Slow Moving Debt Crises*

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Abstract

What circumstances or policies leave sovereign borrowers at the mercy of self-fulfilling increases in interest rates? To answer this question, we study the dynamics of debt and interest rates in a model where default is driven by insolvency. Fiscal deficits and surpluses are subject to shocks but influenced by a fiscal policy rule. Whenever possible the government issues debt to meet its current obligations and defaults otherwise. We show that low and high interest rate equilibria may coexist. Higher interest rates, prompted by fears of default, lead to faster debt accumulation, validating default fears. We call such an equilibrium a slow moving crisis, to distinguish it from rollover crises in which an investors' run precipitates immediate default. We investigate how the existence of multiple equilibria is affected by the fiscal policy rule, the maturity of debt, and the level of debt.

1 Introduction

Yields on sovereign bonds for Italy, Spain and Portugal shot up dramatically in late 2010 with nervous investors suddenly casting the debt sustainability of these countries into doubt. An important concern for policy makers was the possibility that higher interest rates were self-fulfilling. High interest rates, the argument goes, contribute to the rise in debt over time, eventually driving countries into insolvency, thus justifying higher interest rates in the first place.

News coverage reflected the fact that uncertainty about future interest rates and debt dynamics were at the center of investors' concern. For example, a *Financial Times*' report

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on the Italian bond market cites a pessimistic observer expecting "Italian bonds to perform worse than Spanish debt this summer, as investors focus on the sustainability of Italy's debt burden," given Italy's high initial debt-to-GDP ratio. A more optimistic investor in the same report argues that Italy "can cope with elevated borrowing costs for some time particularly when shorter-dated bond yields remain anchored" and that "it's critical to bring these yields down, but there is time for Italy to establish that its policies are working."[1] A number of reports referred to an "Italian Debt Spiral" webpage by Thomson Reuters in which users could compute the primary surplus needed by Italy to stabilize its debt-to-GDP ratio under different scenarios.[2]

Yields subsided in the late summer of 2012 after the European Central Bank's president, Mario Draghi, unveiled plans to purchase sovereign bonds to help sustain their market price. A view based on self-fulfilling crises can help justify such lender-of-last-resort interventions to rule out bad equilibria. Indeed, this notion was articulated by Draghi during the news conference announcing the Outright Monetary Transactions (OMT) bond-purchasing program (September 6th, 2012),

> "The assessment of the Governing Council is that we are in a situation now where you have large parts of the Euro Area in what we call a bad equilibrium, namely an equilibrium where you have self-fulfilling expectations. You may have self-fulfilling expectations that generate, that feed upon themselves, and generate adverse, very adverse scenarios. So there is a case for intervening to, in a sense, break these expectations [...]"

If this view is correct, a credible announcement to do "whatever it takes" is all it takes to rule out bad equilibria, no bond purchases need to be carried out. To date, this is exactly how it seems to have played out. There have been no purchases by the ECB and no countries have applied to the OMT program.

In this paper we investigate the possibility of self-fulfilling crises in a dynamic sovereign debt model in which investors use a simple forecasting model to form expectations about future debt sustainability. Calvo (1988) first formalized the feedback between interest rates and the debt burden, showing that it opens the door to multiple equilibria.[3] Our contribution is to cast this feedback mechanism in a dynamic setting, focusing on the conditions for multiple equilibria.

[1] "Investors wary of Italy's borrowing test", *Financial Times*, July 16, 2012.

[2] The widget can be found at http://graphics.thomsonreuters.com/11/07/BV_ITDBT0711_VF.html

[3] For recent extensions of this framework applied to the European crisis see Corsetti and Dedola (2011) and Corsetti and Dedola (2013).

In our model, default is driven by insolvency and occurs only when the government is unable to finance debt payments. The government faces random shocks to its budgetary needs and attempts to finance itself by selling bonds to a large group of risk-neutral investors. We assume that government's policy is described by a fiscal rule, as in the literature on debt sustainability (Bohn, 2005; Ghosh et al., 2011) and in work studying the interaction of fiscal and monetary policy (Leeper, 1991). The fiscal rule specifies deficits and surpluses as functions of exogenous shocks and endogenous state variables, such as the debt level. Default occurs when the government's need for funds exceeds its borrowing capacity. The borrowing capacity, in turn, is limited endogenously by the probability of future default.

We begin by studying a case where only short-term debt is allowed. We show that the equilibrium bond price function (mapping the state of the economy into bond prices) and borrowing capacity is uniquely determined in this case. However, this does not imply that the equilibrium is unique. Multiplicity still arises in this case from what we call a *Laffer curve effect:* revenue from a bond auction is non-monotone in the amount of bonds issued. Since the borrower needs to target a given level of revenue, there are multiple bond prices consistent with an equilibrium.

We then turn to a model with a flexible debt maturity. Interestingly, with long-term bonds, the price function and borrowing capacity are no longer uniquely determined, instead, lower bond prices in the future feed back into current bond prices. This highlights an intertemporal coordination problem among investors, since they must now worry about future market conditions. By implication, even if all current investors were gathered in a room, in an effort to coordinate their actions, this will not prevent the bad equilibrium.

Along a bad equilibrium the government faces higher interest rates, leading to increased debt accumulation. This raises the probability of insolvency and default, ultimately justifying investors' demands for a higher interest rates. We call such self-fulfilling high interest rate equilibria "slow moving crises" to capture the fact that it develops over time through the accumulation of debt. The label helps distinguishes this type of crisis from rollover debt crises, which have been extensively studied in the literature, starting with Giavazzi and Pagano (1989), Alesina et al. (1992), Cole and Kehoe (2000) and more recently in Conesa and Kehoe (2012) and Aguiar et al. (2013). A rollover crisis is essentially a "run" on the borrowing government by current investors, who pull out of the market entirely, leading to a failed bond auction and triggering immediate default.[4] We

[4] Chamon (2007) argues that the coordination problem leading to a "run" could be prevented in practice by the manner in which bonds are underwritten and offered for purchase to investors by investment banks.

see rollover crises and slow moving crises as complementary ingredients to interpret turbulence in sovereign debt markets.[5] Indeed, rollover crises are also possible in our model, but for most of the paper we leave them aside to focus on slow moving crises.

How can slow moving crises be avoided? We identify a safe region of initial conditions and parameters for which the equilibrium is unique. The equilibrium is unique whenever debt is low enough. This result is intuitive since low debt mutes the feedback from interest rates to the cost of debt service. The equilibrium is also unique for fiscal rules that actively respond by reducing deficits when debt rises. This responsiveness directly counters the feedback effect from rising debt. Finally, longer debt maturity also helps guarantee a unique equilibrium. Shorter maturities require greater refinancing which potentiates the effects of high interest rates on debt accumulation.

A noteworthy feature of slow moving crises is that the existence of both a good and bad equilibrium may be transitory, in the sense that, if one goes down the path of a bad equilibrium for a sufficiently long time, debt may reach a level at which there exists a unique continuation equilibrium with high interest rates; the bad equilibrium may set in. Although the debt crises is initially triggered by self-fulfilling pessimistic expectations, the government becomes trapped into a bad outcome, due to the poor fundamentals it develops. The government may blame the vagaries of the market for some time, but eventually the market's mistreatment does real and irreparable damage. By implication, policy attempts to rule out bad equilibria must be put in place swiftly to avoid going down a bad equilibrium's detrimental path for too long.

Relative to the most recent literature on sovereign debt crises, which builds on the formalizations in Eaton and Gersovitz (1981) and Cole and Kehoe (2000), our approach differs in two ways.

The first difference is that for most of the analysis we model the government as following a fiscal rule, rather than model it as an optimizing agent. This modeling choice is not essential for the emergence of slow moving crises (as we show in Section 6) but we think it is useful for several reasons. First, it allows us to focus on the coordination problem between investors, since, as we show, multiple equilibria may arise even when the government is not strategic. Second, fiscal policy rules allow us to consider situations with partial commitment. For example, a government may promise efforts to increase surpluses when debt rises, but to a limited degree due to political economy constraints. The end-product of these considerations may be embedded in the fiscal policy rule. Un-

[5] Failed tesobonos auctions during Mexico's 1994 crises provided a motivation for the rollover crises literature. In the recent case of Italy, on the other hand, market participants seem clearly worried about adverse debt dynamics and bond yield, suggesting the forces at work in a slow moving crisis.

derstanding the positive implications of different policy rules constitutes an important first step towards a normative analysis. Third, although making fiscal policy endogenous is desirable, it may be difficult to capture in stylized optimization problems a number of constraints and biases coming from the political process. Finally, at a more practical level, fiscal policy rules seem descriptive of the debt-sustainability forecasting models actually used by market participants; we also hope to provide a bridge to the academic literature estimating fiscal rules.

The more fundamental difference with most of the current literature is in our timing assumptions in the debt market. The typical way sovereign-debt models are set up assumes borrowers can commit, within a period, to the amount of bonds issued, in keeping with the convention introduced by Eaton and Gersovitz (1981). As a side effect, this rules out slow moving crises, because it allows the borrower to always select the path of lower debt accumulation. Instead, we follow Calvo (1988) and assume that the government's resource needs from the financial market are determined first and that it then adjusts bond issuances to meet these financing needs, given market prices. This makes the size of the issuance endogenous to bond prices. As we show, this timing assumption is crucial to capturing slow moving crises, both in models with fiscal rules and in optimizing models.

At first glance, it may appear intuitively reasonable to adopt the standard timing assumption, letting borrowers commit to the amount of bonds issued. Certainly in the very short run, say, during any given market transaction or offer, the issuer is able to commit to a given bond size offering. However, this is not the relevant time frame. To see why, consider a borrower showing up to market with some given amount of bonds to sell. If the price turns out to be lower than expected, the borrower may quickly return to offer additional bonds for sale in order to make up the difference in funding. The important point here is that the overall size of the bond issuance remains endogenous to the bond price.

In Section 6 we provide a microfoundation for our timing assumption, by studying two explicit game-theoretic models of an optimizing government issuing bonds in repeated rounds. The first game is cast in continuous time and assumes a potentially unlimited number of bond auctions within each "period". The government loses all ability to commit to its bond issuance, since it can always reverse or increase issuances in the next round. The subgame-perfect equilibrium of this game provides an explicit microfoundation for the timing assumption used in the rest of the paper and shows how our approach can be extended to models with an optimizing government. The second game, set in discrete time, features only a finite number of periods and rounds, but introduces preferences that are not additively separable. Lower funds acquired in the market today increase the desire or need for funds tomorrow. This seems directly relevant for such things as infrastructure spending, but it may also capture elements of the financing of payroll, where a temporary shortfall in payments may be possible but must be repaid eventually. For this game, we show that there may be multiple subgame-perfect equilibria with different bond prices, similar to the equilibria we isolate in this paper. The second game shows that our form of multiplicity can arise even in environments where the government has some ability to commit to bond issuances, as long as one captures intertemporal linkages.

2 Short-Term Debt

In this section we introduce the basic model that we build on in later sections. We start by assuming that all borrowing is short term, that the primary surplus is completely exogenous and that there is zero recovery after default. All these assumptions are relaxed later.

2.1 Borrowers and Investors

Time is discrete with periods $t = 1, 2, \ldots, T$. A finite horizon is not crucial, but makes arguments simpler and ensures that multiplicity is not driven by an infinite horizon.

Government. The government generates a sequence of primary fiscal surpluses $\{s_t\}$, representing total taxes collected minus total outlays on government purchases and transfers (s_t is negative in the case of a deficit). We take the stochastic process $\{s_t\}$ as exogenously given and assume it is bounded above by $\bar{s} < \infty$. Let $s^t = (s_1, s_2, \ldots, s_t)$ denote a history up to period t. In period t, s_t is drawn from a continuous c.d.f. $F\left(s_t | s^{t-1}\right)$.[6]

The government attempts to finance $\{s_t\}$ by selling non-contingent debt to a continuum of investors in competitive credit markets. Absent default, the government budget constraint in period $t < T$ is

$$q_t(s^t) \cdot b_{t+1}(s^t) = b_t(s^{t-1}) - s_t, \tag{1}$$

where b_t represents debt due in period t and q_t is the price of a bond issued at t that is

[6] In applications it will be convenient to make the Markov assumption and write $F(s_t \mid s_{t-1})$, but at this point nothing is gained by this restriction.

due at $t+1$. In the last period, $b_{T+1}(s^T) = 0$ and avoiding default requires

$$s_T \geq b_T(s^{T-1}).$$

We write this last period constraint as an inequality, instead of an equality, to allow larger surpluses than those needed to service the debt. Of course, the resulting slack would be redirected towards lower taxes or increased spending and transfers, but we abstain from describing such a process.[7]

We assume that the government honors its debts whenever possible, so that default occurs only if the surplus and potential borrowing are insufficient to refinance outstanding debt. For now, we assume that if a default does occur bond holders lose everything; this assumption will be relaxed later. Let $\chi(s^t) = 1$ denote full repayment and $\chi(s^t) = 0$ denote default.

Our focus is on the debt dynamics during normal times or during crises leading up to a default. Consequently, we characterize the outcome up to the first default episode and abstain from describing the post-default outcomes. Specifically, for any realization of surpluses $\{s_t\}_{t=0}^T$ we only specify the outcome for debt and prices $b_{t+1}(s^t)$ and $q_t(s^t)$ in period t if $\chi(s^\tau) = 1$ for all $\tau \leq t$.[8] Similarly, one can interpret s_t as the surplus in periods t prior to default; default may alter future surpluses, but we need not model this fact to solve for the evolution of debt before default.[9]

Investors and Bond Prices. Each period there is a group of wealthy risk-neutral investors that compete in the credit market and ensure that the equilibrium price of a short term debt equals

$$q_t(s^t) = \beta \mathbb{E}[\chi_{t+1}(s^{t+1}) \mid s^t].$$

2.2 Equilibrium in Debt Markets

An equilibrium specifies $\{b_{t+1}(s^t), q_t(s^t), \chi_t(s^t)\}$ such that for all histories s^t with no current or prior default—i.e., with $\chi_\tau(s^\tau) = 0$ for all s^τ prior or equal to s^t—the government budget constraint (1) holds and the bond price satisfies $q_t(s^t) = \beta \mathbb{E}[\chi_{t+1}(s^{t+1}) \mid s^t]$.

[7]It is best not to interpret the finite horizon literally. One can imagine, instead, that the last "period" T represents an infinite continuation of periods. As long as all uncertainty is realized by T one can collapse the remaining periods from T onwards into the last period.

[8]This is possible because we abstract from modeling government welfare. In models where default is the result of an optimizing government, future variables enter its decision.

[9]Perhaps default alters future primary surpluses—for example, if creditors punish debtors or if defaulting debtors adjust taxes and spending to the new financial circumstances.

In addition, default occurs only when inevitable, a notion formalized by the following backward-induction argument.

In the last period the government repays if and only if $s_T \geq b_T$. The price of debt equals

$$q_{T-1} = \beta\left(1 - F\left(b_T | s^{T-1}\right)\right) \equiv Q_{T-1}(b_T, s^{T-1}).$$

Define the *maximal debt capacity* by[10]

$$m_{T-1}(s^{T-1}) \equiv \max_{b'} Q_{T-1}(b', s^{T-1})b',$$

where b' represents next period's debt, b_T in this case.

The government seeks to finance $b_{T-1} - s_{T-1}$ in period $T-1$ by accessing the bond market. This is possible if and only if

$$b_{T-1} - s_{T-1} \leq m_{T-1}(s^{T-1}).$$

We assume that whenever this condition is met the government does indeed manage to finance its needs and avoid default; otherwise, when $b_{T-1} - s_{T-1} > m_{T-1}(s^{T-1})$, the government defaults on its debt.

Turning to period $T-2$, investors anticipate that the government will default in the next period whenever $s_{T-1} < b_{T-1} - m_{T-1}(s^{T-1})$. Thus, the bond price equals

$$q_{T-2} = \beta \Pr\left(s_{T-1} \geq b_{T-1} - m_{T-1}(s^{T-1}) | s^{T-2}\right) \equiv Q_{T-2}(b_{T-1}, s^{T-2}).$$

The maximal debt capacity in period $T-2$ is then

$$m_{T-2}(s^{T-2}) \equiv \max_{b'} Q_{T-2}(b', s^{T-2})b'.$$

Again, default is avoided if and only if $b_{T-2} - s_{T-2} \leq m_{T-2}(s^{T-2})$. The probability of this event determines bond prices in period $T-3$.

Continuing in this way we can solve for the debt limits and price functions in all earlier periods by the recursion

$$m_t(s^t) = \max_{b'} \beta \Pr\left(s_{t+1} \geq b' - m_{t+1}(s^{t+1}) \mid s^t\right) \cdot b'$$

[10]The maximum is well defined because the function involved is continuous and we can restrict the maximization to $0 \leq b \leq \bar{s}$, since $b < 0$ yields negative values and $b > \bar{s}$ yields zero.

and the associated price functions

$$Q_t(b', s^t) \equiv \beta \Pr\left(s_{t+1} \geq b' - m_{t+1}(s^{t+1}) \mid s^t\right).$$

Returning to the conditions for an equilibrium sequence $\{b_{t+1}(s^t), q_t(s^t), \chi_t(s^t)\}$, we require that for all histories s^t where $b_t(s^{t-1}) - s_t \leq m_t(s^t)$ that $\chi_t(s^t) = 1$ and $b_{t+1}(s^t)$ solve

$$Q_t(b_{t+1}(s^t), s^t) \cdot b_{t+1}(s^t) = b_t(s^{t-1}) - s_t. \tag{2}$$

Interestingly, both the maximal debt capacity function $\{m\}$ and the price functions $\{Q\}$ are uniquely determined. As we show next, this does not imply that the equilibrium path for debt is unique.

3 Self-Fulfilling Crises

In this section we study the short-term debt model set out in the previous section. We first show that there are multiple equilibria, with different self-fulfilling interest rates and debt dynamics. We then extend the model by including a recovery value and by allowing surpluses to react to debt levels.

3.1 Multiple Equilibria in the Basic Model

Define the correspondence

$$B_t(b, s^t) = \{b' \mid Q_t(b', s^t) b' = b - s_t\}.$$

Note that $B_t(b, s^t)$ is nonempty if and only if $b \leq m_t\left(s^t\right) + s_t$. Whenever $b < m_t\left(s^t\right) + s_t$ the set $B_t(b, s^t)$ contains at least two values. This follows because the Laffer curve $Q_t(b', s^t)b'$ attains a strictly positive maximum $m_t\left(s^t\right)$ for some $b' \in [0, T\bar{s}]$ yet $Q_t(b', s^t)b' \to 0$ as both $b' \to 0$ and $b' \to \infty$. In other words, the Laffer curve is initially increasing for low b' but eventually decreases and converges to zero for $b \to \infty$. The maximum is interior because higher debt increases the probability of default, destroying bond holder value.

Define the selection with the lowest debt

$$\underline{B}_t(b, s^t) = \min B_t(b, s^t),$$

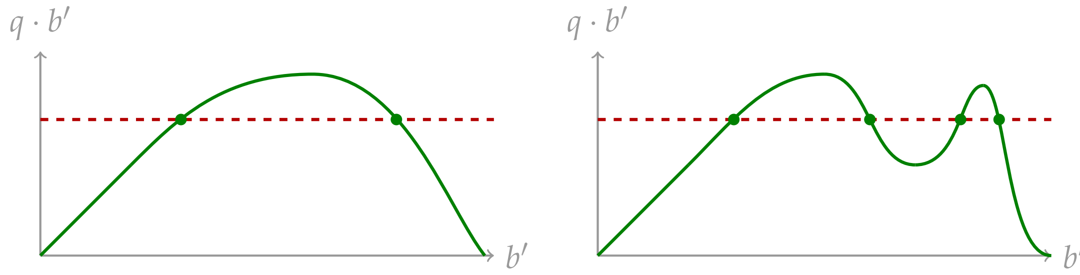


Figure 1: The revenue function without recovery value. Left panel shows a case with 2 equilibria; right panel shows a case with 4 equilibria.

and let $\{\underline{b}_{t+1}(s^t)\}$ to be the path generated by

$$\underline{b}_t(s^{t-1}) = \underline{B}_t(\underline{b}_{t+1}(s^t), s^t).$$

Eaton and Gersovitz (1981) and much of the subsequent literature on sovereign debt proceed by implicitly selecting this low debt equilibrium outcome. We are concerned with the possibility of other outcomes with higher debt.

Proposition 1 (Multiplicity)**.** *Any sequence for debt $\{b_{t+1}\}$ satisfying*

$$b_{t+1}(s^t) \in B_t(b(s^{t-1}), s^t)$$

until $B_t(b(s^{t-1}), s^t)$ is empty is part of an equilibrium. In any equilibrium

$$b_{t+1}(s^t) \geq \underline{b}_{t+1}(s^t) \qquad \textit{for all } s^t.$$

If $b_1 - s_1 < m_1(s_1)$ or $T \geq 3$ then there are at least two equilibrium paths.

Figure 1 plots two possibilities for the Laffer curve $Q_t(b', s^t)b'$. The left panel shows a case with a unique local maximum, while the right panel displays a case with several local maxima. The government needs to finance $b_t - s_t$, represented by the dashed horizontal line. In the left panel there are two values of b_{t+1} that raise the required revenue. In the right panel there are four solutions. The bond price q_t corresponds to the slope of a ray going through the origin and the equilibrium point, so that larger values of b_{t+1} correspond to lower prices, i.e. higher interest rates. Thus, one could translate the figure into the quantity and price space (b_{t+1}, q_t), with the demand from investors being a translation of the Laffer curve and the supply curve from the government being a translation of the financing need, i.e. $b_{t+1} = \frac{b_t - s_t}{q_t}$. Both the demand and supply schedules are downward sloping and may intersect twice.

Equilibrium points where the Laffer curve is locally decreasing are "unstable" in the Walrasian sense that a small increase in the price of bonds reduces the supply by more than the demand, creating excess demand. These equilibria are also pathological on other grounds. First, they are also unlikely to be stable with respect to most forms of learning dynamics. Second, Frankel, Morris and Pauzner (2003) show that global games do not select such equilibria. Finally, these equilibria lead to counterintuitive comparative statics: an increase in the current debt level b_t increases the bond price q_t, i.e. higher financing needs lower the equilibrium interest rate.

For all these reasons, one may discard unstable equilibria and only consider equilibrium points where the Laffer curve is locally increasing. Adopting the stability criterion, the panel on the left features a unique equilibrium, while the panel on the right features two equilibria. In the second case, the high-debt equilibrium is sustained by a higher interest rate that is self fulfilling: a lower bond price forces the government to sell more bonds to meet its financial obligations; this higher debt leads to a higher probability of default in the future, lowering the price of the bond and justifying the pessimistic outlook. This two-way feedback between high interest rates and debt sustains multiple equilibria.

In the model with short term debt, multiple stable equilibria can only be obtained with primitives that generate a Laffer curve with multiple peaks. As we shall see, this conclusion is not robust to the introduction of longer term debt.

What are the effects of inherited debt b_t? Along stable equilibria that are continuous in inherited debt, an increase in b_t raises the current interest rate as well and the entire path of bond issuances and interest rates. Higher debt also increases the potential for multiple equilibria. In the right panel of Figure 1, a sufficient reduction in the initial debt level b_t shifts the dashed red line downwards and eliminates bad equilibria.

This creates a feedback, from past equilibria selection, into the future. For example, if a bad equilibrium is selected at t, this raises b_{t+1} and the interest rate at $t+1$ even if the good equilibrium is expected to be played at $t+1$. It also raises the potential for equilibrium multiplicity. Notably, in the present model, there is no feedback running in the opposite direction, from the future to the present. Our backward induction shows that the correspondence B_t, maximum debt capacity m_t, and the Laffer curve $Q_t(b', s^t)b'$ are all defined independently of the equilibrium selection in future periods. In particular, expectations of a bad equilibrium selection in the future, have no effects in the current period. As we shall see in Section 4, this conclusion is not robust to the introduction of longer term debt.

3.2 Relation to the Literature

The presence of a two-way feedback between interest rates and the probability of repayment was first explored in Calvo (1988) in a two-period model: in the second period, a government partially defaults on debt to mitigate the costs from taxation; in the first period, it decides how much financing it requires. Calvo assumes a "price taking" assumption similar to the one adopted here, opening the possibility to multiple equilibrium interest rates.

In contrast, most of the sovereign debt literature, e.g. Eaton and Gersovitz (1981), Cole and Kehoe (2000) and Arellano (2008), assume that the government can choose the amount of bonds issued b_{t+1} each period, rather than its financing needs. Equivalently, the government is assumed to commit to adjusting its short run financing needs, instead of its bond issuance, if bond prices turn out differently than expected. This commitment assumption has the, perhaps unintended, consequence of selecting the low-debt equilibrium in Figure 1.

One may formalize the different commitment assumptions as different timings in a one-shot game between the government and investors. In the first game, the timing has investors first bid and the government decides how much to issue at the lowest bid.[11] In the second game the timing is inverted: the government chooses how many bonds to issue; investors then bid over these bonds.

In Section 6, we formulate a dynamic game that offers a microfoundation for our choice. Within a stage, we adopt the second timing. However, the government lacks commitment across stages and cares about total resources obtained across all stages. As a result, the end outcome of this dynamic game resembles the first timing. The crucial assumption is lack of commitment over future bond issuances.

Turning to another issue, our discussion of instability raises the possibility of unraveling towards an immediate rollover crisis. Consider the left panel of Figure 1. Suppose we start immediately to the right of the bad, unstable equilibrium. At that point, there is an excess supply of bonds and we can consequent downward adjustment in bond prices, pushing further and further to the right. This continues until we reach, or asymptote towards, a level of bond issuance associated with a zero probability of repayment. This suggests the presence of a third, stable equilibrium, with a zero bond price and default in period t.[12] In other words, a rollover crises where pessimistic investor expectations force

[11] Importantly, they are restricted to bidding a single price and absorbing any quantity at that price, i.e. perfectly horizontal demand schedules.

[12] To allow for this possibility we must relax the assumption that default only happens when there is no issuance b' that allows the government to finance $b_t - s_t$.

immediate default and this current default makes repayment in the future impossible, justifying investors' pessimistic expectations.

Rollover crises are the focus of Cole and Kehoe (2000), where a bad equilibrium corresponds to a situation in which at zero bond prices the government prefers to default today. It is interesting that rollover crises can also occur in our model, despite the modeling differences note above. However, for the remainder of the paper, we will ignore rollover crises and focus on slow moving crises.

3.3 Recovery Value

We now add a recovery value for debt by assuming that after a default event debtors can seize a fraction $\phi \in [0,1)$ of the available surplus, so that

$$Q_{T-1}(b_T, s^{T-1}) = \beta \left(1 - F\left(b_T | s^{T-1}\right)\right) + \frac{\beta}{b_T}\phi \int_0^{b_T} s_T \, dF\left(s_T | s^{T-1}\right).$$

One may interpret the fact that $\phi < 1$ as representing the inefficiencies inherent in the default process, much as the costs involved in bankrupcy.

Defining the revenue function

$$G(b_T, s^{T-1}) \equiv Q_{T-1}(b_T, s^{T-1}) b_T = \beta \left(1 - F\left(b_T | s^{T-1}\right)\right) b_T + \beta\phi \int_0^{b_T} s_T \, dF\left(s_T | s^{T-1}\right),$$

and note that the slope

$$\frac{\partial}{\partial b_T} G(b_T, s^{T-1}) = \beta \left(1 - F\left(b_T | s^{T-1}\right)\right) - \beta(1-\phi) f\left(b_T | s^{T-1}\right) b_T$$

may be positive or negative. However, in the limit as $b_T \to \infty$ we have

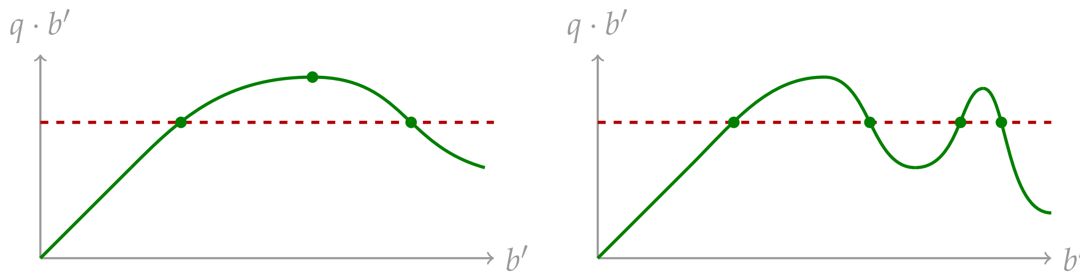
$$G(b_T, s^{T-1}) \to \beta \mathbb{E}[s_T | s^{T-1}] > 0,$$

implying a safe region with a unique equilibrium. Identical calculation apply is earlier periods.

Proposition 2. *Suppose a positive recovery value $\phi > 0$ and fix any history s^t. For low enough debt $b_t(s^{t-1})$ there is a unique solution for $b_{t+1}(s^t)$ in*

$$Q_t(b_{t+1}(s^t), s^t) \cdot b_{t+1}(s^t) = b_t(s^{t-1}) - s_t.$$

Figure 2 illustrates the situation. In both panels, for high debt there may still be mul-



(a) A case with 2 equilibria.

(b) A case with 4 equilibria.

Figure 2: The revenue function with recovery value displaying two equilibrium points.

tiple equilibria, but for sufficiently small debt only the good side of the Laffer curve is available.

3.4 Fiscal Rules

When debt rises, governments tend to make efforts to stabilize it. To capture this we can assume that surpluses are depend on the current level of debt as follows:

$$s_t \sim F(s_t \mid s^{t-1}, b_t).$$

Fiscal policy rules of this kind are commonly adopted in the literature studying solvency (e.g. Bohn, 2005; Ghosh et al., 2011) as well as the literature studying the interaction of monetary and fiscal policy (e.g. Leeper, 1991).

The recursion defining an equilibrium is similar

$$m_t(s^t) = \max_{b'} \beta \Pr\left(s_{t+1} \geq b' - m_{t+1}(s^{t+1}) | s^t, b'\right) b',$$

$$Q_t(b', s^t) \equiv \beta \Pr\left(s_{t+1} \geq b' - m_{t+1}(s^{t+1}) | s^t, b'\right).$$

Fiscal rules may have an important impact on debt limits $m_t(s^t)$ as well as on the existence of multiple equilibria. We will explore these effects in the next section, using an extended model with a more general bond maturity.

4 Long-Term Debt

We now extend the model to long term bonds. This is important for a number of reasons. First, in most advanced economies the bulk of the stock of debt is in the form of relatively long-term bonds, rather than three-month or one-year bonds (Arellano and Ramanarayanan, 2012). For example, the average maturity of sovereign debt for Greece, Spain, Portugal and Italy was 5-7 years over the 2000-2009 period. Second, it is widely believed that short term debt exposes a country to debt crises and that long debt provides greater protection. In the context of a rollover crises, Cole and Kehoe (1996) find support for this view. How does debt maturity affect the possibility of slow moving debt crises in our model?

The time horizon is infinite $t = 0, 1, \ldots$ However, we assume that all uncertainty is resolved in some period $T < \infty$ and that for $t \geq T$ the surplus is constant and equal to its value at T.[13] This allows us to solve for equilibria by backwards induction, as before.

We assume that the government issues bonds with geometrically decreasing coupons: a bond issued at t promises to pay a sequence of coupons $\kappa, (1-\delta)\,\kappa, (1-\delta)^2\,\kappa, ...$ where $\delta \in (0,1)$ and $\kappa > 0$ are fixed parameters. This well-known formulation of long-term bonds is useful because it avoids having to carry the entire distribution of bonds of different maturities (see Hatchondo and Martinez, 2009). A bond issued at $t-j$ is equivalent to $(1-\delta)^j$ bonds issued at t. Using this transformation, the vector of outstanding bonds can be summarized by single quantity, b_t, measured in units of newly issued bonds. Likewise, we only keep track of a corresponding single price, denoted by q_t.

The budget constraint is

$$q_t(s^t) \cdot (b_{t+1}(s^t) - (1-\delta)\, b_t(s^{t-1})) = \kappa b_t(s^{t-1}) - s_t.$$

The difference between the coupon payments and the primary surplus must be covered by selling newly issued bonds. Since past bonds are equivalent to $(1-\delta)\, b_t$ newly issued bonds, the revenue from selling new bonds equals $q_t\,(b_{t+1} - (1-\delta)\, b_t)$.

The current surplus is affected by last period's surplus and the level of current debt

$$s_t \sim F(s_t \mid s_{t-1}, b_t).$$

We drops the potential dependence on the past history s^{t-2}.

To allow for some positive recovery in the event of default, we assume that when

[13]Equivalently, the surplus may fluctuate, but what matters is that the present value of this surplus at T equal s_T in annuity value.

default occurs investors, as a group, recover $v\left(s_t\right)$. The pricing condition takes the form

$$q_t = \beta E_t\left[\kappa + (1-\delta)\, q_{t+1} | \text{No default at } t+1\right] \Pr\left[\text{No default at } t+1\right] + \frac{1}{b_{t+1}} E_t\left[v\left(s_{t+1}\right) | \text{Default at } t+1\right] \Pr\left[\text{Default at } t+1\right].$$

The second term is divided by b_{t+1}, reflecting the fact that each bond holder gets a proportion of the recovery value $v(s_{t+1})$.

If default is avoided, the price of long-term bonds for $t \geq T$ equals

$$q^* \equiv \frac{\beta\kappa}{1-\beta\left(1-\delta\right)} = 1,$$

where we adopt the normalization $\kappa = 1/\beta - 1 + \delta$ so that $q^* = 1$.

The government is solvent and default is avoided for $t \geq T$ if and only if

$$s_T \geq \kappa b_T - \delta b_T = r b_T,$$

where $r \equiv \kappa - \delta = \frac{1}{\beta} - 1$.[14] The price of the bond in period $T-1$ is then

$$q_{T-1} = Q_{T-1}\left(b_T, s_{T-1}\right) \equiv 1 - F(rb_T | s_{T-1}, b_T) + \frac{\beta}{b_T}\int_{-\infty}^{rb_T} v\left(s_T\right) dF\left(s_T | s_{T-1}, b_T\right).$$

The maximal revenue from debt issuance at $T-1$

$$m_{T-1}\left(b_{T-1}, s_{T-1}\right) \equiv \max_{b_T}\left\{Q_{T-1}\left(b_T, s_{T-1}\right)\left(b_T - (1-\delta)b_{T-1}\right)\right\},$$

will serve as a debt limit. Default can be avoided at $T-1$ if and only if

$$b_{T-1} \leq m_{T-1}\left(b_{T-1}, s_{T-1}\right) + s_{T-1}.$$

Let R_{T-1} denote the set of pairs $\left(b_{T-1}, s_{T-1}\right)$ where this inequality holds.

Unlike the case with short-term date, we must select a value for b_T satisfying

$$Q_{T-1}\left(b_T, s_{T-1}\right)\left(b_T - \left(1-\delta\right) b_{T-1}\right) = \kappa b_{T-1} - s_{T-1}, \tag{3}$$

to continue the induction. Let $B_T\left(b_{T-1}, s_{T-1}\right)$ denote a selection for b_T as function of $\left(b_{T-1}, s_{T-1}\right)$ with domain R_{T-1}.

[14] Once again, the budget constraint is written as an inequality in the last period. Of course, if the inequality holds with slack we can interpret the true surplus as adjusting to reach equality.

The recursion for earlier periods $t \leq T-2$ is similar. Given Q_{t+1}, m_{t+1}, R_{t+1}, B_{t+2}, we compute the price function

$$Q_t\left(b_{t+1}, s_t\right) = \beta \int_{R_{t+1}} \left(\kappa + (1-\delta) Q_{t+1}\left(B_{t+2}\left(b_{t+1}, s_{t+1}\right), s_{t+1}\right)\right) dF\left(s_{t+1} \mid s_t, b_{t+1}\right) + \frac{\beta}{b_{t+1}} \int_{R_{t+1}^c} v\left(s_{t+1}\right) dF\left(s_{t+1} \mid s_t, b_{t+1}\right), \quad (4)$$

the debt limit function

$$m_t\left(b_t, s_t\right) \equiv \max_{b_{t+1}} Q_t\left(b_{t+1}, s_t\right) \cdot \left(b_{t+1} - (1-\delta) b_t\right)$$

the set $R_t = \{(b_t, s_t) \mid b_t \leq m_t(b_t, s_t) + s_t\}$ of repayment and a new selection $B_{t+1}(b_t, s_t)$ function

$$Q_t\left(B_{t+1}(b_t, s_t), s_t\right) \cdot \left(B_{t+1}(b_t, s_t) - (1-\delta)\, b_t\right) = \kappa b_t - s_t. \quad (5)$$

This defines a recursion from $(Q_{t+1}, m_{t+1}, R_{t+1}, B_{t+2})$ to (Q_t, m_t, R_t, B_{t+1}). Continuing in this way we can compute the entire sequence $\{Q_t, m_t, R_t, B_{t+1}\}$ for $t = 0, 1, \ldots$

The dynamics for debt can be solved forward by iterating on

$$b_{t+1}(s^t) = B_{t+1}(b_t(s^{t-1}), s_t)$$

until

$$(b_t, s_t) \notin R_t$$

at which point default occurs.

With long-term bonds, the maximal revenue m_t, the repayment set R_t and the bond price Q_t all depend on the selection rule used in future periods $t' > t$. Thus, a country's debt capacity is affected by investors' expectations about the future, because the expectation of higher bond issuances in the future depress the value of current bonds. This introduces a feedback between the future and the present that was absent with short term debt.

Laffer Curves. With long-term bonds we distinguish two different types of coordination failure among investors. The first is the case in which the country could reduce the amount of bonds issued and still be able to cover its financing needs $\kappa b_t - s_t$, if all the investors who are purchasing bonds at date t bid a higher price for these bonds. This is the case in which the expression

$$Q_t\left(b_{t+1}, s_t\right) \cdot \left(b_{t+1} - (1-\delta)\, b_t\right) \tag{6}$$

is a decreasing function of b_{t+1} at $b_{t+1}(s^t)$. The second is the case in which all the investors who are purchasing bonds at date t and all the investors who purchased bonds in the past would get a higher expected repayment if they coordinated on reducing the face value of the debt b_{t+1}. This is the case in which the expression

$$Q_t\left(b_{t+1}, s_t\right) \cdot b_{t+1} \tag{7}$$

is a decreasing function of b_{t+1} at $b_{t+1}(s^t)$. We call the first the "issuance Laffer curve" and the second the "stock Laffer curve". Notice that a country can very well be on the decreasing side of the stock Laffer curve and yet still be on the increasing side of the issuance Laffer curve.

5 Applications

We now apply the tools developed in the last section by making specific assumptions about the stochastic process of the primary surplus. We consider both a stationary and a non-stationary environment. In the context of these environments, we construct examples that capture our notion of a slow moving crisis. Both applications are developed in continuous time for the sake of convenience. In the stationary case, continuous time is useful as it allows us to characterize equilibria in a standard phase diagram.

The first objective of this section is to illustrate the dynamics of a slow moving crisis when long-term bonds are present. In particular, we want to characterize the debt levels for which multiplicity is present and show how debt dynamics are such that multiplicity appears during the build-up phase of the crisis. Initially, there is both a good equilibrium with a high bond price for the bond and a bad one with a low price. At some point in time the continuation equilibrium becomes unique: the bad equilibrium path features a high probability of default because of the high debt accumulated, but there is no other equilibrium. Likewise, along the good equilibrium path debt is low and eventually the only equilibrium features a low probability of default. The second objective is to show how the fiscal rule, the initial debt level, and debt maturity affect the presence of multiplicity.

5.1 A stationary model

Time is continuous and runs forever. Investors are risk neutral and have discount factor r. Bonds issued at time t pay a flow coupon $\kappa \exp\{-\delta(\tau - t)\}$ at each $\tau > t$. These bonds are the continuous-time analog of the long-term bonds introduced in the previous section. Let $b(t)$ to denote the stock of outstanding bonds in terms of equivalent bonds issued at time t. We adopt the normalization $\kappa = r + \delta$ so that absent default risk the bond price equals $q^* = 1$.

The primary surplus process evolves in two stages. In the first stage, the primary surplus s is a deterministic function of the stock of bonds outstanding

$$s = h(b),$$

where h is a weakly increasing function that reaches its maximum $\bar{s} > 0$ for $b \geq \bar{b}$, i.e. $h(b) = \bar{s}$ for $b \geq \bar{b}$.

At a Poisson arrival rate λ we reach the second stage, where all uncertainty is resolved and the present value S of the country's future primary surpluses is drawn from a continuous distribution with c.d.f. $F(S)$ and finite support $[\underline{S}, \overline{S}]$. If S is greater than or equal to the accumulated debt b there is no default and the bond price equals 1. If, instead, $S < b$, the bond holders share equally the recovery value ϕS, with $\phi < 1$. The bond price upon entering the second stage, but immediately before the resolution of uncertainty, is given by

$$q = \Psi(b) \equiv 1 - F(b) + \phi \int_0^b \frac{S}{b} \, dF(S). \tag{8}$$

We can now focus on the dynamics of q and b in the first stage of the surplus process, before the resolution of uncertainty. Under our assumptions, default never occurs in the first stage. So the bond pricing equation is

$$(r + \delta) q = \kappa + \lambda(\Psi(b) - q) + \dot{q}, \tag{9}$$

which is the continuous time analog of (4) and captures the fact that the return on the bond includes the coupon κ, the capital gain $\Psi(b) - q$ when uncertainty is revealed, and the capital gain $\dot{q}$ before uncertainty is revealed. The government budget constraint is

$$q\left(\dot{b} + \delta b\right) = \kappa b - h(b) \tag{10}$$

which is the continous time equivalent of (5). Therefore, the model dynamics before the resolution of uncertainty can be found solving the ODEs (9) and (10).

5.1.1 Steady states

The steady states can be found by plotting the loci $\dot{q} = 0$ and $\dot{b} = 0$ as in Figure 3. The locus $\dot{q} = 0$ is given by

$$q = \frac{\kappa + \lambda\Psi(b)}{r + \delta + \lambda} \tag{11}$$

and is downward sloping since $\Psi'(b) < 0$. The interpretation is easy: a larger stock of bonds implies a smaller probability of repayment after the resolution of uncertainty and thus lower bond prices. The locus $\dot{b} = 0$ is given by

$$q = \frac{\kappa b - h(b)}{\delta b} \tag{12}$$

and its slope depends on the slope of the fiscal rule. Namely, it is decreasing if and only if

$$h'(b) > \kappa - \delta q. \tag{13}$$

This condition is violated if the fiscal rule is insensitive to debt, so that $h'(b) = 0$. Although this may appear to be a standard benchmark case, a more common assumption is that fiscal rules stabilize debt. For example, in economies without default risk it is common to assume $h'(b) > r$ around a steady state. This other benchmark amounts to a Ricardian regime or an active fiscal policy in the terminology of Leeper (1991).

The average cost of servicing a dollar of debt in steady state is $\kappa - \delta q$ because the government needs to pay the coupon and can issue δ new bonds to replace depreciated old bonds. Therefore, inequality (13) ensures that an increase in the stock of debt leads to an increase in the primary surplus greater than the increase in the average cost of servicing the debt. The relative slopes of the two curves in Figure 3 play an important role when we turn to the dynamics.

Figure 3 displays an example with two steady states. In the low-debt steady state, bond prices are high, debt issuances δqb cover large fraction of the coupon payments κb and the country can run a low primary surplus, consistently with its policy rule. In the high-debt steady state bond prices are low, so debt issuances cover a smaller portion of coupon payments and the country needs to run a larger surplus.

To solve our ODEs we consider must impose some boundary or transversality conditions. We consider two types of conditions: either the economy converges to a steady state, or it converges to a path with a constant value of debt

$$bq = \hat{v} \equiv \frac{\bar{s} + \lambda\Psi\left(\bar{S}\right)\bar{S}}{r + \lambda},$$

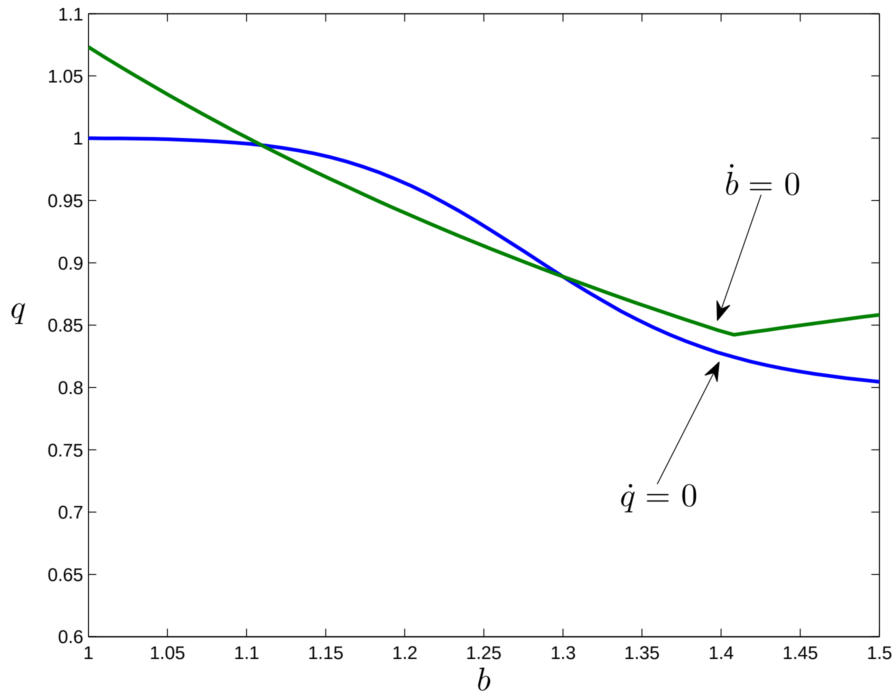


Figure 3: Steady states

with ever growing stock of debt $b \to \infty$ and collapsing bond prices $q \to 0$. The path with ever growing debt can be characterized analytically using the fact that $h(b)$ and S are bounded above and these bounds become binding for b large enough. Along this explosive debt path, the primary surplus is constant at $\bar{s}$ and lenders expect certain default as soon as stage 2 is reached. Nonetheless, the government is able to service the cost of debt in excess of $\bar{s} > 0$ by issuing new debt and diluting old lenders. Note that, although debt explodes, the value of debt is constant.

The path with exploding debt is the unique equilibrium whenever b is larger than some cutoff $\hat{b}$.[15] Thus, this path can be thought of as a boundary condition at $\hat{b}$. One can entertain alternative upper boundary conditions. For example, we could assume that if debt b reaches some high enough level, $\hat{b}$, then some form of renegotiation occurs between the lenders and the sovereign borrower. This would pin down the value of debt at $\hat{b}$ and we could solve the ODE system using such a boundary condition.

[15]The cutoff is

$$\hat{b} = \max\left\{\bar{s}, \overline{S}, \frac{1}{\kappa}\left[\bar{s} + \delta \frac{\bar{s} + \lambda \Psi\left(\overline{S}\right)\overline{S}}{r+\lambda}\right]\right\}.$$

5.1.2 Stability

A necessary and sufficient condition for saddle-path stability of a steady state is given by the following lemma.

Lemma 1. *A steady state with positive debt is locally saddle-path stable iff at the steady-state (b, q) the $\dot{b} = 0$ locus is downward sloping and steeper than the $\dot{q} = 0$ locus or, equivalently, iff*

$$h'(b) > \kappa - \delta q - \frac{\delta\lambda}{r + \delta + \lambda}\Psi'(b)\, b. \tag{14}$$

Proof. Using steady-state conditions, the Jacobian can be written as

$$J = \begin{bmatrix} \frac{\kappa - h'(b)}{q} - \delta & -\frac{\delta b}{q} \\ -\lambda\Psi'(b) & r + \delta + \lambda \end{bmatrix}.$$

A necessary and sufficient condition for a saddle is a negative determinant of J, i.e., $J_{11}J_{22} < J_{12}J_{21}$. Since $J_{12} < 0$ and $J_{22} > 0$, this is equivalent to $-J_{11}/J_{12} < -J_{21}/J_{22}$, which means that the $\dot{b} = 0$ locus is downward sloping and steeper than the $\dot{q} = 0$ locus. Condition (14) comes from rearranging. □

As mentioned above, in a steady state with no default risk we have $q = 1$ and $\Psi'(b) = 0$, so condition (14) boils down to $h'(b) > r$. This condition ensures that the total surplus (primary minus interest payments) is increasing in b, a standard stability condition for fiscal rules in the literature on fiscal and monetary policy, e.g. Leeper (1991). With positive default risk the interpretation is similar, but now the cost of servicing debt is $\kappa b - \delta q b$ instead of rb and the expression on the right hand side of (14) is equal to the derivative of $\kappa b - \delta q b$ with respect to b, after substituting for q the steady state pricing condition (11). Notice that this condition is stronger than condition (13) precisely because it includes the effect that changes in b have on bond prices. Condition (13) means that the $\dot{b} = 0$ locus is downward sloping, while condition (14) means that the $\dot{b} = 0$ locus is steeper than the $\dot{q} = 0$ locus.

Given the model non-linearities, having a saddle-path stable equilibrium is not enough to rule out multiple steady states or multiple equilibria. In fact, the following proposition shows that multiple steady states as in Figure 3 are a common feature of this model.

Proposition 3. *If there is a saddle-path stable steady state with positive debt, then there are multiple steady states and at least one of them is not a saddle.*

Proof. Consider the functions on the right-hand sides of (11) and (12), which are both continuous for $b > 0$. If there is a saddle-path stable steady state at b', the second function is steeper, from Lemma 1, and so is below the first function at $b' + \epsilon$ for some $\epsilon > 0$. Taking limits for $b \to \infty$ the the second function yields $q \to \kappa/\delta$ and the first yields

$$q \to \frac{\kappa + \lambda\Psi\left(\overline{S}\right)}{r + \delta + \lambda} < \frac{\kappa}{\delta},$$

where the inequality can be proved using $\Psi\left(\overline{S}\right) < 1$ and $\kappa = r + \delta$. Therefore, the second function is above the first for some b'' large enough. The intermediate value theorem implies that a second steady state exists in $(b' + \epsilon, b'')$. □

5.1.3 Multiple equilibria

Figure 4 shows the phase diagram for our example. The blue and green solid lines are the $\dot{q} = 0$ and $\dot{b} = 0$ loci as in Figure 3. The dashed lines are the equilibrium paths: the dashed purple line is the path converging to the low-debt steady state at $b = 1.11$, the dashed red line is the equilibrium path leading to ever increasing debt. In this example, the high-debt steady state is a source and there are no equilibrium paths leading to it. Therefore, generically, the equilibrium either converges to the low-debt steady state or diverges to ever growing debt. Moreover, the Jacobian has complex eigenvalues at the high-debt steady state, so the equilibrium paths have spiral-like dynamics near that steady state. This opens the door to multiple equilibria.

If the initial value of b is in the interval between the two dotted black lines, the model features multiple equilibria, as both equilibrium paths can be reached. However, if the economy spends enough time on the good or on the bad equilibrium path, eventually the debt level exits the region of multiplicity and the continuation equilibrium is unique. So a crisis initially driven by bad expectations can eventually turn into an insolvency crisis from which it is impossible to transition back to low debt.

The presence of multiple steady states does not necessarily imply that multiple equilibria are possible. For example, in Figure 5 we see an example with two steady states in which the high-debt steady state is a source with two real eigenvalues. In this case, the model displays a bifurcation around the high-debt steady state, meaning that small changes in the initial conditions for debt can lead to dramatically different debt dynamics. This is a formal example of the idea that debt dynamics may display a "tipping point", i.e., reach a threshold above which concerns about sustainability drastically alter the debt dynamics, as suggested by Greenlaw et al. (2013).

The following lemma gives a sufficient condition for equilibrium multiplicity.

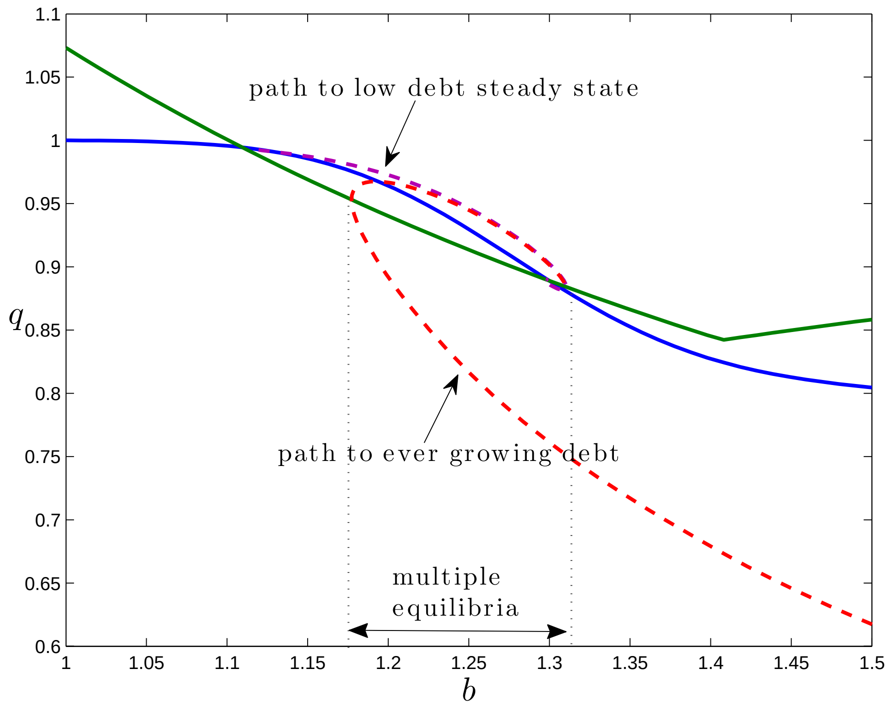


Figure 4: Phase diagram

Lemma 2. *If a steady-state* (b, q) *the following inequality holds*

$$4\delta\lambda\Psi'(b)\,qb + \left(\delta q + h'(b) + \lambda\Psi(b)\right)^2 < 0,$$

the model features multiple equilibria.

The condition in the lemma is only sufficient because multiple equilibria can arise even if the steady-state Jacobian has positive real eigenvalues, since the non-linearity of the system can still lead to a path non-monotone in b leading to the ever-growing debt path.

5.1.4 An example based on the case of Italy

We now adapt the model and choose parameters to capture the dynamics of bond prices and government debt for Italy in the Summer of 2012. It is easy to re-interpret our model in terms of debt-to-GDP and primary-surplus-to-GDP ratios. The only equation that needs to be modified is the government budget constraint which becomes

$$q\left(\dot{b} + (\delta + g)\,b\right) = \kappa b - h(b),$$

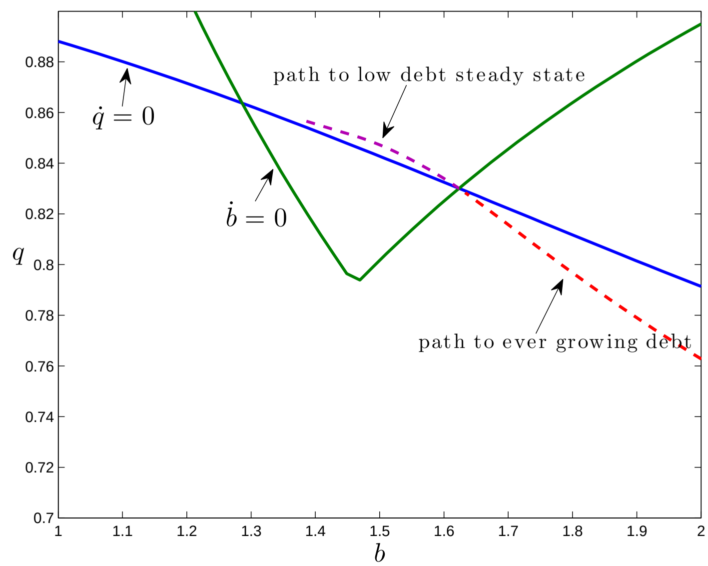


Figure 5: Multiple steady states, unique equilibrium

where g is the growth rate of GDP. It is also useful to allow for a different interest rate in the long run, assuming that after the realization of uncertainty the interest rate is given by the random variable $\tilde{r}$. This only affects the calculation of the Ψ function which becomes

$$\Psi(b) \equiv \int_{S \geq b} \frac{\kappa}{\tilde{r} + \delta} dF(\tilde{r}, S) + \phi \int_{S<b} \frac{S}{b} dF(\tilde{r}, S),$$

where F is the joint distribution of $\tilde{r}$ and S.

The fiscal rule is chosen to fit the observed relation between the debt-to-GDP ratio and the primary surplus in the period 1988-2012, as plotted in Figure 6. We assume the fiscal surplus is bounded above at 6% and use the rule

$$s = \min\{\alpha_0 + \alpha_1 b, \bar{s}\}$$

with $\alpha_0 = -0.13, \alpha_1 = 0.135, \bar{s} = 0.06$. We choose $\delta = 1/7$ to match the average maturity of italian government debt, which is about 7 years. We choose $r = 3\%$ to match the 10 year nominal bond yield in Germany in 2011 and $g = 2\%$ to capture nominal growth equal to the ECB inflation target of 2% plus zero real growth. We assume that the interest rate after the resolution of uncertainty is $\tilde{r} = 5\%$. We choose $\phi = 0.5$ and assume that S is normally distributed. The mean and variance of S are chosen so that in the low-debt steady state debt is essentially safe (with a spread of 10 basis points) and so that the high-debt steady state is $b = 1.3$.

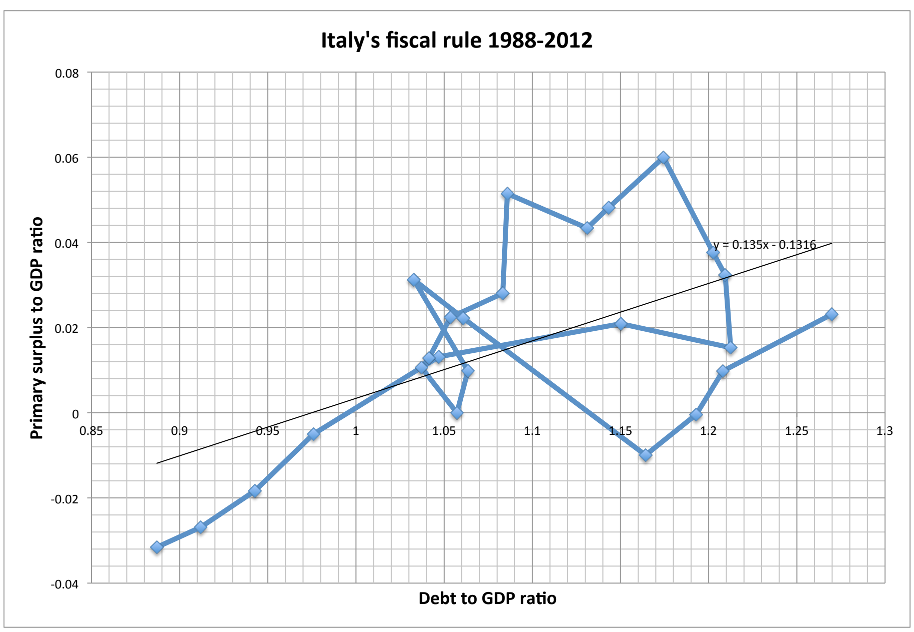


Figure 6: Estimated fiscal rule for Italy

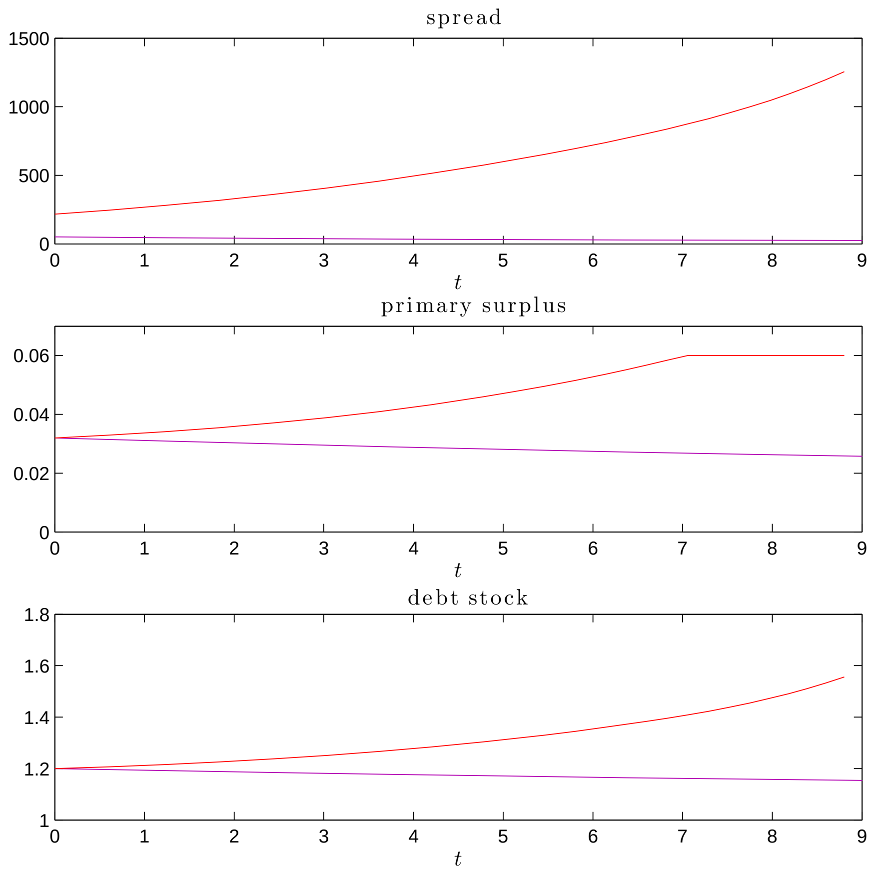


Figure 7: Two equilibrium paths starting at $b = 1.2$

We start the economy at $b = 1.2$, which corresponds to Italy's debt-to-GDP ratio in 2011. Given the parameters above, the model features multiple equilibria and $b = 1.2$ is in the multiplicity region. The dynamics of bond spreads, of the primary surplus and of debt-to-GDP are plotted in Figure 7. We can then imagine Italy following a path of slow debt reduction, as in the purple-line equilibrium. At date 0, investors' sentiment shift to the bad equilibrium path, spreads jump from 50bp to 220bp.

[...]

5.2 A non-stationary model

After time T, the country's long-run surplus is constant at $s(t) = rS$, where S drawn randomlyfrom a continuous distribution with c.d.f. $F(S)$. We assume $\kappa = r + \delta$ so the bond price under no default equals 1. Between times 0 and T, the country surplus evolves deterministically following the differential equation

$$\dot{s} = -\lambda\left(s - \alpha\left(b - b^*\right)\right). \tag{15}$$

The country has some target debt level b^*, when current debt exceeds the target the country adjusts its fiscal surplus towards the value $\alpha\left(b-b^*\right)$. The speed of adjustment to the target surplus is determined by the parameter λ. A larger coefficient α implies a more aggressive response to high debt. After time T, the country's long-run surplus is constant at $s\left(t\right)=rS$, where S is the long-run present value of surplus which is drawn randomly at time T from a continuous distribution with c.d.f. $F(S)$.

We focus on cases in which default never occurs before time T. Therefore, the bond price satisfies the differential equation

$$\left(r+\delta\right)q=\kappa+\dot{q}, \tag{16}$$

and the government's budget constraint is

$$q\left(\dot{b}+\delta b\right)=\kappa b-s. \tag{17}$$

To characterize the equilibria, we proceed as follows. The initial values for the debt stock and for the surplus, $b\left(0\right)$ and $s\left(0\right)$, are given. Choosing an initial value $q\left(0\right)$ we can then solve forward the system of ODEs in s, q, b given by (15), (16) and (17) and find the terminal values $b\left(T\right)$ and $q\left(T\right)$. If these values satisfy (8) we have an equilibrium. It is convenient to represent this construction graphically in terms of two loci for the terminal value of debt $b\left(T\right)$ and the terminal value of debt $q\left(T\right)b\left(T\right)$. In Figure 8 we plot two curves. The curve with an interior maximum is a Laffer curve similar to the one analyzed in Section 3, showing the relation between $b\left(T\right)$ and $q\left(T\right)b\left(T\right)$ implied by the bond pricing equation (8), namely

$$q\left(T\right)b\left(T\right)=\left(1-F(b\left(T\right))\right)b(T)+\phi\int_0^{b(T)}S\,dF(S). \tag{18}$$

The downward sloping curve plots the values of $b\left(T\right)$ and $q\left(T\right)b\left(T\right)$ that come from solving the ODEs (15), (16) and (17) for different values of the initial price $q\left(0\right)$. The curves are plotted for a numerical example with the following parameters:

$$T=10,\quad \delta=1/7,\quad r=0.02,\quad \phi=0.7,\quad \log S\sim N\left(0.3,0.1^2\right).$$

Taking the time period as a year, we consider a country in which uncertainty will be resolved in 10 years and the average debt maturity is 7 years. The risk-free interest rate is 2% and the recovery rate in case of default is 70%. The distribution of the present value of surplus, after uncertainty is resolved has mean 1.357 and standard deviation 0.136. The

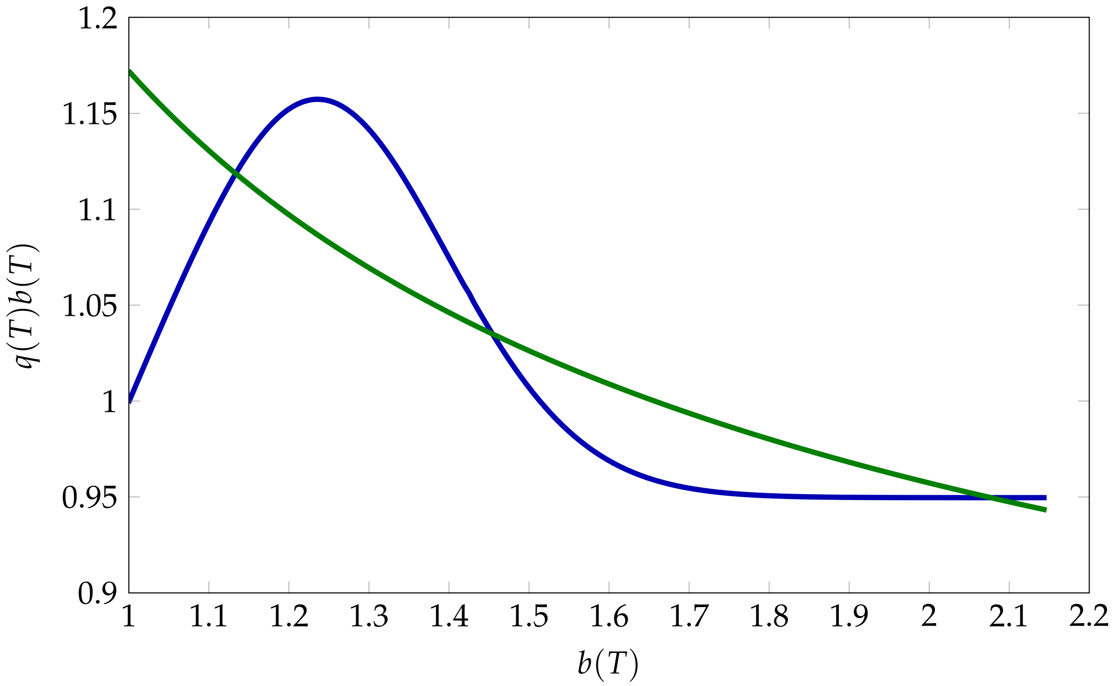


Figure 8: Three equilibria in example economy.

initial conditions are

$$s\left(0\right) = -0.1, \quad b\left(0\right) = 1,$$

and the fiscal policy parameters are

$$\lambda = 1, \quad \alpha = 0.02, \quad b^* = 0.$$

Figure 8 shows the presence of three equilibria. Note that both the first and third equilibrium are "stable", under various notions of stability discussed earlier. Thus, the model with long-term debt features multiple stable equilibria even when the Laffer curve is singled peaked. Figure 9 shows the dynamics of the primary surplus, debt and bond prices for the two stable equilibria, which we term "good" (solid lines) and the "bad" (dashed lines).

The model captures various features of recent episodes of sovereign market turbulence. Sovereign bond spreads experience a sudden and unexpected jump, in moving from the good to the bad equilibrium. The debt-to-GDP ratio increases slowly but steadily. Auctions of new debt issues do not show particular signs of illiquidity, yet, interest rates climb along with the level of debt. Large differences in debt dynamics appears gradually, as bond prices diverge and a larger fraction of debt is issued at crisis prices.

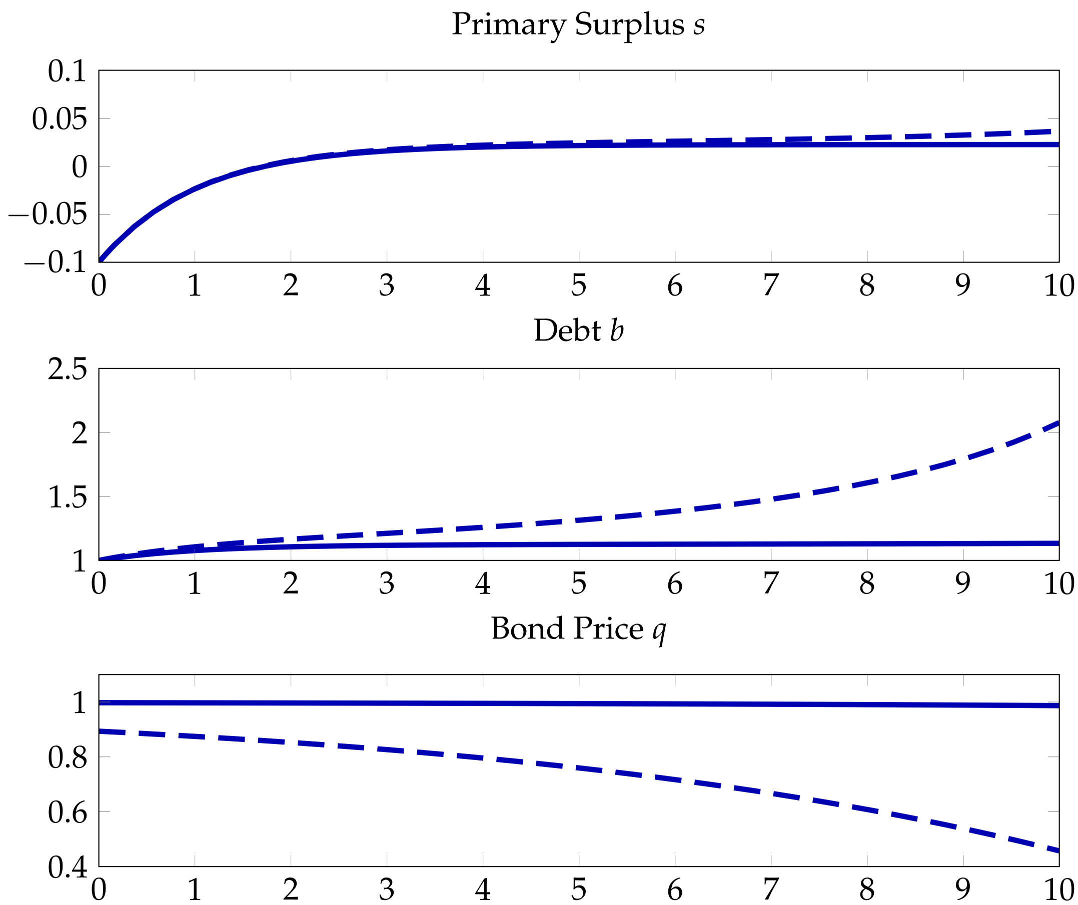


Figure 9: Dynamics of surplus, debt and bond price in good (solid line) and bad (dashed line) equilibrium.

A characteristic feature of a slow moving crisis is that multiplicity plays out in the early phase of a crisis. This is unlike the case of rollover crises, where multiplicity in the rollover crisis occurs in the terminal phase that ultimately triggers default. In our model, instead, along either equilibrium path, multiplicity eventually disappears.

Figure 10 illustrates this point. It overlays Figure 8, with four new dashed lines. Each dashed line corresponds to a different time horizon and initial debt condition. In particular, we plot them for $t = 1.2$, and $t = 2.9$ and use as initial conditions the values of $s(t)$ and $b(t)$ reached under the good and the bad equilibrium paths shown in Figure 9 (which coincide at $t = 0$). Notice that at $t = 1.2$ multiplicity is still present, so it is possible, for example, for the economy to follow the bad path between $t = 0$ and $t = 1.2$ and then to switch to a good path.[16] However, at $t = 2.9$ a switch is no longer possible. There are two reasons multiplicity disappears as we approach T. First, the remaining time hori-

[16]Clearly, the switch needs to be unexpected for prices to be in equilibrium between $t = 0$ and $t = 1.2$.

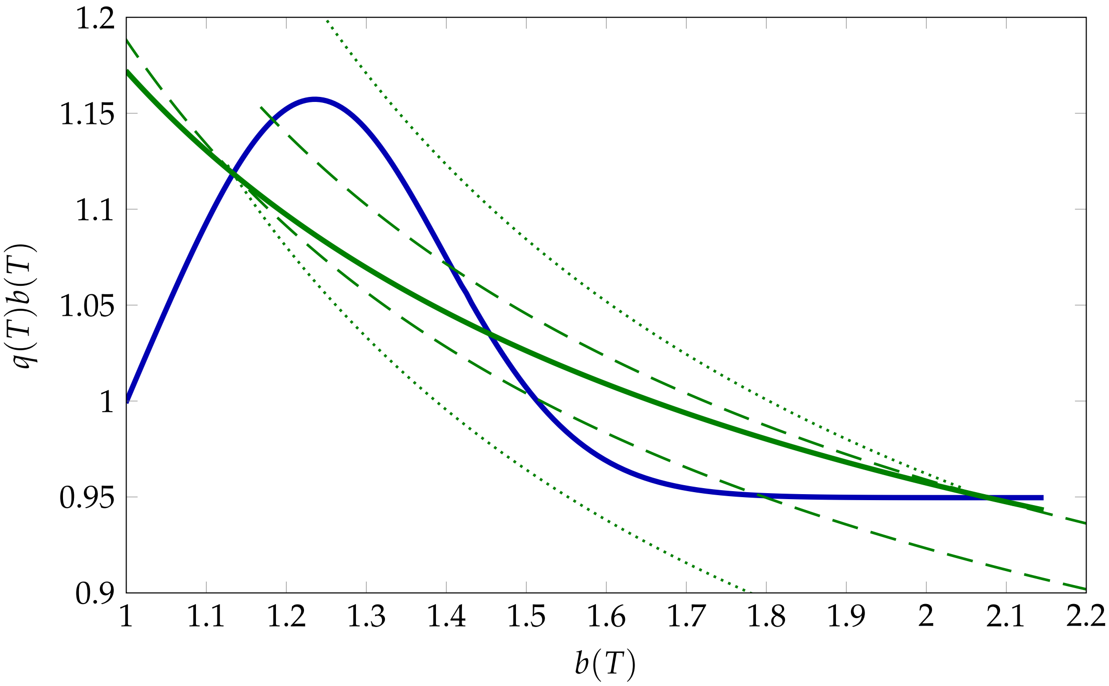


Figure 10: Solid green line shows $t = 0$, dashed green line $t = 1.2$, dotted green line $t = 2.9$.

zon shrinks, leaving less time to accumulate or decumulate debt. Second, debt may have reached a high enough level to ensure the bad equilibrium, or viceversa.

Fiscal Rules. How does the fiscal policy rule affect the equilibrium or the existence of multiple equilibria? In Figure 11, we look at the effects of increasing α. To better illustrate the power of a more responsive fiscal policy, we adjust the debt target b^* so that for each of the three values of α the country reaches a good equilibrium with the same $q(T)$ and $b(T)$. A sufficiently high value of α rules out the bad equilibrium, because as the investors contemplate the effect of lower bond prices they realize that the government would react more aggressively to a faster increase in b and thus eventually reach a lower level of $b(T)$.

A different way to look at policy rules is to ask how aggressive the rules need to be to make a given initial debt level immune to bad equilibria. In particular, in Figure 12 we look at the parameter space (α, b_0) and divide it into four regions, making no adjustments to b^*. In the red region there is a single equilibrium, in the bottom portion debt is low and on the good side of the Laffer curve, while in the upper portion (above pink region) the unique equilibrium lies on the bad side of the Laffer curve. There are three equilibria in the pink region, just as in our calibrated example. In the yellow region no equilibrium

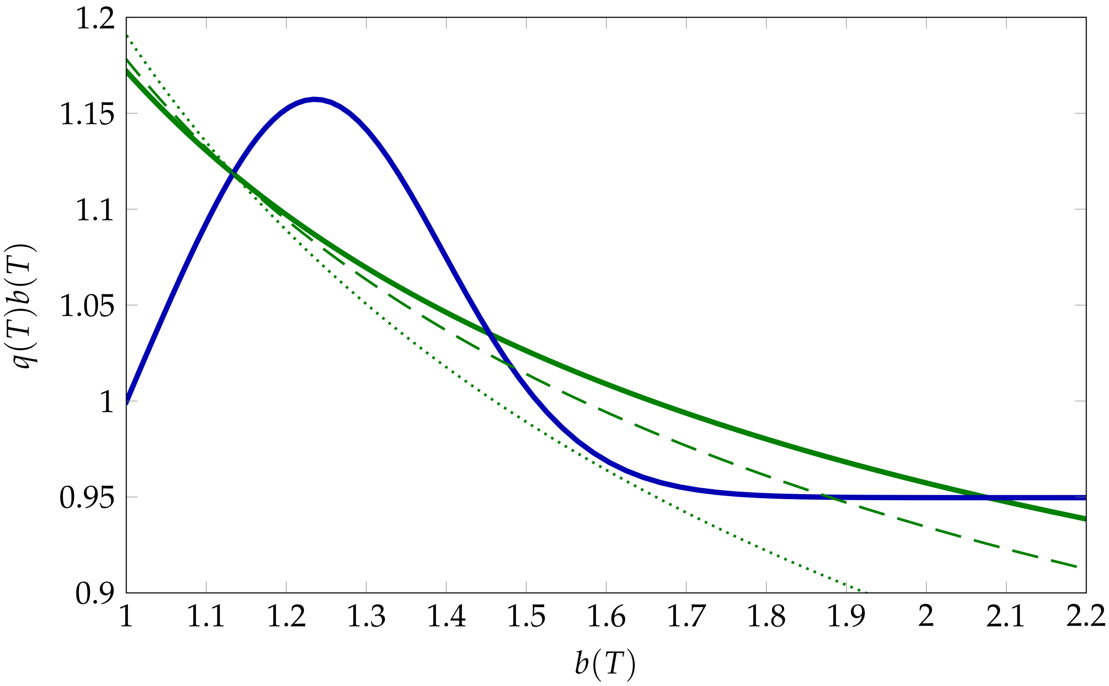


Figure 11: Solid green line $\alpha = 0.02$, dashed green line $\alpha = 0.03$, dotted green line $\alpha = 0.05$.

with debt exists, implying immediate default at $t = 0$.

Consider for example, the case $\alpha = 0.01$ in the graph, in which four cases are possible. For low levels of b_0, we get a unique equilibrium on the increasing portion of the Laffer curve (lower portion of the red region). For higher levels of b_0, we have three equilibria, as depicted in Figure 8 (pink region). For even higher levels of b_0, we have a unique equilibrium again, but this time on the decreasing portion of the Laffer curve. Finally, for very high values of b_0, there is no equilibrium without default.

Debt Maturity. Consider next the impact of debt maturity, captured by δ. Figure 13 shows the effects of varying δ around our benchmark value, while adjusting b^* to keep the same low-debt equilibrium. A longer maturity, with a low enough value for δ, leads to a unique equilibrium. Intuitively, shorter maturities require greater refinancing, increasing the exposure to self-fulfilling high interest rates. The debt burden of longer maturities, in contrast, is less sensitive to the interest rate.

Figure 14 is similar to Figure 12, but over the parameter space (δ, b_0) instead of (α, b_0). Again, we divide the figure into four regions. There are three equilibria in the pink region, just as in our calibrated example. In the red region there is a single equilibrium. In the

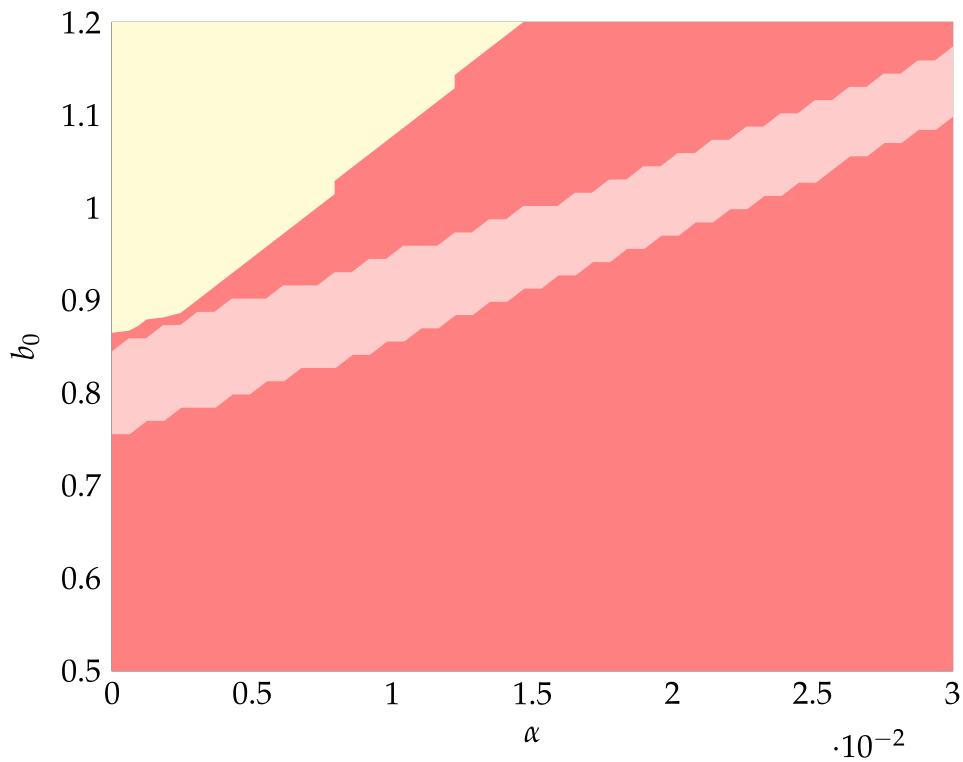


Figure 12: Regions with unique equilibrium (red), three equilibria (pink) or immediate default (yellow).

bottom portion of the red region the equilibrium lies on the good side of the Laffer curve, while in the upper portion (above the pink region) it is on the bad side of the Laffer curve. In the yellow region no equilibrium exists, implying immediate default at $t = 0$.

In the figure, for given δ, the good equilibrium is unique for low enough levels of debt b_0. For a given initial debt b_0, a longer maturity for debt, a lower value for δ, also leads to a unique good equilibrium (lower red region). Shorter maturities, higher values for δ, may place the economy in the intermediate "danger zone" (pink region) with 3 equilibrium values for the interest rate. Still higher values for δ may lead to a unique bad equilibrium (upper red region) or to non-existence prompting immediate default (upper right, yellow region).

We conclude that according to Figure 12, shorter maturities place the borrower in danger: in some cases vulnerable to a possible bad equilibrium, in others certain of a bad equilibrium and in still others in an immediate situation of default.

Slow Moving Crises and Rollover Crises. It is interesting to compare the slow moving crisis in our model to liquidity induced crises featured in Cole and Kehoe (2000) and

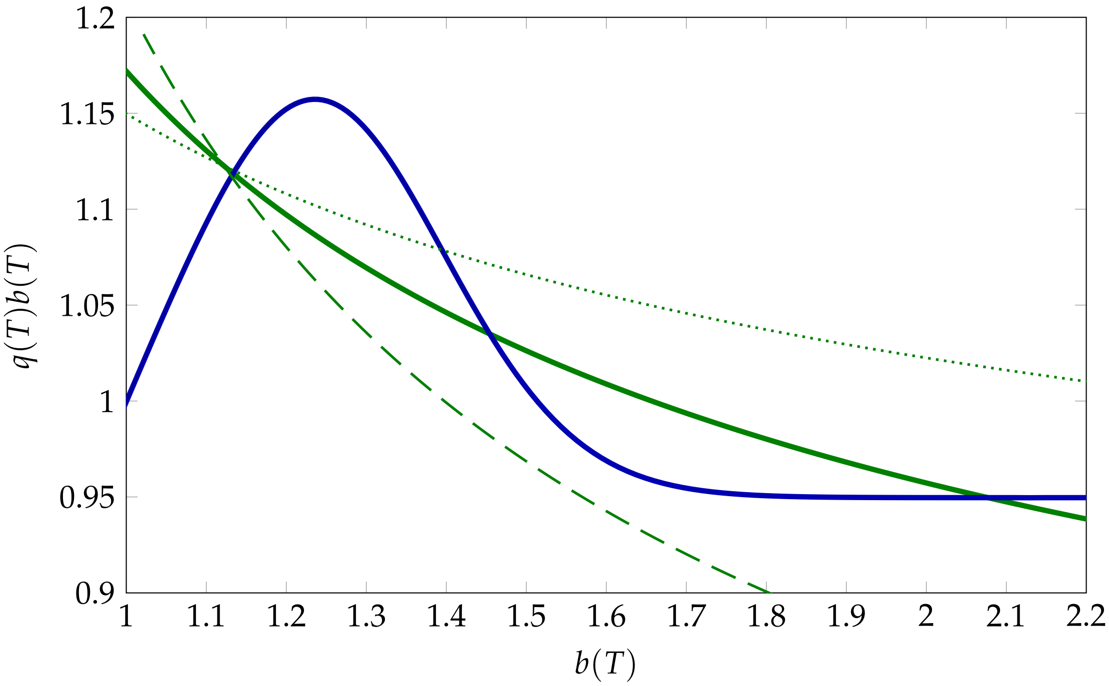


Figure 13: Solid green line $\delta = 1/7$, dashed green line $\delta = 1/10$, dotted green line $\delta = 1/5$.

related work, such as Cole and Kehoe (1996) and Conesa and Kehoe (2012). In these models, when debt is high enough borrowers become vulnerable to a run by investors, who may decide not to rollover debt, prompting default. If this run comes unexpectedly, there would be no rise in interest rates, just a sudden crisis, a zero price for bonds and default as in Giavazzi and Pagano (1989) and Alesina et al. (1992). Cole and Kehoe (2000) extended these models by studying sunspot equilibria with a constant arrival probability for the liquidity crisis. When this probability is not zero, the interest rate rises and the government makes an effort to reduce debt to a safe level that excludes investor runs and lowers the interest rate. Thus, high interest rates in liquidity crisis models may be present even with a decreasing path for debt.[17] In contrast, in our model debt rises along the bad, high interest equilibrium path. Indeed, the rising path for debt and higher interest rates are intimately related, the one implying the other.

[17] Conesa and Kehoe (2012) extend liquidity crisis models to include uncertainty in income and find that debt may be increasing in some cases. Nevertheless, high interest rates are driven by the sunspot probability of a run, not by the accumulation of debt.

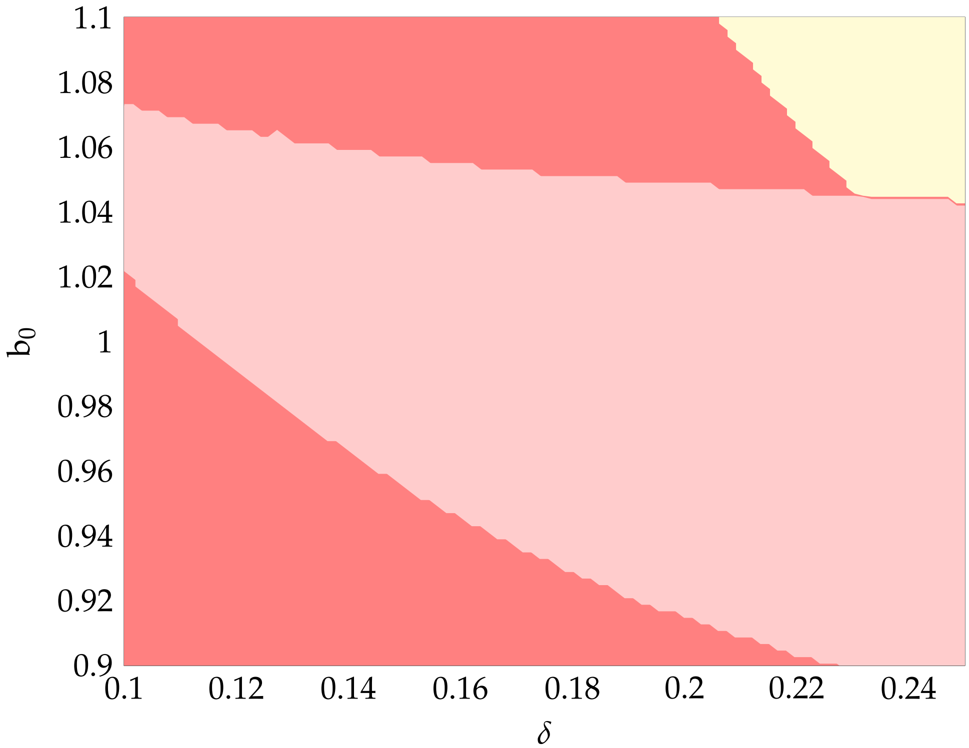


Figure 14: Regions with unique equilibrium (red), three equilibria (pink) or immediate default (yellow).

6 Commitment and Multiplicity

In the previous sections, we have assumed that whenever the government budget constraint can be satisfied at multiple bond prices, all these prices constitute potential equilibria. That is, we have assumed that the government cannot commit to the amount of bonds issued in a given period. In this section, we consider a model in which the government *can* commit to bond issuance in the very short run and yet multiple equilibria arise. The idea is to split a period of the models in the previous sections into shorter subperiods and to assume that the government can only commit to bond issuances in a subperiod. For a concrete example, a period in the model of the previous sections could be interpreted as a month, in which the government borrowing needs are determined by fiscal policy decisions that adjust slowly, while the subperiods may be different days in which auctions of Treasury bonds can take place. The government can commit to sell a fixed amount of bonds in each auction, but cannot commit to run future auctions if it hasn't reached its objective in terms of resources raised.

Given the purposes of this section, we will work with fully specified games in which the government's behavior is derived from maximization.

6.1 A Game with No Commitment

Consider a two-period model in which the government's objective function is

$$u(s) + \beta V(b),$$

where s is current primary surplus and b is the stock of bonds issued in the first period, to be repaid in the second period. Both u and V are decreasing, differentiable and concave functions. We could interpret u as the payoff resulting from a full specification of the benefits of public expenditure and the costs of taxation and V as as the expectation of a value function in an optimizing model with an infinite horizon.

The government also has a stock of bonds b_- inherited from the past that it needs to repay at the end of the first period. Thus, in the first period it must finance $b_- - s$ from outside investors.

There is a continuum of risk neutral atomistic investors with discount factor β. Because of risk neutrality and because all bond holders are treated equally, only the expected payment by the government in the second period need be specified to price bonds. If the total debt owed to investors is b, their total expected repayment is given by the function $G(b)$. Naturally, $G(0) = 0$. Each bond obtains an expected repayment equal to $G(b)/b$ and we assume that $G(b)/b$ is a non-increasing function.[18] The function G encapsulates all the relevant considerations regarding repayment, including the probability of default as well as the recovery value in the case of default and how these vary with the level of indebtedness. Note that this framework could capture strategic default or moral hazard by the government, as all these consideration can be embedded in V and G.

The first period is divided into infinite rounds $i = 1, 2, \dots$ and the government can run an auction in each round. We can think of auctions taking place in real time at $t = 0, 1/2, 2/3, 3/4, \dots$ At $t = 1$, the government collects the revenue from all these auctions, uses them to pay $b_- - s$ and the payoff $u(s)$ is realized. Finally, at $t = 2$ the payoff $V(b)$ is realized. Letting d_i denote bond issuances in round i, total bonds issued in period 1 are then

$$b = \sum_{i=0}^{\infty} d_i.$$

At each round i the investors bid price q_i for the issuance d_i.

[18] This is guaranteed, for example, if $G(b)$ is concave.

The crucial assumption we make is that in each auction the government cannot commit to the size of debt issuances in future auctions. We capture this by studying this setup as a game and looking at subgame perfect equilibria. Formally, strategies are described by functions $d_i = D_i(d^{i-1}, q^{i-1})$ and $q_i = Q_i(d^i, q^{i-1})$, where superscripts denote sequences up to round i. An equilibrium requires that:

i. In round i, after any history (d^{i-1}, q^{i-1}), the government strategy D_j for the remaining rounds $j = i, i+1, \ldots$ is optimal, given that future prices satisfy $q_j = Q(d^j, q^{j-1})$ at $j = i, i+1, \ldots$

ii. The price in round i after history (d^i, q^{i-1}) satisfies $Q(d^i, q^{i-1}) = G(\sum_{i=0}^{\infty} d_i) / \sum_{i=0}^{\infty} d_i$ where $\{d_i\}$ is computed using the government strategy D_j for $j = i, i+1, \ldots$ and future bond prices Q_j for $j = i+1, \ldots$

For an equilibrium to be well defined, the sequence $\{d_i\}$ must be summable both in equilibrium and after any possible deviation. Moreover, since investors are atomistic, the only restriction on prices is that they be consistent with expected repayment, which in turn is determined by total debt issued. Observe that along an equilibrium path the bond price is constant across rounds $q_i = q^*$. We can then denote by (s^*, b^*, q^*) an equilibrium outcome of the game in terms of primary surplys, total debt issued and bond price. The main result of this section is a tight characterization of all possible equilibrium outcomes.

Proposition 4. *A triplet (s^*, b^*, q^*) is the outcome of a subgame perfect, pure strategy equilibrium if and only if*

$$(s^*, b^*) \in \arg\max_{s,b} u(s) + \beta V(b) \qquad s.t. \qquad s + b_- = q^* b$$

and

$$q^* = \beta \frac{G(b^*)}{b^*}.$$

Proof. We start with the sufficiency part, by constructing an equilibrium which implements the desired outcome. The equilibrium pricing function sets the price $Q\left(d^i, q^{i-1}\right) = q^*$ for any history (d^i, q^{i-1}) with $q^{i-1} = \{q^*, ..., q^*\}$. The strategy of the government is to issue $b^* - \sum_{j=0}^{i} d_j$ after any history with $q^{i-1} = \{q^*, ..., q^*\}$.[19] The resulting equilibrium play is that the government issues b^* in the first auction and no further auction takes place. Since at each round the price is independent of the amount of bonds issued, the

[19] It is not difficult to complete the description of the equilibrium constructing continuation strategies after histories with $q_i \neq q^*$. However, given the atomistic nature of investors, these off-equilibrium paths are irrelevant for the borrower's maximization problem.

government cannot gain by changing its bonds issuances. Investors, on the other hand, expect that if the government deviates and offers anything other than $b^* - \sum_{j=0}^{i} d_j$ in the current round, it will adjust its issuances in the next round so as to reach the debt level b^*. This justifies their bid being independent of the amount of bonds issued in the current round.

Turning to the necessity part, suppose we have an equilibrium with outcome (s, b) and define $q = (b_- - c)/b$. We want to prove that $q = MRS \equiv V'(b)/u'(s)$. Suppose, towards a contradiction, that we have a proposed equilibrium where instead $q \neq MRS$. For concreteness suppose $q > MRS$. The other case is symmetric.

In equilibrium the borrower is supposed to exit with (b, c) at some round. Suppose that instead, upon reaching this round, the government considers a deviation, does not exit and instead issues a small extra $\varepsilon > 0$ amount of debt in the next round, for a current total of

$$\tilde{b} = b + \varepsilon.$$

The market must then respond with a price $\tilde{q}$ for this round. The current price yields a current revised consumption

$$\tilde{c} = c + \tilde{q}\varepsilon.$$

In the equilibrium of the ensuing sub-game, the price in all future rounds must be constant and given by $\tilde{q} = \tilde{q}^* = G(\tilde{b}^*)$ where $\tilde{b}^*$ is the end outcome for debt following this sub-game. The associated end outcome for consumption is then $\tilde{c}^* = c + \tilde{q}(\tilde{b}^* - b)$.

The following inequalities hold

$$u(c) + V(b) \geq u(\tilde{c}^*) + V(\tilde{b}^*) \geq u(\tilde{c}) + V(\tilde{b}). \tag{19}$$

The first inequality follows because (b, c) is an equilibrium outcome. The second because otherwise in the sub-game the borrower would prefer to stop after the initial deviation. We can now prove that the end outcome of the sub-game cannot have more total debt than the initial deviation, that is, $\tilde{b}^* \leq \tilde{b}$. Suppose, by contradiction, that $\tilde{b}^* > \tilde{b}$. Let

$$\lambda = \frac{\tilde{b} - b}{\tilde{b}^* - b} \in (0, 1),$$

and notice that

$$(\tilde{b}, \tilde{c}) = (1 - \lambda)(b, c) + \lambda(\tilde{b}^*, \tilde{c}^*).$$

Strict concavity and the first inequality in (19) then imply $u(\tilde{c}^*) + V(\tilde{b}^*) < u(\tilde{c}) + V(\tilde{b})$, which contradicts the second inequality in (19).

Since the function $G(b)/b$ is non-increasing in b, $\tilde{b}^* \leq \tilde{b}$ implies

$$\tilde{q} = \frac{G(\tilde{b}^*)}{\tilde{b}^*} \geq \frac{G(\tilde{b})}{\tilde{b}}.$$

By choosing an initial deviation with $\varepsilon > 0$ small enough, the borrower can ensure that the lower bound on $\tilde{q}$ is arbitrarily close to q, since $G(\tilde{b})/\tilde{b} \to q$ as $\tilde{b} \to b$. But then, since $q > MRS$, this implies that along this deviation the borrower can sell bonds at a price $\tilde{q} > MRS$, which implies $u(\tilde{c}) + V(\tilde{b}) > u(c) + V(b)$. Therefore, if the proposed equilibrium satisfies $q > MRS$, there is a profitable deviation by the borrower, a contradiction. □

The assumption that time is perfectly divisible, so that a further auction round is always available, delivers equilibria with outcomes that are equivalent to that of a price-taking government. The price-taking government in the first condition of Proposition 3 is the polar opposite of a government that can fully commit to b' and solve

$$\max_{s,b'} u(s) + \beta V(b') \qquad \text{s.t.} \qquad s + b = Q(b')b'$$

with $Q(b') = \beta G(b')/b'$. This is the assumption typically adopted in the literature. Instead, we assume that the government can commit to bond issuances in each round, but find that the outcome is *as if* it lacked any such commitment.

It is of interest to look at intermediate cases in which some degree of commitment is available. In the remainder of this section, we explore one such case using a simple example.

6.2 A Game with Partial Commitment

We now turn to a game with partial commitment. For simplicity, we focus on a simple three-period model. Our results show that the possibility to raise funds in future rounds of issuance can jeopardize the borrower's attempt to stay away from the wrong side of the Laffer curve.

6.2.1 The Game

There are three periods, $t = 0, 1, 2$. Debt is long-term and is a promise to pay 1 at date 2. In period 0, the government chooses how many bonds b_1 to sell. Next, an auction takes place and risk neutral investors bid q_0 for the bonds, the government receives $q_0 b_1$ from the investors and uses it to finance spending[20]

$$g_0 = q_0 b_1.$$

In period 1, the government chooses b_2, the investors bid q_1, the government then raises $q_1 (b_2 - b_1)$ and uses it to finance spending

$$g_1 = q_1 (b_2 - b_1).$$

Finally, in period 2 the surplus s is randomly drawn from an exponential distribution with CDF $F(s) = 1 - e^{-\lambda s}$ on $[0, \infty)$. The government repays if $s \geq b_2$, defaults otherwise and there is no recovery.

The government objective is to maximize

$$\alpha \min \{g_0, \overline{g}\} + \theta \min \{g_0 + g_1, \overline{g}\} + \int_{b_2}^{\infty} (s - b_2) \, dF(s),$$

that is, the government needs to finance a target level of spending $\overline{g}$ and has a preference for early spending. The parameter $\theta > 1$ captures the loss from not meeting the target $\overline{g}$, α captures the gain from early spending, g_0 and g_1 are restricted to be non-negative.[21] Investors are risk neutral and do not discount future payoffs.

6.2.2 Strategies and Equilibrium

The government's strategy is given by a b_1 and a function $B_2(b_1, q_0)$ that gives b_2 for each past history (b_1, q_0). The investors' strategy is given by two functions $Q_0(b_1)$ and $Q_1(b_1, q_0, b_2)$.

We analyze subgame perfect equilibria moving backward in time, starting from period 1. In period 1, investors are willing to pay

$$Q_1(b_1, q_0, b_2) = 1 - F(b_2),$$

given the stock b_2 of government bonds. In period 1, given the stock of bonds b_1 and the

[20] In following the timing of the game, one could find a bit confusing the fact that the government first chooses the issuance b_1 and then the investors choose the price q_0. But we stick to the subscripts to keep the notation consistent throughout the paper.

[21] Both assumptions seem reasonable. For example, investment spending on infrastructure requires some total outlay over an extended time horizon, but with a preference for early completion. As another example consider the payment of government wages. Suppose payment can be delayed, if needed, but at a cost, because workers are impatient and demand compensation.

price q_0, the government solves

$$\max_{g_1, b_2} \theta \min \{g_0 + g_1, \overline{g}\} + \int_{b_2}^{\infty} (s - b_2)\, dF(s) \tag{20}$$

subject to

$$g_1 = (1 - F(b_2))(b_2 - b_1)$$

and $g_0 = q_0 b_1$. The solution to this problem gives us the best response $B_2(b_1, q_0)$. Going back to period 0, investors' optimality requires

$$q_0 = 1 - F(B_2(b_1, q_0)). \tag{21}$$

We will construct equilibria in which a solution to (21) always exists. However, depending on the value of b_1, there may be multiple values of q_0 that solve (21). Let $\mathcal{Q}_0(b_1)$ be a map that selects a solution of (21) for each b_1 and let $\mathcal{B}_2(b_1) = B_2(b_1, \mathcal{Q}_0(b_1))$ denote the associated value of b_2.[22] To check that the choice of b_1 at date 0 is optimal, we need to check that it maximizes

$$\alpha \min \{[1 - F(\mathcal{B}_2(b_1))]\, b_1, \overline{g}\} + \theta \min \{[1 - F(\mathcal{B}_2(b_1))]\, \mathcal{B}_2(b_1), \overline{g}\} + \int_{\mathcal{B}_2(b_1)}^{\infty} (s - \mathcal{B}_2(b_1))\, dF(s).$$

6.2.3 Multiple equilibria

We now proceed to show that multiple equilibria are possible under some parametric assumptions. We begin by characterizing the government optimal behavior $B_2(b_1, q_0)$ at $t = 1$, for given values of b_1 and q_0.

Lemma 3. *Given q_0 and b_1, the optimal choice of b_2 must satisfy either*

$$q_0 b_1 + (1 - F(b_2))(b_2 - b_1) < \overline{g}$$

and

$$\theta (1 - \lambda (b_2 - b_1)) = 1$$

or

$$q_0 b_1 + (1 - F(b_2))(b_2 - b_1) = \overline{g}$$

and

$$\theta (1 - \lambda (b_2 - b_1)) \geq 1.$$

[22] We could easily extend the analysis to allow a stochastic selection of equilibria.

Proof. It is easy to show that in equilibrium we always have $q_0 b_1 \leq \overline{g}$. Therefore, the marginal benefit of increasing b_2 is

$$\begin{aligned}\theta\left(1-F\left(b_{2}\right)-f\left(b_{2}\right)\left(b_{2}-b_{1}\right)\right)-\left(1-F\left(b_{2}\right)\right) &= \\ \left(1-F\left(b_{2}\right)\right)\left[\theta\left(1-\lambda\left(b_{2}-b_{1}\right)\right)-1\right]\end{aligned}$$

if $g_0 + (1 - F(b_2))(b_2 - b_1) < \overline{g}$ and 0 otherwise. The statement follows immediately. $\square$

The Laffer curve for total debt issued in this game is given by

$$(1 - F(b))\, b = e^{-\lambda b} b.$$

We assume

$$\overline{g} < \max_b e^{-\lambda b} b = (\lambda e)^{-1} \tag{22}$$

so in equilibrium the government can reach the target $\overline{g}$. Under assumption (22) there are two solutions to

$$e^{-\lambda b} b = \overline{g},$$

which we label $\underline{b}$ and $\overline{b}$. The two solutions satisfy $\underline{b} < 1/\lambda < \overline{b}$. Assume also that

$$\theta\left(1 - \lambda \underline{b}\right) > 1, \tag{23}$$

which implies that the government has a sufficiently strong incentive to spend in periods 0 and 1. Define the cutoff

$$\hat{b}_1 = \overline{b} - \frac{1}{\lambda}\left(1 - \frac{1}{\theta}\right) \in \left(0, \overline{b}\right), \tag{24}$$

where the inequalities follows from $\overline{b} > 1/\lambda$ and $\theta > 1$ (from (23)).

We can now characterize the continuation equilibria that arise after the choice of b_1 by the government at date 0, that is, we look for candidates for the equilibrium selections $\mathcal{Q}_0(b_1)$ and $\mathcal{B}_2(b_1)$. We first consider the case in which b_1 is below the cutoff $\hat{b}_1$.

Lemma 4. *If $b_1 < \hat{b}_1$ there is a unique continuation equilibrium, with $b_2 = \underline{b}$.*

Proof. The equilibrium exists because $(1 - F(b_2))\, b_2 = \overline{g}$ at $b_2 = \underline{b}$ and assumption (23) implies $\theta(1 - \lambda(b_2 - b_1)) > 1$ for any $b_1 \geq 0$. To prove uniqueness notice that we cannot have $b_2 \in (\underline{b}, \overline{b})$ in equilibrium, otherwise $e^{-\lambda b_2} b_2 > \overline{g}$, we cannot have $b_2 \geq \overline{b}$, otherwise $\theta(1 - \lambda(b_2 - b_1)) < 1$, and we cannot have $b_2 < \underline{b}$, otherwise $e^{-\lambda b_2} b_2 < \overline{g}$ and $\theta(1 - \lambda(b_2 - b_1)) > 1$ (always using Lemma 3). $\square$

Lemma 5. *If $b_1 \geq \hat{b}_1$ there are two continuation equilibria, one with $b_2 = \underline{b}$ and one with $b_2 = \overline{b}$.*

Proof. The good equilibrium exists as in the previous claim. The second equilibrium exists because $b_1 \geq \hat{b}_1$ is equivalent to

$$\theta \left(1 - \lambda \left(\overline{b} - b_1\right)\right) \geq 1.$$

$\square$

The previous two lemmas imply that the following is a possible selection for continuation equilibria

$$\mathcal{B}_2\left(b_1\right) = \begin{cases} \underline{b} & \text{if } b_1 \leq \hat{b}_1 \\ \overline{b} & \text{if } b_1 > \hat{b}_1 \end{cases}. \tag{25}$$

Now we can go back to period 0 and study the government's optimization problem when the continuation equilibria are selected as in (25). The government chooses b_1 to maximize

$$\alpha e^{-\lambda \mathcal{B}_2(b_1)} b_1 + \theta \min \left\{ e^{-\lambda \mathcal{B}_2(b_1)} \mathcal{B}_2\left(b_1\right), \overline{g} \right\} + \frac{1}{\lambda} e^{-\lambda \mathcal{B}_2(b_1)}.$$

The government faces a trade-off here. If it chooses $b_1 \leq \hat{b}_1$ it ensures that in the continuation game investors will expect low issuance of bonds in period 1 and so only $\underline{b}$ bonds will be eventually issued, keeping the government on the good side of the Laffer curve. However, to keep b_1 low the government foregoes the benefits from early spending α. In particular, choosing $0 \leq b_1 \leq \hat{b}_1$ we have

$$\alpha e^{-\lambda \underline{b}} b_1 + \theta \overline{g} + \frac{1}{\lambda} e^{-\lambda \underline{b}}.$$

While choosing $\hat{b}_1 < b_1 \leq \overline{b}$ we have

$$\alpha e^{-\lambda \overline{b}} b_1 + \theta \overline{g} + \frac{1}{\lambda} e^{-\lambda \overline{b}}.$$

Clearly, the only possible optimal choices are $b_1 = \hat{b}_1$ and $b_1 = \overline{b}$. It is optimal to choose $b_1 = \overline{b}$ if

$$\alpha e^{-\lambda \overline{b}} \overline{b} + \frac{1}{\lambda} e^{-\lambda \overline{b}} > \alpha e^{-\lambda \underline{b}} \hat{b}_1 + \frac{1}{\lambda} e^{-\lambda \underline{b}}.$$

Using (24) to substitute for $\hat{b}_1$ in this inequality we obtain the following proposition. Define the cutoff

$$\hat{\alpha} \equiv \frac{1}{\lambda} \frac{e^{-\lambda \underline{b}} - e^{-\lambda \overline{b}}}{\overline{g} - e^{-\lambda \underline{b}} \left(\overline{b} - \frac{1}{\lambda}\left(1 - \frac{1}{\theta}\right)\right)}$$

if the expression at the denominator is positive and let $\hat{\alpha} = \infty$ otherwise.[23]

Proposition 5. *If $\alpha > \hat{\alpha}$ there is an equilibrium in which the stock of bonds is constant at $b_1 = b_2 = \bar{b}$, on the wrong side of the Laffer curve.*

The game also admits a good equilibrium in which $\mathcal{B}_2(b_1) = \underline{b}$ for all b_1. Notice that also in this good equilibrium all bonds are issued at date 0, and we have $b_1 = b_2 = \underline{b}$. Therefore, bond issuance in period 1 only matters for off-the-equilibrium-path dynamics. However, off-the-equilibrium-path dynamics are crucial to determine the amount of bonds the government issues in the first period.

The government can commit not to issue more bonds than b_2 in period 2, given that it is the final date before the resolution of uncertainty. So the government will never reach a b_2 such that a reduction in b_2 can increase current revenues, in other words, it will always be on the increasing side of the issuance Laffer curve:

$$1 - \lambda(b_2 - b_1) \geq 0. \tag{26}$$

However, this condition is not enough to rule out an equilibrium with total debt on the wrong side of the Laffer curve, because the slope of the stock Laffer curve is $1 - \lambda b_2$, which can be negative in spite of (26) if $b_2 - b_1$ is small. Moreover, the government at date 0 cannot try to move away from the bad equilibrium by reducing b_1 below $\bar{b}$, because, if it does, the market expects the government to issue $\bar{b} - b_1 > 0$ at date 1, and therefore the pricing function $\mathcal{Q}_0(b_1)$ is flat for b_1 near $\bar{b}$. The only option is to reduce b_1 all the way to $\hat{b}_1$, which is enough to eliminate the bad equilibrium. But this is too costly in terms of delayed spending.

7 Concluding Remarks

Based on our analysis it seems difficult to dismiss the concern that a country may find itself in a self-fulfilling "bad equilibrium" with high interest rates. In our model, bad equilibria are not driven by the fear of a sudden rollover crisis, as commonly modeled in the literature following Giavazzi and Pagano (1989), Alesina et al. (1992) and Cole and Kehoe (1996) and others. Thus, the problems these "bad equilibria" present are not resolved by attempts to rule out such investor runs. Instead, high interest rates can be self fulfilling because they imply a slow but perverse debt dynamic. Our results highlight the

[23] It is easy to find combinations of model parameters that ensure $\hat{\alpha} < \infty$.

importance of fiscal policy rules and debt maturity in determining whether the economy is safe from the threat of crises.

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