

# The Optimal Maturity of Government Debt

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## Abstract

We study a Ramsey planner who chooses a distorting tax on labor and manages a portfolio of bonds of different maturities in the representative agent economy with aggregate shocks. We show that covariances of holding period returns of these bonds with the primary deficit are the key statistics that determine the optimal composition of Ramsey portfolio. We document properties of these moments in the U.S. data and calibrate a version of a neoclassical model with Epstein-Zin preferences that matches these moments. The optimal portfolio does not short any bond, allocates approximately equal share of portfolio in bonds of different maturity with a slight tilt towards longer maturities when the outstanding debt is large, and requires little re-balancing in response to aggregate shocks. These prescriptions stand in marked contrast with the prescriptions of standard models used in the business cycle literature. We show that the difference in the results is driven by the counterfactual asset pricing implications of such models.

# 1 Introduction

A central insight of portfolio theory is that the composition of the optimal investment portfolio is determined by the covariance between returns of the securities available to the investor with her income and expenditure shocks. The optimal portfolio uses the fluctuations in returns to hedge investor's shocks and smooth her consumption stream. As the choice of the maturity structure of government debt is also a portfolio problem, the behavior of returns on debts of different maturities with shocks to government revenues and expenditures determines the optimal maturity structure. The canonical macroeconomic models that are used to study the optimal maturity structure are notoriously bad at capturing the behavior of asset prices observed in the data. This makes interpreting the normative prescriptions of such models problematic.

In this paper we re-examine the normative prescriptions for the optimal management of the maturity of government debt in a canonical representative agent Ramsey model. First, we show theoretically that covariances between certain measures of returns on debts of various maturities with the innovations to the present value of government revenues and expenditures determine the optimal choice of maturity structure in such an economy. We then document the key stylized facts about such covariances in the U.S. in the post WWII period. Finally, we calibrate a version of this economy with Epstein-Zin preferences to match these stylized facts, develop numerical methods to computationally characterize the portfolio problem and study the optimal maturity structure.

We find that given the observed behavior of debt returns it is impossible for the government to fully hedge its shocks and as the result the value of outstanding debt fluctuates with the aggregate shocks. The optimal debt portfolio is long in both long-term and short-term debt with the optimal portfolio composition depending on the value of the outstanding debt. In our baseline calibration with one- and five-year bonds, the optimal share of 5 year bonds in the portfolio is about 45% when debt to GDP ratio is about 30 percent, and it raises to 55% when debt to GDP ratio is about 160 percent. These ratios do not change much in response to typical business-cycle frequency shocks as the optimal portfolio requires little re-balancing between the maturities.

These results stand in marked contrast with the prescriptions of the canonical Ramsey models, such as Buera and Nicolini (2004) and Angeletos (2002). Those

models predict that the optimal choice of maturities of government debt can fully hedge government shocks, but do so by taking extremely long and short positions in debts of different maturities. The standard prescription of such models is that the government should issue long-term debt valued tens or even hundreds times of GDP and take offsetting short positions in the short-term debt of similar magnitude. The optimal portfolio is then massively re-balances after each shock, again with the change in debt positions often of the order of tens of times of GDP. Furthermore, the composition of the optimal portfolio is very sensitive to the available maturities.

The key reason for the difference in our results comes from the discrepancies in the behavior of debt returns in the data and in the canonical Ramsey models. In the data the returns on government debts are quite volatile and a substantial part of that volatility is unrelated to the shocks in government revenues or expenditures over business cycle frequencies. As a result, holding large positions in any given maturity is risky for the government, and the government's ability to hedge its revenues and expenditures shocks is limited. To eliminate that risk, the government holds similar positions in debts of different maturities, smoothly changing the duration of the outstanding government debt as it accumulates or decreases the total outstanding debt. In contrast, canonical Ramsey models assume that debt returns are determined solely by aggregate shocks that affect government revenues and expenditures. This assumption generates fluctuations in returns on government debt that are, at the same time, small and highly correlated with these shocks. As a result, the government can fully hedge its shocks using small fluctuation in returns by leveraging and taking huge short and long positions in debts of different maturities. Shocks to returns that are uncorrelated with government financial needs would make holding such positions extremely risky.

Our paper builds on a large Ramsey literature on optimal taxation and debt management over the business cycle, going at least to the work of Barro (1979) and Lucas and Stokey (1983). Buera and Nicolini (2004) and Angeletos (2002) are the two benchmark specifications of those models to incomplete market settings with debt of different maturities to which we are most closely related. Faraglia et al. (2012) and Debortoli et al. (2016 forthcoming) extend this framework by restricting government's ability to re-balance its portfolio and commit respectively. Lustig et al. (2008) allow for nominal bonds of several maturities but impose portfolio restrictions that prevent the government from going short in any maturity. Farhi (2010) allows

the government to hold a risky asset in a form of capital. All these papers use the same general specification of preferences and shocks as standard Ramsey models, they also fail to match the behavior of returns observed in the data.

Our work also builds on papers by Bohn (1990) and Bhandari et al. (2017a) who showed how the insights from portfolio theory extend to the prescriptions about optimal security trading by the government, and emphasized the role of covariances of the securities available to the government and shocks. We adapt general theoretical insights of those papers to study the optimal maturity structure of government debt.

We also build on a large empirical and theoretical literature in macro and finance about the behavior of debt returns. That literature has documented that returns in the data are quite volatile, not very correlated with standard macroeconomic shocks, and features an increasing yield curve. For example Campbell and Shiller (1991), Campbell (1995), Cochrane and Piazzesi (2005) and Ludvigson and Ng (2009). Several authors, such as Piazzesi and Schneider (2006), Bansal and Shaliastovich (2013), Wachter (2006), and Albuquerque et al. (2016) proposed general equilibrium frameworks that are consistent with these facts. We extend the findings of this literature to study the relationship between the returns and the innovations to the present value of government revenue and expenditure shocks, which our theoretical framework predicts to be the key measure that determines the optimal debt composition. We also build on Albuquerque et al. (2016) in constructing a general equilibrium model that is consistent with the empirical asset pricing patterns.

To the best of our knowledge, we are the first to develop numerical methods to study incomplete market economies with Epstein-Zin preferences. Karantounias (2013), an early predecessor to our work, solves the Ramsey problem with Epstein-Zin preferences when markets are complete. We build on some of his insights, but our incomplete market economies is substantially more complicated, with several additional state variables requiring additional computational techniques. To that end, we extend the techniques developed in Evans (2014) and Bhandari et al. (2017b) to use perturbation theory to sequentially approximate the optimal Ramsey plan around the current level of government debt. In doing so we also build on insights in Guu and Judd (2001) and Devereux and Sutherland (2011) to approximate the dynamic portfolio problem of the Ramsey planner.

The rest of the paper is organized as follows. In Section 2 we present a simple Ramsey framework that solves for the approximated optimal government portfolio in

closed form. It shows the role that covariances of debt returns play in determining the optimal debt composition. Section 3 presents the empirical facts about those covariances. Section 4 presents our general framework and numerical solution methods. Section 5 shows the results.

## 2 Theoretical motivation

In this section we used a simplified model to highlight the key considerations that shape the optimal portfolio problem of a canonical Ramsey planner. Time is discrete, and the horizon is infinite. The uncertainty in this economy stems from a exogenous Markov process  $s_t$  with shocks taking values in a finite set. Let  $s^t = (s_0, s_1, \dots, s_t)$  denote the finite history of shocks up until time  $t$ . The economy is populated by a representative agent which consumes  $c_t(s^t)$ , works  $l_t(s^t)$  hours and pays a linear income tax rate  $\tau_t(s^t)$ . For the remainder of this paper we will use the notation  $x_t$  to denote the random variable  $x_t(s^t)$ . The resource constraint for this economy is given by

$$c_t + g_t = \theta_t l_t. \quad (1)$$

The representative household and the government trade  $K$  securities available in zero net supply. Security  $k$ , when bought at time  $t - 1$ , has price  $q_{t-1}^k$  and gives a payoff  $p_t^k$  at time  $t$ . All securities are assumed to be exponentially decaying rates  $\delta^k$ . Let  $-B_t^k$  be the asset holdings of the government of security  $k$  at time  $t$ . We use a convention that positive values of  $B_t^k$  corresponds to net liability for the government. In later sections of the paper they will denote debts of different maturities. The government budget constraint is given by

$$g_t + \sum_k (p_t^k + (1 - \delta_k) q_t^k) B_{t-1}^k = \tau_t \theta_t l_t + \sum_k q_t^k B_t^k. \quad (2)$$

Denote holding period returns as  $R_t^k \equiv \frac{p_t^k + (1 - \delta^k) q_t^k}{q_{t-1}^k}$ . We use boldface letters to denote the vector corresponding to the  $k$  securities, e.g.  $\mathbf{R}_t$  denoting the vector of returns  $R_t^1, \dots, R_t^K$ .

Households rank consumption and labor plans according to a utility function

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \eta_t \left[ \frac{c_t^{1-\rho}}{1-\rho} - \theta_t^{1-\rho} \frac{l_t^{1+\phi}}{1+\phi} \right] \quad (3)$$

where  $\eta_t$  is the preference shock. The shock process follows

$$\begin{bmatrix} \ln \frac{\theta_t}{\theta_{t-1}} \\ \ln \frac{g_t}{\theta_t} \\ \mathbf{p}_t \\ \eta_t \end{bmatrix} = \begin{bmatrix} \bar{\theta} \\ \bar{g} \\ \mathbf{1} \\ 1 \end{bmatrix} + \sigma \begin{bmatrix} \epsilon_{\theta,t} \\ \epsilon_{g,t} \\ \boldsymbol{\epsilon}_{p,t} \\ \epsilon_{\eta,t} \end{bmatrix} \equiv \boldsymbol{\mu} + \sigma \mathbf{s}_t,$$

where  $\sigma$  is a positive scalar,  $\boldsymbol{\mu}$  is a  $(K+3) \times 1$  vector and vector  $\mathbf{s}_t$  represents shocks that are mean zero and i.i.d. over time but can be arbitrarily correlated to each other.

Our choice of preference, shocks processes and asset structure is chosen to simplify the exposition. The analysis of any finitely lived security extends without changes but requires more involved notion. Our insights extend to more general preference specifications and shock processes, in particular like those we consider in the quantitative section of the paper.

We study the classical Ramsey problem of choosing the optimal sequences of taxes and debt portfolios  $\{\tau_t, \mathbf{B}_t\}_t$  to finance the stochastic process  $\{g_t\}_t$  so that the resulting competitive equilibrium maximizes (3). Since our economy is growing, it will be convenient to renormalize several period  $t$  variables by the TFP shock  $\theta_t$  to make the economy stationary. Let  $\hat{c}_t = c_t/\theta_t$ ,  $\hat{g}_t = g_t/\theta_t$  and  $\hat{B}_t^k \equiv q_t^k B_t^k/\theta_t$ . Here  $\hat{B}_t^k$  corresponds to the market value of government holdings of security  $k$  normalized by  $\theta_t$ . Then our constraints can be re-written as

$$\hat{g}_t + e^{-g_{\theta,t}} \sum_k R_t^k \hat{B}_{t-1}^k = \tau_t l_t + \hat{B}_t, \quad (4)$$

$$\hat{B}_t = \sum_k \hat{B}_t^k, \quad (5)$$

$$\hat{c}_t + \hat{g}_t = l_t, \quad (6)$$

where  $g_{\theta,t} = \bar{\theta} + \sigma \epsilon_{\theta,t}$ . Finally, let the normalized primary deficit be defined as  $\hat{X}_t \equiv \hat{g}_t - \tau_t l_t$ .

## 2.1 Quasi-linear preferences

The analysis is particular straightforward if we assume quasilinear preferences,  $\rho = 0$ , and no preference shocks,  $\eta_t = 1$  for all  $t$ . In this case  $R_t^k$  follows stochastic process that is purely determined by the stochastic process for  $p_t^k$  and is independent of other variables. The intra-temporal optimality condition of the household is

$$l_t^\phi = 1 - \tau_t. \quad (7)$$

Therefore,  $\{\hat{c}_t, l_t, \tau_t, \hat{B}_t, \hat{\mathbf{B}}_t\}_t$  is a competitive equilibrium if and only if they satisfy constraints (4) - (7). The optimal Ramsey problem can then be solved recursively as

$$V(\hat{B}_-) = \max_{\hat{c}, l, \hat{\mathbf{B}}} \mathbb{E} e^{-g\theta(s)} \left\{ l(s) - \frac{l(s)^{1+\phi}}{1+\phi} + \beta V(\hat{B}(s)) \right\}$$

subject to

$$e^{-g\theta(s)} \sum_k \hat{B}^k R^k(s) + l(s)^{1+\phi} = l(s) - \hat{g}(s) + \hat{B}(s),$$

$$\sum_k \hat{B}^k = \hat{B}_-,$$

where we used to the feasibility constraint to substitute for  $\hat{c}$ .

This problem can be analyzed along the lines we developed in Bhandari et al. (2017a). In particular we consider second order Taylor expansion of the policy functions to this Bellman equation with respect to  $\sigma$  and assume that the discount factor is small in a sense that  $1 - \beta$  is of the same order as  $\sigma$ . This approach yields tractable linear approximations of policy functions that allows closed form characterization of the optimal portfolio. Let  $X_{\tau,t} = g_t - \theta_t \tau l(\tau)$  be the primary deficit at a given tax rate  $\tau$ ,  $PV_t(X_\tau)$  be the present discount value of primary deficits,

$$PV_t(X_\tau) = \mathbb{E}_t \sum_j \beta^j X_{\tau,t+j},$$

and let  $\hat{P}V_t(X_\tau) \equiv PV_t(X_\tau)/\theta_{t-1}$ . We use  $\mathbb{C}[\mathbf{R}, \mathbf{R}]$  to denote the unconditional covariance matrix of the stochastic process for returns  $\mathbf{R}_t$  and  $\mathbb{C}[\mathbf{R}, PV(X_\tau)]$  the vector of covariances of  $\mathbf{R}_t$  with  $PV_t(X_\tau)$ .

**Proposition 1.** *The optimal debt portfolio  $\mathbf{B}_{t-1} = \theta_{t-1} \hat{\mathbf{B}}_{t-1}$  is given by*

$$\hat{\mathbf{B}}_{t-1} = -\mathbb{C}[\mathbf{R}, \mathbf{R}]^{-1} \mathbb{C}[\mathbf{R}, \hat{PV}(X_\tau)] + \frac{\mathbb{C}[\mathbf{R}, \mathbf{R}]^{-1} \mathbf{1}}{\mathbf{1}^\top \mathbb{C}[\mathbf{R}, \mathbf{R}]^{-1} \mathbf{1}} \left( \hat{B}_{t-1} + \mathbf{1}^\top \mathbb{C}[\mathbf{R}, \mathbf{R}]^{-1} \mathbb{C}[\mathbf{R}, \hat{PV}(X_\tau)] \right) + \mathcal{O}(\sigma, (1-\beta)) \quad (8)$$

where  $\tau$  satisfies  $B_{t-1} = \beta \mathbb{E}_{t-1}[PV_t(X_\tau)]$ .

The easiest way to understanding this proposition is to consider the variance minimization problem at  $t - 1$  given total debt  $B_{t-1}$

$$\min_{\mathbf{B}} \text{Var}_{t-1} \left[ \sum_k B^k R_t^k + PV_t(X_\tau) \right]$$

subject to

$$\sum_k B^k = B_{t-1},$$

where  $\tau$  satisfies  $B_{t-1} = \beta PV_t(X_\tau)$ . Denote the solution to this variance minimization problem by  $\mathbf{B}_{t-1}^*$ . The next proposition shows that the optimal portfolio characterized in Proposition 1 is approximately given by  $\mathbf{B}_{t-1}^*$ .

**Proposition 2.** *The optimal debt portfolio satisfies*

$$\mathbf{B}_{t-1} = \mathbf{B}_{t-1}^* + \mathcal{O}(\sigma, (1 - \beta)).$$

Thus, Proposition 1 shows that the optimal Ramsey portfolio is chosen so as to minimize risk, i.e. to choose the portfolio composition to maximally offset fluctuations in the present discounted value of primary deficits at a constant tax rate. The intuition for this result is as follows. The government would like to minimize the fluctuations in the deadweight loss of taxes and for this reason would like to keep the tax rate as smooth as possible. Shocks to government revenues and expenditures, and hence shocks to the present value of primary deficits, hinder government's ability to satisfy its budget constraint at constant tax rate. The optimal portfolio is chosen so as to minimize the effect of shocks.

An important insight that emerges from equation (8) is that co-movements of returns with the shocks to government expenditures and revenues are the critical parameters that determines the optimal composition of the Ramsey portfolio. To



illustrate this, we consider a simple example, which will also prove helpful in later sections to understand the optimal portfolios that emerge in the models of Angeletos (2002) and Buera and Nicolini (2004) and to contrast with our results. Suppose the government has access to two assets with returns  $R_t^1$  and  $R_t^2$ . For simplicity of exposition, assume  $R_t^1 = \frac{1}{\beta}(1 + \alpha_1 \hat{P}\hat{V}_t(X_\tau))$  and  $R_t^2 = \frac{1}{\beta}(1 + \alpha_2 \hat{P}\hat{V}_t(X_\tau) + \epsilon_t)$  where  $\alpha_2 < \alpha_1 \leq 0$  and  $\epsilon_t$  is orthogonal to  $\hat{P}\hat{V}_t(X_\tau)$ .<sup>1</sup> From Proposition 1, the optimal portfolio will be given by

$$\hat{\mathbf{B}}_{t-1} = \begin{bmatrix} -1/\alpha_1 \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{\alpha_2^2 \sigma_{\hat{P}\hat{V}}^2 - \alpha_1 \alpha_2 \sigma_{\hat{P}\hat{V}}^2 + \sigma_\epsilon^2}{(\alpha_1 - \alpha_2)^2 \sigma_{\hat{P}\hat{V}}^2 + \sigma_\epsilon^2} \\ \frac{\alpha_1^2 \sigma_{\hat{P}\hat{V}}^2 - \alpha_1 \alpha_2 \sigma_{\hat{P}\hat{V}}^2}{(\alpha_1 - \alpha_2)^2 \sigma_{\hat{P}\hat{V}}^2} \end{bmatrix} (\hat{B}_{t-1} + 1/\alpha_1),$$

where  $\sigma_\epsilon^2$  is the variance of  $\epsilon_t$  and  $\sigma_{\hat{P}\hat{V}}^2$  is the variance of  $\hat{P}\hat{V}_t(X_\tau)$ . Consider two extreme cases. First, suppose that the returns are highly correlated with the shocks to the primary surplus, in the sense that  $\sigma_\epsilon^2 \approx 0$ . In this case the optimal portfolio is given by

$$\hat{\mathbf{B}} \approx \begin{bmatrix} \frac{\alpha_2}{\alpha_1 - \alpha_2} \hat{B} - \frac{1}{\alpha_1 - \alpha_2} \\ \frac{\alpha_1}{\alpha_2 - \alpha_1} \hat{B} - \frac{1}{\alpha_2 - \alpha_1} \end{bmatrix}.$$

As  $\alpha_2 \rightarrow 0$ , this portfolio diverges to  $\begin{bmatrix} -\infty \\ \infty \end{bmatrix}$ . This example is constructed to capture the key mechanism behind findings of Buera and Nicolini (2004) that the optimal government portfolio of the government issues huge amount of the long debt (tens or even hundreds times of GDP) and take the offsetting short position in the short-term debt. In our example securities 1 and 2 approximate the behavior of the short-term and long-term bond in their economy. As we discuss in more details in Section 4, standard calibrations of Ramsey problems imply that bond returns are highly correlated with government shocks ( $\sigma_\epsilon^2 \approx 0$ ), the holding period returns on long-term bonds decline in response to a temporary adverse shock by more than returns on short-term bonds ( $\alpha_2 < \alpha_1 \leq 0$ ) but bond returns overall are very smooth given the size of standard business cycle shocks ( $\alpha_2$  is close to zero). Given such return process, the government can hedge all of its shocks to primary deficits, but to do so requires taking a huge positive position in long debt and a negative position in the short debt. Such highly levered holding of the long bond amplifies its small negative co-movement with the shocks to primary deficit and provides substantial insurance.

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<sup>1</sup>This will be the case if  $R_t^1 = \frac{1}{\beta}(1 + a_1 \epsilon_{g,t} + a_2 \epsilon_{\theta,t})$  and  $R_t^2 = \frac{1}{\beta}(1 + \alpha(a_1 \epsilon_{g,t} + a_2 \epsilon_{\theta,t}) + \epsilon_t)$ .

Alternatively, consider the case when much of the variability of assets returns is not driven by the shocks to the primary deficit, in the sense that  $\sigma_\epsilon^2$  as large. As  $\sigma_\epsilon^2 \rightarrow \infty$ , the optimal portfolio converges to  $\begin{bmatrix} \hat{B}_{t-1} \\ 0 \end{bmatrix}$ . Fluctuations in returns that are orthogonal to the shocks to the primary deficit makes holding such security risky and the optimal portfolio in the economies with high  $\sigma_\epsilon^2$  looks very different from those with low  $\sigma_\epsilon^2$ . Thus the correlations of returns with primary deficit is the critical ingredient in determining the optimal portfolio.

## 2.2 Recursive Ramsey problem with isoelastic preferences

The discuss above extends to economies with risk-averse consumers. Consider now general isoelastic preferences (3). The implementability constraint can be written as

$$\begin{aligned} & e^{-g\theta,t} \left( \sum_k \hat{c}_t^{-\rho} \eta_t \left[ \frac{p_t^k + (1 - \delta^k)q_t^k}{\beta \mathbb{E}_{t-1} \eta_t e^{-\rho g\theta,t} \hat{c}_t^{-\rho} [p_t^k + (1 - \delta^k)q_t^k]} \right] \hat{c}_{t-1}^{-\rho} \eta_{t-1} \hat{B}_{t-1}^k \right) + \eta_t l_t^{1+\phi} \\ & = \hat{c}_t^{-\rho} \eta_t \hat{B}_t + \hat{c}_t^{1-\rho} \eta_t. \end{aligned}$$

Now substitute for  $\mathcal{B}_t^k = c_t^{-\rho} \eta_t \hat{B}_t^k$ ,  $\hat{\mathcal{B}}_t = c_t^{-\rho} \eta_t \hat{B}_t$ ,  $\mathcal{R}_{t-1,t}^k = \hat{c}_t^{-\rho} \eta_t \left[ \frac{p_t^k + (1 - \delta^k)q_t^k}{\beta \mathbb{E}_{t-1} e^{-\rho g\theta,t} \eta_t \hat{c}_t^{-\rho} [p_t^k + (1 - \delta^k)q_t^k]} \right] c_{t-1}^{-\rho}$  and  $\mathcal{X}_t = \eta_t l_t^{1+\phi} - \hat{c}_t^{1-\rho} \eta_t$  so that we can write these conditions as

$$\begin{aligned} & e^{-g\theta,t} \left( \sum_k \mathcal{R}_{t-1,t}^k \hat{\mathcal{B}}_{t+1}^k \right) + \mathcal{X}_t = \hat{\mathcal{B}}_t, \\ & \sum_k \hat{\mathcal{B}}_t^k = \hat{\mathcal{B}}_t, \\ & \tilde{q}_t^k = \beta \mathbb{E}_t \eta_{t+1} e^{-\rho g\theta,t+1} \hat{c}_{t+1}^{-\rho} [p_{t+1}^k + (1 - \delta^k)q_{t+1}^k], \\ & \mathcal{R}_{t-1,t}^k = \frac{U_{c,t} p_t^k + (1 - \delta^k) \tilde{q}_t^k}{\tilde{q}_{t-1}^k}. \end{aligned}$$

Let This problem is recursive in  $(\hat{\mathcal{B}}_{t-1}, \tilde{\mathbf{q}}_{t-1})$ , where  $\tilde{\mathbf{q}}_t = \{\tilde{q}_t^k\}_k$ . We can simplify it further if we observe that given a tax rate  $\tau$ , consumption,  $\hat{c}_\tau(s)$ , and labor,  $l_\tau(s)$ , in any state  $s$  are pinned down by

$$\hat{c}(s) + g(s) = l(s),$$

$$\hat{c}(s)^{-\rho} (1 - \tau(s)) = l(s)^\phi.$$

Thus, the Ramsey problem can be written recursively as

$$\max_{\tilde{q}(s), \hat{\mathcal{B}}(s), \tau(s), \hat{\mathcal{B}}_-} V(\hat{\mathcal{B}}_-, \tilde{\mathbf{q}}_-) = \mathbb{E} e^{[(1-\rho)g_\theta(s)]} \left\{ \eta(s) \left( \frac{\hat{c}_{\tau(s)}(s)^{1-\rho}}{1-\rho} - \frac{l_{\tau(s)}(s)^{1+\phi}}{1+\phi} \right) + \beta V(\hat{\mathcal{B}}(s), \tilde{\mathbf{q}}(s)) \right\}$$

subject to

$$\mathcal{R}_{\tau(s)}^k(s) = \frac{[\eta(s) \hat{c}_{\tau(s)}(s)^{-\rho} p^k(s) + (1 - \delta^k) \tilde{q}^k(s)]}{\tilde{q}^k_-},$$

$$\tilde{q}^k_- = \beta \mathbb{E} [e^{-\rho g_\theta(s)} (\eta(s) \hat{c}_{\tau(s)}(s)^{-\rho} p^k(s) + (1 - \delta^k) \tilde{q}^k(s))], \quad (9)$$

$$\sum_k e^{-g_\theta(s)} \mathcal{R}_{\tau(s)}^k(s) \hat{\mathcal{B}}^k = -\mathcal{X}_{\tau(s)}(s) + \hat{\mathcal{B}}_- \quad \text{for } s, s_-, \quad (10)$$

$$\hat{\mathcal{B}}_- = \sum_k \hat{\mathcal{B}}^k.$$

Using the same approximation techniques as in the previous section we obtain

**Proposition 3.** *The optimal debt portfolio  $\mathcal{B}_{t-1} = \theta_{t-1} \hat{\mathcal{B}}_{t-1}$  is given by*

$$\begin{aligned} \hat{\mathcal{B}}_{t-1} &= -\mathbb{C}[\mathcal{R}_\tau, \mathcal{R}_\tau]^{-1} \mathbb{C}[\mathcal{R}_\tau, PV(\mathcal{X}_\tau)] \\ &+ \frac{\mathbb{C}[\mathcal{R}_\tau, \mathcal{R}_\tau]^{-1} \mathbf{1}}{\mathbf{1}^\top \mathbb{C}[\mathcal{R}_\tau, \mathcal{R}_\tau]^{-1} \mathbf{1}} \left( \hat{\mathcal{B}}_{t-1} + \mathbf{1}^\top \mathbb{C}[\mathcal{R}_\tau, \mathcal{R}_\tau]^{-1} \mathbb{C}[\mathcal{R}_\tau, P\dot{V}(\mathcal{X}_\tau)] \right) + \mathcal{O}(\sigma, (1 - \beta)) \end{aligned} \quad (11)$$

where  $\tau$  satisfies  $\mathcal{B}_{t-1} = \beta \mathbb{E}_{t-1}(PV_t(\mathcal{X}_\tau))$ .

Just like the optimal portfolio in the quasi-linear economy, the optimal portfolio  $\theta_{t-1} \hat{\mathcal{B}}_{t-1}$  minimizes the variance in the present value of the effective shocks,  $\text{Var}_{t-1} [\sum_k \mathcal{B}^k \mathcal{R}_t^k + PV_t(\mathcal{X}_\tau)]$  subject to  $\sum_k \mathcal{B}^k = \mathcal{B}_{t-1}$ . The adjective ‘‘effective’’ needs to be used (and all variables in the formula adjusted by the marginal utility of consumption) because the marginal cost of public funds now vary with the state of the economy. This cost of raising revenues is proportional to the marginal utility of consumption, necessitating the adjustment. Observe, however, that the main insights of Section 2.1 continue to hold when consumers are risk-averse: the optimal portfolio of the government minimizes fluctuations in the present value of primary deficits and portfolio’s composition depends critically on the covariances of securities returns with shocks to the deficits.

### 3 Bond return data

In this section we use US bond market data and document patterns of holding period returns on bonds of several maturities. As discussed in the previous section, the covariance properties of the returns with shocks that drive fiscal needs are key in determining the optimal portfolio. We will use the moments documented in this section as an input to the quantitative exercise where we numerically compute the optimal portfolio in a calibrated economy.

Our baseline results are compiled using bond prices data from the Fama-Bliss discount bond series in Center for Research in Security Prices (CRSP) database. This series records prices for artificial discount bonds with one to five years to maturity at a monthly frequency from 1952. Let  $q_t^n$  be the price of a bond of maturity  $n$  years at beginning of the year  $t$  and  $P_t$  be the corresponding price level<sup>2</sup> We define the annual real holding period return as

$$R_{t,t+1}^n = \left( \frac{q_{t+1}^{n-1}}{q_t^n} \right) \left( \frac{P_t}{P_{t+1}} \right).$$

The holding period returns are plotted in figure I and the covariance matrix of the returns with output-growth and expenditure/output are tabulated in Table II. We see that (a) returns are correlated across maturities and (b) mean returns as well as the volatility of returns are increasing in the length of maturity and (c) returns are not correlated with output growth. These patterns are also documented and studied in several other papers such as Campbell (1995) etc.

We next use a simple regression of returns on output growth, primary deficit/output and inflation to measure the part of the variation in returns that is unrelated to the fiscal needs.<sup>3</sup>

$$\log R_{t,t+1}^n = \alpha_0 + \alpha_y \text{output growth} + \alpha_g \text{deficits/output} + \alpha_\pi \text{inflation} + \epsilon_{t+1} \quad (12)$$

The OLS estimates are summarized in table II. We see that across maturities  $R^2$  are in the range of 7% – 15% which shows that the orthogonal component in returns

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<sup>2</sup>We use the GDP deflator series from NIPA for  $P_t$

<sup>3</sup>We include inflation in the regression because in the current version of the paper we study real bonds and abstract from monetary policy. In this way we hope to get a conservative estimate of how large is the component of returns that is orthogonal to the government's fiscal or monetary needs.

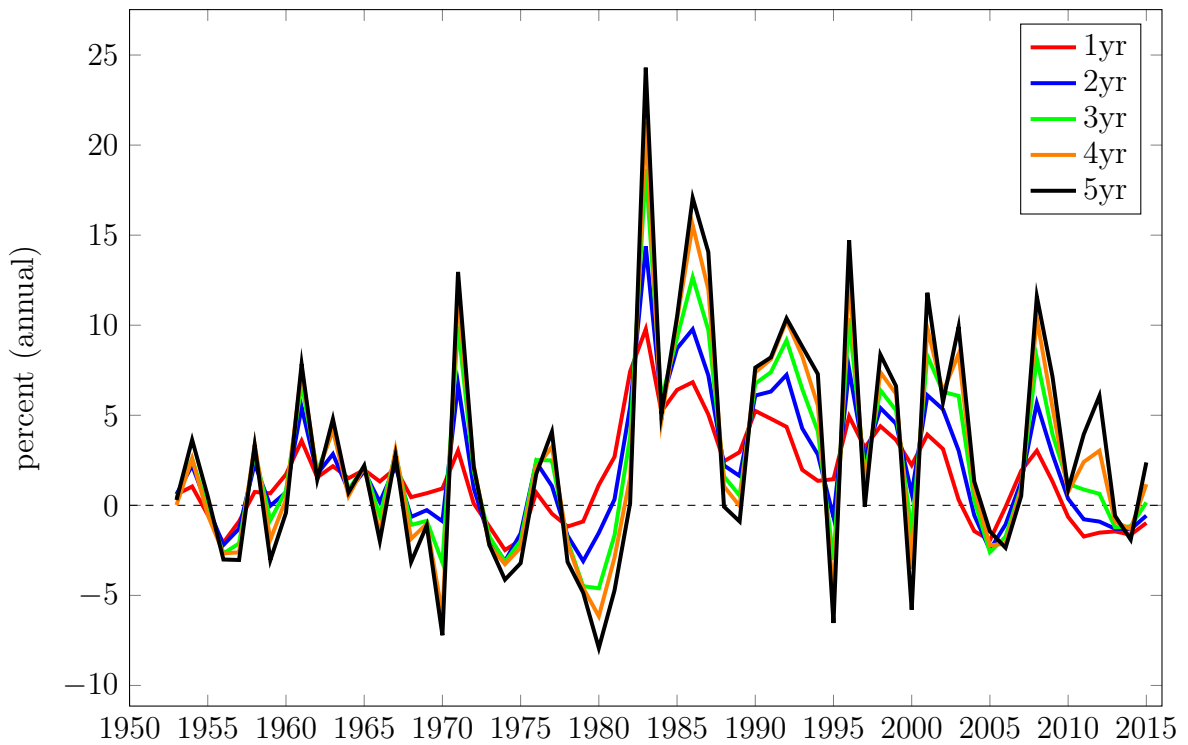


Figure I: Time series for annualized holding period real returns on bonds using Fama-Bliss data

Maturity	mean	1 yr	2 yr	3 yr	4 yr	5 yr	exp /output	output growth
1 yr	1.64	6.79	8.41	9.68	10.42	10.99	-0.84	0.22
2 yr	2.09	8.41	12.51	15.93	18.28	20.19	-2.08	0.21
3 yr	2.49	9.68	15.93	21.46	25.46	28.76	-3.25	0.34
4 yr	2.70	10.42	18.28	25.46	30.99	35.51	-4.09	0.20
5 yr	2.86	10.99	20.19	28.76	35.51	41.33	-5.10	0.21
exp/output	16.99	0.22	0.21	0.34	0.20	0.21	-0.43	5.32
output growth	3.14	-0.72	-0.13	0.44	1.01	1.58	0.33	-1.87

Table I: Descriptive statistics: means and covariance matrix for annualized holding period real returns on bonds using Fama-Bliss data

	1 yr	2 yr	3 yr	4 yr	5 yr
constant	5.54 (1.96)	4.66 (2.66)	3.54 (3.43)	2.63 (4.1)	1.58 (4.61)
output growth	-0.09 (0.16)	-0.07 (0.22)	0.00 (0.29)	0.00 (0.34)	0.03 (0.39)
deficits/ output	-0.31 (0.17)	-0.09 (0.22)	0.12 (0.3)	0.30 (0.34)	0.05 (0.40)
inflation	-0.20 (0.16)	-0.47 (0.21)	-0.71 (0.27)	-0.9 (0.32)	-1.11 (0.36)
$R^2$	0.07	0.08	0.11	0.13	0.16

Table II: OLS estimates for regression (12). Standard errors are in parenthesis.

is large. The estimated coefficients on output growth are all statistically insignificant and except for the one year maturity, the coefficients on deficits/output are also insignificant. These results are largely consistent with predictability regressions that appear in Fama and Bliss (1987), Cochrane and Piazzesi (2005), Ludvigson and Ng (2009). For example, Ludvigson and Ng (2009) use sophisticated dynamic latent factor methods and extract “macro factors” from a large data set of macro variables and use them to predict returns. In specifications where they only use the macro factors they find about 75-80% of the variation in returns is orthogonal to their macro factors. We read the findings in II and related evidence as suggesting that variations in returns are largely driven by predictable movements discount rates that are uncorrelated to macroeconomic aggregates like output growth and deficits.

## 4 Quantitative analysis

A common choice in RBC literature is to use separable expected utility preferences with CRRA form. However, it is well known from the finance literature on the “equity premium puzzle” that for reasonable parameter choices such preferences are not able to generate high excess returns on risky assets or volatility of these returns across time with smooth risk free rates (see, for example, Hansen and Singleton (1982); Mehra and Prescott (1985); Hansen and Jagannathan (1991)). The induced pricing kernel with CRRA preferences is function of aggregate consumption growth which is relatively smooth at business cycle frequencies. As we showed in Propositions 1 and 3, the composition of the optimal Ramsey portfolio crucially depend on the covariance properties of bond returns. Thus, it is crucial to adopt preference specification that implies realistic asset pricing properties.

We follow Albuquerque et al. (2016) and adopt their specification of recursive

Epstein-Zin preferences augmented with the discount factor shocks. As Albuquerque et al. (2016) show, such a model matches a range of empirical facts about returns on bonds and equities such as the excess returns for the long bonds, an upward sloping term structure, variations in the pricing kernel and bond returns that have realistic covariances with aggregate output.<sup>4</sup> While Albuquerque et al. (2016) consider an endowment economy, we further extend this specification by introducing the disutility of labor along the lines of Karantounias (2013). Specifically, we use the following recursion to value stochastic streams of consumption and leisure:

$$V_{t-1} = \mathbb{E}_{t-1} [W_t^{1-\gamma}]^{\frac{1}{1-\gamma}},$$

$$W_t \equiv \left( (1-\beta)\eta_t \left[ c_t^{1-\rho} - (1-\rho)\theta_t^{1-\rho} \frac{n_t^{1+\phi}}{1+\phi} \right] + \beta V_t^{1-\rho} \right)^{\frac{1}{1-\rho}}.$$

Here  $\rho$  measures the inverse of the intertemporal elasticity of substitution and  $\gamma$  measures risk aversion. A special case of this specification that sets  $\rho = \gamma$  recovers the isoelastic form that we used in Section 2.2.

Defining  $U(c, l) = \eta \left[ \frac{c^{1-\rho}}{1-\rho} - \theta^{1-\rho} \frac{n^{1+\phi}}{1+\phi} \right]$ , the first order necessary conditions of the household problem imply that

$$U_{n,t} = (1 - \tau_t)\theta_t U_{c,t},$$

and that assets are priced with by the stochastic discount factor

$$S_{t+1} \equiv \beta m_t^{\frac{\rho-\gamma}{1-\gamma}} \frac{U_{c,t+1}}{U_{c,t}},$$

where  $m_{t+1} = \frac{V_{t+1}^{1-\gamma}}{\mathbb{E}_{t-1} V_{t+1}^{1-\gamma}}$ . Following the previous sections, we construct the normalized variables  $\hat{c}_t = \frac{c_t}{\theta_t}$ ,  $\hat{g}_t = \frac{g_t}{\theta_t}$ ,  $\hat{W}_t = \frac{W_t}{\theta_t}$ ,  $\hat{V}_t = \frac{V_t}{\theta_t}$ ,  $\hat{B}_t = \frac{B_t}{\theta_t}$ , and  $\hat{B}_t^k = \frac{B_t^k}{\theta_t}$ .

We assume that the government has access to  $K$  consols that pays a riskless coupon 1 in each period and decay with rates  $\{\delta^k\}_k$ . Thus, the holding period return

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<sup>4</sup>Other alternatives include Piazzesi and Schneider (2006), Bansal and Shaliastovich (2013) also estimate models with stochastic volatility and correlated shocks to inflation and consumption growth with Epstein and Zin (1989) preferences, Wachter (2006) who use habit-persistence as in Campbell and Cochrane (1999). We conjecture as long as the models are calibrated to match the covariance matrix of returns, their implications on the optimal portfolios would also be the quantitatively similar. We plan to expand on this in future work.

on security  $k$  is given  $R_t^k = \frac{1+(1-\delta^k)q_t^k}{q_{t-1}^k}$ .

After multiplying the budget constraint by  $U_{c,t}$  we obtain the implementability constraint for the planners problem as

$$\sum_k \frac{e^{-g_{\theta,t}} \hat{\mathcal{B}}_{t-1}^k \eta_t (\hat{c}_t^{-\rho} + (1-\delta^k) \tilde{q}_t^k)}{\beta \mathbb{E}_{t-1} \left[ m_t^{\frac{\rho-\gamma}{1-\gamma}} e^{-\rho g_{\theta,t}} (\hat{c}_t^{-\rho} + (1-\delta^k) \tilde{q}_t^k) \right]} = \eta_t \hat{c}_t^{1-\rho} - \eta_t n_t^{1+\phi} + \hat{\mathcal{B}}_t.$$

Here  $\tilde{q}_t^k$  is the marginal utility weighted price of the geometrically decaying coupon bond with decay rate  $\delta^k$ , which must satisfy

$$\tilde{q}_{t-1}^k = \beta \mathbb{E}_{t-1} \left[ m_t^{\frac{\rho-\gamma}{1-\gamma}} e^{-\rho g_{\theta,t}} (\hat{c}_t^{-\rho} + (1-\delta^k) \tilde{q}_t^k) \right].$$

The scaled planner's problem can therefore be written recursively as follows<sup>5</sup>

$$\hat{V}_{t-1}(\hat{\mathcal{B}}_{t-1}, \tilde{\mathbf{q}}_{t-1}, \eta_t) = \max_{\hat{W}_t, m_t, \hat{c}_t, n_t, \tilde{\mathbf{q}}_t, \hat{\mathcal{B}}_t} \mathbb{E}_{t-1} \left[ \left( e^{g_{\theta,t}} \hat{W}_t \right)^{1-\gamma} \right]^{\frac{1}{1-\gamma}} \quad (13)$$

subject to

$$\hat{\mathcal{B}}_t = \sum_k \frac{e^{-g_{\theta,t}} \hat{\mathcal{B}}_{t-1}^k \eta_t (\hat{c}_t^{-\rho} + (1-\delta^k) \tilde{q}_t^k)}{\beta \mathbb{E}_{t-1} \left[ m_t^{\frac{\rho-\gamma}{1-\gamma}} e^{-\rho g_{\theta,t}} (\eta_t \hat{c}_t^{-\rho} + (1-\delta^k) \tilde{q}_t^k) \right]} - \eta_t \hat{c}_t^{1-\rho} + \eta_t n_t^{1+\phi} \quad (14a)$$

$$\tilde{q}_{t-1}^k = \beta \mathbb{E}_{t-1} \left[ m_t^{\frac{\rho-\gamma}{1-\gamma}} e^{-\rho g_{\theta,t}} (\hat{c}_t^{-\rho} + (1-\delta^k) \tilde{q}_t^k) \right] \quad (14b)$$

$$m_t = \frac{\left( e^{g_{\theta,t}} \hat{W}_t \right)^{1-\gamma}}{\mathbb{E}_{t-1} \left[ \left( e^{g_{\theta,t}} \hat{W}_t \right)^{1-\gamma} \right]} \quad (14c)$$

$$n_t = \hat{c}_t + \hat{g}_t \quad (14d)$$

$$\hat{W}_t = \left( (1-\beta) \eta_t \left[ \hat{c}_t^{1-\rho} - (1-\rho) \frac{n_t^{1+\phi}}{1+\phi} \right] + \beta \hat{V}_t(\hat{\mathcal{B}}_t, \tilde{\mathbf{q}}_t, \eta_{t+1})^{1-\rho} \right)^{\frac{1}{1-\rho}} \quad (14e)$$

$$\hat{\mathcal{B}}_{t-1} = \sum_k \hat{\mathcal{B}}_{t-1}^k \quad (14f)$$

$$\log(\eta_{t+1}) = \rho_\eta \log(\eta_t) + \epsilon_{\eta,t} \quad (14g)$$

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<sup>5</sup>Following Albuquerque et al. (2016) we assume that the discount factor shock,  $\eta_t$ , is known at time  $t-1$ .



In the following section we document the computational techniques we use to solve this Ramsey planner's problem.

## 4.1 Computational Method

Solving numerically problem 16 is difficult with the conventional numerical techniques because the state space consists of  $K$  endogenous variables. To overcome this curse of dimensionality to adopt the numerical methods developed in Evans (2014) and Bhandari et al. (2017b). Here we briefly outline main ideas behind this approach. Let  $\mu_t$ ,  $\lambda_t$ ,  $\nu_t$ , and  $\xi_t$  be the Lagrange multipliers associated with constraints (14a)-(14d) respectively. Problem (13) is recursive in the effective debt  $\hat{\mathcal{B}}_{t-1}$ , the discount factor shock  $\eta_t$ , and the marginal utility weighted prices of the long maturity bonds  $\tilde{\mathbf{q}}_{t-1}$ . Following Marcet and Marimon (1994), we replace the vector  $\tilde{\mathbf{q}}_{t-1}$  with its co-state  $\lambda_{t-1}$  and search for solutions that are recursive in  $\hat{\mathcal{B}}_{t-1}$ ,  $\lambda_t$ ,  $\eta_t$ . We begin by the separating variables as:

- $\mathbf{z}_{t-1} \equiv \{\lambda_{t-1}, \eta_t\}$ ,
- $\mathbf{y}_t \equiv \{\hat{\mathcal{B}}_t, \eta_t, \lambda_t, \hat{c}_t, n_t, \mathcal{R}_t, \hat{W}_t, m_t, \nu_t, \xi_t, \mu_t, \hat{V}_{t-1}, \hat{V}_{x,t-1}, \tilde{\mathbf{q}}_{t-1}\}$
- $\mathbf{e}_t \equiv \{\mathbb{E}_{t-1}[\mathcal{R}_t], \mathbb{E}_{t-1}[\mathcal{R}_t \mu_t], \mathbb{E}_{t-1}[W_t^{1-\gamma}], \mathbb{E}_{t-1}[m_t \nu_t]\}$
- $\boldsymbol{\epsilon}_t \equiv \{\epsilon_{\theta,t}, \epsilon_{g,t}, \epsilon_{\eta,t}\}$ .

Here,  $\mathcal{R}_t^k = m_t^{\frac{\rho-\gamma}{1-\gamma}} e^{-\rho g_{\theta,t}} (\eta_t \hat{c}_t^{-\rho} + (1-\delta^k) \tilde{q}_t^k)$  are the effective payoffs for each asset. We stack the first order conditions of the Ramsey problem are listed in Appendix A into the functions  $f$ ,  $g$  and  $F$  that satisfy

$$\mathbf{e}_{t-1} = \mathbb{E}_{t-1}[f(\mathbf{y}_t)]. \quad (15a)$$

$$g(\mathbf{e}_{t-1}) = 0. \quad (15b)$$

$$F(\hat{\mathcal{B}}_{t-1}, \mathbf{z}_{t-1}, \mathbf{y}_t, \mathbf{e}_{t-1}, \mathbf{y}_{t+1}, \hat{\mathcal{B}}_{t-1}, \sigma \boldsymbol{\epsilon}_t) = 0 \quad (15c)$$

In our problem  $f$  is given by the definition of  $\mathbf{e}_{t-1}$  above,  $g$  is given by equation (28) which are the first order conditions with respect to  $\{\hat{\mathcal{B}}_{t-1}^k\}_{k \geq 2}$  and  $F$  is given by the remaining equations (20a)-(27). The optimal allocation can be represented by

functions  $(y, B)$

$$\begin{aligned} \mathbf{y}_t &= y(\hat{\mathcal{B}}_{t-1}, \mathbf{z}_{t-1}, \sigma\epsilon_t|\sigma) \\ \hat{\mathcal{B}}_{t-1} &= B(\hat{\mathcal{B}}_{t-1}, \mathbf{z}_{t-1}|\sigma) \end{aligned}$$

that solve the system of equations (15).

We approximate  $(y, B)$  around  $\hat{\mathcal{B}}_{t-1} = \hat{\mathcal{B}}_-$  and  $\sigma = 0$  using a second order Taylor expansion of the form:

$$\begin{aligned} y(\hat{\mathcal{B}}_-, z_-, \sigma\epsilon|\sigma) &= \bar{y}(\hat{\mathcal{B}}_-) + y_z(\hat{\mathcal{B}}_-)(\mathbf{z}_- - \bar{\mathbf{z}}) + y_\epsilon(\hat{\mathcal{B}}_-)\sigma\epsilon + \frac{1}{2} \left( y_{zz}(\hat{\mathcal{B}}_-)(\mathbf{z}_- - \bar{\mathbf{z}})(\mathbf{z}_- - \bar{\mathbf{z}}) \right. \\ &\quad \left. + 2y_{z\epsilon}(\hat{\mathcal{B}}_-)(\mathbf{z}_- - \bar{\mathbf{z}})\sigma\epsilon + y_{\epsilon\epsilon}(\hat{\mathcal{B}}_-)\sigma\epsilon\sigma\epsilon + y_{\sigma\sigma}(\hat{\mathcal{B}}_-)\sigma^2 \right) + \mathcal{O}(\sigma^3) \\ &\equiv \check{y}(\mathcal{B}_-, z_-, \sigma\epsilon_t|\sigma) + \mathcal{O}(\sigma^3) \end{aligned} \tag{16}$$

and

$$\begin{aligned} B(\hat{\mathcal{B}}_-, z_-|\sigma) &= \bar{\mathcal{B}}(\hat{\mathcal{B}}_-) + B_z(\hat{\mathcal{B}}_-)(\mathbf{z}_- - \bar{\mathbf{z}}) + B_\sigma(\hat{\mathcal{B}}_-)\sigma + \mathcal{O}(\sigma^2) \\ &\equiv \check{B}(\hat{\mathcal{B}}_-, z_-|\sigma) + \mathcal{O}(\sigma^2). \end{aligned} \tag{17}$$

Here  $\bar{y}(\hat{\mathcal{B}}_-)$  satisfies  $\bar{y} = y(\hat{\mathcal{B}}_-, \bar{z}(\hat{\mathcal{B}}_-), 0|0)$ ,  $\bar{z}(\hat{\mathcal{B}}_-) = I_y^z \bar{y}(\hat{\mathcal{B}}_-)$  where  $I_y^z$  is a matrix that selects variables  $\mathbf{z}$  from the vector  $\mathbf{y}$ , and  $\bar{\mathcal{B}}(\hat{\mathcal{B}}_-) = \lim_{\sigma \rightarrow 0} B(\hat{\mathcal{B}}_-, \bar{z}(\hat{\mathcal{B}}_-)|\sigma)$ . The remaining coefficients are derivatives of  $y$  and  $B$  evaluated at the non-stochastic steady state associated with  $\hat{\mathcal{B}}_{t-1} = \hat{\mathcal{B}}_-$  and  $\sigma = 0$ .

The immediate hurdle in applying a perturbation method for problem with portfolio choice is that at  $\sigma = 0$ , all assets earn the same returns and so  $\bar{\mathcal{B}}(\hat{\mathcal{B}}_-)$  is not pinned down when solving for non stochastic steady state. Furthermore all the derivative terms in expansion (16) and (17) are evaluated at  $\hat{\mathcal{B}}_{t-1} = \hat{\mathcal{B}}_-$  and  $\sigma = 0$  and implicitly depend on the choice of the steady state portfolio  $\bar{\mathcal{B}}(\hat{\mathcal{B}}_-)$ . To see this notice that non-stochastic steady state Ramsey plan,  $\bar{y}(\hat{\mathcal{B}}_-)$ , can be found by solving the system of equations

$$F(\hat{\mathcal{B}}_-, I_y^z \bar{y}, f(\bar{y}), \bar{y}, \hat{\mathcal{B}}, 0) = 0 \tag{18}$$

for any choice of  $\hat{\mathcal{B}}$ , and the function  $g$  given by (15b) will be identically 0 when  $\sigma = 0$ .

In the appendix, we show that with sufficient smoothness of  $y$  and  $B$ , that  $\bar{\mathbf{B}}(\hat{\mathbf{B}}_-)$  must solve the non-linear equation

$$g_e \mathbb{E} [f_{yy}(y_{\epsilon\epsilon}, y_{\epsilon\epsilon})] = 0, \quad (19)$$

where  $y_{\epsilon}$  implicitly depends on the choice of  $\bar{\mathbf{B}}$ . A special feature of (19) is that only the first order derivatives of  $y$  are required to determine  $\bar{\mathbf{B}}$ . In the same spirit we can show a corresponding result for  $B_z$  and  $B_{\sigma}$  that says that they satisfy a linear equation which depends on only second order derivatives of  $y$ .

Using these insights we proceed along the following steps

1. Given current scaled effective debt,  $\hat{\mathbf{B}}_{t-1}$  and states  $\mathbf{z}_{t-1}$ , obtain  $\bar{y}$  and  $\bar{\mathbf{B}}$  by solving (18) and (19).
2. Use total differentiation of equations (15) around  $(\bar{y}, \bar{\mathbf{B}})$  to solve for the remaining derivatives in (16) and (17) following the Appendix.
3. Use expansions in (16) and (17) to simulate the policy rules for one period and obtain  $y_t$ .
4. Restart from 1 with a new level of effective debt  $\hat{\mathbf{B}}_t$  and states  $\mathbf{z}_t$ .

A sample path for the optimal Ramsey allocation is then simulated by repeating steps 1-4 until a desired simulation length has been reached.

## 5 Quantitative analysis

### 5.1 Calibration

We now turn to the quantitative analysis of the optimal portfolio in a calibrated economy. Our strategy is to choose parameters that describe preferences and shocks so that our model economy with a tax policy  $\tau_t = \bar{\tau}$  produces moments that match the patterns in returns that we outlined in the previous section along with other business cycle facts for the US. After estimating these parameters we compute the Ramsey allocation.<sup>6</sup>

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<sup>6</sup>We have experimented with several tax rules, for example specifications where tax rates depend on output growth, expenditures and debt levels. We estimated these specifications using observed average marginal tax rates as in Barro and Redlick (2011). Our parameter estimates for preferences and shocks are not too sensitive to what tax rules we use.

The stochastic process for the shocks is parameterized as

$$\log \frac{\theta_t}{\theta_{t-1}} = g_{\theta,t}, \quad \log \frac{g_t}{\theta_t} = \hat{g}_t$$

and

$$\begin{bmatrix} g_{\theta,t} \\ \hat{g}_t \\ \log \eta_t \end{bmatrix} = \mu + A \begin{bmatrix} g_{\theta,t-1} \\ \hat{g}_{t-1} \\ \log \eta_{t-1} \end{bmatrix} + \Sigma \begin{bmatrix} \epsilon_{\theta,t} \\ \epsilon_{g,t} \\ \epsilon_{\eta,t} \end{bmatrix}$$

We restrict  $A$  so that  $g_{\theta,t}$  and  $\hat{g}_t$  are serially uncorrelated after conditioning on  $\{\eta_j\}_{j=0}^t$ .<sup>7</sup> We use  $K = 2$  and set  $\delta^2 = 0.2$  capturing a maturity of the 5 year bond.<sup>8</sup>

We estimate  $\beta, \gamma, \rho, \phi$  along with  $\mu, A, \Sigma$  using a simulated method of moments procedure. Our target moments are listed in table IV. For a given choice of parameters we simulate 10,000 samples of length 50 and compute a distance metric that weights all the moments equally, with the caveat that deviations of means and standard deviations from the targets are measured as percent deviation. Our model fit is described in “Model” column of table IV. The underlying estimates are in table III.

Our estimation favors the Epstein-Zin specification with fairly persistent and large discount rate shocks. We find a large risk aversion around 23 and IES of 3. The model replicates excess returns on long bond and correlation patterns in returns and output growth, expenditures.

## 5.2 Results

In this section we describe the optimal portfolio management for the government. There are two central findings: (a) the optimal portfolio is long in both the short maturity asset and the long long maturity asset with about equal shares, and (b) there is little re-balancing of the portfolio in response to shocks.

Figure IV plots the portfolio holdings in the short and the long maturity bond respectively as a function of the debt to gdp ratio. The share of the total debt in the

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<sup>7</sup>This implies that  $\mu = \begin{bmatrix} \mu_\theta \\ \bar{g} \\ 0 \end{bmatrix}$ ,  $A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \rho_\eta \end{bmatrix}$  and  $\Sigma = \begin{bmatrix} \sigma_\theta & 0 & \sigma_{\eta\theta} \\ 0 & \sigma_g & \sigma_{\eta g} \\ \sigma_{\eta\theta} & \sigma_{\eta g} & \sigma_\eta \end{bmatrix}$

<sup>8</sup>We chose a two bond specification as this is the most transparent model of maturity management. 5 year bonds are chosen as these are the longest bonds for which Fama-Bliss data is available. Our numerical methods extend straightforwardly to the portfolio problems with more bonds, or with bonds of finite maturity.

Parameters	Values
Preferences:	
$\beta$	0.98
$\phi$	2
$\rho, \gamma$	0.29, 23.22
Shocks:	
$\bar{g}, \sigma_g$	0.4, 0.023
$\mu_\theta, \sigma_\theta$	0.015, 0.02
$\rho_\eta, \sigma_\eta$	0.98, 0.03
$\sigma_{\eta g}, \sigma_{\eta \theta}$	0.21, 0.03

Table III: Estimated parameters

Holding period returns	Data	Model	Macro quantities	Data	Model
mean 1 yr bond	1.63%	1.39%	mean output growth	1.50%	1.50%
mean 5 yr bond	2.85%	2.40%	std. output growth	2.31%	2.32%
std 1 year bond	2.61%	3.14%	mean exp. output ratio	17.00%	16.00%
std. 5 year bond	6.42%	4.00%	std govt. exp growth	4.17%	4.11%
corr 1yr, 5yr	0.66	0.80	corr output, govt. exp	-0.06	0.00
corr 1/5 yr, output growth	0.04, 0.02	0.02, 0.03	Frisch elasticity	-	0.5
corr 1/5 yr, govt. exp growth	0.11, 0.15	0.09, 0.13			

Table IV: Targeted moments and model fit

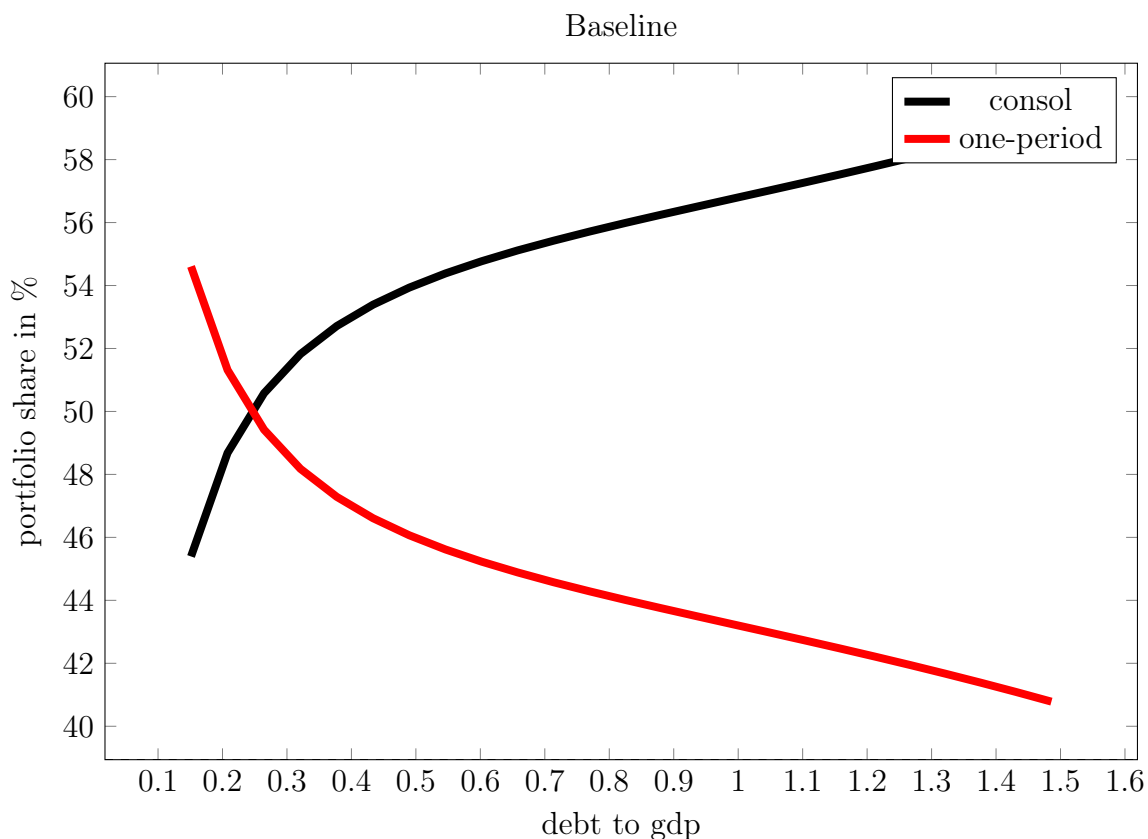


Figure II: Percentage of total debt held in the short maturity bond and the long maturity bond.

long maturity bond ranges between 46% and 58% as we vary debt between 30% and 120% of output.

The key intuition for why we get such a portfolio comes from the workings of formula 11. The returns induced by the optimal allocations retain the patterns of returns in the data, i.e., they are volatile and uncorrelated with shocks that drive the primary deficits. A portfolio with large holdings especially in the long maturity has an disadvantage of exposing the governments budget constraint to large fluctuations in returns induced by discount factor shocks. Hence it is optimal to hold moderate positions. Furthermore, the returns on the long maturity bonds are negatively correlated with marginal utility of consumption. This implies that the volatility of effective returns is lower than the volatility of returns for the long bond and larger for the short bond. Through the lens of effective returns, the two securities are more alike in their variability and this feature is reflected in the approximately 50 – 50

share that the optimal portfolio displays.

Next we turn to how the portfolios are re-balanced in response to shocks. In figure III we plot the impulse response of total debt, share of the total debt in the long maturity and tax rate to a one standard deviation shock to  $\epsilon_\theta$ ,  $\epsilon_g$  and  $\epsilon_\eta$ . We see that the portfolio is fairly stable in response to all these shocks. For example, after a 2% increase productivity growth, the government reduces debt to gdp ratio by 1.5%, with almost no change in the portfolio structure. Secondly, for all shocks tax rates are fairly smooth but not constant. This is because with two assets, the planner cannot implement the complete markets allocation.

### 5.2.1 Comparison to conventional RBC models

The stochastic process for productivity and expenditure shocks in our model as well as parameters of the Frisch elasticity of labor supply and the IES are very similar to those typically used in the Ramsey models, such as Chari et al. (1994), Buera and Nicolini (2004), Schmitt-Grohe and Uribe (2004), Farhi (2010), Faraglia et al. (2012). The crucial difference is that in our model risk aversion is not equal to the IES, and a substantial fluctuation in asset returns are driven by shocks orthogonal to short run productivity and expenditure shocks. The conventional parameterizations can be captured in our model by setting  $\gamma = \rho$  and  $\sigma_\eta = 0$ . In this section we contrast our findings with those obtained under such calibrations. We refer to our model specification as the “baseline calibration” and a version in which we set  $\gamma = \rho = 2$  and  $\sigma_\eta = 0$  as the “RBC calibration”. This will also help to explain why our conclusions about portfolio management differ from those in Buera and Nicolini (2004), Faraglia et al. (2012) or Debortoli et al. (2016 forthcoming).

In figure IV, we overlay the optimal portfolio as a function of the debt to gdp ratio for the RBC calibration on top of the graph for the baseline calibration. The optimal portfolio now is about 10 to 20 times larger with large long positions in the 5 yr bond and offsetting short positions in the one year bond. In figure V we plot the impulse responses for the RBC calibration and compare them to the baseline calibration. The top panel of figure V shows that in response to a 2% increase in productivity, the response of total debt and tax rates is similar across both settings but the response of the portfolio holdings are about 10 times larger in the RBC calibration.

The reason behind the dramatic differences in portfolio positions can be inferred from table V where we compare the covariance matrix of effective returns and the

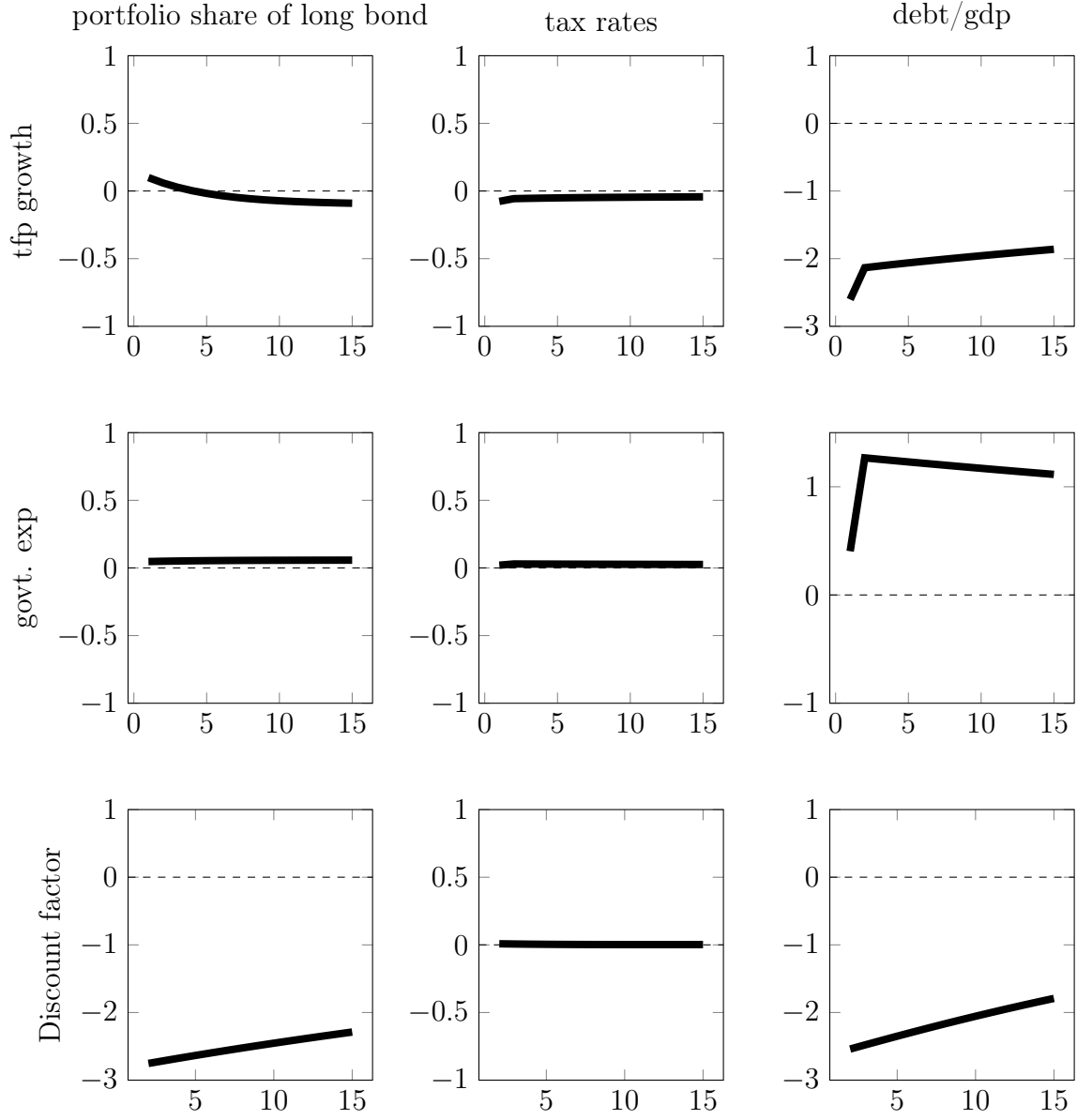


Figure III: Impulse response functions for baseline calibration for a one standard deviation shock to  $\epsilon_\theta, \epsilon_g, \epsilon_\eta$  respectively. The units on the y-axis are in percentage points.



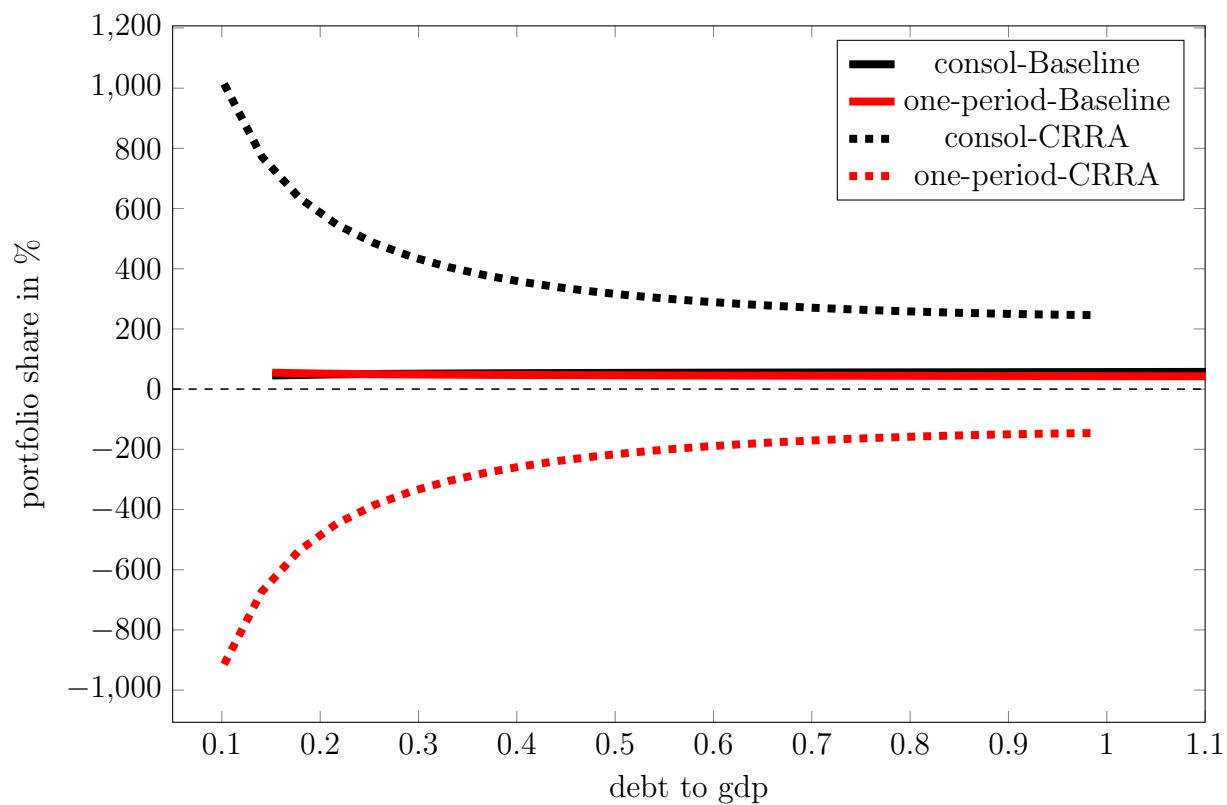


Figure IV: Percentage of total debt held in the short maturity bond and the long maturity bond for the baseline calibration (solid lines) and the CRRA-no discount factor shocks specification (dashed lines)

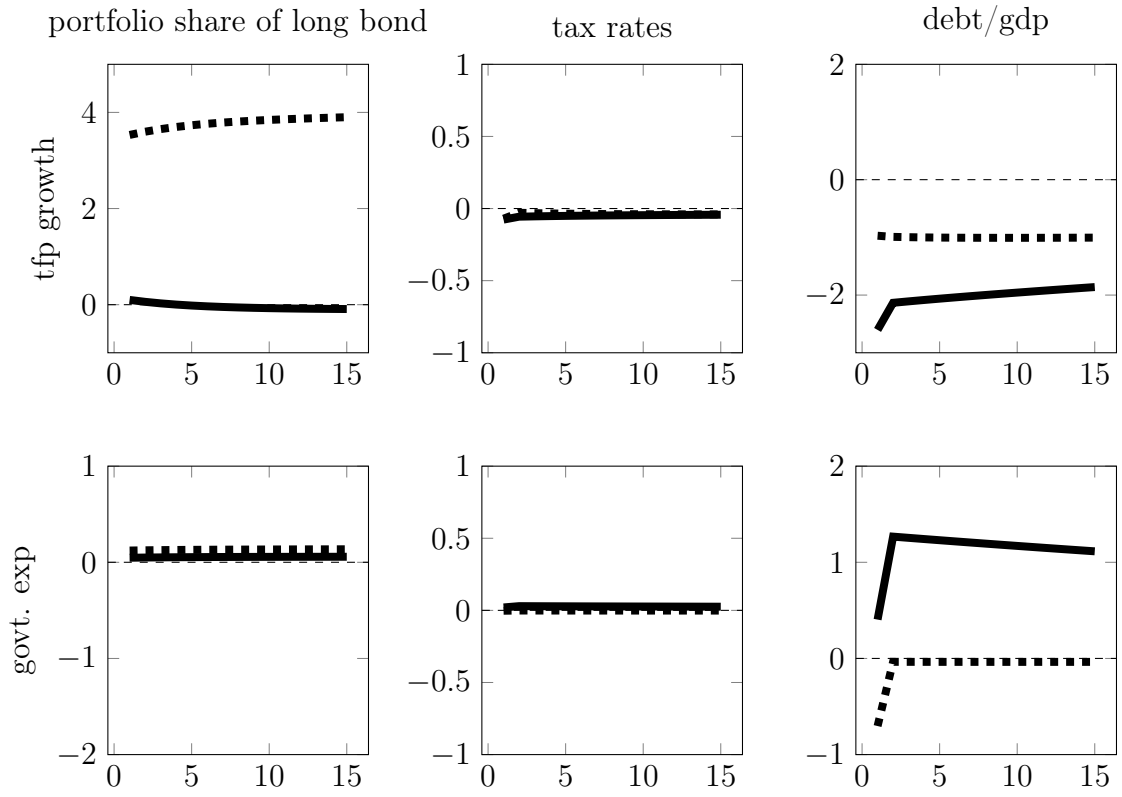


Figure V: Impulse response functions for CRRA-no discount factor shock specification (dashed) and baseline calibration (solid) for a one standard deviation shock to  $\epsilon_\theta, \epsilon_g$  respectively. The units on the y-axis are in percentage points.

Baseline Calibration	short bond	long bond	deficits	RBC Calibration	short bond	long bond	deficits
short bond	75.01	69.59	6.58	short bond	1.01	0.24	4.22
long bond	69.59	71.07	3.49	long bond	0.24	0.05	0.97
deficits	6.58	3.49	20.12	deficits	4.22	0.97	42.83

Table V: Covariance matrix of effective returns and innovations to present value of effective primary deficits

present value of effective deficits across both calibrations. Compared to the baseline calibration, the RBC calibration features both a lower volatility of effective returns and a much greater correlation of effective returns with innovations to the present value of effective deficits. This higher correlation provides the planner a much greater ability to hedge innovations to the present value of deficits. For instance, in figure V, the volatility of tax rates and debt is much lower in the RBC calibration relative to the baseline calibration. Combined with low volatility of returns, the RBC planner necessarily has to take much larger portfolio positions in order to achieve this hedging.

## 6 Conclusion

In this paper we revisit the predictions of Ramsey models for optimal management of government portfolios. For a range of settings, we show that the covariance of marginal utility adjusted or effective returns with primary deficits is a key determinant of the portfolio structure. Using US data on bond prices, we document that holding period returns on bonds are volatile and have low correlation with macro aggregates like output growth, primary deficits etc. We adopt the lessons from the asset pricing literature and use recursive preference and richer shocks processes with the goal of studying optimal policy in a setting that is consistent with observed returns. We find that the optimal portfolio loads evenly on short and long maturities and is fairly stable with respect to business cycle shocks. A byproduct of our analysis is a computational method allows for fast and reliable solutions for Ramsey models with arbitrarily rich asset markets.

Our point of departure is a body of work that dealt with this topic in canonical Real Business Cycle type settings which feature expected utility preferences and exogenous fluctuations in productivity or government purchases. A common pre-

scription from these models is for the government to issue large amounts of debt in the long maturities with offsetting positions in the shorter maturities. Furthermore, these models also suggest that the government should actively re-balance its portfolios. We show that the failure of RBC models to match bond returns and more generally dynamics of risk premia is exactly what drives their implications about portfolio management.

A natural extension to our exercise is to allow for inflation and study joint portfolio and monetary policy in the direction of Lustig et al. (2008). We leave this for future work.

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## A First Order Conditions of The Ramsey Problem

In this appendix we document the first order conditions of the Ramsey problem. Define  $U(\hat{c}_t, n_t) = \eta_t \left[ \frac{\hat{c}_t^{1-\rho}}{1-\rho} - \frac{n_t^{1+\phi}}{1+\phi} \right]$  and its respective derivatives by  $\hat{U}_{c,t}$ ,  $\hat{U}_{n,t}$ ,  $\hat{U}_{cc,t}$ , and  $\hat{U}_{nn,t}$ . We begin by documenting the following derivatives of  $\hat{W}_t$  with respect to  $\hat{c}_t$ ,  $n_t$ ,  $\hat{\mathcal{B}}_t$ , and  $\tilde{\mathbf{q}}_t$ :

$$\begin{aligned}\hat{W}_{c,t} &= (1 - \beta)\eta_t \hat{W}_t^{-\rho} \hat{U}_{c,t} \\ \hat{W}_{n,t} &= -(1 - \beta)\eta_t \hat{W}_t^{-\rho} \hat{U}_{n,t} \\ \hat{W}_{\mathcal{B},t} &= \beta \hat{W}_t^{-\rho} \hat{V}_t^{-\rho} \hat{V}_{\mathcal{B},t} \\ \hat{W}_{\mathbf{q},t} &= \beta \hat{W}_t^{-\rho} \hat{V}_t^{-\rho} \hat{V}_{\mathbf{q},t}.\end{aligned}$$

After multiplying the implementability constraint of the Ramsey problem by  $e^{(1-\rho)g_{\theta,t}} m_t^{\frac{\rho-\gamma}{1-\gamma}}$  and defining  $\mathcal{R}_t^1 = m_t^{\frac{\rho-\gamma}{1-\gamma}} e^{-\rho g_{\theta,t}} \hat{U}_{c,t}$  and  $\mathcal{R}_t^k = m_t^{\frac{\rho-\gamma}{1-\gamma}} e^{-\rho g_{\theta,t}} (\hat{U}_{c,t} + (1 - \delta^k) \tilde{q}_t^k)$  for  $k \geq 2$  the recursive version of the Ramsey problem becomes

$$\hat{V}_{t-1}(\hat{\mathcal{B}}_{t-1}, \tilde{\mathbf{q}}_{t-1}, \eta_t) = \max_{\hat{W}_t, m_t, \hat{c}_t, n_t, \tilde{\mathbf{q}}_t, \hat{\mathcal{B}}_t, \hat{\mathcal{B}}_{t-1}} \mathbb{E}_{t-1} \left[ \left( e^{g_{\theta,t}} \hat{W}_t \right)^{1-\gamma} \right]^{\frac{1}{1-\gamma}}$$

subject to

$$\hat{\mathcal{B}}_t e^{(1-\rho)g_{\theta,t}} m_t^{\frac{\rho-\gamma}{1-\gamma}} = \frac{(\hat{\mathcal{B}}_{t-1} - \sum_{k \geq 2} \hat{\mathcal{B}}_{t-1}^k) \mathcal{R}_t^1}{\beta \mathbb{E}_{t-1} [\mathcal{R}_t^1]} + \sum_{k \geq 2} \frac{\hat{\mathcal{B}}_{t-1}^k \mathcal{R}_t^k}{\beta \mathbb{E}_{t-1} [\mathcal{R}_t^k]} - (\hat{c}_t \hat{U}_{c,t} + n_t \hat{U}_{n,t}) e^{(1-\rho)g_{\theta,t}} m_t^{\frac{\rho-\gamma}{1-\gamma}} \quad (20a)$$

$$\tilde{q}_{t-1}^k = \beta \mathbb{E}_{t-1} [\mathcal{R}_t^k] \quad (20b)$$

$$m_t = \frac{\left( e^{g_{\theta,t}} \hat{W}_t \right)^{1-\gamma}}{\mathbb{E}_{t-1} \left[ \left( e^{g_{\theta,t}} \hat{W}_t \right)^{1-\gamma} \right]} \quad (20c)$$

$$n_t = \hat{c}_t + \hat{g}_t \quad (20d)$$

$$\hat{W}_t = \left( (1 - \beta)\eta_t \left[ \hat{c}_t^{1-\rho} - (1 - \rho) \frac{n_t^{1+\phi}}{1 + \phi} \right] + \beta \hat{V}_t(\hat{\mathcal{B}}_t, \tilde{\mathbf{q}}_t, \eta_{t+1})^{1-\rho} \right)^{\frac{1}{1-\rho}} \quad (20e)$$

$$\log(\eta_{t+1}) = \rho_\eta \log(\eta_t) + \epsilon_{\eta,t} \quad (20f)$$



Letting  $\mu_t$ ,  $\lambda_{t-1}$ ,  $\nu_t$ , and  $\xi_t$  be the multipliers of constraints (20a)-(20d) respectively, the envelope conditions for the planner's problem imply

$$\hat{V}_{\mathcal{B},t-1} = \frac{\mathbb{E}_{t-1}[\mathcal{R}_t^1 \mu_t]}{\beta \mathbb{E}_{t-1}[\mathcal{R}_t^1]} \quad (21)$$

$$\hat{V}_{\mathcal{q},t-1} = -\lambda_{t-1}. \quad (22)$$

The first order conditions with respect to  $\hat{\mathcal{B}}_t$ ,  $\hat{c}_t$ ,  $n_t$ ,  $\hat{\mathcal{q}}_t$ , and  $m_t$  are then given by

$$e^{(1-\gamma)g_{\theta,t}} \hat{V}_{t-1}^\gamma \hat{W}_t^{-\gamma} \hat{W}_{\mathcal{B},t} - m_t^{\frac{\rho-\gamma}{1-\gamma}} e^{(1-\rho)g_{\theta,t}} \mu_t + \frac{(1-\gamma)e^{(1-\gamma)g_{\theta,t}} \hat{W}_t^{-\gamma} \hat{W}_{\mathcal{B},t}}{\mathbb{E}_{t-1} \left[ \left( e^{g_{\theta,t}} \hat{W}_t \right)^{1-\gamma} \right]} (\nu_t - \mathbb{E}_{t-1}[m_t \nu_t]) = 0 \quad (23)$$

$$\begin{aligned} & e^{(1-\gamma)g_{\theta,t}} \hat{V}_{t-1}^\gamma \hat{W}_t^{-\gamma} \hat{W}_{c,t} + \frac{(1-\gamma)e^{(1-\gamma)g_{\theta,t}} \hat{W}_t^{-\gamma} \hat{W}_{c,t}}{\mathbb{E}_{t-1} \left[ \left( e^{g_{\theta,t}} \hat{W}_t \right)^{1-\gamma} \right]} (\nu_t - \mathbb{E}_{t-1}[m_t \nu_t]) \\ & + \left( \frac{(\hat{\mathcal{B}}_{t-1} - \sum_{k \geq 2} \hat{\mathcal{B}}_{t-1}^k) \mathcal{R}_{c,t}^1}{\beta \mathbb{E}_{t-1}[\mathcal{R}_t^1]} + \sum_{k \geq 2} \frac{\hat{\mathcal{B}}_{t-1}^k \mathcal{R}_{c,t}^k}{\beta \mathbb{E}_{t-1}[\mathcal{R}_t^k]} \right) (\mu_t - \beta \hat{V}_{\mathcal{B},t-1}) \\ & - (\hat{c}_t \hat{U}_{cc,t} + \hat{U}_{c,t}) e^{(1-\rho)g_{\theta,t}} m_t^{\frac{\rho-\gamma}{1-\gamma}} \mu_t + \beta \sum_{k \geq 2} \lambda_{t-1}^k \mathcal{R}_{c,t}^k - \xi_t = 0 \end{aligned} \quad (24)$$

$$e^{(1-\gamma)g_{\theta,t}} \hat{V}_{t-1}^\gamma \hat{W}_t^{-\gamma} \hat{W}_{n,t} - (n_t \hat{U}_{nn,t} + \hat{U}_{n,t}) e^{(1-\rho)g_{\theta,t}} m_t^{\frac{\rho-\gamma}{1-\gamma}} \mu_t + \xi_t = 0 \quad (25)$$

$$\begin{aligned} & e^{(1-\gamma)g_{\theta,t}} \hat{V}_{t-1}^\gamma \hat{W}_t^{-\gamma} \hat{W}_{q^k,t} + \frac{(1-\gamma)e^{(1-\gamma)g_{\theta,t}} \hat{W}_t^{-\gamma} \hat{W}_{q^k,t}}{\mathbb{E}_{t-1} \left[ \left( e^{g_{\theta,t}} \hat{W}_t \right)^{1-\gamma} \right]} (\nu_t - \mathbb{E}_{t-1}[m_t \nu_t]) \\ & + \frac{\hat{\mathcal{B}}_{t-1}^k \mathcal{R}_{q^k,t}^k}{\beta \mathbb{E}_{t-1}[\mathcal{R}_t^k]} (\mu_t - \beta \hat{V}_{\mathcal{B},t-1}) + \beta \lambda_{t-1}^k \mathcal{R}_{q^k,t}^k = 0 \end{aligned} \quad (26)$$

$$- \left( \frac{(\hat{\mathcal{B}}_{t-1} - \sum_{k \geq 2} \hat{\mathcal{B}}_{t-1}^k) \mathcal{R}_{m,t}^1}{\beta \mathbb{E}_{t-1}[\mathcal{R}_t^1]} + \sum_{k \geq 2} \frac{\hat{\mathcal{B}}_{t-1}^k \mathcal{R}_{m,t}^k}{\beta \mathbb{E}_{t-1}[\mathcal{R}_t^k]} \right) \beta \hat{V}_{\mathcal{B},t-1} + \sum_{k \geq 2} \beta \mathcal{R}_{m,t}^k \lambda_{t-1}^k - \nu_t = 0 \quad (27)$$

while the first order conditions with respect to the portfolio choice  $\hat{\mathbf{B}}_{t-1}$  are

$$\frac{\mathbb{E}_{t-1} [\mathcal{R}_t^k \mu_t]}{\mathbb{E}_{t-1} [\mathcal{R}_t^k]} - \frac{\mathbb{E}_{t-1} [\mathcal{R}_t^1 \mu_t]}{\mathbb{E}_{t-1} [\mathcal{R}_t^1]} = 0. \quad (28)$$