On Greedy Algorithms and Approximate Matroids

a riff on Paul Milgrom's "Prices and Auctions in Markets with Complex Constraints"

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A 50-Year-Old Puzzle

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Why should economists care?: real-world market design often requires heuristics.

1. *Computational reasons.* Not enough time/computational power to solve exactly. [Nisan/Ronen 99, Lehmann/O'Callaghan/Shoham 99]
2. *Economic reasons.* Descending clock implementations require "reverse greedy" algorithms. [Milgrom/Segal 14]

Two Example Problems

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Milgrom-Segal/FCC Incentive Auction: each station has valuation for its current license.

- goal: select subset of stations to max welfare
  - winners keep their licenses
  - subject to repacking winners into target # of channels

General Formalism

Packing problem: ground set X, collection C of subsets (C satisfies "free disposal/downward-closure").

- each x in X has a nonnegative value v_x
- goal: choose S from C to maximize $\Sigma_{x \text{ in } S}\, v_x$

Examples: (all NP-hard)

- knapsack
- single-minded bidders (LOS99) [≈ independent set]
- station repacking (MS14) [≈ graph coloring]

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Definition: (X,C) is a *matroid* if .... [omitted]

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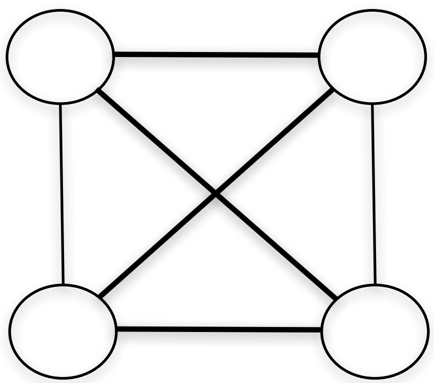
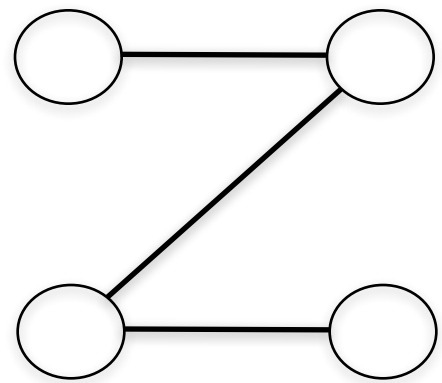
Example: acyclic subgraphs



Non-example: matchings



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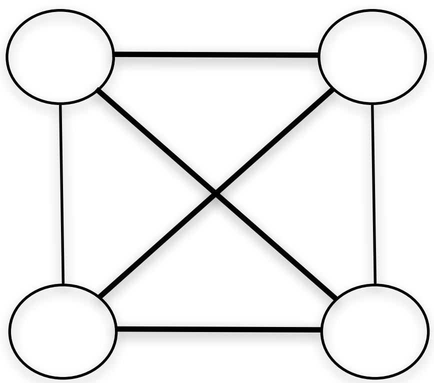

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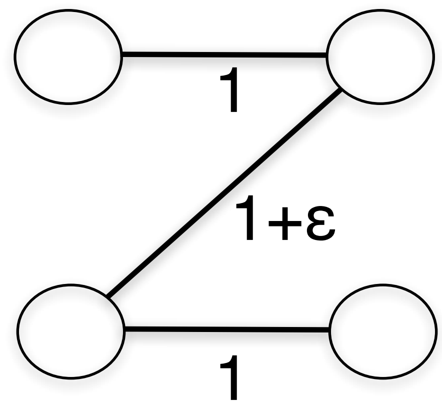

Fact: greedy algorithm always optimal iff matroid.

Approximate Matroids/Substitutes?

Substitutability index: [Milgrom]

$$\rho(C) \coloneqq \max_{matroid\,.R \subseteq C} \left(\min_{X \in C} \left(\max_{\substack{X' \subseteq X \\ X' \in R}} \frac{|X'|}{|X|} \right) \right)$$

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Heuristic #1: Define R^* = argmax above. Optimize over R^* (e.g., using greedy) instead of over C.

Theorem: [Milgrom] for *every* C, worst-case (over v_x's) approximation is exactly ρ(C).

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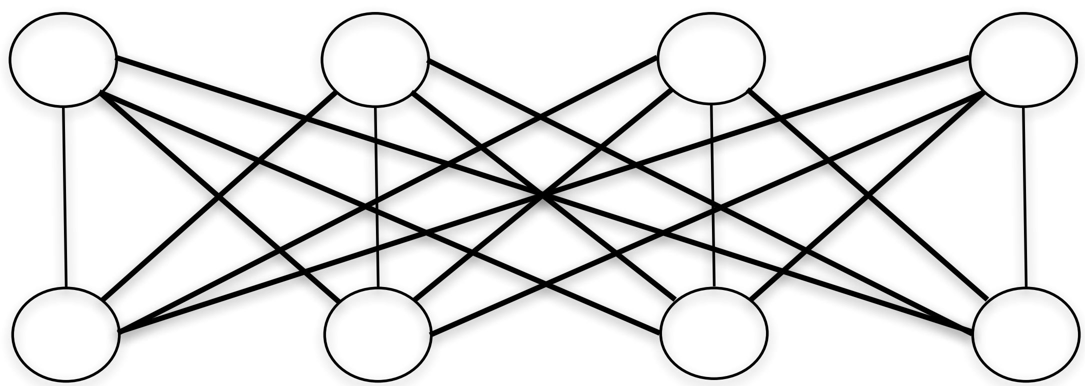

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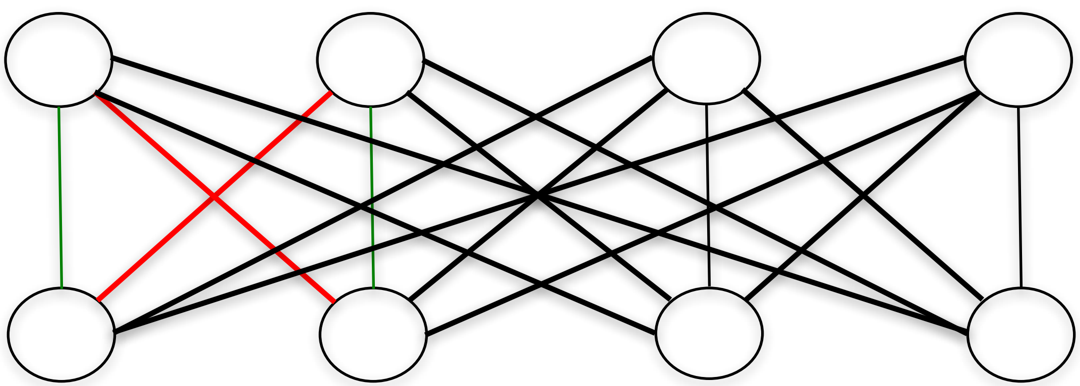

substitutability index of matchings in $K_{n,n}$ = $1/n$

An Alternative Parameterization

Rank quotient: [Korte/Hausmann 78]

$$\alpha(C) := \min_{X, X' \text{ maximal in } C} \frac{|X'|}{|X|}$$

Heuristic #2: Run greedy algorithm w.r.t. C.

- one pass through elements from highest to lowest
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