Discussion of “Structural GARCH: The Volatility-Leverage Connection” by Rob Engle and Emil Siriwardane

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The Basic Framework

Modigliani-Miller Proposition II:

\[ R_a = \frac{D}{V} R_d + \frac{E}{V} R_e \Rightarrow R_e = \frac{D}{E} (R_a - R_d) \]

- Equity volatility increases linearly with leverage
- If debt is risky, the relationship is more complex
- Use option-pricing model (Merton, 1974)
The Basic Framework

Company XYZ

<table>
<thead>
<tr>
<th>Assets</th>
<th>Liabilities</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_T$</td>
<td>$E_T = \max [ A_T - X, 0 ]$</td>
</tr>
<tr>
<td>$A_T$</td>
<td>$D_T = \min [ A_T, X ]$</td>
</tr>
</tbody>
</table>

- Equity is a call option on the firm’s assets
- Option-pricing model provides valuation formulas
The Basic Framework

- Specific case of geometric Brownian motion for $A_t$

\[ dA_t = \mu A_t dt + \sigma A_t dW_t \]
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- Itô’s formula yields the exact dynamics of $E_t$

\[
dE_t = \frac{\partial E_t}{\partial t} dt + \frac{\partial E_t}{\partial A_t} dA_t + \frac{1}{2} \frac{\partial^2 E_t}{\partial A_t^2} (dA_t)^2
\]

\[
\frac{dE_t}{E_t} = \frac{1}{E_t} \left( \frac{\partial E_t}{\partial t} + \mu A_t \frac{\partial E_t}{\partial A_t} + \frac{1}{2} \sigma^2 A_t^2 \frac{\partial^2 E_t}{\partial A_t^2} \right) dt + \sigma \frac{A_t}{E_t} \frac{\partial E_t}{\partial A_t} dW_t
\]

- $\lambda = \text{LM}(A_t/X, 1, \sigma, r, \tau)$ is the option elasticity or “gearing”

\[
\text{LM}(A_t/X, 1, \sigma, r, \tau) = \frac{1}{1 - (X/A_t) \exp(-r\tau) \Phi(d_2)/\Phi(d_1)}
\]
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Call Option Elasticity

$A_t$
The Basic Framework

- Now assume a more general asset-price process

\[
dA_t = \mu(A_t, t)A_t dt + \sqrt{h(A_t, t)}A_t dW_t
\]

\[
r_{E,t} = LM_{t-1} r_{A,t}
\]

\[
r_{A,t} = \sqrt{h_{A,t}} \varepsilon_{A,t}, \quad \varepsilon_{A,t} \sim D(0, 1)
\]

\[
h_{A,t} = \omega + \alpha \left( \frac{r_{E,t-1}}{LM_{t-2}} \right)^2 + \gamma \left( \frac{r_{E,t-1}}{LM_{t-2}} \right) 1_{r_{E,t-1} < 0} + \beta h_{A,t-1}
\]

\[
LM_{t-1} = \left[ \Delta_{t-1}^{BS} \times g^{BS} \left( E_{t-1}/D_{t-1}, 1, \sigma_{A,t-1}^f, \tau \right) \times \frac{D_{t-1}}{E_{t-1}} \right]^{\phi}
\]
From a formal perspective, not a fully “structural” specification

- e.g., suppose asset prices followed a CEV process

\[ dA_t = \mu(A_t, t)A_t dt + \sigma A_t^\delta dW_t, \quad \delta < 1 \]

\[ E_t = A_t \sum_{k=0}^{\infty} g(\theta A_t^{2\delta - 2}, k + 1) G\left(\theta (X e^{-r\tau})^{-2\delta + 2}, k + 1 - \frac{1}{2\delta - 2}\right) - \]

\[ X e^{-r\tau} \sum_{k=0}^{\infty} g(\theta A_t^{2\delta - 2}, k + 1 - \frac{1}{2\delta - 2}) G\left(\theta (X e^{-r\tau})^{-2\delta + 2}, k + 1\right) \]

\[ g(a, b) \equiv e^{-a} a^{b-1} / \Gamma(b), \quad G(c, b) \equiv \int_c^{\infty} g(a, b) da, \quad \Gamma(b) \equiv \int_0^{\infty} e^{-x} x^{b-1} dx \]

\[ \text{LM} = \frac{A_t \partial E_t}{E_t \partial A_t} A_t^{\delta - 1} \]
The benefit of BSM is specificity, not flexibility

Consistency of BSM functions with GARCH?

If flexibility is the goal, then:

1. Use implied volatilities, e.g., VIX
2. Use nonparametric option-pricing model (Ait-Sahalia and Lo, 1998) to estimate SPD from options, which yields LM
3. Use M&M II and multiple factors

\[
\text{Var}[R_e] = \left(1 + \frac{D}{E}\right)^2 \text{Var}[R_a] + \left(\frac{D}{E}\right)^2 \text{Var}[R_d] - 2\frac{D}{E} \text{Cov}[R_a, R_d]
\]

- Estimate GARCH models with D/E, CDS, etc.
Structural Interpretation

- Complex capital structure makes this more complex
  - Coupon bonds are compound options (Geske, 1977)
  - Bond indenture provisions (Black and Cox, 1976)
  - Particularly problematic for financial firms

- Potential issue with option-pricing approach
  - Pricing model hinges on dynamic completeness, i.e., dynamic replication of option payoff via trading strategy
  - Market for corporate assets are relatively illiquid
  - Delta-hedging strategy may be very expensive, i.e., option-pricing formula may be a poor approximation (important for systemic risk measurement)
Leverage Effect


\[ \log(\frac{\sigma_t}{\sigma_{t-1}}) = \alpha + \lambda R_{t-1} + \epsilon_t \]

- Hasanhodzic and Lo (2013):

January 2, 1973 to December 31, 2010 (241 firms)

<table>
<thead>
<tr>
<th>Statistic</th>
<th>Min</th>
<th>Median</th>
<th>Mean</th>
<th>Max</th>
<th>Std</th>
</tr>
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<tbody>
<tr>
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<td>0.00</td>
<td>0.00</td>
<td>0.09</td>
<td>0.02</td>
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<td>(1.41)</td>
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<tr>
<td>Adj. ( R^2 )</td>
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<td>1.1%</td>
<td>2.7%</td>
<td>22.1%</td>
<td>4.5%</td>
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**Leverage Effect**

- But leverage effect is the same for all-equity-financed companies!

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<tr>
<th>Dataset</th>
<th>Sample Size</th>
<th>Statistic</th>
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<td>-0.01</td>
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During periods of financial distress, BSM option-pricing model for firm’s equity may be inadequate

Complex capital structure is particularly relevant, e.g., AIG’s credit support agreements

A structural alternative is Merton’s jump-diffusion model:

\[ dA_t = \mu A_t dt + \sigma A_t dW_t + ZA_t dN_t , \quad E[\log Z] \ll 0 \]

- For lognormal Z, LM can be derived in closed-form; compare and contrast to the pure diffusion case

A reduced-form alternative is regime-switching
Another measure of systemic risk might be the option’s gamma:

\[
\frac{\partial^2 E_t}{\partial A_t^2} = \frac{1}{A_t \sigma \sqrt{\tau}} \phi(d_1)
\]

\[
d_1 \equiv \frac{\log(A_t/X) + (r + \sigma^2/2)\tau}{\sigma \sqrt{\tau}}
\]