|\mu) := \frac{1}{n} \sum_{i=0}^{n_w} \sum_{j \in \mu_w(i)} 1\{X_i \leq x, Z_j \leq z\}
\]

Then for any sequence of stable matchings \( \mu^*_n \), \( \hat{F}_n(x, z|\mu^*_n) \) converges to a measure \( F \) with the density \( f(x, z) \) corresponding to the fixed point for the continuum game.
Interpretation

To summarize, this limit has four important qualitative features:

1. all agents can choose among a large number of matching opportunities as the market grows, where

2. similar agents face similar options in terms of the inclusive value of their choice sets. Nevertheless,

3. the asymptotic sequence presumes that the option of remaining single remains sufficiently attractive relative to that rich set of potential matches.

4. The number of distinct matchings may in fact be very large, but from the outsider’s perspective the different matchings become observationally equivalent in the limit.

We find that these features are consistent with the idea of a matching market becoming “thick” as the number of participants becomes large.
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Identification of Surplus

In order to analyze identification from the limiting model, define the marital pseudo-surplus of a match as the sum of the deterministic parts of random payoffs,

\[ W(x, z) := U(x, z) + V(z, x) \]

Note that, in the absence of a common numeraire, the relative scales of men and women’s preference are normalized by a multiple of the conditional standard deviation of random utilities given \( x_i = x \) and \( z_j = z \).
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- We can now express the distribution in terms of \( W(x, z) \) alone:

\[
f(x, z) = \exp\left\{ W(x, z) + \gamma_w + \gamma_m \right\} \frac{w(x)m(z)}{(1 + \Gamma_w^*(x))(1 + \Gamma_m^*(z))} \quad x \in \mathcal{X}, z \in \mathcal{Z}
\]

- Also, the fixed-point equations defining \( \Gamma_w^* \) and \( \Gamma_m^* \) can be rewritten as

\[
\Gamma_w^*(x) := \int \frac{\exp\{ W(x, s) + \gamma_m \} m(s)}{1 + \Gamma_m^*(s)} ds \\
\Gamma_m^*(z) := \int \frac{\exp\{ W(s, z) + \gamma_w \} w(s)}{1 + \Gamma_w^*(s)} ds
\]

- Hence, we can’t identify \( U(x, z) \) and \( V(z, x) \) separately, but only the surplus function \( W(x, z) \).
Starting from the formula for the limiting p.d.f., we can use differencing arguments to eliminate the inclusive values $\Gamma_w^*(x)$ and $\Gamma_m^*(z)$ using information on the shares of unmatched individuals.

Specifically, we can recover the matching surplus function from

$$\log f(x, z) - \log f(x, \ast) - \log f(\ast, z) = W(x, z)$$

Knowledge of the surplus function $W(x, z)$ is sufficient for predicting counterfactual distributions as we vary the marginal distributions $w(x)$ and $m(z)$.

Separate identification of preference parameters of the male or female side of the market appears to require “exclusion” restrictions on observable types and their interactions.
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Separate identification of preference parameters of the male or female side of the market appears to require “exclusion” restrictions on observable types and their interactions.

Furthermore, we can identify the average inclusive value function from the conditional probabilities of remaining single given different values of $x$,

$$\Gamma_w(x) = \frac{w(x) \exp\{\gamma_w\}}{f(x, \ast)} - 1$$
Welfare Analysis

In addition to prediction of counterfactuals regarding observable characteristics of the realized matches, our model also allows for welfare evaluations of policy interventions.

- The inclusive value is related to the expectation of indirect utility via

\[
E \left[ \max_{j \in M_i} U_{ij} \bigg| X_i = x \right] = \log(1 + \Gamma_w(x)) + \kappa
\]

where \( \kappa \approx 0.5772 \) is Euler’s constant.
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where \( \kappa \approx 0.5772 \) is Euler’s constant.

- If \( s_w(x) := \frac{f(x, \ast)}{w(x) \exp\{\gamma_w\}} \) is the share of women of type \( x \) that remain single, we can express the inclusive value in terms of observable quantities,

\[
\mathbb{E} \left[ \max_{j \in M_i} U_{ij} \mid X_i = x \right] = -\log \frac{f(x, \ast)}{w(x)} + \gamma_w + \kappa = -\log s_w(x) + \text{const}
\]

- Hence, for natural experiments that change the composition of matching markets, we can interpret the difference in the log shares of unmatched individuals directly as the average change in the surplus from participating in the matching market for individuals of a given observable type.
Extension: Matching with Limited Awareness

We can generalize our results to a model in which agents’ knowledge of matching opportunities is more limited:

- Suppose that in a first stage agents meet at random and independently of the realized random matching payoffs, where the probability of a woman of type $x$ meeting a man of type $z$ is given by $r(x, z) \in [0, 1]$. Awareness is assumed to be mutual, i.e. woman $i$ is aware of man $j$ if and only if man $j$ is also aware of woman $i$.

- If a pair $(i, j)$ of a woman and a man are not aware of each other, the random payoffs for a match between them are set to minus infinity, and otherwise equal to their initial values, i.e.

$$
\tilde{U}_{ij} = \begin{cases} 
U_{ij} & \text{if } i \text{ and } j \text{ meet} \\
-\infty & \text{otherwise}
\end{cases} \quad \tilde{V}_{ji} = \begin{cases} 
V_{ji} & \text{if } i \text{ and } j \text{ meet} \\
-\infty & \text{otherwise}
\end{cases}
$$
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\end{cases}, \quad \tilde{V}_{ji} = \begin{cases} 
V_{ji} & \text{if } i \text{ and } j \text{ meet} \\
-\infty & \text{otherwise}
\end{cases}
\]

- If $|\log r(x, z)|$ is bounded, our asymptotic arguments can also be applied to the modified model with a pseudo-surplus function

\[
W(x, z) := U(x, z) + V(x, z) + \log r(x, z)
\]

In particular, $W(x, z)$ is nonparametrically identified from the distribution $f(x, z)$. 

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Suppose that we have a parametric model for the systematic parts of the payoff functions

\[ U(x, z) := U(x, z; \theta) \quad \text{and} \quad V(z, x) := V(x, z; \theta) \]

so that the surplus function is \( W(x, z) = W(x, z; \theta) := U(x, z; \theta) + V(z, x; \theta) \)
Estimation

Suppose that we have a parametric model for the systematic parts of the payoff functions

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so that the surplus function is \( W(x, z) = W(x, z; \theta) := U(x, z; \theta) + V(z, x; \theta) \)

Furthermore, the sample is obtained by

- drawing \( K \) individuals (men or women) at random from the population,
- and recording their characteristics and those of their spouse (if applicable)

Then the limiting matching frequencies lead to the sampling distribution

\[
\begin{align*}
    h_1(x, z) &= \frac{2f(x, z)}{\exp\{\gamma_w\} + \exp\{\gamma_m\}} \\
    h_1(x, *) &= \frac{f(x, *)}{\exp\{\gamma_w\} + \exp\{\gamma_m\}} \\
    h_1(\ast, z) &= \frac{f(\ast, z)}{\exp\{\gamma_w\} + \exp\{\gamma_m\}}
\end{align*}
\]
Estimation

Based on the limiting distribution, the likelihood function for the preference parameter is (up to a constant)

\[
L_K(\theta, \Gamma) := \sum_{k=1}^{K} \left( W(x_{w(k)}, z_{m(k)}; \theta) + \log(2) \mathbb{1}\{w(k) \neq 0, m(k) \neq 0\} \right.
\]
\[
- \log(1 + \Gamma_w(x_{w(k)})) - \log(1 + \Gamma_m(z_{m(k)})) \bigg)
\]

where we can approximate the fixed-point problem determining \( \Gamma \) by the empirical restrictions

\[
\hat{\Psi}_{wK}[\Gamma](x) = \frac{1}{K} \sum_{k=1}^{K} \exp \left\{ \frac{W(x, z_{m(k)}; \theta)}{1 + \Gamma_m(z_{m(k)})} \right\}
\]
\[
\hat{\Psi}_{mK}[\Gamma](z) = \frac{1}{K} \sum_{k=1}^{K} \exp \left\{ \frac{W(x_{w(k)}, z; \theta)}{1 + \Gamma_w(x_{w(k)})} \right\}
\]

Hence the maximum likelihood estimator \( \hat{\theta} \) solves

\[
\max_{\theta, \Gamma} L_K(\theta, \Gamma) \quad \text{s.t.} \quad \Gamma = \hat{\Psi}_K[\Gamma]
\]
Estimation

- We can find the MLE $\hat{\theta}$ by constrained maximization of the log-likelihood, where treating the equilibrium conditions as constraints obviates the need of solving for an equilibrium in $\Gamma$ at every maximization step.
- The Lagrangian for this estimation problem is
  \[ L = L_K(\theta, \Gamma) + \langle \lambda, \Gamma - \hat{\Psi}_K[\Gamma] \rangle \]
- This constrained problem is high-dimensional in that $\Gamma$ generally has to be evaluated at up to $2^K$ different arguments, leading to $2^K$ constraints.
- That system of constraints is not sparse but collinear, and we may adapt the constrained MPEC algorithm proposed and described in Dubé, Fox, and Su (2012), and Su and Judd (2012).
- Presumably better statistical properties than two-step estimator with non-parametric first stage, but also computationally more efficient than a nested fixed point algorithm.
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In order to illustrate the different aspects of the theoretical convergence result, we simulate a very basic version of our model.

Payoff matrices are generated from our random utility model,

\[
U_{ij} = U(x_i, z_j) + \sigma \eta_{ij}
\]
\[
V_{ji} = V(z_j, x_i) + \sigma \zeta_{ji}
\]

where idiosyncratic taste shifters \( \eta_{ij}, \zeta_{ji} \) are i.i.d. draws from a standard normal or extreme value type-I distribution.

Results in the following tables are only for the case of one observable type on each side of the market.

We then find the \( M \)- and \( W \)-preferred matchings using the Gale-Shapley (deferred acceptance) algorithm.

In this setting, the matching probabilities are the same under the \( W \)—and \( M \)—preferred matching by the “Rural Hospital Theorem.”
Simulation Results - Logit Errors

In this table we report (1) the difference in the average size of a woman’s opportunity set between the extremal matchings, $|M_i^W| - |M_i^M|$, (2) the number of women for whom $M_i$ differs across matchings, and (3) the average inclusive values for the $W$— and $M$—preferred matchings.

| n   | $|M_i^W| - |M_i^M|$ | $\#\{M_i^W \neq M_i^M\}$ | $I_w[M_i^W]$ | $I_w[M_i^W] - I_w[M_i^M]$ |
|-----|---------------------|-----------------------------|--------------|-----------------------------|
| 10  | 0.0350              | 0.30                        | 6.73         | 0.08                        |
| 20  | 0.2125              | 3.05                        | 7.50         | 0.35                        |
| 50  | 0.1370              | 5.90                        | 6.92         | 0.14                        |
| 100 | 0.0905              | 8.35                        | 6.77         | 0.07                        |
| 200 | 0.1175              | 21.95                       | 6.82         | 0.06                        |
| 500 | 0.0574              | 27.50                       | 6.87         | 0.02                        |
| 1000| 0.0539              | 52.00                       | 6.85         | 0.01                        |
| 2000| 0.0510              | 92.60                       | 6.86         | 0.01                        |
| 5000| 0.0041              | 102.00                      | 6.87         | 0.00                        |
Constrained MLE

This design specifies binary types for men and women, $X_i, Z_j \in \{0, 1\}$, and systematic utilities

\[ U^*(x_i, z_j) = \theta_0 + \theta_1 x_i + \theta_2 x_i z_j \]
\[ V^*(z_j, x_i) = \theta_0 + \theta_1 z_j + \theta_2 x_i z_j \]

We report the simulated mean and standard error (in parentheses) for the distribution of the MLE:

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<th>$\theta_0$</th>
<th>$\theta_1$</th>
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<td>(0.0688)</td>
<td>(0.0998)</td>
<td>(0.0362)</td>
</tr>
<tr>
<td>(DGP)</td>
<td>0.50</td>
<td>0.50</td>
<td>1.00</td>
</tr>
</tbody>
</table>
Constrained MLE

We also report two summary measures of estimation error,

$$RMSE = \left( n \sum_j \mathbb{E}[(\hat{\theta}_j - \theta_j)^2] \right)^{1/2}$$

and the maximum of studentized bias over components of $\hat{\theta}_j$,

$$SBR := \max_j |Bias(\hat{\theta}_j)| / SE(\hat{\theta}_j)$$

together with bias and standard error for the individual parameter MLE, scaled by $n^{1/2}$.

<table>
<thead>
<tr>
<th>n</th>
<th>$\theta_0$</th>
<th>$\theta_1$</th>
<th>$\theta_2$</th>
<th>RMSE</th>
<th>SBR</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>0.83</td>
<td>-1.16</td>
<td>-1.09</td>
<td>3.48</td>
<td>1.23</td>
</tr>
<tr>
<td>50</td>
<td>-0.27</td>
<td>-0.47</td>
<td>-0.48</td>
<td>5.32</td>
<td>0.30</td>
</tr>
<tr>
<td>100</td>
<td>-0.81</td>
<td>0.22</td>
<td>-0.43</td>
<td>6.21</td>
<td>0.23</td>
</tr>
<tr>
<td>200</td>
<td>-0.75</td>
<td>0.17</td>
<td>-0.53</td>
<td>6.41</td>
<td>0.27</td>
</tr>
<tr>
<td>500</td>
<td>-0.30</td>
<td>0.17</td>
<td>-0.47</td>
<td>5.97</td>
<td>0.26</td>
</tr>
<tr>
<td>1000</td>
<td>-1.26</td>
<td>0.47</td>
<td>-0.50</td>
<td>5.72</td>
<td>0.40</td>
</tr>
<tr>
<td>2000</td>
<td>-0.23</td>
<td>-0.75</td>
<td>-0.42</td>
<td>5.77</td>
<td>0.26</td>
</tr>
</tbody>
</table>
We develop approximations to the distribution of stable matchings, where we find that

- in the limit, a model with independent separable taste shocks satisfies IIA.
- Hence we could represent any distribution supported by a stable matching as an equilibrium in inclusive values.
- We give a fixed-point characterization for the inclusive values as a function of observable types, where
- the resulting distribution of match characteristics is unique in the limit.
- We show how to use the limiting model for identification analysis and (parametric and nonparametric) estimation.