

An End-to-end Argument in Mechanism Design (Prior-independent Auctions for Budgeted Agents) *

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Abstract

This paper considers prior-independent mechanism design, namely identifying a single mechanism that has near optimal performance on every prior distribution. We show that mechanisms with truth-telling equilibria, a.k.a., revelation mechanisms, do not always give optimal prior-independent mechanisms and we define the revelation gap to quantify the non-optimality of revelation mechanisms. This study suggests that it is important to develop a theory for the design of non-revelation mechanisms.

Our analysis focuses on welfare maximization in single-item auctions for agents with budgets and a natural regularity assumption on their distribution of values. The all-pay auction (a non-revelation mechanism) is the Bayesian optimal mechanism; as it is prior-independent it is also the prior-independent optimal mechanism (a 1-approximation). We prove a lower bound on the prior-independent approximation of revelation mechanisms of 1.013 and that the clinching auction (a revelation mechanism) is a prior-independent $e \approx 2.714$ approximation. Thus the revelation gap for single-item welfare maximization with public budget agents is in $[1.013, e]$.

Some of our analyses extend to the revenue objective, position environments, and irregular distributions.

1 Introduction

The *end-to-end principle* in distributed systems advocates environment-independent protocols (for the center) that push environment-specific complexity to the applications (the end points) that use the protocol (Saltzer et al., 1984). This principle enabled the Internet protocols designed for the workloads of the 1980s to continue to succeed with workloads of the 2010s. On the other hand, research in mechanism design (which governs the design of protocols for strategic agents and has application both in computer science and economics) almost exclusively adheres to the *revelation*

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principle which suggests the design of mechanisms where each agent’s best strategy is to truthfully report her preferences. In revelation mechanisms the agents (the end points) have very a simple “report your true preference” strategies and the mechanism (the center) has the complex task of finding an outcome that both enforces this truthfulness property and also obtains a desirable outcome. Unsurprisingly, optimal revelation mechanisms tend to be complex and dependent on the environment. This paper demonstrates that the end-to-end argument has bite in mechanism design by showing that non-revelation mechanisms are strictly better than revelation mechanisms for a canonical mechanism design problem.

In prior-independent mechanism design, it is assumed that the agents’ preferences are drawn from a distribution that is not known to the designer. The goal of prior-independent mechanism design is to identify mechanisms that are good approximations to the optimal mechanism that is tailored to the distribution of preferences. Specifically, a mechanism is sought to minimize the ratio of its expected performance to the expected performance of the optimal mechanism in worst case over distributions from which the preferences of the agents are drawn. This notion is a standard one that has been applied to revenue maximization (Dhangwatnotai et al., 2010; Roughgarden et al., 2012; Fu et al., 2015; Allouah and Besbes, 2018), multi-dimensional mechanism design (Devanur et al., 2011; Roughgarden et al., 2015), makespan minimization (Chawla et al., 2013), mechanism design for risk-averse agents (Fu et al., 2013), and mechanism design for agents with interdependent values (Chawla et al., 2014a). In none of these scenarios is the optimal prior-independent mechanism known; cf. Fu et al. (2015) and Allouah and Besbes (2018).

The revelation principle suggests that if there is a mechanism with a good equilibrium outcome, there is a mechanism where truthtelling achieves the same outcome in a truthtelling equilibrium. Due to the revelation principle, much of the theory of mechanism design is developed for revelation mechanisms, i.e., ones where truthtelling is an equilibrium. The proof of the revelation principle is simple: A revelation mechanism can simulate the equilibrium strategies in the non-revelation mechanism to obtain the same outcome as a truthtelling equilibrium, i.e., agents report true preferences to the revelation mechanism, it simulates the agent strategies in the non-revelation mechanism, and it outputs the outcome of the simulation. For Bayesian non-revelation mechanisms (where the agents’ preferences are drawn from a prior distribution), the agents’ equilibrium strategies are a function of the prior and thus the corresponding revelation mechanism constructed via the revelation principle is not prior-independent. Thus, the restriction to revelation mechanisms is not generally without loss for prior-independent mechanism design. Non-revelation mechanisms, on the other hand, are widely used in practice and frequently have easily to identify and natural equilibria (e.g., in rank-based auctions, see Chawla and Hartline, 2013). Our proof of a non-trivial revelation gap – that the prior-independent approximation factor of the best non-revelation mechanism is better than that of the best revelation mechanism – gives concrete motivation for a theory of mechanism design without the revelation principle.

It is not hard to invent pathological scenarios where there is a non-trivial revelation gap. Instead, this paper considers the canonical environment of welfare maximization for agents with budgets and shows such a gap even for distributions on preferences that satisfy a standard regularity property. Moreover, the environment in which we exhibit the revelation gap suggests the end-to-end principle: the agents can easily implement the optimal outcome in the equilibrium of a simple mechanism, while revelation mechanisms that satisfy the constraints must be complex and either prior-dependent or non-optimal.

Main Results. Our analysis focuses on welfare maximization in a canonical single-item environment with ex ante symmetric budget constrained agents, i.e., each agent’s value is drawn independently and identically from an unknown distribution and the agent cannot make payments that exceed a known and identical budget (cf. Maskin, 2000). Our main treatment of this problem will make a simplifying assumption that the distribution follows a regularity property that implies that the Bayesian optimal mechanism has a nice form (Pai and Vohra, 2014). Our results require a symmetric environment, i.e., an i.i.d. distribution and identical budget. A number of our results extend to the objective of revenue (Laffont and Robert, 1996), to position environments as popularized as a model for ad auctions (Devanur et al., 2013), and beyond regular distributions (Devanur et al., 2013). For clarity the main results are described first for welfare maximization, single-item environments, and regularly distributed agent values.

The main challenge in demonstrating a revelation gap is that it is difficult to identify prior-independent optimal mechanisms, cf. Fu et al. (2015). Though the question has been considered, the prior literature has no examples of optimal prior-independent mechanisms for non-trivial environments.¹ Our non-trivial revelation-gap theorem follows from three results. First, the all-pay auction (from the literature, defined below) has a unique equilibrium that is Bayesian optimal and it is prior-independent. Second, we obtain a lower bound on the ability of a prior-independent revelation mechanism to approximate the Bayesian optimal mechanism by identifying the dominant strategy incentive compatible mechanism that is Bayesian optimal for the uniform distribution. The performance of this mechanism is strictly worse than that of the Bayesian optimal mechanism (which is Bayesian incentive compatible); specifically the gap is 1.013. Third, we show that the dominant strategy incentive compatible clinching auction (from the literature, defined below) is an $e \approx 2.72$ approximation to the Bayesian optimal mechanism. Combining the upper and lower bounds we see a revelation gap between 1.013 and e .² The first result follows naturally from the literature; the second and third results are the main technical contributions of the paper.

Three auctions are at the forefront of our study. The *all-pay auction* solicits bids, assigns the item to the highest agent, and charges all agents their bids. The *clinching auction* (Ausubel, 2004; Dobzinski et al., 2008; Goel et al., 2015) is an ascending price auction that can be thought of as allocating a unit measure of lottery tickets: a price is offered in each stage, each agent specifies the measure of tickets desired at the given price, each agent is allocated a number of tickets that is equal to the minimum of her demand and the measure of remaining tickets if this agent is only allowed to buy tickets after all other agents have bought as much as they desire first.³ The *middle-ironed clinching auction* – which we identify as the optimal dominant strategy incentive compatible mechanism – behaves like the clinching auction except that values that fall within a middle range are ironed. The allocation that an agent in this middle range receives is the average over the original allocation of for middle range values in the clinching auction. This averaging results in the the budget binding later and more efficient outcomes than in the original clinching auction.

The second step, mentioned above, is to obtain a lower bound on the prior-independent ap-

¹Contemporaneously with our results, Allouah and Besbes (2018) showed that the second-price auction is the prior-independent optimal revelation mechanism for revenue when the agents’ values are distributed according to a monotone hazard rate distribution.

²To better appreciate the magnitude of this lower bound, notice that it is demonstrated for two agents with uniformly distributed values where the optimal expected welfare (even without budgets) is $2/3$ and the lottery mechanism (which gives the item to a random agent) has expected welfare $1/2$ and is a $4/3 \approx 1.33$ approximation.

³For example, at a price of 0 all agents would want to buy all the tickets, but the agent that arrives last gets no tickets, thus no agents get any tickets at this price; the price increases.

proximation of a revelation mechanism. Our analysis begins with the observation that a prior-independent revelation mechanism must be Bayesian incentive compatible for every distribution. For two agents, this condition is equivalent to being dominant strategy incentive compatible. We ask whether there is a gap between the Bayesian optimal dominant strategy and Bayesian incentive compatible mechanism. The comparison between optimal dominant strategy and Bayesian incentive compatible mechanism is standard for multi-dimensional mechanism design problems, e.g., see Gershkov et al. (2013) and Yao (2017); we are unaware of previous studies of this phenomenon for single-dimensional agents with non-linear preferences. We answer this question positively by writing the dominant strategy mechanism design problem as a linear program and solving it by identifying a dual solution that proves the optimality of the middle-ironed clinching auction, cf. Pai and Vohra (2014) and Devanur and Weinberg (2017). The identified gap gives a lower bound on the approximation factor of the optimal prior-independent mechanism.

The third step, mentioned above, proves that the prior-independent approximation factor of the clinching auction is at most e and resolves in the affirmative an open question from Devanur et al. (2013). Our proof follows from a novel adaptation of a standard method for approximation results in mechanism design where an auction’s performance is compared to the upper bound given by the ex ante relaxation, in this case, the welfare of the optimal mechanism that sells one item in expectation over the random draws of the agents’ values (i.e., ex ante) rather than for all draws of the agents’ values (i.e., ex post). This method was introduced by Chawla et al. (2007), formalized by Alaei (2011, 2014), generalized by Alaei et al. (2013), and employed in many subsequent analyses.

Extensions. A number of our results extend beyond regular distributions, single-item environments, and the welfare objective as described above. These extensions all require that the environment be symmetric, specifically, that the agents’ values are independent and identically distributed and their budgets are identical.

For irregular distributions the welfare-optimal auction is not generally the all-pay auction; moreover, it does not generally have a prior-independent implementation. We prove that the all-pay auction is a prior-independent two approximation. Both the regular and irregular prior-independent optimality and approximation results for the one-item all-pay auction extend to the all-pay position auction.

The degradation of the approximation factor by a factor of two for irregular distributions extends to the single-item clinching auction which is an e approximation for regular distributions (as described above) and a $2e$ approximation for irregular distributions.

For the revenue objective (Laffont and Robert, 1996), with appropriate definition of regularity, the n -agent single-item all-pay auction is a prior-independent $n/(n - 1)$ approximation to the revenue optimal auction (cf. Bulow and Klemperer, 1996).

Important Directions. The most general direction suggested by this work is for a systematic development of non-revelation mechanism design. Unfortunately, it is not generally helpful to do revelation mechanism design and then try to go from the suggested revelation mechanism to a practical and simple non-revelation mechanism. There is a nascent literature on this topic. Papers working to develop a theory of non-revelation mechanism design include Chawla and Hartline (2013), which proves the uniqueness and optimality of equilibria in symmetric rank-based auctions; Chawla et al. (2014b, 2016), which gives data driven methods for optimizing non-revelation

mechanisms in symmetric environments; and Hartline and Taggart (2016), which gives a theory for non-revelation sample complexity and the design of approximately optimal non-revelation mechanisms in asymmetric environments.

While the literature has many interesting approximation bounds for prior-independent mechanism design. Rarely have the prior-independent optimal mechanism been identified. Moreover, the prior-independent approximation factors achievable tend to be surprising; for example, Fu et al. (2015) show that the second-price action is not the optimal prior-independent mechanisms for two-agent revenue maximization with agents with regularly distributed values.⁴ The literature lacks general techniques for answering this question.

We have observed that there is a very simple prior-independent optimal mechanism for welfare maximization in symmetric environments for agents with identical budgets. This mechanism, namely the all-pay auction, achieves its optimal outcome in Bayes-Nash equilibrium. The general question of identifying prior-independent non-revelation mechanisms that optimize a desired objective, like welfare or revenue, needs to be asked with care. Without restrictions to this question, it is asked and answered in the literature on non-parametric implementation theory, see the survey of Jackson (2001). This literature shows that arbitrarily close approximations, called “virtual implementations”, to the Bayesian optimal mechanism can be implemented by an uninformed principal. The mechanisms in this literature tend to be sequential – where agents interact in multiple rounds – and require agents to make reports about their own preferences and crossreports about their beliefs on other agents’ preferences. Our perspective on these results is that they take the model of Bayes-Nash equilibrium too literally and the resulting cross-reporting mechanisms are both fragile and impractical. One approach for ruling out these mechanisms is to restrict attention to mechanism formats that are commonly occurring in practice. Specifically, in the general *winner-pays-bid* format: agents bid, an allocation rule maps bids to a set of winners, and all winners pay their bids; in the general *all-pay* format: agents bid, an allocation rule maps bids to a set of winners, and all agents pay their bids. There may be other restricted formats that are also interesting for specific scenarios, e.g., the seller-offer mechanisms that are prevalent as real estate exchange mechanisms (Niazadeh et al., 2014).

Finally, there are still many gaps in our understanding of auctions for identically distributed agents with common budgets. For welfare, the bounds in this paper show that the clinching auction’s approximation factor for the welfare objective is in $[1.03, e]$ for regular distributions and $[2, 2e]$ for irregular distribution. Sharpening these bounds is an open question. Moreover, we conjecture that the clinching auction is also a prior-independent constant approximation for the revenue objective (with i.i.d. public-budget regular distributions). We also leave open a number of questions with regard to the Bayesian optimal dominant strategy incentive compatible mechanism for agents with budgets. We conjecture that the welfare optimality of the middle-ironed clinching auction extends from uniform distributions to regular distributions. We leave open the question of a similar result for the revenue objective, even for the special case of uniform distributions. There are specific issues that prevent straightforward generalization of our approach of using the dual to certify the optimality of the middle-ironed clinching auction for these questions.

Other Related Work. There is a significant area of research analyzing the performance of simple non-revelation mechanisms in equilibrium, a.k.a., the *price of anarchy*. Generally these

⁴Allouah and Besbes (2018) show that with more restrictive monotone hazard rate distributions, the second-price auction is an optimal prior-independent revelation mechanism.

mechanisms are prior-independent and the aim of the literature, e.g. Syrgkanis and Tardos (2013), is to demonstrate that they are approximately efficient. On the other hand, for welfare maximization in many of the studied environments, there is a DSIC revelation mechanism that is (exactly) efficient and, thus, there is no revelation gap. Though this literature focuses on the analysis rather than the design of mechanisms, two conclusions for mechanism design are: (a) that a simple *revenue covering property* is sufficient (Hartline et al., 2014), necessary (Dütting and Kesselheim, 2015), and potentially optimizable; and (b) that this property (and also a more general *smoothness property*) is closed under composition, i.e., when multiple independent mechanisms are run simultaneously (Syrgkanis and Tardos, 2013). For a surveys of these and other results see Roughgarden et al. (2017) and Chapter 6 of Hartline (2016).

For agents with budgets, approximation mechanisms have been studied from both a design and analysis perspective for the *liquid welfare* benchmark proposed by Chawla et al. (2011), Syrgkanis and Tardos (2013), and Dobzinski and Paes Leme (2014). The liquid welfare benchmark is the optimal surplus of a feasible allocation when each agent’s contribution to the surplus is the maximum of her value for her allocation and her budget. These and subsequent papers show that simple mechanisms have welfare that approximate the liquid welfare benchmark. Unfortunately, when evaluated under the formal study of benchmarks for mechanism design developed by Hartline and Roughgarden (2008) and summarized in Chapter 7 of Hartline (2016), the liquid welfare does not satisfy a key property. Specifically, there are i.i.d. distributions where the expected welfare of the Bayesian optimal mechanism is arbitrarily larger than the expected optimal liquid welfare. This bound follows because liquid welfare is at most the sum of the agent budgets which can be arbitrarily close to zero and, in such cases, is unrelated to the welfare possible by a mechanism.⁵ Thus, testing mechanisms for their worst-case approximation factor with respect to liquid welfare does not necessarily separate good mechanisms from bad mechanisms.

Organization. In Section 2 we give the preliminaries of our setup. In Section 3 we analyze the prior-independent approximation factor of the clinching auction for public-budget regular agents. In Section 4 we derive the Bayesian optimal DSIC mechanism for two agents with value uniformly distributed and show it is the middle-ironed clinching auction. In Section 5, we prove that the revelation gap is a constant. In Section 6, we analyze the prior-independent approximation factor of the all-pay auction and the clinching auction for irregular agents. In Section 7, we analyze the approximation ratio of winner-pays-bid mechanisms. In Section 8, we analyze the prior-independent revenue approximation of the all-pay auction for public-budget regular agents. In Appendix A, we give an ascending implementation of the clinching auction with price jumps (a generalization of the middle-ironed clinching auction). In Appendix B, we give a geometric framework for deriving Bayesian optimal mechanisms for agents with budgets and apply this framework to solve for the Bayesian optimal winner-pays-bid mechanism.

2 Preliminaries

Model for auctions with budgets We consider mechanisms for agents with independent and identically distributed values and identical public budgets. The budget is denoted by B . For

⁵For example, in a single-item environment with budgets identically equal to zero and agent values identically equal to one (trivially an i.i.d. distribution); the lottery, which allocates the item to a random agent for free, has welfare one while the liquid welfare is zero.

allocation $x \in [0, 1]$ and payment $p \in \mathbb{R}$, an agent with value $v \in \mathbb{R}$ has utility $vx - p$ if p is at most the budget B and utility $-\infty$ otherwise. In other words, the agent cannot under any circumstances pay more than her budget. The agents' values are drawn independently and identically from distribution F with support $[0, h]$.

Denote the strategy function of an agent by $s(\cdot)$ where $s(v)$ is the bid made by the agent when her value is v . A bid profile is $\mathbf{b} = (b_1, \dots, b_n)$. A mechanism is given by mapping from bids to allocations and payments which we will denote by $\tilde{\mathbf{x}}(\mathbf{b}) = (\tilde{x}_1(\mathbf{b}), \dots, \tilde{x}_n(\mathbf{b}))$ and $\tilde{\mathbf{p}}(\mathbf{b}) = (\tilde{p}_1(\mathbf{b}), \dots, \tilde{p}_n(\mathbf{b}))$. The outcome of the mechanism $(\tilde{\mathbf{x}}, \tilde{\mathbf{p}})$ and strategy profile $\mathbf{s} = (s_1, \dots, s_n)$ on a profile of agent values \mathbf{v} is denoted by allocation rule $\mathbf{x}(\mathbf{v}) = \tilde{\mathbf{x}}(\mathbf{s}(\mathbf{v}))$ and payment rule $\mathbf{p}(\mathbf{v}) = \tilde{\mathbf{p}}(\mathbf{s}(\mathbf{v}))$.

The auction designer typically faces a feasibility constraint that restricts the allocations that can be produced. For example, a single-item auction requires that the number of agents allocated is at most one, i.e., $\sum_i x_i(\mathbf{v}) \leq 1$. A position environment generalizes a single item auction and is given by a sequence of decreasing probabilities $\mathbf{w} = (w_1, \dots, w_n)$ where without loss of generality the number of positions is equal to the number of agents. The probability that an agent is allocated if assigned to position j is w_j . A mechanism then can assign agents to positions (deterministically or randomly) and this process and the position probabilities induce allocation probabilities $x_i(\mathbf{v})$.

Basic auction theory A Bayes-Nash equilibrium (BNE) in the mechanism $(\tilde{\mathbf{x}}, \tilde{\mathbf{p}})$ is a profile of agent strategies \mathbf{s} where each s_i maps a value to a bid that is a best response to the other strategies and the common knowledge that values are drawn i.i.d. from distribution F . The mechanism $(\tilde{\mathbf{x}}, \tilde{\mathbf{p}})$ and strategy profile \mathbf{s} induce for each agent an interim allocation rule $x_i(v) = \mathbf{E}_{\mathbf{v}_{-i}}[x_i(v, \mathbf{v}_{-i})]$. We will consider only symmetric distributions and symmetric auctions. In such auctions, Chawla and Hartline (2013) show that all equilibria are symmetric, thus it is without loss to drop the subscript and refer to the interim allocation rule and payment rule as (x, p) . The Myerson (1981) characterization of BNE requires that (a) the interim allocation $x(v)$ is monotone non-decreasing and (b) the interim payment $p(v) = v \cdot x(v) - \int_0^v x(t) dt$. Condition (b) is known as the payment identity. A mechanism is Bayesian incentive compatible (BIC) if it induces a BNE where each agents strategy is reporting her value truthfully. A mechanism is interim individual rational (IIR) if the interim allocation is non-negative for all value.

A dominant strategy equilibrium (DSE) in the mechanism $(\tilde{\mathbf{x}}, \tilde{\mathbf{p}})$ is a profile of agent strategies \mathbf{s} where each s_i maps a value to a bid that is a best response regardless of what other agents are doing. The characterization of DSE follows from the BNE characterization: (a) the allocation $x(v_i, \mathbf{v}_{-i})$ is monotone non-decreasing in v_i and (b) the payment $p(v_i, \mathbf{v}_{-i}) = v_i \cdot x(v_i, \mathbf{v}_{-i}) - \int_0^{v_i} x(t, \mathbf{v}_{-i}) dt$. A mechanism is dominant strategy incentive compatible (DSIC) if it induces a DSE where each agents strategy is reporting her value truthfully. A mechanism is ex-post individual rational (ex-post IR) if the allocation is non-negative for all value profile.

Optimal auctions with budgets Laffont and Robert (1996) and Maskin (2000) for the revenue and welfare objectives, respectively, characterize the Bayesian optimal mechanisms for agents with public budgets. With the following regularity assumptions on the distribution, defined distinctly for revenue and welfare, the optimal mechanism has a nice form.

Definition 2.1. *A single-dimensional public budget agent is public-budget regular for welfare if her cumulative distribution function $F(\cdot)$ is concave; she is public-budget regular for revenue if additionally $v - \frac{1-F(v)}{f(v)}$ is non-decreasing.*

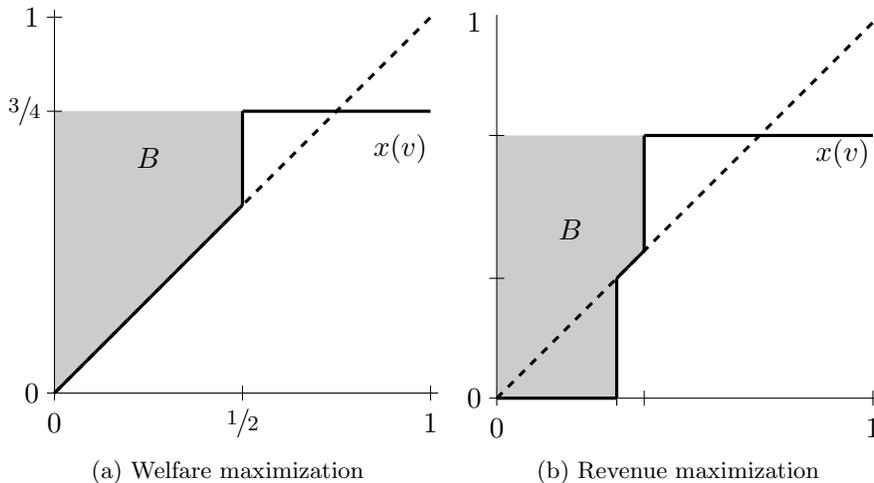


Figure 1: Depicted are the interim allocation rules of the welfare-optimal and revenue-optimal mechanisms for two agents with uniform values on $[0, 1]$. In each figure the highest-bid-wins allocation rule is depicted with a dashed line.

The results of Laffont and Robert and Maskin can be reinterpreted, à la Alaei et al. (2013), as solving a single-agent interim optimization problem that is given by an interim constraint $x^*(\cdot)$. An interim allocation is interim feasible under the interim constraint $x^*(\cdot)$ if for all values $v \in [0, h]$, the probability of allocating item to an agent with value greater than v with allocation rule $x(\cdot)$ is at most that with allocation rule x^* , i.e. $\int_v^h x(t) dF(t) \leq \int_v^h x^*(t) dF(t)$. In many cases solution to these interim optimization problems will take the form of the original constraint with ironed interval and reserve. Ironing on arbitrary interval $[v^\dagger, v^\ddagger]$ corresponds to the distribution weighted averaging as follow, $x(v) = \int_{v^\dagger}^{v^\ddagger} x^*(t) dF(t)$ for all $v \in [v^\dagger, v^\ddagger]$. Reserve at value v^\dagger corresponds to rejecting all value below v^\dagger as follows, $x(v) = 0$ for all $v \in [0, v^\dagger]$. An important allocation constraint is that given by the *highest-bid-wins* allocation rule. The highest-bid-wins allocation rule for n agents and with values from cumulative distribution function F is $x^*(v) = F^{n-1}(v)$, e.g., for two agents with uniform values it is $x^*(v) = v$.

Theorem 2.1 (Laffont and Robert, 1996; Maskin, 2000; Alaei et al., 2013). *For public-budget regular i.i.d. agents and interim allocation constraint $x^*(\cdot)$, the welfare-optimal single-agent mechanism allocates as by $x^*(\cdot)$ except that values in $[v^\dagger, h]$ are ironed for some v^\dagger and the revenue-optimal single-agent mechanism additionally reserve prices values in $[0, v^\dagger]$ for some v^\dagger ; payments are given deterministically by the payment identity.*

For single-item environments, one possible implementation of Theorem 2.1 is the all-pay auction. The all-pay auction has a unique Bayes-Nash equilibrium which is identical to outcome described in Theorem 2.1 for the allocation constraint given by the highest-bid-wins allocation rule.

Definition 2.2 (all-pay auction). *The all-pay auction is a mechanism $(\tilde{\mathbf{x}}, \tilde{\mathbf{p}})$ where $\tilde{\mathbf{x}}(\cdot)$ allocates item to the agent with highest bid with tie broken at random and $\tilde{\mathbf{p}}(\cdot)$ charges each agent their bid, i.e. $\tilde{p}_i(\mathbf{b}) = b_i$.*

Theorem 2.2 (Maskin, 2000). *For public-budget regular i.i.d. agents, the all-pay auction is welfare optimal.*

3 Welfare Approximation of the Clinching Auction

In this section, we study a prior-independent revelation mechanism called the clinching auction in single-item environments. Dobzinski et al. (2008) gave the following formulation of the clinching auction and characterized properties of its outcome. See Figure 3b.

Definition 3.1 (clinching auction). *The clinching auction maintains an allocation and price-clock starting from zero. The price-clock ascends continuously and the allocation and budget are adjusted as follows.*

1. *Agents whose values are less than price-clock are removed and their allocation is frozen.*
2. *The demand of any remaining agent is the remaining budget divided by the price-clock.*
3. *Each remaining agent clinches (and adds to their current allocation) an amount that corresponds to the largest fraction of their demand that can be satisfied when all other remaining agents are first given as much of their demand as possible.*
4. *The budget and allocation are updated to reflect the amount clinched in the previous step.*

Proposition 3.1 (Dobzinski et al., 2008). *For public-budget agents, the clinching auction always allocates all items, is ex-post IR, and is DSIC.*

Lemma 3.2 (a special case of Dobzinski et al., 2008). *In single-item environment, for public-budget agents with budget B and value profile \mathbf{v} , and let k be the largest integer such that*

$$v_{(k)} \geq B \cdot k \quad \text{where } v_{(k)} \text{ is the } k\text{-th highest value in value profile } \mathbf{v}.$$

Then, the agents with highest $(k - 1)$ values win with same probability greater or equal to $\frac{1}{k}$ and the agent with the k -th highest value wins with the remaining probability.

We use the following approach to show that the clinching auction is an ϵ -approximation for public-budget regular agents. We relax the feasibility constraint to an ex ante constraint and show that the optimal mechanism that sells to each agent with ex ante probability $1/n$ simply posts a price (of exactly B assuming that the budget binds) for a chance to win the item (Lemma 3.3, below). This simple form of mechanism is closely approximated by the clinching auction which sells k lotteries of $1/k$ probability (full details given subsequently). A key property is that in this clinching auction with lotteries, the budget does not bind with constant probability. The probability that the budget does not bind in the clinching auction with lotteries allows a lower bound on the allocation probability in the clinching auction which allows its welfare to be compared to the ex ante relaxation.

Consider the welfare-optimal auction. Since agents are symmetric, each agent will win with ex ante probability exactly $\frac{1}{n}$ in the welfare-optimal auction where n is the number of agents. We replace the feasibility constraint that ex post allocation cannot allocate more than one item (i.e. $\sum_{i \in [n]} \tilde{x}_i(\mathbf{v}) \leq 1$ for all \mathbf{v}) with a $\frac{1}{n}$ ex ante constraint that each agent cannot be allocated more than $\frac{1}{n}$ in expectation (i.e. $\mathbf{E}_v[x(v)] \leq 1/n$). Ex ante optimal mechanisms for agents with public budgets were proposed and studied by Alaei et al. (2013).

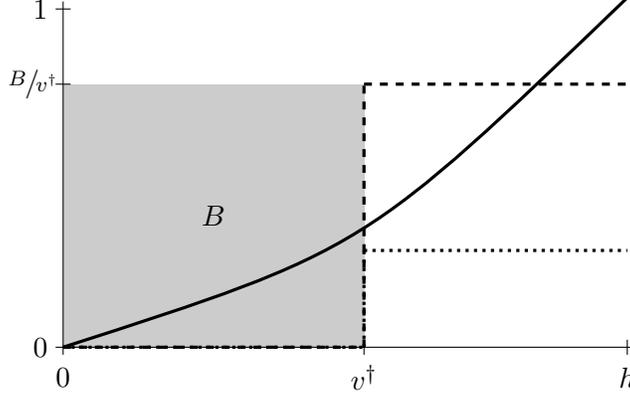


Figure 2: The allocation rules of the ex ante relaxation (dashed), an $1/e$ -fraction of the ex ante relaxation (dotted), and the clinching auction with lotteries (solid) are depicted. The clinching auction with lotteries pointwise exceeds an $1/e$ -fraction of the ex ante relaxation.

Lemma 3.3 (Alaei et al., 2013). *For public-budget regular i.i.d. agents with budget B , the ex ante welfare-optimal mechanism is either:*

- i. Budget binds: Post the price B for allocation probability $\frac{B}{v^\dagger} \leq 1$ with v^\dagger set to satisfy $\frac{1}{n} = \frac{B}{v^\dagger}(1 - F(v^\dagger))$. Values $v \geq v^\dagger$ select the lottery.*
- ii. Allocation probability binds: Post price $v^\dagger = F^{-1}(1 - \frac{1}{n})$ for allocation probability one.*

We build the connection between the clinching auction and the ex ante optimal mechanism by considering the an additional auction: the clinching auction with lotteries Clinch_k which allocates k lotteries with winning probability $1/k$ per lottery, using the clinching auction framework under the same public budget. Lemma 3.4 below shows that by selecting an appropriate k , the probability that an agent with value v^\dagger wins in the clinching auction with lotteries Clinch_k is at least an e fraction of the probability that the agent (with value v^\dagger) wins in the ex ante relaxation. See Figure 2.

Lemma 3.4. *For public budget i.i.d. agents, at value v^\dagger defined in Lemma 3.3, there exists $k \in [n]$, such that the interim allocation of the clinching auction with lotteries $x_{\text{Clinch}_k}(v^\dagger)$ is an e -approximation of the interim allocation of the ex ante optimal mechanism $x_{\text{PP}}(v^\dagger)$, i.e., $x_{\text{Clinch}_k}(v^\dagger) \geq 1/e \cdot x_{\text{PP}}(v^\dagger)$.*

Proof. Denote the notation $v_{(j:m)}$ as the j -th order statistic among m i.i.d. random variables from distribution F .

We denote the posted pricing in Lemma 3.3 as PP. Let $k_0 = 1/x_{\text{PP}}(h)$ where $x_{\text{PP}}(h)$ is the interim allocation at the highest value h . By the construction of PP, $F(v^\dagger) = 1 - \frac{k_0}{n}$ and $v^\dagger \leq B \cdot k_0$ (equality holds when the budget binds in PP). Let $k = \lceil k_0 \rceil$ be the smallest integer which is greater or equal to k_0 . Consider the clinching auction Clinch_k which allocates k lotteries with winning probability $1/k$ per lottery, using the clinching auction framework under public budget B .

First, fix an arbitrary agent and fix her value to be v^\dagger , we consider the following event \mathcal{E} : in Clinch_k , this agent with value v^\dagger is one of the highest k valued agents and the budget does not bind. Recall that when the budget does not bind, the highest k agents in Clinch_k each receive

lotteries (with allocation probability $1/k$) and pay the value of the $(k+1)$ -st highest agent divided by k (i.e. $v_{(k+1:n)}/k$). The budget bids in Clinch_k if and only if $v_{(k+1:n)}/k \leq B$ and we can lower bound the lower bound the probability of the event \mathcal{E} as follows,

$$\begin{aligned} \Pr[\mathcal{E}] &= \Pr\left[(v_{(k:n-1)}/k \leq B) \wedge (v_{(k:n-1)} \leq v^\dagger)\right] \\ &= \Pr\left[(v_{(k:n-1)}/k \leq v^\dagger/k_0) \wedge (v_{(k:n-1)} \leq v^\dagger)\right] \\ &= \Pr\left[v_{(k:n-1)} \leq v^\dagger\right] \\ &= \sum_{i=0}^{k-1} \binom{n-1}{i} \left(\frac{k_0}{n}\right)^i \left(\frac{n-k_0}{n}\right)^{n-1-i}. \end{aligned}$$

Above, the third line is derived from the second line using the definition of $k \geq k_0$. Denote by $x^\mathcal{E}$ and $x^\bar{\mathcal{E}}$ the allocation rule x conditioned on the events \mathcal{E} and $\bar{\mathcal{E}}$, respectively. The interim allocation for Clinch_k at value v^\dagger can be lower bounded as follows.

$$\begin{aligned} x_{\text{Clinch}_k}(v^\dagger) &= x_{\text{Clinch}_k}^\mathcal{E}(v^\dagger) \cdot \Pr[\mathcal{E}] + x_{\text{Clinch}_k}^{\bar{\mathcal{E}}}(v^\dagger) \cdot \Pr[\bar{\mathcal{E}}] \\ &\geq x_{\text{Clinch}_k}^\mathcal{E}(v^\dagger) \cdot \Pr[\mathcal{E}] \\ &= \frac{k_0}{k} \cdot x_{\text{PP}}(h) \cdot \Pr[\mathcal{E}] \\ &\geq \frac{1}{e} \cdot x_{\text{PP}}(h). \end{aligned}$$

The final inequality follows because the term $\frac{k_0}{k} \cdot \Pr[\mathcal{E}]$ achieves the minimum at $1/e$ when $k_0 = k = 1$ and n goes to infinity. \square

We now prove our main theorem about the approximation ratio for the clinching auction.

Theorem 3.5. *For public-budget regular i.i.d. agents, the clinching auction is an e -approximation to the welfare-optimal mechanism.*

Proof. By Lemma 3.3 the interim allocation rule of the ex ante optimal mechanism is a step function that steps at value v^\dagger . By Lemma 3.4, at value v^\dagger , the allocation rule of the clinching auction with lotteries is an e -approximation to that of the ex ante optimal mechanism. The allocation rule of the clinching auction with lotteries is monotone, so its allocation rule is an e -approximation to that of the ex ante optimal mechanism at every value. Consequently, the expected welfare of the clinching auction with lotteries is at least an e -approximation to that of the ex ante relaxation. See Figure 2.

Finally, Lemma 3.2 implies that for every ex post value profile, the welfare of the clinching auction is at least that of the clinching auction with lotteries. \square

For public-budget regular i.i.d. agents, the all-pay auction is optimal while the clinching auction is not, since the budget binds for more value profiles in the clinching auction than in the all-pay auction. Based on this, we give a 1.03 lower bound of the approximation ratio for the clinching auction and leave the actual approximation ratio as an open problem.

Lemma 3.6. *There exists the instance of public-budget regular agents where the clinching auction is a 1.03-approximation of the welfare-optimal mechanism.*

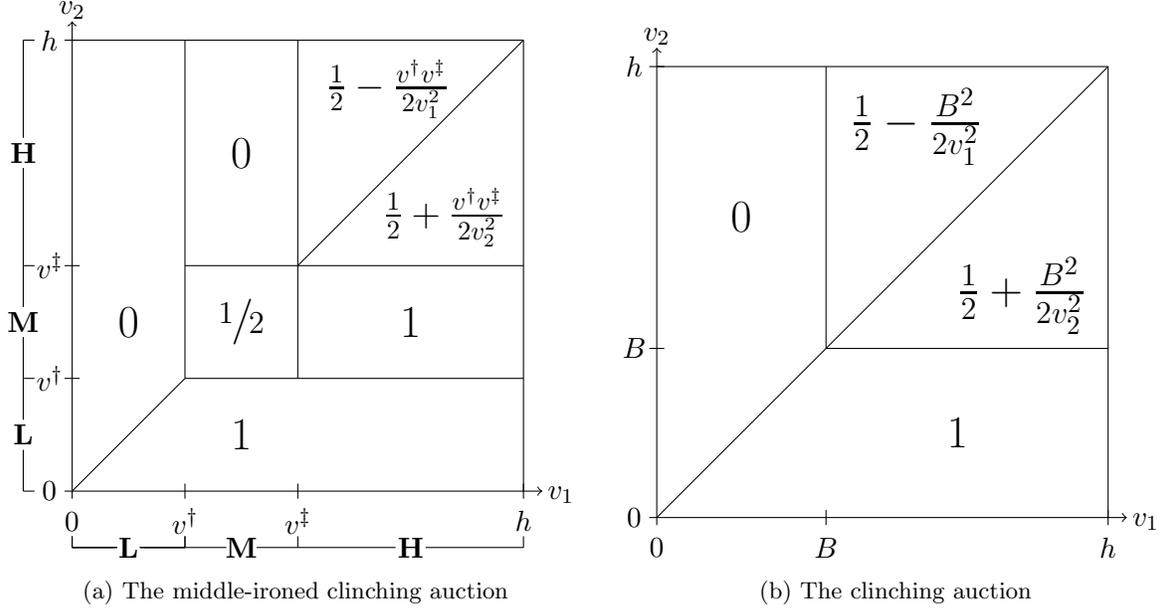


Figure 3: The comparison of the allocation rule $x_1(v_1, v_2)$ for the middle-ironed clinching auction and the clinching auction. In the middle-ironed clinching auction, for the values in interval \mathbf{M} can be thought as “ironed”, i.e. an agent receives the same outcome for any value $v \in \mathbf{M}$.

Proof. Consider a simple setting: there are 2 public-budget regular agents with value drawn uniformly from $[0, h]$ and the budget $B = 1$.

By Theorem 2.2, the all-pay auction is welfare-optimal for public-budget regular agents. The interim allocation rule of it is $x(v) = \frac{v}{h}$ if $v \leq 2$ and $x(v) = \frac{h+2}{2h}$ otherwise. The expected welfare of all-pay auction is $(3h^3 + 6h^2 - 12h + 8)/6h^2$.

The interim allocation rule of the clinching auction is $x(v) = \frac{v}{h}$ if $v \leq 1$ and $x(v) = \frac{h+2}{2h} - \frac{1}{2v^2}$ otherwise. The expected welfare of the clinching auction is $(3h^3 + 6h^2 - 3h - 6h \ln h - 2)/6h^2$.

By setting $h = 4.04$, it optimizes the ratio as 1.03. \square

4 Bayesian Optimal DSIC Mechanism

In Theorem 2.2, the all-pay auction is welfare-optimal under public-budget regular distribution. Hence, applying the revelation principle to the all-pay auction, it produces a Bayesian optimal revelation mechanism. This mechanism is BIC but not DSIC. In this section, we characterize the optimal DSIC mechanism for two agents with uniformly distributed values. We obtain a lower bound on its approximation ratio with the BIC optimal mechanism.

We first introduce the middle-ironed clinching auction (for two agents).

Definition 4.1. *The two-agent middle-ironed clinching auction is parameterized by $v^\dagger \leq B$ and $v^\ddagger = 2B - v^\dagger$ and its outcome is highest-bid-wins on values less than v^\dagger , a fair lottery on values in $[v^\dagger, v^\ddagger]$, and the clinching auction on values exceeding v^\ddagger ; a precise formulation for two-agents is given in Figure 3a and a general formulation is given in Appendix A.*

For two-agents case, the middle-ironed clinching auction allocates the item efficiently except

for value profiles in **MM** (both agents with values in **M**) or **HH** (both agents with values in **H**). For the value profile in **MM**, it randomly allocates the item to one of the agent with probability $\frac{1}{2}$ with payment $\frac{v^\dagger}{2}$. For the value profile in **HH**, it allocates the item such that the budget binds for the agent with higher value and the allocation rule depends on the lower value only. Figure 3b depicts the allocation rule of the clinching auction for comparison. The middle-ironed clinching auction can be implemented with an ascending price via a generalization of the clinching auction that allows for price jumps which we develop in Appendix A (this generalization is non-trivial).

We will show that by selecting the proper thresholds v^\dagger and v^\ddagger , the middle-ironed clinching auction is the Bayesian optimal DSIC mechanism for two agents with uniformly distributed values. An intuition behind the optimality of the middle-ironed clinching auction is as follows: Dobzinski et al. (2008) show that for two public budget agents, the clinching auction is the only Pareto optimal (i.e. there is no outcome which is weakly better for all agents and strictly better for one agent) and DSIC auction. Moreover, after the price increases past the point where the budget binds, a differential equation governs the allocation of any DSIC mechanism. Our goal is to optimize expected welfare rather than satisfy Pareto optimality. Sacrificing welfare for lower-valued agents by ironing can delay the budget from binding and enable greater welfare from higher-valued agents. From our proof of optimality, it is sufficient to only iron one region in the middle of value space.

Theorem 4.1. *For two public-budget agents with budget B and value uniformly drawn from $[0, h]$, Bayesian optimal DSIC mechanism is the middle-ironed clinching auction with some thresholds v^\dagger and v^\ddagger .*

The approach of the proof is to write down our problem as a linear program (primal), assume the middle-ironed clinching auction to be the solution, and then construct the dual program with a dual solution which witnesses the optimality of the primal solution by complementary slackness. This approach is reminiscent of that of Pai and Vohra (2014) and Devanur and Weinberg (2017); however, our multi-agent DSIC constrained program presents novel challenges and for this reason we only solve the problem of two agents and uniform distributions.

We first solve a discrete version of the problem. Then, we solve the continuous version as the limit from the discrete version. Consider the value distribution with finite value space $[h] = \{1, 2, \dots, h\}$ with equal probability each. We begin by writing down the optimization program for welfare maximization among all possible DSIC mechanism.

$$\begin{aligned} \sup_{(\mathbf{x}, \mathbf{p})} \quad & \sum_{v_1, v_2 \in [h]} (v_1 \cdot x_1(v_1, v_2) + v_2 \cdot x_2(v_1, v_2)) \cdot \frac{1}{h} \cdot \frac{1}{h} \\ \text{s.t.} \quad & (\mathbf{x}, \mathbf{p}) \text{ are DSIC, ex-post IR, and feasible} \\ & (\mathbf{x}, \mathbf{p}) \text{ is budget balanced} \end{aligned}$$

By the characterization of dominant strategy equilibrium, we simplify this optimization program into a linear program as follows,

$$\begin{aligned} \max_{(\mathbf{x}, \mathbf{p}) \geq 0} \quad & \sum_{v_1, v_2 \in [h]} v_1 \cdot x(v_1, v_2) \\ \text{s.t.} \quad & h \cdot x(h, v_2) - \sum_{t=1}^h x(t, v_2) \leq B && \text{for all } v_2 \in [h] && \text{[Budget Constraint]} \\ & x(v_1, v_2) + x(v_2, v_1) \leq 1 && \text{for all } v_1, v_2 \in [h] && \text{[Feasibility Constraint]} \\ & x(v_1, v_2) \leq x(v_1 + 1, v_2) && \text{for all } v_1 \in [h - 1], v_2 \in [h] && \text{[Monotonicity Constraint]} \end{aligned}$$

where we assume $x_1(a, b) = x_2(b, a) = x(a, b)$ for all $a, b \in [h]$ since it is an agent-symmetric linear program.⁶

Additionally, we relax the monotonicity constraint by replacing it with $x(v_1, v_2) \leq x(h, v_2)$ which is common for Bayesian mechanism design with public budget agents.

$$x(v_1, v_2) \leq x(h, v_2) \quad \text{for all } v_1 \in [h-1], v_2 \in [h] \quad [\text{Relaxed Monotonicity Constraint}]$$

Then we write down the corresponding dual program. Let $\{\Lambda(v_2)\}_{v_2 \in [h]}$ denote the dual variables for budget constraints; $\{\beta(v_1, v_2)\}_{v_1, v_2 \in [h]}$ denote the dual variables for feasibility constraints (for simplicity, we use both $\beta(v_1, v_2)$ and $\beta(v_2, v_1)$ to denote the same dual variable); and $\{\mu(v_1, v_2)\}_{v_1 \in [h-1], v_2 \in [h]}$ denote the dual variables for monotonicity constraints. The dual program is,

$$\begin{aligned} \min_{(\Lambda, \beta, \mu) \geq 0} \quad & \sum_{v_2 \in [h]} B \cdot \Lambda(v_2) + \frac{1}{2} \sum_{v_1, v_2 \in [h]} \beta(v_1, v_2) \\ \text{s.t.} \quad & -\Lambda(v_2) + \beta(v_1, v_2) + \mu(v_1, v_2) \geq v_1 && \text{for all } v_1 \in [h-1], v_2 \in [h] && [x(v_1, v_2)] \\ & (h-1)\Lambda(v_2) + \beta(h, v_2) - \sum_{t=1}^{h-1} \mu(t, v_2) \geq h && \text{for all } v_2 \in [h] && [x(h, v_2)] \end{aligned}$$

We give a short overview of the plan to solve the program. For each possible thresholds v^\dagger, v^\ddagger chosen in the middle-ironed clinching auction, we first construct a solution in dual which satisfies the complementary slackness with this middle-ironed clinching auction as a solution in primal. These induced dual solutions may be infeasible. Next, we will show that there exists a pair of thresholds v^\dagger, v^\ddagger which induces a feasible dual solution. This feasible dual solution witnesses the optimality of the middle-ironed clinching auction.

We will partition the dual variables into following five areas (**L***, **MM**, **HH**, **MH** and **HM**) as in Figure 4; and construct the dual solution for them separately. We denote λ as the discrete derivative of the dual variable Λ , i.e. $\lambda(v) = \Lambda(v) - \Lambda(v+1)$.

Λ in **L:** Since the budget constraints do not bind, by complementary slackness,

$$\Lambda(v) = 0 \text{ for all } v \in \mathbf{L}.$$

β, μ in **L*:** Let (v, v') be a value profile in area **L*** such that $v \geq v'$. By complementary slackness on $x(v, v')$, $\beta(v, v') + \mu(v, v') - \Lambda(v') = v$ if $v < h$; $\beta(v, v') - \sum_{t=1}^{h-1} \mu(t, v') + (h-1)\Lambda(v') = v$ otherwise (i.e. $v = h$). We let

$$\beta(v, v') = v \text{ and } \mu(v, v') = 0.^7$$

Since the relaxed monotonicity constraint does not bind at $x(v', v)$, i.e. $x(v', v) < x(h, v)$, the corresponding dual variable is

$$\mu(v', v) = 0.$$

⁶ Note the program in terms of $x(a, b)$ is asymmetric.

⁷ An intuition here is: μ are the dual variables for the relaxed monotonicity constraint and can be thought as indicators of ironing. Though the monotonicity constraint binds, this is not because of ironing but binding allocation (i.e. $x(\cdot) \leq 1$). Therefore, we set μ as zero.

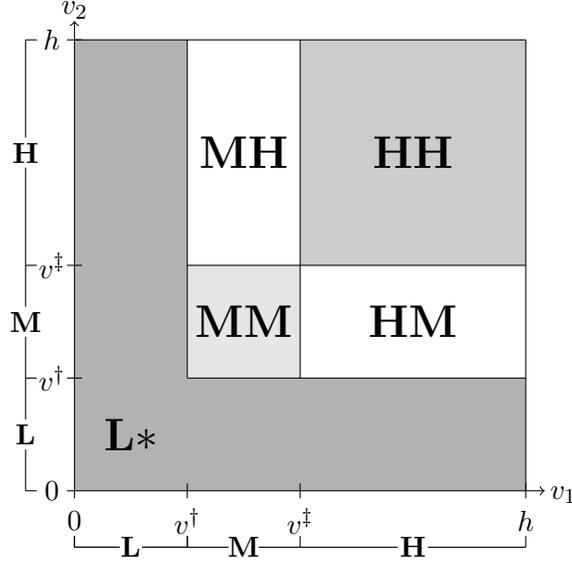


Figure 4: We partition the dual variables into \mathbf{L}^* (at least one agent with value in \mathbf{L}), \mathbf{HH} (both agents with values in \mathbf{H}), \mathbf{MM} (both agents with values in \mathbf{M}), \mathbf{MH} and \mathbf{HM} (one agent with value in \mathbf{M} and the other with value in \mathbf{H}) five areas.

β, μ in \mathbf{HH} : Let (v, v') be a value profile in area \mathbf{HH} such that $v \geq v'$. Since both agents win with non-zero probability, by complementary slackness on $x(v, v')$ and $x(v', v)$, the corresponding dual constraints bind. Since the relaxed monotonicity constraint does not bind at $x(v', v)$, the monotonicity dual variable is

$$\mu(v', v) = 0.$$

The binding dual constraint of $x(v', v)$ is $\beta(v', v) - \Lambda(v) + \mu(v', v) = v'$. Hence,

$$\beta(v, v') = \beta(v', v) = v' + \Lambda(v).^8$$

The binding dual constraint of $x(v, v')$ is $\beta(v, v') - \Lambda(v') + \mu(v, v') = v$. Note the relaxed monotonicity constraint is tight for (v, v') . Hence,

$$\mu(v, v') = v - v' + \Lambda(v') - \Lambda(v).$$

Here we write β, μ as terms of Λ . In the next paragraph, we will solve for Λ .

Λ in \mathbf{H} : Let $v \in \mathbf{H}$. Consider the binding dual constraint of $x(h, v)$, $(h - 1)\Lambda(v) + \beta(h, v) - \sum_{t=1}^{h-1} \mu(t, v) = h$. Notice that by complementary slackness, $\mu(t, v) = 0$ for all $t \leq v$. Plugging β and μ as terms of Λ into the these dual constraints of $x(h, v)$, we can solve for Λ as

$$\lambda(v) = \frac{h - v}{v} \text{ for all } v \in \mathbf{H} \text{ and } \Lambda(h) = 0.^9$$

⁸Recall that $\beta(v, v')$ and $\beta(v', v)$ denote the same dual variable.

⁹Recall that λ is the discrete derivative of dual variables Λ , so $\Lambda(v) = \sum_{t=v}^{h-1} \lambda(t)$.

β, μ in MM and Λ in M: Let (v, v') be a value profile in area **MM** such that $v \geq v'$. Since the relaxed monotonicity constraints do not bind for either $x(v, v')$ or $x(v', v)$, the corresponding dual variables are

$$\mu(v, v') = \mu(v', v) = 0.$$

The binding dual constraints of $x(v, v')$ implies $\beta(v, v') = v' + \Lambda(v)$. On the other hand, the binding dual constraints of $x(v', v)$ implies $\beta(v', v) = v + \Lambda(v')$. Recall that $\beta(v, v')$ and $\beta(v', v)$ denote the same variable, hence,

$$\begin{aligned} \lambda(v) &= -1 \text{ for all } v \in \mathbf{M} \setminus \{v^\dagger - 1\},^{10} \\ \beta(v, v') &= \Lambda(v^\dagger) + \lambda(v^\dagger - 1) + v + v' - v^\dagger. \end{aligned}$$

β, μ in MH and HM: Let (v, v') be a value profile in area **HM** such that $v > v'$. With the similar argument for area **HH**,

$$\begin{aligned} \mu(v', v) &= 0 \text{ and } \mu(v, v') = v - v' + \Lambda(v') - \Lambda(v), \\ \beta(v, v') &= v' + \Lambda(v) \text{ if } v < h. \end{aligned}$$

Plugging μ as terms of Λ into the binding dual constraint of $x(h, v')$,

$$\beta(h, v') = (h - 1)(v^\dagger - v') + 1 + (v^\dagger - 1)\lambda(v^\dagger - 1).$$

With the analysis above, we construct the following dual solution which satisfies complementary slackness with the middle-ironed clinching auction as a solution in primal,

$$\begin{aligned} \Lambda(v_2) &= \begin{cases} 0 & \text{if } v_2 < v^\dagger \\ \sum_{k=v^\dagger}^{h-1} \frac{h-k}{k} + v_2 - v^\dagger + 1 + \lambda(v^\dagger - 1) & \text{if } v^\dagger \leq v_2 < v^\ddagger \\ \sum_{k=v_2}^{h-1} \frac{h-k}{k} & \text{if } v_2 \geq v^\ddagger \end{cases} \\ \beta(v_1, v_2) &= \begin{cases} v_1 & \text{if } v_1 \geq v_2, v_2 < v^\dagger \\ \sum_{k=v^\dagger}^{h-1} \frac{h-k}{k} + v_1 + v_2 - v^\dagger + 1 + \lambda(v^\dagger - 1) & \text{if } v_1 \geq v_2, v^\dagger \leq v_2 < v^\ddagger, v_1 < v^\ddagger \\ \sum_{k=v_1}^{h-1} \frac{h-k}{k} + v_2 & \text{if } v_1 \geq v_2, v^\dagger \leq v_2 < v^\ddagger, v^\ddagger \leq v_1 \leq h - 1 \\ (h - 1)(v^\dagger - v_2) + 1 + (v^\dagger - 1)\lambda(v^\dagger - 1) & \text{if } v_1 \geq v_2, v^\dagger \leq v_2 < v^\ddagger, v_1 = h \\ \sum_{k=v_1}^{h-1} \frac{h-k}{k} + v_2 & \text{if } v_1 \geq v_2, v^\ddagger \leq v_2 \end{cases} \\ \mu(v_1, v_2) &= \begin{cases} 0 & \text{if } v_2 < v^\dagger \\ 0 & \text{if } v^\dagger \leq v_2 < v^\ddagger, v_1 < v^\ddagger \\ v_1 - v^\dagger + \sum_{k=v^\dagger}^{v_1-1} \frac{h-k}{k} + 1 + \lambda(v^\dagger - 1) & \text{if } v^\dagger \leq v_2 < v^\ddagger, v_1 \geq v^\ddagger \\ 0 & \text{if } v_2 \geq v^\ddagger, v_1 \leq v_2 \\ v_1 - v_2 + \sum_{k=v_2}^{v_1-1} \frac{h-k}{k} & \text{if } v_2 \geq v^\ddagger, v_1 > v_2 \end{cases} \end{aligned} \tag{1}$$

Lemma 4.2. *For the middle-ironed clinching auction with arbitrary thresholds v^\dagger and v^\ddagger , the dual solution (1) satisfies the complementary slackness.*

¹⁰ Complementary slackness does not pin down $\lambda(v^\dagger - 1)$. We leave it as a variable and identify it later when we choose the thresholds v^\dagger, v^\ddagger to ensure that the dual solution is feasible.

Proof. The complementary slackness is directly implied by the construction above. \square

Though the this dual solution satisfies the complementary slackness, it may be infeasible. Therefore, we argue that there exists some thresholds v^\dagger , v^\ddagger and $\lambda(v^\ddagger - 1)$ under which the dual solution is feasible.

Lemma 4.3. *There exists v^\dagger, v^\ddagger and $\lambda(v^\ddagger - 1)$ such that the constructed dual solution (1) is feasible.*

Proof. We define function $Z(v) = 2v - 2B - 2 - \sum_{k=v}^{h-1} \frac{h-k}{k}$ to simplify the argument. Notice that $\Lambda(v) = \sum_{k=v}^{h-1} \frac{h-k}{k}$ in the dual solution (1) if $v \in \mathbf{H}$.

Due to complementary slackness, all dual constraints corresponding to some $x(v, v') > 0$ bind, so they are satisfied automatically. Hence, to ensure the constructed dual solution is feasible, there remain four groups of constraints which need to be satisfied. For each group of constraints, there is a ‘‘pivotal’’ constraint such that if it is satisfied, all constraints in that group is satisfied. We list these four groups of constraints and ‘‘pivotal’’ constraint for each group below,

All dual constraints of $x(v, v')$ where $v \in \mathbf{L}$ and $v' \in \mathbf{M}$: The pivotal constraint is the dual constraint of $x(v^\dagger - 1, 1)$, which can be simplified as

$$\lambda(v^\ddagger - 1) \leq Z(v^\dagger).$$

All dual constraints of $x(v, v')$ where $v \in \mathbf{L}$ and $v' \in \mathbf{H}$: The pivotal constraint is the dual constraint of $x(v^\dagger - 1, v^\ddagger)$, which can be simplified as

$$-1 \leq Z(v^\dagger).$$

All dual constraints of $x(v, h)$ where $v \in \mathbf{M}$: The pivotal constraint is the dual constraint of $x(v^\dagger - 1, h)$, which can be simplified as

$$\lambda(v^\ddagger - 1) \leq \frac{h}{v^\ddagger - 1} - 1.$$

$\Lambda, \mu, \beta \geq 0$: The pivotal constraint is $\Lambda(v^\dagger) \geq 0$, which can be simplified as

$$\lambda(v^\ddagger - 1) \geq Z(v^\dagger) - 1.$$

We now show how to relate v^\dagger , v^\ddagger and $\lambda(v^\ddagger - 1)$ to satisfy the four inequalities identified above.

Notice that when $v^\dagger = 1$ and $v^\ddagger = 2B + 1$, the interval \mathbf{L} becomes empty. In that case, the first and second groups of constraints disappear. The combination of these four inequalities is equivalent to

i. $v^\ddagger = 2B + 1$ and $Z(v^\dagger) \leq \frac{h}{v^\ddagger - 1}$; or

ii. $-1 \leq Z(v^\dagger) \leq \frac{h}{v^\ddagger - 1}$.

Without loss of generality, we assume that Condition (i) does not hold and then argue that Condition (ii) holds in this case.

The construction of $Z(\cdot)$ implies the following two facts,

- (a) if $Z(v) < -1$, then $Z(v+1) < \frac{h}{v}$;
(b) if $Z(v) > \frac{h}{v-1}$, then $Z(v-1) > -1$.

If we think of interval $(-1, \frac{h}{v-1})$ as a “window”, these two facts say that if a point $Z(v)$ is on the left hand side of this window (i.e. $Z(v) < -1$), then the next point $Z(v+1)$ cannot jump to the right hand side of the window (i.e. it is either in the window or still on the left hand side of the window) and vice versa. Notice that $Z(B+1) = -\sum_{k=B+1}^{h-1} \frac{h-k}{k} \leq 0 < \frac{h}{B}$ and our assumption (i.e. Condition (i) does not hold) implies $Z(2B+1) > \frac{h}{2B} > -1$. Thus, there exists v^\ddagger such that $-1 \leq Z(v^\ddagger) \leq \frac{h}{v^\ddagger-1}$, i.e. Condition (ii) holds. \square

The construction of the dual solution which satisfies feasibility and complementary slackness witnesses the optimality of the middle-ironed clinching auction. We offer the following discrete version of Theorem 4.1.

Theorem 4.4. *For two public-budget agents with value uniformly distributed from $\{1, \dots, h\}$, the Bayesian optimal DSIC mechanism is the middle-ironed clinching auction for some thresholds v^\ddagger and v^\ddagger .*

We now focus on continuous uniform distribution with value space $[0, h]$. Again, we write the problem as an optimization program as follows,

$$\begin{aligned} & \max_{(x,p) \geq 0} \int_0^h \int_0^h v_1 \cdot x(v_1, v_2) dv_2 dv_1 \\ & \text{s.t.} \\ & \quad h \cdot x(h, v_2) - \int_0^h x(t, v_2) dt \leq B \quad \text{for all } v_2 \in [0, h] \quad [\text{Budget Constraint}] \\ & \quad x(v_1, v_2) + x(v_2, v_1) \leq 1 \quad \text{for all } v_1, v_2 \in [0, h] \quad [\text{Feasibility Constraint}] \\ & \quad x(v_1, v_2) \leq x(h, v_2) \quad \text{for all } v_1, v_2 \in [0, h] \quad [\text{Relaxed Monotonicity Constraint}] \end{aligned}$$

Proof of Theorem 4.1. Discretize the value space $[0, h]$ into $\{\epsilon, 2\epsilon, \dots, m\epsilon\}$ where $m\epsilon = h$ with density $\frac{1}{m}$ each. Define \mathcal{X}_ϵ to be the class of all possible DSIC, ex-post IR, budget balanced allocations such that each value $v \in [(k-1)\epsilon, k\epsilon]$ must be ironed for all $k = 1, \dots, h$. By the construction of \mathcal{X}_ϵ , the allocation function x^ϵ in Theorem 4.4 indeed solves $\max_{x \in \mathcal{X}_\epsilon} \int_0^h \int_0^h v_1 \cdot x(v_1, v_2) dv_2 dv_1$ after rescaling both value space and budget by $\frac{1}{\epsilon}$.

Let \mathcal{X} be the class of all possible DSIC, ex-post IR, budget balanced allocations. Notice that \mathcal{X}_ϵ converges to \mathcal{X} pointwise and that both are compact subsets of the L_1 space defined by uniform measure. The operator $T(x) = \int_0^h \int_0^h v_1 \cdot x(v_1, v_2) dv_2 dv_1$ is a bounded linear operator from the L_1 space of allocation function to \mathbb{R} . Therefore, T achieves its maximum on each set \mathcal{X}^ϵ and \mathcal{X} .

The pointwise convergence ensures that

$$\lim_{\epsilon \rightarrow 0} \max_{x \in \mathcal{X}^\epsilon} \int_0^h \int_0^h v_1 \cdot x(v_1, v_2) dv_2 dv_1 = \max_{x \in \mathcal{X}} \int_0^h \int_0^h v_1 \cdot x(v_1, v_2) dv_2 dv_1$$

Since $T(x)$ is a bounded linear operator and $\{x^\epsilon\}$ has a pointwise limit,

$$\lim_{\epsilon \rightarrow 0} x^\epsilon \in \{\operatorname{argmax}_{x \in \mathcal{X}} \int_0^h \int_0^h v_1 \cdot x(v_1, v_2) dv_2 dv_1\}.$$

Thus, we see that Theorem 4.1 holds. \square

Based on Theorem 4.1, we compare the performance between the DSIC mechanism and the welfare-optimal BIC mechanism.

Lemma 4.5. *There exists the instance of public-budget regular agents where the welfare-optimal DSIC mechanism is a 1.013-approximation to the welfare-optimal BIC mechanism.*

Proof. Consider two agents with values drawn uniformly from $[0, h]$ where $h \geq 5.5$ and the budget $B = 1$. By Theorem 4.1, the welfare-optimal DSIC mechanism in this case is the middle-ironed clinching auction with $v^\dagger = 0$ and $v^\ddagger = 2$. The welfare-optimal BIC mechanism is the all-pay auction (applying the revelation principle). By computing the welfare for both mechanisms under this distribution, and setting $h = 5.5$, it optimizes the ratio as 1.013. \square

5 Revelation Gap

In the literature, prior-independent mechanisms have been shown to approximate the Bayesian optimal mechanism for many objectives. Interestingly, except when the solution is trivial, none of the approximation mechanisms in the literature are known to be optimal. The formal question of optimal prior-independent mechanism design is the following:

$$\beta = \min_{\mathcal{M} \in \text{MECHS}} \max_{F \in \text{DISTS}} \frac{\mathbf{E}_{\mathbf{v} \sim F}[\text{OPT}_F(\mathbf{v})]}{\mathbf{E}_{\mathbf{v} \sim F}[\mathcal{M}(\mathbf{v})]}.$$

In this definition, $\mathbf{E}_{\mathbf{v} \sim F}[\mathcal{M}(\mathbf{v})]$ is the equilibrium performance of mechanism \mathcal{M} on distribution F and OPT_F is the optimal mechanism for given objective on distribution F . Importantly in this definition, the family of mechanism MECHS may be restricted from all mechanisms and the family of distribution DISTS may be restricted from all distributions.

As discussed in the introduction, the revelation principle is not without loss for prior-independent mechanism design. Based on this idea, we introduce the concept of revelation gap.

Definition 5.1. *The revelation gap is the ratio of the prior-independent approximation of incentive compatible mechanisms to the prior-independent approximation of (generally non-revelation) mechanisms.*

In this section, we consider welfare maximization with public-budget regular agents. With the results in the previous sections, we give both the upper bound and the lower bound of revelation gap for this objective.

Theorem 5.1. *For public-budget regular i.i.d. agents, the revelation gap for welfare maximization is at most e . Specifically, this upper bound considers prior-independent DSIC, ex-post IR mechanisms.*

Proof. This upper bound is given by considering the clinching auction which is a prior-independent DSIC and ex-post IR mechanism. Theorem 3.5 says that the clinching auction is an e -approximation to the welfare-optimal mechanism for public-budget regular agents. Thus, the revelation gap is at most e . \square

For the lower bound, we use the result in Section 4 where we solve the welfare-optimal DSIC mechanism for two agent with uniformly distributed values. Note that for two-agent environments,

the DSIC ex-post IR constraints are equivalent to prior-independent BIC and IIR constraints.¹¹ With more than two agents, this equivalence does not generally hold.

Lemma 5.2. *For two i.i.d. agents, a mechanism is Bayesian incentive compatible and interim individual rational for all i.i.d. distributions if and only if it is dominant strategy incentive compatible and ex-post individually rational.*

Proof. The direction that DSIC implies BIC for all i.i.d. distribution is trivial by the definition. To show the other direction, for arbitrary value v , consider the distribution which puts the whole mass on v . These distributions break the interim constraints in BIC into the ex-post constraints in DSIC for every value profiles. Hence, BIC for all i.i.d. distribution implies DSIC for two agents setting. \square

Theorem 5.3. *For public-budget regular i.i.d. agents, the revelation gap for welfare maximization is at least 1.013.*

Proof. This lower bound is given by considering the all-pay auction and the middle-ironed clinching auction.

As the characterization in Section 2, the all-pay auction is a prior-independent mechanism. Theorem 2.2 says that the all-pay auction is welfare-optimal for public budget regular agents. Hence, the prior-independent approximation of the all-pay auction is 1.

Next, we show that the prior-independent approximation of Bayesian incentive compatible mechanisms is at least 1.013. Theorem 4.1 says that the middle-ironed clinching auction is Bayesian optimal DSIC mechanism for two agents with values drawn uniformly from $[0, h]$. Since for two agents case, the DSIC property is equivalent to the BIC for all i.i.d. distribution property, Lemma 4.5 suggests that the prior-independent approximation of incentive compatible mechanisms is at least 1.013.

Thus, the revelation gap for welfare maximization is at least 1.013. \square

6 Welfare Approximation for Irregular Distribution

In this section, we analyze the welfare approximation of the all-pay auction and the clinching auction for public budget agents without regularity assumption.

The main technique we use is the following lemma which relaxes the budget constraint to another constraint which upper bounds the winning probability of the highest value, i.e. $x(h)$ where h is the highest value in the support of the distribution.

Lemma 6.1. *Given any interim constraint x^* and budget B , let v^\dagger be the value where the budget binds in x^* after ironing from v^\dagger to h , i.e. $v^\dagger \cdot z^*(v^\dagger) - \int_0^{v^\dagger} x^*(t)dt = B$ where $z^*(v^\dagger) = \frac{1}{1-F(v^\dagger)} \int_{v^\dagger}^h x^*(t)dF(t)$, the averaging winning probability for value beyond v^\dagger in allocation x^* . Any interim feasible and budget balanced allocation x satisfies*

$$x(h) \leq 2z^*(v^\dagger).$$

¹¹Prior-independent BIC and IIR mechanisms are the mechanisms which are BIC and IIR for all i.i.d. distributions. This property is stronger than BIC (for a single distribution) but generally weaker than DSIC.

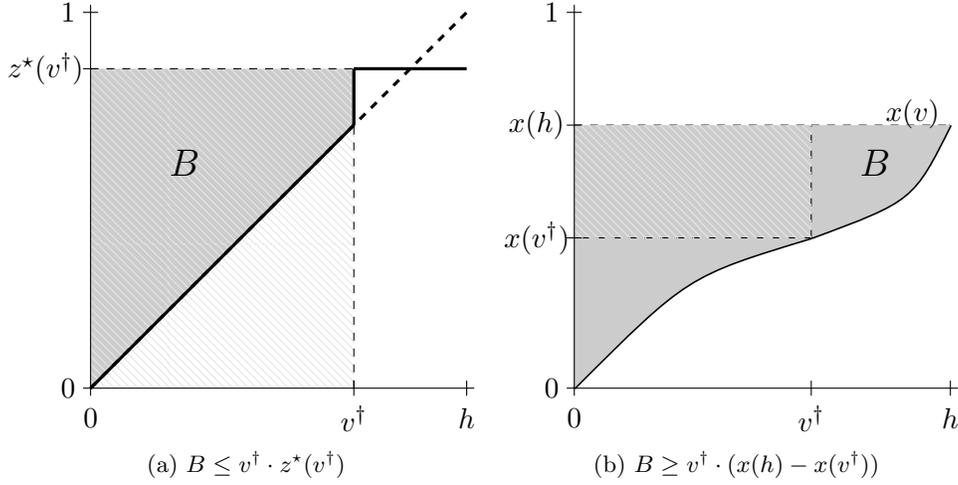


Figure 5: Proofs by picture of the upper bound and lower bound on budget B .

Proof. Recall that v^\dagger is the value where the budget binds in x^* after ironing from v^\dagger to h . Thus,

$$B = v^\dagger \cdot z^*(v^\dagger) - \int_0^{v^\dagger} x^*(t) dt \leq v^\dagger \cdot z^*(v^\dagger).$$

On the other hand, suppose x is budget balance,

$$B \geq h \cdot x(h) - \int_0^h x(t) dt \geq v^\dagger \cdot (x(h) - x(v^\dagger))$$

Suppose x is interim feasible,

$$x(v^\dagger) \leq \frac{1}{1 - F(v^\dagger)} \int_{v^\dagger}^h x(t) dt \leq \frac{1}{1 - F(v^\dagger)} \int_{v^\dagger}^h x^*(t) dt = z^*(v^\dagger).$$

Combine the inequalities above,

$$x(h) \leq x(v^\dagger) + z^*(v^\dagger) \leq 2z^*(v^\dagger). \quad \square$$

The All-pay Auction

First, we discuss the performance of all-pay auction for the irregular distribution. Pai and Vohra (2014) show that the welfare-optimal interim allocation is both ironing top interval and perhaps ironing some other intervals in the middle. It turns out that even though the all-pay auction only irons the top interval, its welfare only suffers a modest loss.

Theorem 6.2. *For public-budget i.i.d. agents, the all-pay auction is a 2-approximation to the welfare-optimal mechanism.*

Proof. Applying Lemma 6.1, we relax the budget constraint to the constraint that $x(h) \leq 2z^*(v^\dagger)$.

Denote x^0 as the welfare-optimal interim feasible and budget balanced allocation and x as the welfare-optimal interim feasible allocation under the relaxed constraint, then $\mathbf{welfare}[x^0] \leq \mathbf{welfare}[x]$.

Since x maximizes the welfare under interim feasibility constraint x^* , it allocates as by x^* except that values in $[v^\dagger, h]$ are ironed. The threshold v^\dagger is selected such that $x(h) = 2z^*(v^\dagger)$. By definitions of v^\dagger and v^\ddagger , we know $v^\dagger \leq v^\ddagger$. Consider x for values below and beyond v^\dagger separately. For value below v^\dagger , the expected welfare $\int_0^{v^\dagger} v \cdot x(v) dF(v) = \int_0^{v^\dagger} v \cdot x^*(v) dF(v)$. For value beyond v^\dagger , the expected welfare $\int_{v^\dagger}^h v \cdot x(v) dF(v) \leq \int_{v^\dagger}^h v \cdot 2z^*(v^\dagger) dF(v)$.

Notice that v^\dagger coincides with the threshold in the all-pay auction, and the all-pay auction allocates as by x^* except value beyond v^\dagger win with probability $z^*(v^\dagger)$. Thus, $\mathbf{welfare}[x] \leq 2 \cdot \mathbf{welfare}[\text{All-pay}]$. \square

In fact, the 2-approximation bound is tight.

Lemma 6.3. *There exists the instance where the welfare of the all-pay auction is half of welfare-optimal mechanism.*

Proof. Consider the following single-item instance with budget $B = 1$. There are $N + 1$ agents with valuation distribution

$$v = \begin{cases} N - \epsilon & \text{w.p. } \frac{1}{N+1}, \\ N & \text{w.p. } \frac{N-1}{N+1}, \\ N^3 & \text{w.p. } \frac{1}{N+1}. \end{cases}$$

In the all-pay auction, the interim allocation rule irons values N and N^3 ,

$$x(v) = \begin{cases} \delta & \text{if } v = N - \epsilon, \\ \frac{1-\delta}{N} & \text{if } v = N \text{ or } N^3, \end{cases} \quad \text{where } \delta = \left(\frac{1}{N+1}\right)^{N+1} \rightarrow 0.$$

where the expected welfare is roughly $N + 1$.

Notice that in the all-pay auction, the mechanism uses almost all budget to distinguish values $N - \epsilon$ and N whose contribution to the expected welfare is almost the same. Therefore, consider the auction which irons values $N - \epsilon$ and N , and moves some winning probability from value N^3 to values $N - \epsilon, N$ as follows,

$$x(v) = \begin{cases} \frac{N-1}{N(N+1)} & \text{if } v = N - \epsilon \text{ or } N, \\ \frac{2}{N+1} & \text{if } v = N^3. \end{cases}$$

where the expected welfare is roughly $2N + 1$.

Let $N \rightarrow \infty$ and $\epsilon \rightarrow 0$, the expected welfare from the all-pay auction is exactly half of the expected welfare from the optimal auction. \square

The Clinching Auction

To analyze the welfare approximation of the clinching auction for irregular distributions, we follow almost the same argument as for regular distributions. The only difference in the argument is that the ex ante welfare-optimal mechanism may not be a simple posted price for a probabilistic allocation. However, by Lemma 6.1, such a posted pricing is still a 2-approximation.

Lemma 6.4. *For public-budget i.i.d. agents, the posted pricing described in Lemma 3.3 is a 2-approximation to the ex ante welfare-optimal mechanism.*

Proof. Consider the interim constraint $x^*(v) = 1$ if $F(v) \geq 1/n$ and $x^*(v) = 0$ if $F(v) < 1/n$. Applying the similar argument in Theorem 6.2 with Lemma 6.1, the lemma holds. \square

The following corollary is combines Lemma 6.4 with Theorem 3.5.

Corollary 6.5. *For public-budget i.i.d. agents, the clinching auction is a 2e-approximation to the welfare-optimal mechanism.*

7 Welfare Approximation of Losers-pay-nothing Mechanisms

Both the all-pay and clinching auctions discussed previously allocate probabilistically and agents make payments even when they lose. These mechanisms contrast with the first-price and second-price auctions which have the property that losers pay nothing. In this section we will show that losers-pay-nothing mechanisms do not give good approximations to the welfare-optimal mechanism.

Our approach is to prove three results. First, we show that when optimizing over losers-pay-nothing mechanisms, it is without loss in the objective to consider only winner-pays-bid mechanisms. Second, we show that for public-budget regular agents and a single-item the the welfare-optimal winner-pays-bid mechanism is highest bid wins, i.e., it is the first-price auction. Third, we show that there is a public-budget regular distribution where the welfare of the first-price auction is a linear factor from the welfare of the optimal mechanism (which is the all-pay auction). Combining these results we see that losers-pay-nothing mechanisms are not good approximation mechanisms.

The main difference between winner-pays-bid and all-pay mechanisms is that the budget binds over a larger range of values for winner-pays-bid mechanisms. Consequently the allocation rule of winner-pays-bid mechanisms is further from the efficient highest-value-wins allocation rule.

Winner-pays-bid versus Loser-pays-nothing Mechanisms

Mechanisms can map bids to probabilistic outcomes. Denote the random bits accessed by a mechanism by π . Our previously defined allocation and payment rules take expectation over this randomization, i.e., $\tilde{\mathbf{x}}(\mathbf{b}) = \mathbf{E}_\pi[\tilde{\mathbf{x}}^\pi(\mathbf{b})]$ and $\tilde{\mathbf{p}}(\mathbf{b}) = \mathbf{E}_\pi[\tilde{\mathbf{p}}^\pi(\mathbf{b})]$ where both $\tilde{\mathbf{x}}^\pi$ and $\tilde{\mathbf{p}}^\pi$ are deterministic functions. Specifically $\tilde{\mathbf{x}}^\pi(\mathbf{b}) \in \{0, 1\}^n$.

Definition 7.1. *A loser-pays-nothing mechanism $(\tilde{\mathbf{x}}, \tilde{\mathbf{p}})$ satisfies $\tilde{p}_i^\pi(\mathbf{b}) = 0$ if $\tilde{x}_i^\pi(\mathbf{b}) = 0$ for all agents i and random bits π .*

Definition 7.2. *A winner-pays-bid mechanism $(\tilde{\mathbf{x}}, \tilde{\mathbf{p}})$ satisfies $\tilde{p}_i^\pi(\mathbf{b}) = b_i \tilde{x}_i^\pi(\mathbf{b})$ for all agents i and random bits π .*

Lemma 7.1. *For public budget agents, a winner-pays-bid mechanism is optimal among all loser-pays-nothing mechanisms.*

Proof. Let (\mathbf{x}, \mathbf{p}) be the BNE allocation and payment rules of any loser-pays-nothing mechanism. First, disregarding the budget constraints of the agents, there is a winner-pays-bid mechanisms with the same BNE allocation and payment rules. This fact is a straightforward consequence of the characterization of Bayes-Nash equilibrium, e.g., see Chawla and Hartline (2013). Second, as these

two mechanisms have the same interim allocation rule, namely \mathbf{x} ; the payment-identity requires that they have the same interim payment rule, namely \mathbf{p} . Losers pay nothing in both mechanisms. Thus, the expected payment of an agent i with a value v_i conditioned on winning is the same in the two mechanisms. In the winner pays bid mechanism the agent's payment conditioned on winning is deterministically equal to its conditional expectation. In the original loser-pays-nothing mechanism the maximum payment in the support of the conditional payment distribution is no lower than its expectation. Consequently, the disregarded budget constraints are satisfied by the constructed winner-pays-bid mechanism and it obtains the same objective value. \square

Optimal Winner-pays-bid Mechanisms

Recall that for i.i.d. public-budget regular agents, the optimal mechanism allocates efficiently except that an interval of highest-valued agents are ironed and payments are deterministic functions of values. Specifically, in a single-item environment, the all-pay auction is optimal. We will show below that, restricting the mechanism to be winner-pays-bid, the optimal mechanism has the same form. It allocates efficiently except that an interval of highest values agents are ironed. Specifically, in a single-item environment, the first-price auction is optimal. Notice that the first-price auction is the winner-pays-bid highest-bid-wins mechanism.

Theorem 7.2. *For i.i.d. public-budget regular agents, the welfare-optimal winner-pays-bid mechanism is the highest-bid-wins mechanism, i.e. the first-price auction.*

The formal proof of Theorem 7.2 is given in Appendix B. It can be proved using standard methods in Bayesian mechanism design with budgets, e.g., Laffont and Robert (1996) and Maskin (2000). We give simpler approach that is based on the Alaei et al. (2013) reduction to ex ante pricing. Note that the approach of Alaei et al. (2013) cannot be directly applied as public-budget agents do not have linear utility and the optimal revenues for non-linear agents do not satisfy an important linearity property. We write the problem as a optimization program, use the Lagrangian relaxation to move the budget constraint into the objective, and then observe that the Lagrangian relaxed objective does satisfy the linearity property of Alaei et al. (2013).

Lowerbound for Winner-pays-bid Mechanisms

With Theorem 7.2, we compare the performance of the welfare-optimal winner-pays-bid mechanism and the welfare-optimal mechanism. For i.i.d. public-budget regular agents and a single-item, this comparison is between the first-price auction and the all-pay auction.

Lemma 7.3. *For n i.i.d. public-budget agents, the first-price auction is at best an $(\frac{1}{4}n - o(n))$ -approximation to the all-pay auction.*

Proof. Consider n agents with budget $B = (1 - \frac{1}{e})\frac{1}{n}$ and value distributed from the following distribution F with density function

$$f(v) = \begin{cases} n-1 & \text{if } v \in [0, \frac{1}{n}] \\ \frac{1}{n-1} & \text{if } v \in (\frac{1}{n}, 1]. \end{cases}$$

By Theorem 7.2, the welfare-optimal winner-pays-bid mechanism is the first-price auction where each agent bids B if her value is beyond some v^\dagger to satisfy $\frac{p_{\text{first-price}}(v^\dagger)}{x_{\text{first-price}}(v^\dagger)} = B$. In this setting, as n goes to infinity, v^\dagger goes to B , and the expected welfare goes to $((1 - \frac{1}{2e})(1 - \frac{1}{e}) + \frac{1}{2})\frac{1}{n} + o(\frac{1}{n})$.

The welfare-optimal mechanism is the all-pay auction where each agent bids B if her value is beyond some v^\dagger such that $p_{\text{all-pay}}(v^\dagger) = B$. In this setting, as n goes infinity, v^\dagger goes to $\frac{1}{n}$, and the expected welfare goes to $(1 - \frac{1}{e})\frac{1}{2} + o(1)$.

Thus, as n goes to infinity, the expected welfare of the first-price auction is an $\frac{1}{4}n$ -approximation to the all-pay auction. \square

It is easy to see that this linear approximation is tight up to constant factors. Specifically, the n -agent lottery, which allocates the item to a random agent without payments, is trivially an n -approximation.

8 Revenue Approximation of the All-pay Auction

In this section, we analyze the approximation ratio of the all-pay auction for public-budget regular agents with the revenue-optimal mechanism. In the preliminaries, Theorem 2.1 shows that the revenue-optimal mechanism irons the top and sets a reserve with the combined effect that the budget binds for the highest value. From the equivalence of the all-pay auction (Theorem 2.2) and the welfare-optimal auction (Theorem 2.1), the all-pay auction irons top values to decrease the payment of highest value to meet the budget. Even though the value intervals ironed at top are different between all-pay auction and optimal auction (specifically the all-pay auction irons less than the optimal auction), the all-pay auction is still a good approximation to the revenue-optimal mechanism. Our analysis follows Kirkegaard's 2006 proof of the main theorem from Bulow and Klemperer (1996). These results show that, without budgets, the revenue of the second-price auction approximates the optimal revenue for i.i.d. and regular agents. The revenue-optimal mechanism allocates the item to the highest agent whose value exceeds a specific reserve price while the second-price auction allocates the item to the highest agent with no reserve. As is crucial for Kirkegaard's proof, the second-price auction can be thought as the revenue-optimal mechanism under the constraint that the item must be allocated.

Theorem 8.1 (Bulow and Klemperer 1996). *For $n \geq 2$ regular i.i.d. agents with linear utilities, the expected revenue of the second-price auction is an $\frac{n}{n-1}$ -approximation to the revenue-optimal mechanism.*

For agents with public budget, the all-pay auction plays the similar role as the second-price auction.

Theorem 8.2. *For $n \geq 2$ public-budget regular i.i.d. agents, the all-pay auction is an $\frac{n}{n-1}$ -approximation to the revenue-optimal mechanism.*

Proof. First, we introduce a common approach of the revenue analysis in Bayesian mechanism design. Myerson (1981) defined the virtual valuation for sum of payments as $\phi(v) = v - \frac{1-F(v)}{f(v)}$ and proved that the expected payment $\mathbf{E}[p(v)]$ of an agent in any BNE is equal to her expected virtual surplus $\mathbf{E}[\phi(v)x(v)]$. We use this concept in the following argument.

Denote value v^\dagger as the threshold where the all-pay auction starts to iron.

Consider the optimal mechanism which is (i) budget balanced, (ii) irons the interval $[v^\dagger, h]$, and (iii) always sells the item for n public-budget regular i.i.d. agents.

We claim that the optimal auction under these three requirements is the all-pay auction by analyzing the virtual surplus.¹² For public-budget regular agents, their virtual value is monotone non-decreasing. Consider the virtual surplus from values v below and above v^\dagger separately. For values $v \geq v^\dagger$, requirement (ii) upper bounds the allocation $x(v)$ to be at most as the all-pay auction (otherwise, the budget constraint will be violated). For values $v \leq v^\dagger$, the all-pay auction always allocates the item to the agent with highest virtual value. Thus, the all-pay auction maximizes the virtual surplus under these three requirements.

On the other hand, consider the following auction LB:

1. run the n -agent revenue-optimal budget-balanced auction on the first $n - 1$ agents and a fake agent;
2. if the auction does not sell the item to anyone within the first $n - 1$ agents (i.e. the auction does not sell the item or sells it to the fake agent), give the item to the n -th agent for free.

LB is budget balanced and always sells the item for n public-budget regular i.i.d. agents. Notice that the revenue-optimal auction for public budget n agents irons more than the all-pay at the top. Hence, LB satisfies the three requirements. The expected revenue from it is $\frac{n-1}{n}$ fraction of the expected revenue from the revenue-optimal auction for n public-budget regular i.i.d. agents.

We conclude that the expected revenue of the all-pay auction is at least the expected revenue of LB which is $\frac{n-1}{n}$ fraction of the revenue-optimal mechanism. Thus, the all-pay auction is an $\frac{n}{n-1}$ approximation to the revenue-optimal mechanism. \square

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¹²Under Requirement (i) and Requirement (iii), the all-pay auction is already optimal. We introduce Requirement (ii) to simplify the argument.

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A Clinching Auction with Price Jumps

In this section, we introduce the clinching auction with price jumps. In the standard clinching auction, with a continuous increasing price-clock, excess demand decreases continuously to the point where supply equals demand and the market clears. With a price jump, which leads to a strict drop of demands, the standard clinching auction may leave some supply unallocated. Therefore, to clear the market, the clinching auction with price jumps will need to reallocate some amount of units at the pre-jump price after a price jump. We first focus on the clinching auction with price jumps for agents with identical budgets. The results can be extended to agents with distinct budgets, which we will discuss at the end of this section.

To formally describe this reallocation, suppose the price-clock jumps from v^\dagger to v^\ddagger . Consider the state $\mathcal{C} = (s, S^\dagger, B)$ at price v^\dagger after the clinching step (i.e. Step 3 in Definition 3.1) where s is the current supply remaining, S^\dagger is the agents with values at least v^\dagger (let $k^\dagger = |S^\dagger|$), and B is the current budget of the active agents.¹³ When the price jumps to v^\ddagger , active agents with values below v^\ddagger (“low-valued” agents) will quit, while active agents with values at least v^\ddagger (“high-valued” agents) will stay in the auction. Denote by S^\ddagger the set of high-valued agents and by $k = |S^\ddagger|$ the number of high-valued agents. With pre-jump state $\mathcal{C} = (s, S^\dagger, B)$ and k agents remaining after the jump, define h_k^C and l_k^C as the additional supply allocated at the low price to high- and low-valued agents, respectively.

In the following discussion, we fix an arbitrary state $\mathcal{C} = (s, S^\dagger, B)$ with $k^\dagger = |S^\dagger|$ active agents, drop the superscript of h_k^C and l_k^C , and consider h_k and l_k constrained to the following polytope:

$$\begin{aligned}
\text{IC: } \forall k \in \{1, \dots, k^\dagger\} & \quad h_k = l_{k-1}, \\
\text{BB: } \forall k \in \{0, \dots, k^\dagger\} & \quad h_k, l_k \leq B/v^\dagger, \\
\text{NN: } \forall k \in \{0, \dots, k^\dagger\} & \quad h_k, l_k \geq 0, \\
\text{MC: } \forall k \in \{0, \dots, k^\dagger\} & \quad k h_k + (k^\dagger - k) l_k + \frac{k}{v^\dagger} (B - v^\dagger h_k) \geq s, \\
\text{LS: } \forall k \in \{0, \dots, k^\dagger\} & \quad k h_k + (k^\dagger - k) l_k \leq s.
\end{aligned} \tag{2}$$

The constraints above are, respectively, incentive compatibility (IC), budget balance (BB), non-negative consumption (NN), market clearing (MC), and limited supply (LS). The IC constraint requires that the amount of supply which an agent gets at price v^\dagger does not depend on whether the agent stays or quits during the price jump. The left-hand side of the incentive compatibility (IC)

¹³In the model considered in this paper where initially the agents have identical budgets, the remaining budgets of all active agents remain identical throughout the execution of the clinching auction.

constraint is the additional allocation quantity at price v^\dagger if an agent stays during the price jump, while the right hand side is the additional allocation quantity at price v^\ddagger if she quits. Since the two quantities are equal, active agents with values in $[v^\dagger, v^\ddagger)$ prefer to quit after the clinching step at price v^\dagger while agents with value at least v^\ddagger prefer to stay in the auction. The market clearing (MC) constraint states that the reallocated supply at the low price v^\dagger (the first two terms) plus the quantity demanded by the high-valued agents at the high price v^\ddagger (the third term) must be at least the supply. The limited supply (LS) constraint states that the amount allocated at the low price for any number k of high-valued agents must not exceed the supply.

Consider the problem of selecting a point in polytope (2) to optimize the expected welfare under the value distribution F . First notice that, since the state $\mathcal{C} = (s, S^\dagger, B)$ is induced by the clinching auction, the total demand at price v^\dagger under budget B exceeds the remaining supply s , i.e., $k^\dagger B/v^\dagger \geq s$; setting $h_k = l_k = s/k^\dagger$ for all $k \in [k^\dagger]$ is feasible; and, thus, polytope (2) is not empty. The expected welfare of the clinching auction is complicated to express; we instead consider the objective of minimizing, within the constraints of polytope (2), the expected supply reallocated to low-valued agents, i.e., $\sum_{k=0}^{k^\dagger} l_k \pi^{k^\dagger - k} (1 - \pi)^k$ where $\pi = \frac{F(v^\ddagger) - F(v^\dagger)}{1 - F(v^\dagger)} = \Pr_{v \sim F}[v < v^\ddagger \mid v \geq v^\dagger]$ is the probability an agent has a low value. Based on this reallocation, we formally define a clinching auction with price jumps and show that it clears the market, is ex-post IR, and is DSIC.

Definition A.1. *The clinching auction with price jumps maintains an allocation and price-clock starting from zero. Before and after each price jump point, the price-clock ascends continuously and the allocation and the budget are adjusted as in the standard clinching auction. When the price-clock jumps from v^\dagger to v^\ddagger the following steps are taken:*

1. *run the standard clinching steps on price-clock v^\dagger and the current budgets and let the subsequent state be $\mathcal{C} = (s, S^\dagger, B)$ with $k^\dagger = |S^\dagger|$;*
2. *increase the price-clock to v^\ddagger and let $k^\ddagger = |S^\ddagger|$ be the number of agents remaining in the auction;*
3. *solve for $\{h_k, l_k\}_{k \in [k^\ddagger]}$ to minimize the expected quantity reallocated to the low-valued agents in the polytope (2);*
4. *allocate h_{k^\ddagger} units at price v^\dagger to each of the k^\ddagger agents that stay after the price jump, allocate l_{k^\ddagger} units at price v^\dagger to each of the $k^\dagger - k^\ddagger$ agents that quit during the price jump, and adjust all the agents' budgets for the amount and price allocated;*
5. *run the standard clinching step with price-clock v^\ddagger and updated budgets.*

Proposition A.1. *The clinching auction with price jumps always clears the market.*

Proof. If the price-clock increases continuously, the demand decreases continuously. When the total demands meet the supply remaining, Dobzinski et al. (2008) show that the standard clinching auction halts and the market clears. For the clinching auction with price jumps, when the price-clock goes through a price jump, the market clearing constraints that define polytope (2) guarantee that the total demands are at least the supply remaining. Thus, the clinching auction with price jumps clears the market. \square

Proposition A.2. *The clinching auction with price jumps satisfies ex-post IR, DSIC, and budget balance.*

Proof. Dobzinski et al. (2008) show that the standard clinching auction is ex-post IR, DSIC, and budget balanced. For the clinching auction with price jumps, when the price-clock goes through a price jump from v^\dagger to v^\ddagger , the IC constraints that define polytope (2) guarantee that the agents with values at most v^\ddagger weakly prefer to quit at price v^\dagger and the agents with values above v^\ddagger prefer to stay at price v^\dagger . Meanwhile, the budget constraints and non-negative consumption constraints that define polytope (2) guarantee that the agents are budget balanced and have non-negative utility after the price jump. \square

For two i.i.d. agents with identical budgets, the clinching auction with price jumps induces the same outcome as the middle-ironed clinching auction (Definition 4.1). For a general number of agents, it is polynomial time solvable. We conjecture that, for i.i.d. distributions and identical budgets, minimizing the expected quantity reallocated to low-valued agents, i.e., the objective described previously, is equivalent to maximizing expected welfare. We leave to future studies the question of whether there is a more succinct characterization of the expected welfare maximizing solution and the generalization to agents with non-identical valuation distributions.

If agents have distinct budgets, the linear program can be generalized by replacing the variables, which corresponded to the reallocation to high- and low-valued agents with a given number $k = |S^\ddagger|$ of high-valued agents, with variables that correspond to the reallocation to each agent i with a given set S^\ddagger of high-valued agents. With this modification to the variables and constraints of polytope (2), the previous argument guarantees the new polytope is non-empty. Notice that there are $O(n \cdot 2^n)$ variables defining the new polytope. It is possible, however, to optimize expected allocation to the low-valued agents subject to this polytope in polynomial time when there are a constant number of distinct budgets; symmetries across agents with identical budgets allow the number of variables in the program to be reduced to a polynomial number. We leave to future studies the problem of identifying a polynomial time algorithm for optimally reallocating the supply during a price jump when there are generally distinct budgets.

B Bayesian Optimal Mechanisms for Budgeted Agents

In this section we give a simple geometric approach for identifying Bayesian optimal mechanisms for budgeted agents. This approach can be used to derive Theorem 2.1, which characterizes the Bayesian optimal mechanisms for revenue and welfare. Recall, that the mechanisms characterized by Theorem 2.1 have deterministic interim payments and are naturally implemented by all-pay mechanisms. In this section, we use the approach to analyze winner-pays-bid mechanisms and identify the Bayesian optimal mechanisms restricted this family, i.e., we prove Theorem 7.2.

The main result of this section is Lemma B.1 which describes the optimal single-agent mechanism for any interim constraint and combines with the fact that in symmetric single-item environments, the optimal multi-agent mechanism is given by the interim constraint that corresponds to the highest-bid-wins allocation rule. Theorem 7.2, restated below, follows.

Lemma B.1. *For a public-budget regular agent and interim allocation constraint $x^*(\cdot)$, the welfare-optimal winner-pays-bid single-agent mechanism allocates as by $x^*(\cdot)$ except that values in $[v^\dagger, h]$ are ironed for some v^\dagger ; payments are given deterministically by the payment identity and the winner-pays-bid framework.*

Theorem 7.2. *For i.i.d. public-budget regular agents, the welfare-optimal winner-pays-bid mechanism is the highest-bid-wins mechanism, i.e. the first-price auction.*

While Lemma B.1 can be proven using the traditional analysis of agents with budgets, e.g., Laffont and Robert (1996); we will give a proof that reduces interim optimization to ex ante optimization. For context, Bulow and Roberts (1989) reduce interim revenue maximization to ex ante revenue maximization for single-dimensional linear agents (i.e. without budgets). Alaei et al. (2013) generalized this approach to linear objectives (which do not require single-dimensional linear agents). A challenge that our proof addresses is that public budget agents do not have linear utility functions and, therefore, welfare maximization is not a linear objective.

Interim Optimization for Linear Objectives

We first introduce the *interim optimal payoff*, *quantile space*, and *payoff curves*; payoff curves are a straightforward generalization of revenue curves to non-revenue objectives, cf. Alaei et al. (2013).

Definition B.1. *For any general objective, the interim optimal payoff, given interim allocation constraint x^* , is the payoff of the single-agent mechanism (x, p) that satisfies the interim constraint $\int_v^h x(v) dF(v) \leq \int_v^h x^*(v) dF(v)$ with the highest objective value; denote this optimal payoff by $\mathbf{Payoff}[x^*]$. The objective is linear if the functional $\mathbf{Payoff}[\cdot]$ is linear, i.e., if for any allocations $x = x^\dagger + x^\ddagger$ then $\mathbf{Payoff}[x] = \mathbf{Payoff}[x^\dagger] + \mathbf{Payoff}[x^\ddagger]$.*

Definition B.2. *The quantile q of a single-dimensional agent with value v drawn from distribution F is the measure with respect to F of stronger values, i.e., $q = 1 - F(v)$; the inverse demand curve maps an agent's quantile to her value, i.e., $v(q) = F^{-1}(1 - q)$.*

Definition B.3. *For given ex ante allocation probability q , the single-agent ex ante pricing problem is to find the optimal mechanism with ex ante allocation probability exactly q . The optimal ex ante payoff, as a function of q , is denoted by the payoff curve $\bar{\Phi}(q)$.*

The ex ante pricing problem (Definition B.3) can be solved via the following geometric approach.

Definition B.4. *For any q , denote the price-posting payoff curve, from posting price $v(q)$, by $\Phi(q)$ (with ties broken in favor of higher payoff).*

If the price-posting payoff curve is differentiable and concave, the marginal price-posting payoff, a.k.a., the derivative $\Phi'(q)$, is well defined and monotone non-increasing. Its pointwise optimization leads to an optimal incentive compatible mechanism, cf. Myerson (1981). Specifically, the ex ante pricing problem is solved by posted pricing and $\bar{\Phi} = \Phi$. Otherwise, analogous to the ironing method of Myerson (1981), the ex ante pricing problem is solved by ironing the price-posting revenue curve.

Lemma B.2 (Alaei et al., 2013). *Given any linear payoff objective $\mathbf{Payoff}[\cdot]$, the payoff curve $\bar{\Phi}$, which gives the optimal ex ante pricing as a function of quantile, is given by the concave hull of the price-posting payoff curve Φ .*

For linear objectives, the interim optimization problem (Definition B.1) is solved by reduction to the ex ante optimization problem (Definition B.3).

Lemma B.3 (Alaei et al., 2013). *Given any linear payoff objective $\mathbf{Payoff}[\cdot]$, for any monotone allocation $x^*(\cdot)$ and an agent with any price-posting payoff curve $\Phi(q)$, the expected payoff of agent is upper-bounded her expected marginal payoff of the same allocation rule, i.e.,*

$$\mathbf{Payoff}[x^*] \leq \mathbf{E}[\bar{\Phi}'(q) \cdot x^*(v(q))].$$

Furthermore, this inequality holds with equality if the allocation rule x^* is constant all intervals of values $v(q)$ where $\bar{\Phi}(q) > \Phi(q)$.

This framework allows optimal mechanisms for linear objectives to be characterized. The ex ante optimal mechanism is given by an appropriate ironing of the price-posting payoff curve. The optimal interim allocation, for any interim allocation constraint, is given by ironing the same quantiles as the price-posting payoff curve is ironed. The resulting allocation rule x optimizes $\bar{\Phi}$ pointwise and is constant on intervals of values $v(q)$ where $\bar{\Phi}(q) > \Phi(q)$.

Interim Optimization for Lagrangian Objectives

The difficulty of applying the framework described above is that welfare maximization (also revenue maximization) with budgeted agents is non-linear (Definition B.1). Our approach prove Lemma B.1 is to consider the optimization program for welfare maximization with budgeted agents, Lagrangian relax the budget constraint, and observe that the resulting Lagrangian objective is linear. Then, a characterization of the form of the optimal mechanism for any Lagrangian objective implies the form of the optimal mechanism for the optimal choice of the Lagrangian parameter. Thus, the framework above for optimizing linear objectives can be effectively applied to solve the problem of budgeted agents.

We begin by writing an optimization program for the interim welfare maximization problem for winner-pays-bid mechanisms. The budget constraint for the winner-pays-bid mechanisms is $p(v) \leq Bx(v)$ for all $v \in [0, h]$ where h is the largest value in the support of distribution F . We introduce the following lemma to help simplify the budget constraint.

Lemma B.4. *Given any interim allocation and payment rule (x, p) in BNE with $x(v) > 0$ for $v \in (v^\dagger, h]$, then the ratio $\frac{p(v)}{x(v)}$ is non-decreasing on $(v^\dagger, h]$.*

Proof. By the Myerson (1981) characterization of BNE, the interim allocation rule is non-decreasing and the payment rule satisfies the payment identity, i.e., $p(v) = v \cdot x(v) - \int_0^v x(t)dt$. Suppose $x(v) > 0$ for $v \in (v^\dagger, h]$. Consider the derivative of the ratio $\frac{p(v)}{x(v)}$ with respect to v ,

$$\frac{d}{dv} \left[\frac{p(v)}{x(v)} \right] = \frac{d}{dv} \left[\frac{v \cdot x(v) - \int_0^v x(t)dt}{x(v)} \right] = 1 - 1 + \frac{\int_0^v x(t)dt}{(x(v))^2} x'(v) \geq 0.$$

Thus, the ratio is non-decreasing in value v from v^\dagger to h . □

Combining Lemma B.4 and the fact that the budget constraint holds at value v automatically if the allocation $x(v)$ equal to zero, the budget constraint can be simplified as $p(h) \leq Bx(h)$. We use Lagrangian relaxation to move the budget constraint into the objective, and get a optimization program as follows,

$$\begin{aligned} \max_{(x,p)} \quad & \mathbf{E}[v \cdot x(v)] + \lambda Bx(h) - \lambda p(h) \\ \text{s.t.} \quad & (x, p) \text{ are BIC, IIR,} \\ & \text{and feasible.} \end{aligned} \tag{3}$$

To solve this Lagrangian relaxation program, there will be a correct Lagrangian multiplier λ for which the budget constraint is met with equality. Once the correct value of the Lagrangian

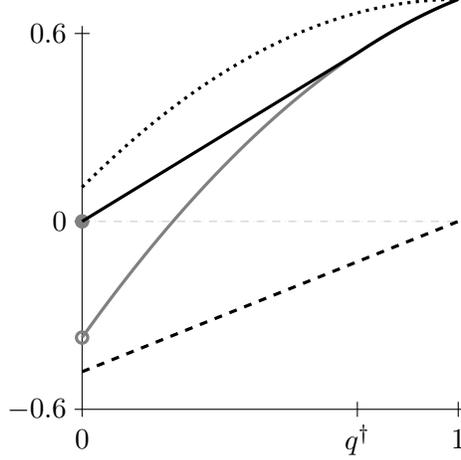


Figure 6: Depicted are the $-\lambda v(q)$ curve (black dashed), the $\int_0^q v(t)dt + \lambda B$ curve (black dotted), the Lagrangian welfare curve $V_\lambda(q)$ (gray solid), and the ironed Lagrangian welfare curve $\bar{V}_\lambda(q)$ (black solid) corresponding to an agent with value uniformly distributed on $[0, 1]$ with budget $B = \frac{1}{4}$ and Lagrangian multiplier $\lambda = \frac{2}{5}$.

multiplier is determined, the welfare-optimal mechanism can be solved using the Lagrangian payoff curve that corresponds to the Lagrangian welfare $\mathbf{E}[v \cdot x(v)] + \lambda Bx(h) - \lambda p(h)$. To prove Lemma B.1, we give a simple characterization of the optimal mechanism for any Lagrangian parameter.

Notice that both surplus, revenue, and the allocation and payments of particular agents are linear objectives. Thus, for a fixed Lagrangian parameter the Lagrangian welfare optimization is a linear objective.

Lemma B.5. *Given any interim allocation and payment rule (x, p) in BNE, for any fixed Lagrangian parameter λ , the Lagrangian welfare $\mathbf{E}[v \cdot x(v)] + \lambda Bx(h) - \lambda p(h)$ is a linear objective.*

Proof. By the definition of linearity, the sum of linear objectives and a scalar multiple of a linear objective are both linear objectives. $x(v)$ is linear for all v , hence, the scalar multiple of allocation $\lambda Bx(h)$, and the expected surplus $\mathbf{E}[v \cdot x(v)]$ are both linear objectives. By the payment identity, the payment is an integral of $x(v)$ and integral is a linear operator, hence, the scalar multiple of price $\lambda p(h)$ is also a linear objective. Thus, the Lagrangian welfare $\mathbf{E}[v \cdot x(v)] + \lambda Bx(h) - \lambda p(h)$ is a linear objective. \square

Consider optimizing the program (3) for a fixed Lagrangian parameter λ . To apply the framework discussed previously, we first construct the price-posting payoff curve. Notice that the identified Lagrangian price-posting welfare curve is discontinuous at $q = 0$ (unless $\lambda = 0$, i.e. when the budget constraint is not binding).

Lemma B.6. *The Lagrangian price-posting welfare curve $V_\lambda(\cdot)$ for a public budget agent satisfies*

$$V_\lambda(q) = \begin{cases} 0 & \text{if } q = 0, \\ \int_0^q v(q) dq - \lambda v(q) + \lambda B & \text{otherwise.} \end{cases}$$

Proof. Consider posting a price $v(q)$ in the quantile space.

For $q > 0$ (strictly positive), the price $v(q)$ is strictly less than the highest value $v(0) = h$, so $p(h) = v(q)$ and $x(h) = 1$. Thus, the Lagrangian objective for $q \in (0, 1]$ is $V_\lambda(q) = \int_0^q v(t)dt - \lambda v(q) + \lambda B$.

For $q = 0$, an agent with the highest value h is indifferent between buying and not buying. If this agent buys, the objective is negative; if this agent does not buy, the objective is zero. Per the definition of the price-posting revenue curve, we break this tie in favor of the objective. \square

Proof of Lemma B.1. If the budget does not bind, it is optimal to allocate the interim constraint $x^*(\cdot)$, since the welfare of any interim feasible allocation is at most the same as the welfare of the interim constraint.

Next, we assume that the budget binds, i.e., $\lambda > 0$. Notice that on $q \in (0, 1]$, the Lagrangian price-posting welfare curve is the constant λB plus the difference between the original welfare curve $\int_0^q v(t)dt$ and the scaled value function $\lambda v(q)$. Since the original welfare curve is always concave, under the public-budget regularity assumption, this Lagrangian price-posting welfare curve is concave on $q \in (0, 1]$. Due to the discontinuity at $q = 0$, and the fact that for sufficient small $q > 0$, $V_\lambda(q)$ is negative, the Lagrangian welfare curve (i.e., solving the ex ante optimization problem) is the Lagrangian price-posting welfare curve with ironing from 0 to some q^\dagger (See Figure 6). Therefore, by Lemma B.3, the welfare-optimal mechanism allocates as by the interim constraint $x^*(\cdot)$ except ironing the top values between $v^\dagger = v(q^\dagger)$ and $h = v(0)$. \square