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Linda M. Schilling*

Abstract

This paper demonstrates that the Modigliani Miller Theorem on capital structure does in general not apply to banks when faced with endogenous liquidity risk in form of bank runs and asset illiquidity. The Modigliani Miller Theorem states that under certain assumptions, firms with different capital structure must have same values if they have identical return distributions (risk class). This paper shows, under endogenous liquidity risk the bank's risk class changes in debt ratio and coupons demanded by depositors such that the Modigliani Miller Theorem can in general not apply when repricing of risk in form of higher coupons is taken into account. In equilibrium, bank value is non-monotone in capital structure. In particular, only the all equity financed bank achieves the highest risk class.

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1 Motivation

We demonstrate that the Modigliani Miller Theorem (Modigliani and Miller, 1958, 1963) does in general not apply to banks which invest in illiquid assets and face endogenous liquidity risk in form of bank runs. Our results hold with or without deposit insurance. The Modigliani Miller Theorem states that in a sufficiently frictionless market[1] the capital structure of a firm (here bank) does not affect the firm's value. The theorem plays a prominent role today in the debate on bank capital regulation (Hellwig, 2010; Hanson et al., 2011; Admati et al., 2013; Myerson, 2014), since according to the Theorem more equity financing imposed by a regulator is socially not costly. Asset illiquidity implies costly bankruptcy. The result, that the Modigliani Miller Theorem does not hold under costly bankruptcy is know in the literature. Still, the connection of the theorem to the literature on endogenous liquidity risk has to the best of our knowledge not been explored yet which is the substance of this paper.

The original Modigliani Miller Theorem is proven via a no arbitrage argument by selecting two firms with distinct capital structures, but of the same risk class and by showing that their values need to be equal, otherwise arbitrage exists. A risk class contains firms with perfectly correlated returns. The derivation of the result proceeds by showing how private investors can replicate firm capital structures by either purchasing shares of the levered company, or by buying shares of an all equity financed firm and borrowing privately. Since return patterns of both firms across states are equal by assumption, different values of the according replicating portfolios would constitute an arbitrage opportunity. Consequently, in competitive, frictionless markets firms of same risk classes have to have same values. The concept of risk classes is important since the no arbitrage proof of the Modigliani Miller Theorem crucially requires that the firms considered are members of the same class. If a change in firm capital structure causes a shift in risk class, returns from investment differ for some states such that in fact the firm becomes more or less valuable which alters her costs of capital.

Can the capital structure of the bank affect the total return distribution from investment (risk class) if the bank stays invested in the same asset? One main takeaway from the financial crises 2007-2009 is that returns from investment are only partially characterized by asset return distributions. A full characterization entails continuation value from investment (asset returns) for buy and hold, liquidation value under premature sales and a bankruptcy state or cut-off state, at which the

[1] No corporate taxation, symmetric information, no bankruptcy, complete capital markets, individuals can borrow at the same rate as firms.

firm switches from liquidation to continuation. Total state contingent return from investment equals liquidation value of assets if the bank goes bankrupt, or equals asset return if investment is continued. When assets are illiquid, liquidation value in general differs from continuation value which results in a jump in the return distribution at the bankruptcy state and a deadweight loss. In particular, when liquidation is not at fair value (continuation value), the bank's risk class depends on the bankruptcy state. Banks which invest in same assets but have distinct bankruptcy states cannot have perfectly correlated returns and thus belong to different risk classes. While there exist extensions of Modigliani Miller to the case of bankruptcy, (Stiglitz, 1969; Merton, 1977; Hellwig, 1981) these papers assume liquidation at fair value (no bankruptcy cost). As Baxter (1967) however already points out, when taking into account bankruptcy costs the value of the firm becomes dependent on capital structure since leverage alters the firm's return distribution, see also Kraus and Litzenberger (1973), Scott Jr (1976), Kim (1978) and Myers (1984).

This paper returns to Baxter's argument and considers bank values when asset liquidation is endogenous and costly. As main contribution of the paper, we characterize bank risk classes and show that capital structure alters risk classes by changing endogenous liquidity risk of the bank. More debt makes the bank more prone to runs which shifts her bankruptcy state. Going one step further, by drawing on the literature on endogenous liquidity risk, we show that bank risk classes also depend on coupons paid to depositors. Interest rate payments are thus more than just transfers since they change the bankruptcy state by altering incentives to roll over debt. The underlying market structure determines the equilibrium change of coupons demanded by depositors and thus the shift in bank's risk class as debt ratio increases since market structure pins down the outside option. In the classic Modigliani Miller setting, by competitiveness of markets, investors may freely trade arbitrary units of debt and equity in banks by borrowing at the risk-free rate and investors' outside option is given by investing in equity and debt of a different firm with same return patterns but distinct capital structure. In the classic setting however all debt is risk-free such that investments in debt of different firms are perfect substitutes and earn the same rate. In particular, investment in debt yields perfectly correlated returns to investment in storage. Once debt becomes risky and thus pays different payoffs for some states, investor's option to not participate (storage) then constitutes a third investment opportunity which cannot be replicated by investing in equity and debt. To concentrate on demonstrating that bank risk classes may change in debt and coupons when accounting for endogenous liquidity risk, we leave the competitive markets framework and consider a monopolistic bank which maximizes equity value where investors invest in debt but have the outside option of storage. By this assumption, we shut down more complex outside options such as investing in other bank's debt and equity. Under a monopolistic bank, as debt increases we contrast the required rise in coupon to maintain depositors' investment in bank debt to the rise in coupon necessary to keep the bank's risk class constant. Only if the bank's risk class stays constant the Modigliani Miller Theorem can apply. Our approach differs from the previous literature on the Modigliani Miller Theorem in that we work in a game theoretic setting, in a monopolistic market, with illiquid assets (costly bankruptcy) and endogenous liquidity risk.[2]

In the literature on endogenous liquidity risk in form of debt runs, it is a common result (Rochet and Vives, 2004; Morris and Shin, 2009; Eisenbach, 2017) that a bank's short-term debt ratio impacts the critical state (bankruptcy cut-off) below which bank runs enforce asset liquidation while holding the asset's return risk fixed. Empirical evidence confirms that runs are sensitive to leverage (Schroth et al., 2014). As debt ratio increases, the critical state goes up, and runs become more likely, by this increasing the riskiness of debt. For this increase in risk, depositors demand a compensation in form of higher coupon payments to continue financing the bank. The required increase in coupon to maintain depositors' participation is given by their marginal rate of substitution since the monopolistic bank[3] maximizes equity value and pays coupons to maintain a utility level equal to depositors' outside option storage. Again, by a standard result in the literature on endogenous liquidity risk (Diamond and Dybvig, 1983; Goldstein and Pauzner, 2005; Rochet and Vives, 2004; Morris and Shin, 2009; Eisenbach, 2017) coupon payments to depositors impact the roll over threshold as well and thus the bank's risk class. As the long-term coupon increases, depositors roll over more often and the bankruptcy cut-off decreases back towards but not necessarily exactly to the initial level. If the raise in coupon demanded by depositors as compensation for the increased debt ratio is such that the cut-off state stays at the initial level, the bank's risk class is unchanged despite costly bankruptcy and the Modigliani Miller Theorem applies, which is in contrast to Baxter (1967) and the classic trade-off theory by Myers (1984). But this is in general not the case. Under endogenous liquidity risk, changes in debt ratio and coupon affect depositors' utility always twofold, directly via a change in payoffs and indirectly via a change in stability (bankruptcy state) since payoffs depend on whether the bank is

[2] Eisenbach (2017) shows how such game theoretic settings can be embedded in general equilibrium frameworks.

[3] In particular, she sets her debt ratio. By this assumption, we leave the classic Modigliani Miller competitive markets framework where investors can trade arbitrary amounts in both equity and debt simultaneously. Instead, investors either invest in the debt contract or do not participate (storage).

liquidated or not. Higher promised coupons benefit depositors only if the bank stays solvent and can actually pay. Higher debt ratios decrease depositors' payoffs contingent on liquidation since asset liquidation value is shared by more debt investors who have a claim. Yet, given the bank continues investment, higher debt ratios do not affect depositors' payoffs. The cut-off state affects how much depositors value higher coupons or disdain higher debt ratios. In particular, the problem of maintaining depositors' participation cannot be considered independently of bank stability (risk class) since changes in coupon and debt affect stability and depositors' utility simultaneously and changes in stability feed back into depositors' utility. If the bankruptcy cut-off shifts in debt although depositors are compensated for the extra risk in form of higher coupons, total risk is altered although risk is priced correctly and the value of the bank changes. Two banks which invest in the same asset but have different short-term debt ratios (capital structures) may have distinct values since the bankruptcy state can both increase or decrease in debt. When grouping firms according to their risk class, the Modigliani Miller Theorem only holds within, but never across risk classes. Since the bank may transition to a different risk class as short-term debt varies, the sets of firms of same risk classes may have only countably many elements such that the Modigliani Miller Theorem in general does not apply.

We divide our analysis into three parts. We first consider run-prone banks which we define as banks that face a liquidity mismatch on their balance sheet such that debt runs may occur. We show, the critical state below which depositors enforce asset liquidation by running increases in debt ratio, but decreases in the long-run coupon on deposits. As the bank raises her debt ratio, the increased risk of runs leads depositors to demand higher coupons ex ante which in return lowers the risk of runs. In general, depositors' marginal rate of substitution of interest rate for debt ratio is not equal to the 'risk-preserving' change in coupon which we define as the coupon which would keep the bankruptcy state constant. Thus, under a monopolistic bank in equilibrium, the critical state may change in debt ratio which leads to a shift in the bank's risk class and the Modigliani Miller Theorem does not apply. Still, we derive a condition under which the value of the bank stays constant in debt despite asset illiquidity, but show that the condition is generically not satisfied. Otherwise, equilibrium bank value either decreases or increases in short-term debt. Next, we consider run-proof banks which we define as banks that exhibit no liquidity mismatch. The riskiness of debt becomes independent of capital structure and depositors demand no additional compensation in form of higher interest rates as debt increases. Yet, depositors withdraw for certain states since the bank's assets are risky. In order to finance withdrawals the bank has to liquidate assets. If asset liquidation is inefficient, the bank loses value the more assets she needs to liquidate i.e. the larger her debt ratio. That is, for a run-proof bank the liquidation cut-offs are constant in capital but the amount the bank needs to liquidate in certain states depends on her debt ratio. Again, the change in debt ratio alters the bank's risk-class, thus Modigliani Miller can never apply to run-proof banks, which is exactly in line with Baxter (1967). In particular, also depositors of run-proof banks withdraw inefficiently often such that the bank drops in her risk class. Last, under complete deposit insurance all run-prone banks are members of the same risk class and the Modigliani Miller Theorem applies within this particular class. The result holds since under complete insurance no depositor withdraws and the bank never liquidates voluntarily due to a debt overhang problem (Myers, 1977). All proceeds would go to depositors and equity value was zero for sure since the economy lacks a profitable reinvestment opportunity for liquidation value, thus the insured bank always continues investment. If the bank is run-proof but insured, depositors never withdraw but the bank liquidates voluntarily in some states. The state below which the bank voluntarily liquidates however undercuts the efficient liquidation state, i.e. the bank liquidates too seldom which is in line with Dewatripont and Tirole (1994). Further, the voluntary liquidation state depends on the debt ratio and again the bank's risk class changes in capital structure, since the bank loses value if she continues investment for states in which liquidation was efficient. Again, the Modigliani Miller Theorem does not apply. With or without deposit insurance, the highest attainable risk class only contains the all equity financed bank. Since the bank's actions are non-contractible, only under all equity financing the bank voluntarily liquidates for all states below the efficient liquidation threshold and otherwise continues investment which implies maximization of bank value. Consequently, the Modigliani Miller Theorem in its generality fails under asset illiquidity and endogenous risk.[4]

Related Literature The papers contributes to the literature strand on bank runs and endogenous liquidity risk which started with the seminal papers by Diamond and Dybvig (1983) and Bryant (1980). Closest to our paper are Goldstein and Pauzner (2005) who analyze optimal risk sharing under partial repayment and Eisenbach (2017) who analyzes efficiency of asset liquidation, both under endogenous liquidity risk in a global game. Also closely related are Morris and Shin (2016), Rochet and Vives (2004) and Allen et al. (2017) who analyze credit risk, respectively intervention

[4] The original Theorem shows equivalence in value of an all equity financed bank to a bank with arbitrary leverage.

by a lender of last resort or government guarantees under endogenous liquidity risk. Allen et al. (2015) analyze costs of capital in a general equilibrium framework with bankruptcy costs and show that equity is costly compared to deposits which opposes Modigliani Miller. Compared to their setting we add endogenous liquidity risk via bank runs. DeAngelo and Stulz (2015) show that high leverage is optimal to banks when depositors pay a liquidity premium on safe, demandable debt and banks can construct safe debt claims via hedging. Since debt is modeled as safe, the possibility of runs and liquidity risk are excluded. Our paper is in spirit close to Dewatripont and Tirole (1994) who analyze control rights and income streams of equity and debt in an incomplete contracting framework. We however add endogenous liquidity risk and concentrate on risk classes. Calomiris and Kahn (1991) analyze the disciplining effect of short-term debt on bank managers. Our paper can be interpreted in this context since coupons depositors demand for higher debt ratios are granted by the bank to keep the roll over threshold low, i.e. to calm depositors and maintain participation. Still, our main focus is on risk classes.

2 The Model

There are three dates, t_0, t_1 and t_2 with no discounting between periods. At time zero, a bank finances a risky asset with equity and short-term debt via demand-deposits. The bank acts in the best interest of and in place of her equity investors, thus the economy has two kinds of agents, the bank and her depositors. The market for deposits and equity are segmented. Equity investors cannot invest in deposits and depositors cannot invest in equity. The idea is that equity investors decide about the measure of deposits the bank accepts to maximize equity value. All agents are risk-neutral. Let $1 - \delta \in (0, 1)$ the measure of equity in place at t_0 and $\delta \in (0, 1)$ the measure of short-term debt the bank decides to additionally raise via demand-deposits in order to finance an investment of one unit. To raise deposits in t_0, the bank offers a demand deposit (debt) contract. For each unit invested in the contract at time zero the bank promises a coupon of one unit if a depositor decides to liquidate the contract prematurely at time one and a long-term coupon $k > 1$ if a depositor decides to stay invested until time two. Depositors in the economy are symmetric at time zero and each endowed with one unit thus the bank's debt is financed by a continuum of measure δ of depositors $[0, \delta]$.[5] The bank invests all funds in a risky asset which requires investment of one unit at time zero (investment is scaleable).

[5] The assumption that each depositor controls exactly one unit in deposits is important such that depositors are small and symmetric in size.

The asset matures at time t_2 and pays off high return H with probability $\theta \sim U[0,1]$ and zero with probability $1-\theta$, where asset return θ is the state of the economy see below. At the interim period t_1, the asset generates no cash flow but can be liquidated. Liquidation of the asset yields liquidation value $l < 1$ per unit invested due to illiquidity.

At time zero, all agents share a common prior about state θ, depositors invest and the state realizes unobservably to all agents. At time one, each depositor $i \in [0, \delta]$ observes noisy private signals

$$\theta_i = \theta + \varepsilon_i, \qquad i \in [0, \delta] \tag{1}$$

about the state (return probability) and then decides whether to roll over her deposit or to withdraw. Depositors cannot liquidate fractions of their deposit. Depositors' strategies are thus measurable mappings from the signal space $[-\varepsilon, 1+\varepsilon]$ into the action space. Here, ε_i are iid random noise terms, independent of θ and distributed according to $U[-\varepsilon, +\varepsilon]$. By correlation, signals convey information about the random asset return probability and about other investors' signals. In particular, they create asymmetry in beliefs of depositors and by this achieve an equilibrium selection (global game).[6] If a depositor withdraws, she has a claim on one unit as promised in the contract. If she rolls over, she has a claim on k units but at time two. Besides the risky asset, there exists a storage technology. Utility debt investors infer from the contract has to exceed one, otherwise depositors do not invest in the bank. Similarly, equity value per invested unit has to exceed return on a direct investment in the asset, otherwise equity investors will not finance the leveraged bank. The risky asset yields higher return than storage $\mathbb{E}[\theta]H > 1$, i.e. $H > 2$.

At time one the bank perfectly observes the state. Depositors observe noisy signals and decide whether to withdraw. By seniority of debt, the bank is obliged to serve withdrawing depositors by liquidating assets. Simultaneously, the bank may decide whether to liquidate additional assets voluntarily even though not enforced by withdrawing depositors. The bank maximizes equity value, thus she liquidates if equity value from liquidation is higher than equity value from continuation. We assume that the bank's actions are not contractible. Denote by $n \in [0,1]$ the endogenous, ex ante random equilibrium proportion of depositors who decide to withdraw at date t_1 (*aggregate action*). The bank has to pay out measure δn in cash to withdrawing depositors. To finance withdrawals at the interim period, the bank can

[6] A further requirement of the Modigliani- Miller Theorem to hold is information symmetry. Since we will consider limit results as the noise term vanishes, there is almost surely information symmetry among depositors at the limit. Still, the assumption of information symmetry can be considered as violated since the asset is illiquid.

either liquidate arbitrarily small fractions of the asset or borrow at zero net interest[7] from the lender of last resort LOLR to prevent liquidating illiquid assets (Bagehot's rule). The LOLR however only provides liquidity assistance to solvent banks, that is, if the bank remains solvent also in absence of the intervention. Since we apply this rule mechanically, the LOLR is no agent, i.e. makes no strategic decisions in this game. The assumption that a LOLR exists is not crucial to the paper, but makes results cleaner since the bank does not have to liquidate assets although no run occurs.[8] We stress this point, since the Modigliani Miller Theorem is known to fail under debt subsidies such as taxation, here though, at the limit, LOLR never has to intervene since the range of states at which intervention takes place is a null set as noise vanishes, thus liquidity assistance is never paid.

The event of a *bank run* is triggered if at date t_1 the measure of short-term funds claimed by withdrawing depositors exceeds liquidation value of the asset l, that is if $n \in [0, 1]$ realizes such that $\delta n > l$. In a run, depositors queue in front of the bank. The bank sequentially serves one unit to her customers until she runs out of assets to liquidate and is then shut down. Not all depositors are served. We assume that the LOLR perfectly observes the state and can evaluate whether the bank could withstand the run also without her intervention. If yes, she pays liquidity assistance to the bank before the bank starts serving depositors in t_1, by this preventing the bank from liquidating illiquid assets to serve customers. This implies, if the proportion of withdrawing depositors realizes such that claims undercut liquidation value of the asset, the bank can access liquidity assistance by LOLR and liquidation of assets is avoided.

Payoffs Debt Investors In a run, since depositors withdraw simultaneously[9], the position in the queue is random. A depositor's probability to be served the original claim of one unit in the queue before the bank runs out of assets is $\frac{l}{\delta n}$ since δn depositors try to withdraw and claim one unit but only l depositors are served. With probability $1 - \frac{l}{\delta n}$ a queuing depositor is not served and obtains zero. The payoff from withdrawing given a run is thus

$$\frac{l}{\delta n} \cdot 1 + \left(1 - \frac{l}{\delta n}\right) \cdot 0 \tag{2}$$

[7]Adding a positive interest rate will not change results.

[8] In particular, the intervention by the LOLR avoids the realization of states in which the bank is liquid but insolvent in t_1 since she excessively had to liquidate assets. A treatment of this setting is carried out in Eisenbach (2017). Our assumption mirrors real world central bank interventions, national central banks in Europe may pay Emergency Liquidity Assistance to illiquid but solvent banks. The assumption of intervention of a LOLR is not crucial to our results.

[9]For instance electronic withdrawal.

Depositors who roll over in a run get zero. In the absence of a run, coupons are as in the contract if the risky asset pays high at time t_2. If at t_2 the asset does not pay with likelihood $1-\theta$, depositors who roll over get zero.

Event/ Action	withdraw	roll-over
no run, $n \in [0, l/\delta]$	1	$\begin{cases} k, & p=\theta \\ 0, & p=1-\theta \end{cases}$
run, $n \in [l/\delta, 1]$	$\frac{l}{\delta n}$	0

Debt investor's utility difference between rolling over debt to period 2 versus withdrawing early in period 1 is given by

$$v(\theta, n) = \begin{cases} \theta k - 1 & \text{,if } n \leq \frac{l}{\delta} \text{ (no run)} \\ -\frac{l}{\delta n} & \text{,if } n > \frac{l}{\delta} \text{ (run)} \end{cases} \tag{3}$$

Payoffs Equity investors At time two, the equity investors receive the residual value of investment net of payments to debt investors. Due to limited liability, equity value cannot become negative. If the bank does not voluntarily liquidate assets and continues investment, equity value per unit invested equals

$$\text{EV} = \begin{cases} \frac{H-\delta n-\delta k(1-n)}{1-\delta} & , p=\theta \text{ (asset pays) , no run} \\ 0 & , p=1-\theta \text{ (asset does not pay) , no run} \\ 0 & \text{, run} \end{cases} \tag{4}$$

since in case of no run, the bank does not need to liquidate assets.[10] If the bank liquidates the asset voluntarily at t_1, equity value is

$$\text{EV} = \max\left(0, \frac{l-\delta n-\delta(1-n)k}{1-\delta}\right) \tag{5}$$

since the bank can reinvest proceeds at t_1 only in storage.

We impose existence of states which yield dominant actions (dominance regions) to obtain an equilibrium selection, see (Morris and Shin, 2001). There are states $\overline{\theta}$ and $\underline{\theta}$ such that if $\theta < \underline{\theta}$, withdrawing is a dominant action whereas if $\theta > \overline{\theta}$ rolling over is the dominant to debt investors. We refer to $[0, \underline{\theta}]$ as the lower dominance region and call $[\overline{\theta}, 1]$ the upper dominance region. The bound $\underline{\theta}$ depends on the specific contract $(1, k)$ set by the bank and is given as the realization of θ such that

[10] Without a LOLR, an without a run equity value was $\max(0, H(1-\delta r/l) - \delta(1-n)k)$ since the bank would need to liquidate fraction $\delta r/l$ of the asset. Again, this assumption is not crucial but keeps the main point of the result cleaner.

$1 = \underline{\theta}\, k$, i.e.

$$\underline{\theta} = \frac{1}{k} \tag{6}$$

For very high states $\theta \geq \overline{\theta}$, we impose that the asset earns return H already in period 1 and with certainty.[11] The coordination problem vanishes for state realizations in the upper or lower dominance region.[12] To ensure that debt investors may receive signals from which they can infer that the state has realized in either of the dominance regions, we assume that noise ε is sufficiently small such that $\underline{\theta}(r, k) > 2\varepsilon$ and $\overline{\theta} < 1 - 2\varepsilon$ hold.

3 Equilibrium Coordination Game

Runs by depositors cause asset liquidation and thus interruption of investment. Liquidation is ex post efficient only if continuation value of investment exceeds liquidation value. Denote by θ_e the cut-off state below which liquidation is inefficient,

$$\theta_e = \frac{l}{H} \tag{7}$$

At the interim period, the bank and depositors simultaneously choose actions and decide whether or not to liquidate assets respectively enforce liquidation by withdrawing. The bank is obliged to serve withdrawing depositors either by liquidating assets or by borrowing from LOLR. The bank may voluntarily liquidate additional assets. All proofs can be found in the appendix. The equilibrium concept is Bayes Nash.

3.1 Run-prone case

In this section we consider banks which exhibit a liquidity mismatch on their balance sheet and are therefore prone to runs at the interim period. That is, the measure of debt that could be reclaimed at the interim period exceeds the liquidation value of the asset $\delta \geq l$. To determine in what states of the world bank assets are liquidated first observe that a run-prone bank has a dominant action to never voluntarily liquidate assets: Since the bank's actions are not contractible, the bank voluntarily liquidates if and only if equity value from liquidation is higher than value from continuation. If

[11] To make this assumption work, the precise return probability of the asset would need to be $p(\theta) = \begin{cases} \frac{\theta}{\overline{\theta}}, & \theta \in [0, \overline{\theta}] \\ 1, & \theta \in [\overline{\theta}, 1] \end{cases}$ with $\theta \in [0,1]$, $\overline{\theta} \in (0,1)$. The constant $\frac{1}{\overline{\theta}}$ however does not alter incentives and as $\overline{\theta} \to 1$, all results apply and it is without loss of generality to consider $p(\theta) = \theta$, $\theta \in [0, 1]$.

[12] When the asset pays off return H at date one already, the bank can always repay all withdrawing debt investors, $H > 1 > \delta n$ for all $n \in [0, 1]$ and debt ratios $\delta \in (0, 1)$.

the bank liquidates the asset, equity value from liquidation equals liquidation value of the asset less principal and interest payments to depositors payable at time one and two. But liquidation value of the asset is already lower than the face value of debt since the bank is run-prone. Also the bank has no reinvestment opportunity for the proceeds besides storage, which does not produce additional net interest between time one and two. Thus, if the bank is run-prone, proceeds from liquidation at time one always undercut debt claims at time one and two combined and equity value is zero. A run-prone bank never voluntarily liquidates assets due to a reverse debt overhang problem (Myers, 1977), all proceeds from liquidation would go to her creditors whereas if she continues she has the chance to earn the high asset return.

Lemma 3.1. *The run-prone bank never voluntarily liquidates assets prematurely.*

Consequently depositors have to withdraw to enforce liquidation. Note that this result is driven by the lack of profitable reinvestment opportunities. We next analyze in which states it is optimal for depositors to withdraw. Debt investors take capital structure of the bank δ and debt contract $(1, k)$ as given.

Proposition 3.1 (Existence and Uniqueness)**.** *The game played by depositors has a unique equilibrium. The equilibrium is in trigger strategies.*

This result is due to Goldstein and Pauzner (2005). There exists a unique threshold signal (trigger) $\theta^* \in [\underline{\theta} - \varepsilon, \overline{\theta} + \varepsilon]$ such that depositors withdraw if they observe a signal below θ^* and roll over if they observe a signal above θ^*. In case they observe the trigger directly, a depositor is indifferent and we assume that she rolls over. The equilibrium threshold depends on the contract coupon k and capital structure δ. Let $n(\theta, \theta^*)$ the proportion of depositors who withdraw (aggregate action) in state θ if the trigger is θ^*.[13] The equilibrium trigger signal θ^* is given as the depositor's private signal which makes her indifferent between rolling over her deposit and withdrawing given her belief about the proportion of withdrawing depositors and the payoff probability of the asset.

Lemma 3.2. *As signals become precise, the equilibrium trigger is given as*

$$\lim_{\varepsilon \to 0} \theta^* = \frac{1 - \ln(n^*(\delta))}{k}, \qquad n^* = \frac{l}{\delta} \tag{8}$$

The run-prone trigger θ^* is monotonically increasing in debt ratio and is monotonically decreasing in coupon.

[13] Since the equilibrium is a symmetric trigger equilibrium played by a continuum of debt investors, the aggregate action is a deterministic function of the random state and the equilibrium trigger signal and the measure of depositors observing signals below the trigger is given as δn.

Thus, the threshold at which depositors enforce liquidation depends on the bank's debt ratio and the coupon set by the bank. These monotonicity results have been derived before in different settings (Diamond and Dybvig, 1983; Rochet and Vives, 2004; Goldstein and Pauzner, 2005; Morris and Shin, 2016; Eisenbach, 2017). Denote by θ_b the *critical state* below which bank runs occur. For a run-prone bank, the critical state is such that

$$n(\theta_b, \theta^*) \equiv n^* = \frac{l}{\delta} \tag{9}$$

For all state realizations below θ_b the aggregate action exceeds $n^* = l/\delta$ and a bank run enforces asset liquidation. As signals become precise, the trigger and the critical state become indistinguishable[14], and we say *bank stability* improves if the trigger decreases and vice versa. The states for which LOLR intervenes with liquidity assistance is given by $[\theta_b, \theta^* + \varepsilon]$ since for states below θ_b the bank is not eligible for assistance while for states above $\theta^* + \varepsilon$ all depositors roll over and assistance is not necessary. As noise vanishes, the interval becomes a set of size zero and LOLR never intervenes. To determine whether runs of depositors improve welfare, we need to determine the relative size of the efficient liquidation state θ_e to the critical state.

Lemma 3.3. *Inefficient runs exist,* $\theta_b > \underline{\theta} > \theta_e$.

For state realizations in the interval $I = (\theta_e, \theta_b)$, continuation value of investment is in expectation higher than liquidation value, but depositors enforce liquidation by withdrawing. For state realizations in I the bank therefore loses value because by asset illiquidity bankruptcy is costly. This is in contrast to Stiglitz (1969) and Hellwig (1981) where in case of bankruptcy, the asset's continuation value goes to debt investors in case of default respectively continuation value equals liquidation value since there exists no bankruptcy cost. As a consequence, welfare here is highest if θ_b is as low as feasible. Function θ_b is however bounded from below by the bound to the lower dominance region $\underline{\theta}$. Intuitively, the result holds for two reasons. First, as shown in (Diamond and Dybvig, 1983) and (Goldstein and Pauzner, 2005), if the bank is run-prone the miscoordination problem among depositors gives rise to excessive runs due to panics so that assets are liquidated inefficiently often. Second however, the fixed ownership structure, which is not present in (Diamond and Dybvig, 1983; Goldstein and Pauzner, 2005) additionally contributed to excessive runs. This is since independently of whether the bank is run-prone or run-proof, depositors do not participate in the upside potential of the asset. If they roll over their deposit, they may only earn coupon k set by the bank instead of H which in

[14] By equation (40), we have $\theta_b = \theta^* + \varepsilon(1 - 2\frac{l}{\delta}) \to \theta^*$ as $\varepsilon \to 0$.

combination with riskiness of bank assets additionally increases the range of inefficient runs (fundamental runs). That is, not only the bank's capital structure but also the bank's long-term interest rate policy contributes to inefficiency. [15]

We define the value of the bank as the combined value of equity, debt and liquidity assistance L. At the limit, the value of liquidity assistance vanishes and bank value equals value of equity and debt as in the case of Modigliani and Miller (1958). Since the bank is financed with δ units in short-term debt and $1-\delta$ units of equity, we have

$$V = \delta\mathbb{E}[EU] + (1-\delta)\mathbb{E}[E] + L \tag{10}$$

where $\mathbb{E}[EU]$ is the value of one unit invested in the debt contract, $\mathbb{E}[E]$ is the value of one unit invested in equity and L is the value of liquidity assistance which vanishes with noise. The value of debt and equity depend on the critical state at which asset liquidation occurs. By the accounting identity, the bank's value of liabilities has to equal her total value of investment (assets), and we obtain

Proposition 3.2. *The value of the run-prone bank equals*

$$V(k,\delta) = l\,\theta_b + H\int_{\theta_b}^{1} \theta\, d\theta \tag{11}$$

where $\theta_b = \theta_b(\delta, k)$ depends on debt ratio and coupon to depositors.

Bank value is the expected total return on bank investment which equals the asset's liquidation value in case of a run and the asset's continuation value given no run occurs. We immediately see, that bank value crucially depends on the size of the critical state by Lemma 3.3 and is therefore influenced by debt ratio and coupon via a shift in depositors' behavior. In particular, coupon and debt payments in the contract not solely constitute transfers to compensate depositors for risk but alter bank value by modifying stability θ_b. These results are already implied by Rochet and Vives (2004) and Morris and Shin (2016). Having derived the total return on bank investment, we can now analyze risk classes to discuss the connection to the Modigliani Miller Theorem.

3.1.1 Risk-classes

Following the definition of Modigliani and Miller, two banks belong to the same risk-class if and only if their random returns on investment are perfectly correlated.

[15] In the setting here, if the ownership structure was hybrid as in (Diamond and Dybvig, 1983; Goldstein and Pauzner, 2005), then due to risk-sharing the bound to the lower dominance region would be at $1/H$, the fixed ownership structure raises the bound further to $1/k$.

In the case of run-prone banks, denote by X the random variable which describes state contingent total return on bank investment, then

$$X(\theta) = \begin{cases} l, & \theta < \theta_b \\ \theta H, & \theta \geq \theta_b \end{cases} \tag{12}$$

As debt ratio increases, by Lemma 3.2 the probability of runs and thus riskiness of debt goes up. In equilibrium, the critical state is therefore given as

$$\theta_b = \theta_b(\delta, k^*(\delta)) \tag{13}$$

where we take into account the repricing of risk in form of higher coupons $k^*(\delta)$ demanded by depositors. The demanded risk-adjustment $k^*(\delta)$ plays a crucial role in this paper since it directly impacts the cut-off state and thus the bank's risk-class, again by Lemma 3.2.

Definition 3.1 (Risk-preserving adjustment of coupons)**.** *We say that coupon adjusts according to the 'risk-preserving rate' if in equilibrium a marginal increase in debt ratio leads to an increase in coupon such that the critical state (bankruptcy state)* $\theta_b(\delta, k(\delta))$ *and thus total bank risk remains constant.*

Assume bank A and bank B invest in the same asset but have different capital structures $\delta_A \neq \delta_B$. Then both banks belong to the same risk class if and only if for all states θ

$$X_A(\theta) = \rho \cdot X_B(\theta) \tag{14}$$

for a constant $\rho \in \mathbb{R}^+$. But this in particular requires that in equilibrium both banks have identical critical states

$$\theta_A(\delta_A, k^*(\delta_A)) = \theta_B(\delta_B, k^*(\delta_B)) \tag{15}$$

despite their distinct debt ratios. If we assume without loss of generality that $\rho = 1$, then for identical critical states we have

$$V_A = \int_0^1 X_A(\theta)\, d\theta = \int_0^1 X_B(\theta)\, d\theta = V_B \tag{16}$$

and both banks have same values.[16] In particular, under asset illiquidity a bank's risk class is not solely described by the asset's return but also by the equi-

[16]In the classic Modigliani Miller setting, the no arbitrage argument requires that the banks have equal returns at every state since 'no arbitrage' is a concept independent of assignments of probabilities. Here, we need to pin down a probability distribution to derive optimal behavior of depositors and thus the bankruptcy state (critical state). Once the trigger equilibrium and critical state are known to all depositors, also bank value here can be analyzed independently of

librium bankruptcy state (critical state) and liquidation value. In contrast, under no bankruptcy costs as in Stiglitz (1969) and Hellwig (1981) the cut-off state does depend on debt but there is no jump in the return distribution since liquidation and continuation value are equal. As a consequence, under no bankruptcy costs, banks which invest in same assets have same risk classes although they may differ in their capital structure. Under asset illiquidity, return from investment jumps in the cut-off, and cut-off state varies in both debt and coupon. Thus, under asset illiquidity here, the debt ratio in general changes the risk class. This argument was pointed out in (Baxter, 1967) and is crucial since the proof of the Modigliani Miller Theorem can only be performed within risk classes and not across. Under endogenous risk however, and this is new to the previous literature on the Modigliani Miller Theorem, the equilibrium risk adjustment of coupons (repricing of debt) also moves the cut-off state as does debt ratio but in opposite direction. We can therefore determine the risk-preserving adjustment in coupon at which bank value would stay constant in debt ratio and contrast this change with the monopoly risk-adjustment of coupon to determine if and when they agree. Since we keep the asset fixed as we alter the bank's capital structure, we have

Lemma 3.4. *Under asset illiquidity and endogenous liquidity risk, the risk class of a run-prone bank is invariant to changes in debt ratio (capital structure) if and only if the equilibrium risk-compensation in form of higher coupons demanded by depositors is such that the critical state remains constant.*

This result opposes (Baxter, 1967) and the literature on trade-off theory (Myers, 1984) according to which the value of the bank has to alter in debt under bankruptcy costs. If in equilibrium, coupon adjusts according to the risk-preserving rate, the Modigliani Miller Theorem holds despite asset illiquidity, the combined value of equity and debt stays constant despite distinct debt ratios.

3.1.2 Risk-adjusted coupons (Repricing of risk)

In the monopoly case, the bank maximizes her equity value subject to depositors' participation constraint. The debt contract needs to yield a utility level at least as high as the return on storage. We would like to analyze the bank's risk-adjustment in coupon due to an incremental increase in debt ratio. If depositors' participation constraint binds in the optimum, the risk adjustment in coupon is given by depositors'

probabilities. Assume bank A and B invest in the same asset but have distinct critical states $\theta_A < \theta_B$. Then both bank's have same returns for all states above θ_B and all states below θ_A. But for states between θ_A and θ_B, bank A has strictly higher returns by Lemma 3.3. Thus, bank A must have higher value.

marginal rate of substitution. It however turns out that depositors' participation constraint may be slack in the optimum which implies that the risk-adjustment can also be determined by marginal equity value. The latter may occur since for given debt ratio, we have

Lemma 3.5. *For every debt ratio, in equilibrium equity value is concave and hump-shaped in coupon.*

Intuitively, as coupon increases there exists a trade-off from the point of view of equity investors since the contract costs to the bank go up but stability improves. By concavity, equity value is either monotone increasing, monotone decreasing or hump-shaped in coupon. Further, as coupon approaches its upper bound, the return of the asset, one can show that equity value undercuts return on a direct investment in the asset and equity investors will not participate, see Lemma 7.1. Consequently, if for some fixed debt ratio equity value monotonically increases in coupon, equity investors do not participate for any coupon. On the other hand, equity value strictly increases in coupon as coupon approaches its lower bound which together with concavity implies that under voluntary participation, equity value is hump-shaped, for given debt ratio there exists a unique interior coupon $k^u(\delta)$ at which equity value is maximized (unconstrained). If depositors' participation constraint is violated at k^u, then the constrained optimal k^* exceeds k^u and is such that depositors' constraint binds since their utility strictly increases in coupon by Lemma 7.2. Otherwise, if depositors' constraint is slack at k^u the optimal coupon is the interior maximizer of equity value $k^* = k^u$. In a nutshell, the risk-adjustment in coupon for an incremental increase in debt is either determined by depositors' marginal rate of substitution if their constraint binds or by the change of the unconstrained maximizer k^u of equity value. We discuss both in the sequel.

Case 1: Binding participation constraint Assume, depositors' participation constraint binds in the optimum. Then, as debt ratio increases depositors demand debt dependent risk-adjustments such that their utility level stays constant. Since the bank's risk class stays constant only if coupon adjusts according to the risk-preserving rate, for Modigliani Miller to hold in equilibrium, depositors' marginal rate of substitution of coupon for debt must equal the risk-preserving adjustment of coupon for debt. Otherwise the equilibrium critical state changes and the risk class shifts. If the critical state increases, total risk goes up, the risk-class drops and bank value goes down. Vice versa, if the critical state decreases.

Let us take a closer look and analyze when the bank's risk-class is invariant to changes in debt. For run-prone banks, as short-term debt ratio increases, more depositors have a claim on debt in period one and riskiness of the bank increases since the critical state goes up by Proposition 3.2. The increase in critical state lowers the bank's risk class and value in a first effect since the critical state always exceeds the efficient liquidation state by Proposition 3.3, thus inefficient liquidation occurs more often. Further, the increased debt ratio affects depositors' utility from the contract in two different ways:

$$EU = \frac{l}{\delta}\theta_b + \int_{\theta_b}^{1} \theta k \, d\theta \tag{17}$$

A rise in debt ratio enters utility (17) directly since given bankruptcy, the asset's liquidation value is shared by more depositors, l/δ decreases. Additionally, there is an indirect feedback effect. Depositors' posterior belief that the bank goes bankrupt increases since the critical state θ_b below which bankruptcy occurs goes up which makes the decrease in payoff given liquidation even worse. It is due to this feedback effect that the problem of maintaining depositors' utility cannot be considered independently of bank stability (risk class). To maintain participation, the bank has to compensate depositors by increasing the coupon k^* at a rate such that depositors' utility is kept constant. But as coupon goes up, not only depositors' utility increases but by Proposition 3.2 also the trigger and thus critical state go down back towards but not necessarily to exactly the original level. This is since also coupon alterations have a twofold effect on depositors' utility. As a direct effect, it increases the payoff from rolling over given the bank continues investment. Indirectly, the feedback effect comes in again, stability improves because the critical state drops thus the increase in coupon benefits depositors even more, see (17). The equilibrium critical state stays constant as debt increases if and only if depositors' marginal rate of substitution of coupon for debt equals the risk-preserving adjustment of coupon for debt. Equivalently, the critical state is invariant in debt if and only if the rate at which bank stability changes in coupon to compensate for the increase in debt is exactly equal to the direct effect a change in debt has on depositors' utility when holding stability fixed, as opposed to the direct effect of an increase in coupons, i.e. if and only if

$$-\frac{\frac{\partial \theta^*}{\partial \delta}}{\frac{\partial \theta^*}{\partial k}} = -\frac{-\frac{l}{\delta^2}\theta_b}{\int_{\theta_b}^{1} \theta \, d\theta} \tag{18}$$

where the numerator on the right hand side is the direct change of depositors' utility (change in payoffs) in debt when ignoring the feedback effect by holding stability fixed while the denominator is the direct change in utility due to a change in coupon again net of the feedback effect, see (17), and the left hand side is the risk-preserving adjustment of coupon for debt such that the bank's risk class stays constant. These feedback effects in utilities are common in games of endogenous liquidity risk, see (Eisenbach, 2017), and here in our case lead to the possibility that a bank's risk class may stay constant in debt despite accounting for bankruptcy and asset illiquidity (Baxter, 1967) if the risk-adjustment in coupon demanded by depositors equals the competitive risk adjustment. The next Theorem tells us when the coupon adjustment under a monopolistic market structure equals the risk-preserving adjustment to keep the risk class constant

Proposition 3.3 (Run-prone,binding). *If the bank is run-prone and depositors' participation constraint binds in the optimum, the risk-class is invariant to changes in short-term debt ratio if and only if stability is at level* $\theta_b = \theta_M$ *in point* $(\delta, k^*(\delta))$ *with*

$$\theta_M(\delta) = \sqrt{\frac{\mathbb{E}[\theta]}{\frac{l}{\delta} + \mathbb{E}[\theta]}} \tag{19}$$

For $\theta_b < \theta_M$ *the risk-class decreases, and for* $\theta_b > \theta_M$ *the risk-class increases in debt.*

Here, $\frac{l}{\delta}$ is the liquidation value per depositor in case of bankruptcy, while $\mathbb{E}[\theta]$ is the expected probability that the asset pays off. Intuitively, if θ_b is high, depositors value the increase in coupon less than if θ_b was low since for the coupon to be actually paid, the state realization needs to exceed θ_b. Therefore, for high critical states the coupon demanded for participation is higher than the coupon necessary to maintain stability. For $\theta_b > \theta_M$, depositors marginal rate of substitution exceeds the risk-preserving adjustment. Thus, in equilibrium bank stability improves in debt by Proposition 3.2, total risk θ_b decreases and the risk-class of the bank and thus her value go up, more short-term debt is socially beneficial. Vice versa for $\theta_b(\delta, k^*(\delta)) < \theta_M$, the bank's risk class drops, total risk increases, and more short-term debt is socially costly. First observe, that the case at which the bank's risk class is invariant to changes in debt is not stable. State θ_M depends on and increases in the bank's debt ratio. If θ_b hits θ_M, $\theta_b = \theta_M$, a marginal increase in debt ratio leaves θ_b constant while θ_M increases and we transition to the case $\theta_b < \theta_M$ in which bank stability and thus the bank's risk class and value deteriorate in debt. Next, see that the case $\theta_b \leq \theta_M$ is absorbing. For $\theta_b < \theta_M$, θ_b becomes increasing in debt, as is θ_M. Even if θ_b catches up with θ_M and the case $\theta_b = \theta_M$ reoccurs, by the same reasoning as above, θ_b will bounce back below the bound θ_M. For $\theta_b > \theta_M$, bank value increases in debt which implies that in equilibrium θ_b decreases and moves towards θ_M which increases. There are two possible cases, either θ_b decreases in debt until it hits θ_M or it decreases for all debt ratios and never hits θ_M. If it never hits, bank value increases in debt for all debt ratios. If it hits, θ_b is pushed below θ_M and stays below θ_M. Last, θ_b seeks θ_M. θ_M monotonically increases in debt. For $\theta_b > \theta_M$, θ_b falls in debt towards θ_M, for $\theta_b < \theta_M$, θ_b climbs in debt after θ_M. Altogether, by the absorption property if the bank's risk-class decreases in debt at some point, it also weakly decreases in debt for all higher debt ratios. Vice versa, if the risk class strictly improves in debt it also improves in debt over all lower debt ratios. Most importantly, the case $\theta_b = \theta_M$ occurs only countably often, thus the risk-class of the bank strictly alters in short-term debt ratio and thus capital structure for uncountably many debt ratios.

Case 2: Slack participation constraint If for given debt ratio, depositors' participation constraint is slack at the equity value maximizing coupon k^u, the equilibrium coupon equals $k^* = k^u$ and the equilibrium change in coupon due to an increase in debt ratio is not given by depositors' marginal rate of substitution but by the change of the unconstrained maximizer k^u. The bank's risk class remains constant if and only if the change in the maximizing coupon is such that the critical state remains constant which is the case if

Proposition 3.4 (Run-prone,slack). *In the run-prone case, if depositors' participation constraint is slack in the optimum, the risk-class is invariant to changes in short-term debt ratio if and only if stability is at level $\theta_b = \theta_{b,N2}$ in point $(\delta, k^u(\delta))$ with*

$$\theta_{b,N2} = \frac{\frac{H}{k}}{(\frac{H}{E[\theta]} - k)} \frac{1-\delta}{\delta} + \sqrt{\left(\frac{\frac{H}{k}}{(\frac{H}{E[\theta]} - k)} \frac{1-\delta}{\delta}\right)^2 + \frac{k}{(\frac{H}{E[\theta]} - k)}} \tag{20}$$

For $\theta_b < \theta_{b,N2}$ bank value deteriorates and the risk-class decreases, and for $\theta_b > \theta_{b,N2}$ the risk-class increases in debt. The barrier $\theta_{b,N2}$ decreases in debt.

Here, $\frac{1-\delta}{\delta}$ is the equity to debt ratio, H/k is the ratio of gross return on investment (asset return) to coupon payable to depositors, while $\frac{H}{E[\theta]} - k$ can be interpreted as a risk-adjusted return calculation to equity investors since the asset earns return H only with probability θ while coupon k needs to be paid for sure. For $\theta_b < \theta_{b,N2}$ bank value deteriorates since the equilibrium adjustment of coupon due to an incremental change in debt is such that the critical state goes up. Intuitively, for low enough critical states, the bank feels safe. For a small increase in debt she adjusts the coupon to depositors not sufficiently upwards to maintain bank stability at a constant level and stability drops. The opposite is the case for critical states above $\theta_{b,N2}$. The bank fears for her stability and adjusts the coupon upwards more than she would need to maintain stability constant, thus stability improves. Further, analogous to the case with a binding participation constraint, since $\theta_{b,N2}$ decreases in debt, the critical state can hit $\theta_{b,N2}$ only countably many times. This is since for $\theta_b = \theta_{b,N2}$, the critical state remains constant while $\theta_{b,N2}$ decreases in debt. Thus, the case $\theta_b > \theta_{b,N2}$ reoccurs and bank stability improves in debt.

Independently of whether the participation constraint binds, for every equilibrium debt ratio θ_{b1} at which the critical state remains constant, there exist two open intervals $(\theta_{b1}, \theta_{b2})$ and $(\theta_{b0}, \theta_{b1})$ for which the critical state is non constant and the bank's risk class alters. How the bank's risk class alters in a specific debt ratio depends on whether depositors' participation constraints binds in the optimum and how the resulting critical state is located in relation to θ_M respectively $\theta_{b,2}$. Instead of determining for each debt ratio whether the constraint binds or not[17], observe that in either case, conditional on a binding or non-binding participation constraint, the set of debt ratios at which the bank's risk class remains constant is countable. The joint of both sets is countable too and constitutes and upper bound to the number of debt ratios for which the equilibrium change in coupon is such that the bank's risk class remains constant. Since countable sets have Lebesgue measure zero, we have argued

Proposition 3.5. *Consider the monopoly case under endogenous liquidity risk and asset illiquidity. For run-prone banks, the set of debt ratios for which in equilibrium an incremental increase in debt ratio leads to a coupon adjustment such that the bank's risk class remains constant has Lebesgue measure zero.*

Now consider a specific risk class, that is consider a set of debt ratios which in equilibrium require coupon payments such that the resulting critical states are identical[18], then

Theorem 3.1. *For run-prone banks, every bank risk class has Lebesgue measure zero.*

Every set of debt ratios which under correctly priced coupons leads to identical bankruptcy states has only countably many elements. Thus, the Modigliani Miller

[17] The cross derivative of equity value in coupon and debt is in general non-monotone. One can however show that for sufficiently high debt ratios, the cross derivative is negative if equity value decreases in coupon. Consequently, for high debt ratios, if depositors' participation constraint binds in the optimum $(\delta, k_D(\delta))$ it also binds for all higher debt ratios, i.e. the participation constraint cannot become slack since the unconstrained maximizer k^u decreases.

[18] That is, we fix an image $\theta_b(\delta, k^*(\delta))$ where $k^*(\delta)$ is the constrained optimum of equity value and consider the set of all debt ratios δ which are mapped at this particular image. Then, all these debt ratios belong to one particular risk class.

Theorem can generically not apply. The result holds since by Proposition 3.5, we can partition the interval of run-prone debt ratios $[l, 1]$ into countably many subintervals where the bounds of the subintervals are determined by those debt ratios at which the bank's risk class remains constant. For debt ratios within each of the subintervals, the risk class of the bank strictly alters. Thus, the number of elements of each risk class is bounded from above by the number of subintervals.

4 Run-proof Case

As the debt ratio of a run-prone bank approaches liquidation value of the asset from above, the bank becomes run-proof. For a run-proof bank, liquidation value of assets covers all debt claims which potentially arise at the interim period and the occurrence of a run is excluded. Debt is risk-free at time one but remains risky at time two due to riskiness of the asset. Consequently, the coordination problem vanishes for run-proof banks, and depositors face a simple decision problem. In this section, we analyze how bank value changes in debt where debt ratio alters in a way that the bank remains run-proof

Lemma 4.1. *In the run-proof case, the equilibrium trigger is given as*

$$\theta^* = \frac{1}{k} = \underline{\theta} \tag{21}$$

and the run-threshold of depositors is independent of debt ratio and decreases in coupon.

For state realizations below $\underline{\theta}$ all depositors withdraw, however the bank stays liquid since she is sufficiently financed with equity $1 - \delta$. Since in the run-proof case the asset's liquidation value exceeds the face value of deposits, at the interim period withdrawals by depositors never enforce full liquidation and the bank needs to decide in what states to liquidate voluntarily the remaining assets to maximize equity value.

Lemma 4.2. *The run-proof bank voluntarily liquidates for all states below the efficient liquidation state θ_e*

The reason for the result is, for states below the critical state the LOLR no longer intervenes with liquidity assistance. Thus, if the bank decides to stay invested she needs to liquidate assets to satisfy depositors' claims. She only liquidates if continuation value of the remaining fraction of the asset undercuts liquidation value, that is only for states below the efficient liquidation state. This case does not occur for run-prone banks since then withdrawals by all depositors enforces complete liquidation of assets and the bank has no choice.

Proposition 4.1. *If the bank is run-proof the bank's value at the limit equals*

$$V^{rp}(\delta, k_D(\delta)) = l\,\theta_e + \int_{\theta_e}^{1} \theta\, H\; d\theta + \int_{\theta_e}^{\underline{\theta}} \delta(1-\theta H/l))\; d\theta \tag{22}$$

The third term $\int_{\theta_e}^{\underline{\theta}} \delta(1-\theta H/l))\; d\theta$ is negative and constitutes the loss in value which occurs by Proposition (3.3) since depositors also withdraw for state above the efficient liquidation state according to their dominance region which requires inefficient liquidation of fraction $\delta n/l$ of the asset. Before we further discuss bank value of run-proof banks we derive the value of the all equity financed bank. As debt ratio goes to zero, the run-proof bank becomes all equity financed and we have

Corollary 4.1. *The value of the all equity financed bank equals*

$$V^e = l\,\theta_e + \int_{\theta_e}^{1} \theta H\, d\theta \tag{23}$$

Since the bank acts in the best interest of equity investors, in the all equity financed case the bank liquidates if and only if liquidation is efficient, that is for states below θ_e. We can now define risk classes of run-proof banks

Risk-classes (run-proof) Denote by X the random variable which describes total return on bank investment, this time for a run-proof bank. From (22), the total return on investment of a run-proof bank equals

$$X(\theta) = \begin{cases} l, & \theta < \theta_e \\ \theta\, H, & \theta \geq \theta_b = \underline{\theta} \\ \delta(1-\theta H/l)), & \theta \in (\theta_e, \underline{\theta}) \end{cases} \tag{24}$$

where coupon k is such that depositors participate at debt ratio δ. For run-proof banks, a rise in debt ratio no longer affects the critical state $\theta_b = \underline{\theta}$ as long as debt ratio is such that the bank remains run-proof. Also, the efficient liquidation state at which the bank liquidates voluntarily is independent of debt. Therefore, to run-proof banks the cut-off states of the bank's risk class are independent of debt ratio. Between the cut-offs however, the bank has to liquidate assets to serve withdrawing investors and liquidation is inefficient. Since the extent of asset liquidation depends on her debt ratio, lost value increases in debt. Thus, the bank's risk class strictly alters in debt ratio if we can show that depositors do not demand a risk-adjustment of coupon for higher debt ratio. But this is the case since depositors' roll over threshold and their utility is independent of debt in the run-proof case. To see this, consider the limit utility to depositors from the contract

$$EU = 1 \cdot \underline{\theta} + \int_{\underline{\theta}}^{1} \theta k \, d\theta \tag{25}$$

As the bank is run-proof, utility no longer directly depends on debt given a run since the bank can always pay the original coupon of one unit. Additionally, the feedback effect via the trigger does not occur for run-proof banks since the trigger (risk of debt) is independent of debt. Thus, since the risk and utility remain constant in debt depositors do not demand higher coupons as compensation for higher debt ratios. Since the return on investment between the cut-offs strictly decreases in debt ratio, the return distribution (risk class) of the run-proof bank alters in debt and Modigliani Miller cannot apply.

Proposition 4.2. *For run-proof banks, the risk- class and value of the bank strictly decrease in debt ratio.*

In particular, every risk class has exactly one element and we obtain as counterpart to Theorem 3.1

Theorem 4.1. *For run-proof banks, every bank risk class has Lebesgue measure zero.*

Further, as a Corollary from Proposition 4.2,

Corollary 4.2. *For a fixed asset, under endogenous liquidity risk the highest attainable risk class only contains the all equity financed bank.*

The result holds, since by Proposition 4.2 the all equity financed bank attains the highest value among all run-proof banks and by Proposition 3.3 the critical state of every run-prone bank exceeds the efficient liquidation state for any debt ratio and for any coupon depositors might demand for compensation. The value of the bank is maximized when financing all investment with equity only, since only then assets are liquidated efficiently. It should be noted that Proposition (4.2) seems driven by the intervention policy of the LOLR. But the result is robust, to assuming for instance that LOLR intervenes with liquidity assistance for all states, see Lemma 7.4.

Theorem (4.2) further tells us, that under endogenous liquidity risk with asset illiquidity the Modigliani Miller Theorem can only apply to run-prone banks if at all. The Theorem never holds in general, since total return of run-proof banks strictly alters in debt.

5 Extension: Deposit Insurance

Under complete deposit insurance, depositors have no incentive to withdraw or are indifferent and ignore their signals since they are compensated by the insurer in case the bank cannot repay deposits. A run-prone bank, by the same reasoning as above, never voluntarily liquidates assets early since equity value from continuation is higher than value from liquidation which is at zero. Since assets are never liquidated, value of the run-prone bank equals

$$V^{i,r} = \int_0^1 \theta \, H \, d\theta \tag{26}$$

The value is independent of debt and coupon payments since no liquidation takes place, and all coupon payments are transfers. Thus, all run-prone, fully insured banks belong to the same risk-class, have same value and the Modigliani Miller Theorem applies.

Proposition 5.1. *Under full deposit insurance, all run-prone banks belong to the same risk class.*

If the bank faces no liquidity mismatch and is run-proof but completely insured, again all depositors roll over. The bank prematurely liquidates assets voluntarily only if time two equity value from liquidation exceeds equity value from continuation. That is for states $\theta < \theta_v(\delta)$ where

$$\theta_v(\delta) = \frac{l - \delta k}{H - \delta k} \tag{27}$$

The value of the insured run-proof bank is thus

$$V^{rp,i} = \int_0^{\theta_v} l \, d\theta + \int_{\theta_v}^1 \theta \, H \, d\theta \tag{28}$$

The voluntary liquidation state θ_v depends on the bank's debt ratio. Intuitively, as the bank's debt ratio increases the bank liquidates less often since proceeds from liquidation net of debt service need to exceed equity value from continuation where continuation yields a chance to earn high asset return. Therefore, while all run-prone, fully insured banks lie in the same risk class, risk classes of run-proof, insured banks alter in capital structure, i.e. decrease in debt and the Modigliani Miller cannot apply.

Proposition 5.2. *Under full deposit insurance, the risk class of run-proof banks strictly decreases in debt ratio .*

As in the uninsured case, risk classes of run-proof banks have only one element and Modigliani Miller cannot apply. Further, the run-proof bank liquidates assets too seldom, $\theta_v(\delta) < \theta_e$, and by this loses value. Still, $\theta_v > 0$ implies that the value of the run-proof fully insured bank exceeds value of the run-prone insured bank which never liquidates, and we have shown

Theorem 5.1. *Under full deposit insurance, the value of every run-proof bank exceeds value of every run-prone bank. The all equity financed bank achieves highest value*

$$V^{i,r} < V^{rp,i} \leq V^{e} \tag{29}$$

By convergence of the state θ_v, the value of the run-proof insured bank approaches the value of the all equity financed bank as debt ratio goes to zero and approaches the value of the run-prone insured bank as debt ratio becomes large. Similar to the case without deposit insurance, the Modigliani Miller Theorem cannot apply in general but under insurance applies within the class of run-prone banks.

6 Conclusion

This paper demonstrates that the Modigliani Miller Theorem does in general not apply to banks which face endogenous liquidity risk when assets are illiquid, independently of deposit insurance. The original no-arbitrage proof of the Modigliani Miller Theorem is conducted by showing that firms of the same risk class but with distinct capital structures must have same values, otherwise arbitrage exists. A risk class contains firms with perfectly correlated returns. If debt is risky, changes in debt require higher coupon payments to depositors to maintain participation. We show that a bank's risk class varies in both the amount of short-term debt financing and coupon demanded for changed risk. Thus, it can be that risk classes contain only small numbers of banks with distinct capital structures although all banks considered invest in the same asset. Therefore, the Modigliani Miller Theorem is in general not applicable since its main requirement, the independence of risk classes from capital structure, is not satisfied for banks. Banks can become more or less valuable as short-term debt ratio alters and their costs of capital change. In particular, more short-term debt can be socially costly or beneficial. While the failure of Modigliani Miller is known for settings which incorporate bankruptcy costs, this paper offers a new perspective from the view point of the game theoretic literature on endogenous liquidity risk.

Our paper deviates from the classic Modigliani Miller setting in various ways to discuss the connection to the literature on endogenous liquidity risk (Diamond and Dybvig, 1983; Goldstein and Pauzner, 2005; Morris and Shin, 2016; Eisenbach, 2017). Crucial for the result to obtain is that the asset carries a liquidity premium such that the bank cannot liquidate prematurely at fair value (continuation value). As a consequence, the total return distribution jumps in the bankruptcy state which depends on debt ratio and coupon. In contrast, Modigliani and Miller assume perfectly safe debt (no bankruptcy) while Stiglitz (1969) allows for bankruptcy but at no costs. Further, we assume non-contractible actions of the bank manager who acts on behalf of equity investors, similar to Dewatripont and Tirole (1994). As a consequence, depositors have to run to enforce liquidation which may result in a deadweight loss. The analysis here reduces bank capital structure to short-term debt and equity. The results remain to hold with some minor changes when substituting part of equity with long-term debt.[19] We assume that a lender of last resort exists to intervene with liquidity assistance at the interim period if the bank is solvent but illiquid. We assume this intervention to maintain clean intuition and formula. The assumption of liquidity assistance paid by the lender of last resort here can be misunderstood as crucial since the Modigliani Miller Theorem is know to not hold under debt subsidies, e.g. in form of tax shield. This is not the case here. All main results continue to hold when not considering intervention.[20] We model the asset's liquidation value as a fixed constant. This is again without loss of generality. Eisenbach (2017) models liquidation value as an endogenous function of the aggregate number of assets sold in the economy, where banks are price takers. Alternatively, Morris and Shin (2016) model the amount of cash the bank can pledge against the asset as function the state. In either case, as long as liquidation value of the asset may in some states deviate from continuation value, our results continue to hold when inefficient asset liquidation exists and reduces bank value. The main deviation of our analysis from the previous literature on the Modigliani Miller Theorem is that we work in a game theoretic setting which requires the allocation of a probability distribution to states of the world to derive optimal behavior of depositors. In contrast, the proof to the original Modigliani Miller Theorem proceeds via no arbitrage which does not require assignments of probabilities. A further deviation is that we work in a monopolistic setting instead of competitive markets. We do so since in the

[19] The bankruptcy state will still vary in debt and coupons.

[20] First, we show that as noise vanishes, the lender of last resort never intervenes. Still, the assumption that LOLR exists alters payoffs for certain states. Without LOLR, the bankruptcy state would still vary in both debt and coupon and the formula for bank value will be the same at the limit. There is one notable exception, without LOLR, the trigger becomes hump-shaped and concave in coupon (interior maximizer).

original Modigliani Miller setting debt is risk-free and equivalent to storage while in our setting debt is risky and storage constitutes an alternative investment to depositors. To concentrate on showing that risk classes may change in debt, we shut down investors' opportunity to invest in equity or other firm's capital to restrict the number of outside options. Since bank risk classes here shift in both, debt ratio and coupons, the outside option plays a crucial role since it determines depositors' compensation for risk. With competition, the following challenge will arise: when allowing investors to trade arbitrary numbers of units in debt and equity, depositors become asymmetrically large which alters the trigger, see Corsetti et al. (2004).

7 Appendix

Proof. [Proposition 3.1] We follow closely the existence and uniqueness proof given in Goldstein and Pauzner (2005).

A: Existence and uniqueness of a trigger equilibrium A Bayesian equilibrium is a strategy profile such that each investor chooses the best action given her private signal and her beliefs about other players actions and strategies of other players. In equilibrium, an investor decides to withdraw when her expected payoff from rolling over versus withdrawing given her signal is negative, decides to roll over when it is positive and is indifferent if the expected payoff is zero. Since investors are identical ex ante, investors strategies can only differ at signals that make an investor indifferent between rolling over and withdrawing. In a trigger equilibrium around trigger signal θ^*, all investors withdraw when they observe signals below θ^* and roll over if they observe signals above θ^*. In case of directly observing θ^* investors are indifferent and we specify here hat they will roll over. A threshold equilibrium around trigger θ^* exists if and only if given that all other investors use a trigger strategy around signal θ^* an investor finds it optimal to also use a trigger strategy around trigger θ^*. Assume, all investors follow a symmetric threshold strategy around trigger signal θ^*, then the proportion of investors who withdraw at each state is deterministic. Denote by $n(\theta, \theta^*)$ the proportion of investors who observe signals below signal θ^* and thus withdraw if the true state is θ.

Let $D(\theta_i, n(\cdot, \theta^*))$ the expected payoff difference from rolling over versus withdrawing when the investor observes signal θ_i, and other investors follow a trigger strategy around θ^*. Given signal θ_i an investor's posterior belief on state θ is uniform on $[\theta_i - \varepsilon, \theta_i + \varepsilon]$. The expected payoff difference therefore is

$$D(\theta_i, n(\cdot, \theta^*)) = \frac{1}{2\varepsilon} \int_{\theta_i - \varepsilon}^{\theta_i + \varepsilon} \left[(k\theta - 1)\ \mathbf{1}_{\{n(\theta,\theta^*) \leq \frac{l}{\delta}\}} - \frac{l}{\delta n(\theta, \theta^*)}\ \mathbf{1}_{\{n(\theta,\theta^*) > \frac{l}{\delta}\}} \right] d\theta \quad (30)$$

where $\mathbf{1}_{\{n(\theta,\theta^*) \leq \frac{l}{\delta}\}}$ is an indicator function which takes value one if and only if the state realizes such that the endogenous proportion of withdrawing investors is below ratio $\frac{l}{\delta}$, that is in the absence of a bank run, while indicator function $\mathbf{1}_{\{n(\theta,\theta^*) > \frac{l}{\delta}\}}$ equals one if and only if the state realizes such that a run occurs. For existence of a trigger equilibrium we need to show that given a signal realization below (above) the threshold which other investors use, the single investor finds it optimal to withdraw (roll over)

$$D(\theta_i, n(\cdot, \theta^*)) < 0 \qquad \text{for all } \theta_i < \theta^* \quad (31)$$

$$D(\theta_i, n(\cdot, \theta^*)) > 0 \qquad \text{for all } \theta_i > \theta^* \quad (32)$$

and existence and uniqueness of a signal θ^* for which an investor is indifferent between rolling over and withdrawing (payoff indifference)

$$0 = D(\theta^*, n(\cdot, \theta^*)) \tag{33}$$

To prove existence and uniqueness of θ^* such that (33) holds, observe that the function $D(\theta^*, n(\cdot, \theta^*))$ is continuous in θ^*. By existence of dominance regions, we have $D(\theta^*, n(\cdot, \theta^*)) < 0$ for signals $\theta^* < \underline{\theta} - \varepsilon$ and $D(\theta^*, n(\cdot, \theta^*)) > 0$ for $\theta^* > \overline{\theta} + \varepsilon$. Thus, together with continuity by the Intermediate Value Theorem there exists at least one θ^* for which (33) holds.

To see uniqueness, since all other investors use a threshold strategy around θ^* and are small, we know the function $n(\theta, \theta^*)$ and can substitute $n(\theta, \theta^*) = \frac{1}{2} + \frac{\theta^* - \theta}{2\varepsilon}$ to derive

$$D(\theta^*, n(\cdot, \theta^*)) = \int_0^{l/(\delta)} (k\theta(n, \theta^*) - 1) \; dn - \int_{l/(\delta)}^1 \frac{l}{\delta n} \; dn \tag{34}$$

where $\theta(n, \theta^*) = \theta^* + \varepsilon(1 - 2n), \theta^* \in [\underline{\theta} - \varepsilon, \overline{\theta} + \varepsilon]$ is the inverse of the function $n(\theta, \theta^*)$. For uniqueness, observe that $D(\theta^*, n(\cdot, \theta^*))$ depends on signal θ^* only via the asset return function $p(\theta)$ which is strictly increasing in signal θ^* for $\theta^* < \overline{\theta} + \varepsilon$. Thus $D(\theta^*, n(\cdot, \theta^*))$ strictly increases in θ^* for $\theta^* \in [\underline{\theta} - \varepsilon, \overline{\theta} + \varepsilon]$ which together with continuity gives us single-crossing.

Next we need to show that withdrawing is a best response if the private signal of an investor realizes below the trigger played by other investors $\theta_i < \theta^*$, that is we need to show (31). Following Goldstein and Pauzner (2005), let $\theta_i < \theta^*$. Decompose the intervals $[\theta_i - \varepsilon, \theta_i + \varepsilon]$ and $[\theta^* - \varepsilon, \theta^* + \varepsilon]$ over which the integrals $D(\theta_i, n(\cdot, \theta^*))$ and $D(\theta^*, n(\cdot, \theta^*))$ are calculated into a potentially empty common part $c = [\theta_i - \varepsilon, \theta_i + \varepsilon] \cap [\theta^* - \varepsilon, \theta^* + \varepsilon]$ and the disjoint parts $d^i = [\theta_i - \varepsilon, \theta_i + \varepsilon] \setminus c$ and $d^* = [\theta^* - \varepsilon, \theta^* + \varepsilon] \setminus c$. Then,

$$D(\theta_i, n(\cdot, \theta^*)) = \frac{1}{2\varepsilon} \int_{\theta \in c} v(\theta, n(\theta, \theta^*)) \; d\theta + \frac{1}{2\varepsilon} \int_{\theta \in d^i} v(\theta, n(\theta, \theta^*)) \; d\theta \tag{35}$$

$$D(\theta^*, n(\cdot, \theta^*)) = \frac{1}{2\varepsilon} \int_{\theta \in c} v(\theta, n(\theta, \theta^*)) \; d\theta + \frac{1}{2\varepsilon} \int_{\theta \in d^*} v(\theta, n(\theta, \theta^*)) \; d\theta \tag{36}$$

Considering (36), the integral $\int_{\theta \in c} v(\theta, n(\theta, \theta^*)) \; d\theta$ has to be negative since by (33) $D(\theta^*, n(\cdot, \theta^*)) = 0$ and since the fundamentals in range d^* are higher than in c as we assumed $\theta_i < \theta^*$ and because in interval $[\theta^* - \varepsilon, \theta^* + \varepsilon]$ the payoff difference $v(\theta, n)$ is positive for high values of θ, negative for low values of θ and satisfies single-crossing. In addition, the function $n(\theta, \theta^*)$ equals one over the interval d^i, since d^i is below $\theta^* - \varepsilon$. Therefore, the integral $\int_{\theta \in d^i} v(\theta, n(\theta, \theta^*))\, d\theta$ is negative too which with (35) implies that $D(\theta_i, n(\cdot, \theta^*))$ is negative. The proof for (32) proceeds analogous.

B No existence of non-monotone equilibria See Goldstein and Pauzner, proof of Theorem 1, part C □

Proof. [Lemma 3.2]

Conditional on observing signal θ_i an investor's posterior on state θ is uniform on $[\theta_i - \varepsilon, \theta_i + \varepsilon]$. Let $n(\theta, \theta^*)$ the proportion of depositors who observe signals below the trigger and thus withdraw (aggregate action) in state θ if the trigger is θ^*.[21] Then, by the error distribution

$$n(\theta, \theta^*) = \begin{cases} \frac{1}{2} + \frac{\theta^* - \theta}{2\varepsilon} & \text{if } \theta \in [\theta^* - \varepsilon, \theta^* + \varepsilon] \\ 1 & \text{if } \theta \leq \theta^* - \varepsilon \\ 0 & \text{if } \theta \geq \theta^* + \varepsilon. \end{cases} \tag{37}$$

The expected payoff from rolling over versus withdrawing given signal $\theta_i = \theta^*$ and threshold θ^* equals

$$D(\theta^*, \theta^*) = \frac{1}{2\varepsilon} \int_{\theta^* - \varepsilon}^{\theta^* + \varepsilon} (\theta\, k - 1)\ \mathbf{1}_{\{n(\theta, \theta^*) \leq n^*\}} - \frac{l}{\delta n} \mathbf{1}_{\{n(\theta, \theta^*) > n^*\}}\, d\theta \tag{38}$$

The marginal investor who observes signal $\theta_i = \theta^*$ has a uniform belief on $n \sim U[0, 1]$ (Laplacian belief, see Morris and Shin (2001)). Substituting for n using (37),

$$0 = \int_0^{n^*} (\theta(n, \theta^*)\, k - 1)\ dn + \int_{n^*}^1 -\frac{l}{\delta n}\, dn \tag{39}$$

where

$$\theta(n, \theta^*) = \theta^* + \varepsilon(1 - 2n) \tag{40}$$

is the inverse of $n(\theta, \theta^*)$ for $\theta \in [\theta^* - \varepsilon, \theta^* + \varepsilon]$. Plugging in $\theta(n, \theta^*)$, since the posterior belief of no run n^* equals liquidity ratio $\frac{l}{\delta}$ we obtain the equilibrium trigger

$$\theta^* = \frac{1 - \ln(n^*(\delta))}{k} - \varepsilon\,(1 - n^*(\delta)) \tag{41}$$

with derivatives which at the limit equal

[21] Since the equilibrium is a symmetric trigger equilibrium played by a continuum of debt investors, the aggregate action is a deterministic function of the random state and the equilibrium trigger signal and the measure of depositors observing signals below the trigger is given as δn.

$$\frac{\partial}{\partial \delta}\theta^* = \frac{1}{k\delta} > 0, \qquad \frac{\partial^2}{\partial \delta^2}\theta^* = -\frac{1}{k\delta^2} < 0 \tag{42}$$

$$\frac{\partial}{\partial k}\theta^* = -\frac{\theta^*}{k} < 0, \qquad \frac{\partial^2}{\partial^2 k}\theta^* = -\frac{\frac{\partial}{\partial k}\theta^*}{k} > 0, \qquad \frac{\partial}{\partial \delta}\frac{\partial}{\partial k}\theta^* = -\frac{1}{k}\frac{\partial \theta^*}{\partial \delta} < 0 \tag{43}$$

□

Proof. [Lemma 3.3] From (41), at the limit $\varepsilon \to 0$

$$\theta^* = \frac{1 - \ln(n^*(\delta))}{k} > \frac{1}{k} = \underline{\theta} > \frac{l}{k} > \frac{l}{H} = \theta_e$$

□

Proof. [Proposition 3.2]

The utility to depositors from the contract equals

$$EU = \int_0^{\theta^*-\varepsilon} \frac{l}{\delta} d\theta + \int_{\theta^*-\varepsilon}^{\theta_b} \frac{l}{\delta n(\theta)} n(\theta)\, d\theta \tag{44}$$

$$+ \int_{\theta_b}^{\theta^*+\varepsilon} n(\theta) + (1 - n(\theta))\, \theta k d\theta + \int_{\theta^*+\varepsilon}^{1} \theta k\, d\theta \tag{45}$$

$$= \theta_b \frac{l}{\delta} + \int_{\theta_b}^{\theta^*+\varepsilon} n(\theta) + (1 - n(\theta))\, \theta k\, d\theta + \int_{\theta^*+\varepsilon}^{1} \theta k\, d\theta \tag{46}$$

The value of debt equals

$$D = \delta \cdot EU = \theta_b l + \int_{\theta_b}^{\theta^*+\varepsilon} \delta n(\theta) + (1 - n(\theta))\, \delta\theta k\, d\theta + \int_{\theta^*+\varepsilon}^{1} \theta\delta k\, d\theta \tag{47}$$

Equity value per unit invested (return on equity) equals

$$EV = \frac{1}{1-\delta} \int_{\theta_b}^{1} \theta \left(H - \delta n(\theta, \theta^*) - (1 - n(\theta, \theta^*))\delta k\right) d\theta \tag{48}$$

Here, $-\delta n(\theta)$ is the repayment to the LOLR who intervenes for states above θ_b and $-(1 - n(\theta))\delta k$ is the repayment of debt claims in period two. LOLR intervenes for states above θ_b when the bank is solvent while for states above $\theta^* + \varepsilon$ all depositors roll over and intervention is not required. Value of liquidity assistance paid equals

$$L = \int_{\theta_b}^{\theta^*+\varepsilon} -\delta n(\theta) + \theta\delta n(\theta)\, d\theta = 2\varepsilon \int_0^{l/\delta} -\delta n + \theta(n,\theta)\delta n\, d\theta \to 0$$

by Lebesgues Dominated Convergence Theorem, as the integrand is bounded. Away from the limit, value of the bank is the sum of the value of debt, equity and liquidity assistance and the equation follows when canceling out all transfers. □

Proof. [Lemma 3.5] From (48)

$$\frac{\partial}{\partial k}EV = \frac{1}{1-\delta}\Big[-\frac{\partial\theta_b}{\partial k}\theta_b(H-l-k(\delta-l)) - \int_{\theta_b}^{1} \theta\delta(1-n(\theta))d\theta \tag{49}$$

$$+\frac{\partial\theta^*}{\partial k}\delta(k-1)\int_{\theta_b}^{1}\theta\,\frac{\partial n}{\partial\theta^*}d\theta\Big] \tag{50}$$

with $n(\theta_b) = \frac{l}{\delta}$. Further, with $\frac{\partial n}{\partial\theta^*} = \frac{1}{2\varepsilon}$ and substituting for $n(\theta,\theta^*)$,

$$\int_{\theta_b}^{1}\theta\,\frac{\partial n}{\partial\theta^*}d\theta = \frac{1}{2\varepsilon}\int_0^{l/\delta}\theta(n,\theta^*)\,2\varepsilon\,dn = \int_0^{l/\delta}(\theta^*+\varepsilon(1-2n))\,dn \tag{51}$$

at the limit, the latter part of the integrand vanishes and we have

$$\frac{\partial}{\partial k}EV = \frac{1}{1-\delta}\Big[-\frac{\partial\theta_b}{\partial k}\theta_b(H-k\delta) - \int_{\theta_b}^{1}\theta\delta(1-n(\theta))d\theta\Big] \tag{52}$$

since at the limit, $\frac{\partial\theta^*}{\partial k} = \frac{\partial\theta_b}{\partial k}$. The first term is the change in equity value due to a change in stability and is positive, the second term is the change in equity value due to an increase in contract costs and is negative. For the second derivatives, note that we can take partial derivatives of (52) directly since away from the limit in (51), any partial derivative of the second part of the integral would vanish as noise goes to zero.

$$\frac{\partial^2}{\partial k^2}EV = \frac{1}{1-\delta}\Big[-\left[\frac{\partial^2\theta_b}{\partial k^2}\cdot\theta_b + \left(\frac{\partial\theta_b}{\partial k}\right)^2\right](H-k\delta) + \delta\frac{\partial\theta_b}{\partial k}\theta_b \tag{53}$$

$$+\int_{\theta_b}^{1}\theta\delta\frac{\partial n}{\partial\theta^*}\frac{\partial\theta^*}{\partial k}d\theta + \frac{\partial\theta_b}{\partial k}\theta_b\delta(1-n(\theta_b))\Big] \tag{54}$$

with $n(\theta_b) = \frac{l}{\delta}$ and again with (51) we can simplify

$$\frac{\partial^2}{\partial k^2}EV = \frac{1}{1-\delta}\left[-\left(\frac{\partial^2\theta_b}{\partial k^2}\cdot\theta_b + \left(\frac{\partial\theta_b}{\partial k}\right)^2\right)(H-k\delta) + 2\delta\,\frac{\partial\theta_b}{\partial k}\theta_b\right] \tag{55}$$

Since $\frac{\partial^2\theta_b}{\partial k^2} = \frac{2\theta^*}{k^2} > 0$ and $\frac{\partial\theta^*}{\partial k} < 0$, all terms are negative and equity value is concave in coupon. As a consequence, for given debt ratio equity value is either monotone increasing, or monotone decreasing or hump-shaped in k. We have

Lemma 7.1. *As coupon approaches its upper bound, equity investors do not participate.*

By Lemma 7.1, for every debt ratio and $k \to H$, equity value undercuts value of a direct investment in the asset, thus in equilibrium for given debt ratio equity values cannot be monotone increasing in coupon since a direct investment in the asset would perform better for all coupons. Consequently, for every debt ratio in equilibrium equity value is either hump-shaped or monotone decreasing in k. We next exclude the latter case. By concavity, if for given debt ratio and equity value decreases in k as $k \to 1$, it also decreases in k for all higher coupons. But

$$\lim_{k\to 1}\frac{\partial}{\partial k}EV = \frac{1}{1-\delta}\Big[-\frac{\partial\theta_b}{\partial k}\theta_b(H-\delta) - \int_{\theta_b}^{1}\theta\delta(1-n(\theta))d\theta\Big] \tag{56}$$

where $-\frac{\partial\theta_b}{\partial k} = \frac{\theta_b}{k} = \theta_b$ as $k \to 1$ and at the limit, $n = 0$ for $\theta > \theta_b$, thus

$$\int_{\theta_b}^{1}\theta\delta(1-n(\theta))d\theta \to \frac{1}{2}\,\delta(1-\theta_b^2) \tag{57}$$

Thus,

$$\lim_{k\to 1}\frac{\partial}{\partial k}EV = \frac{1}{1-\delta}\Big[\theta_b^2(H-\delta) - \frac{1}{2}\,\delta(1-\theta_b^2)\Big] \tag{58}$$

and $\lim_{k\to 1}\frac{\partial}{\partial k}EV < 0$ is equivalent to

$$\theta_b^2 < \frac{\delta}{2H-\delta} \tag{59}$$

but $\frac{\delta}{2H-\delta} < 1$ and as $k \to 1$, we have $\theta_b \to 1 - \ln(l/\delta) > 1$, thus (59) can never hold and $\lim_{k\to 1}\frac{\partial}{\partial k}EV > 0$, thus equity value cannot be monotonically decreasing. Thus, in equilibrium equity value is hump-shaped in coupon k, i.e. there exists an interior maximizer k^u of equity value. □

Proof. [Lemma 7.1] From (48),

Since at the limit $n = 0$ for $\theta > \theta_b$,

$$\lim_{k \to H} EV = \frac{1}{1-\delta} \int_{\theta_b}^{1} \theta\, H(1-\delta)\, d\theta = \int_{\theta_b}^{1} \theta\, H\, d\theta < \int_{0}^{1} \theta\, H\, d\theta \tag{60}$$

where the right hand side equals utility from a direct investment in the asset. □

Proof. [Proposition 3.3] Define coupon $k_D(\delta)$ as the function of debt ratio such that depositors' participation constraint remains binding when altering δ. That is, when considering all run-prone debt ratios $\delta \in [l, 1]$, the coupon k^D may deviate from the constraint optimal k^* since in the optimum depositors' constraint may be slack. Assume for this proof, depositors' participation constraint binds in the optimum, thus we have $k^* = k_D$ and $\theta_b = \theta_b(\delta, k_D(\delta))$. The change of bank value for a change in debt ratio by Proposition 3.2 equals

$$\frac{\partial}{\partial \delta} V(k, \delta) = (l - H\theta_b) \frac{d\theta_b}{d\delta} \tag{61}$$

where the total change in critical state $\theta_b(\delta, k_D(\delta))$ is given as

$$\frac{d\theta_b}{d\delta} = \frac{\partial \theta_b}{\partial \delta} + \frac{\partial \theta_b}{\partial k} \frac{\partial k_D}{\partial \delta} \tag{62}$$

where $\frac{\partial k_D}{\partial \delta}$ is depositors' marginal rate of substitution. The risk-preserving adjustment $k_{\theta_b}(\delta)$, i.e. the coupon at debt ratio δ such that stability is maintained constant at level θ_b satisfies

$$k'(\delta)_{\theta_b} = \frac{\partial k_{\theta_b}}{\partial \delta} = -\frac{\frac{\partial \theta^*}{\partial \delta}}{\frac{\partial \theta^*}{\partial k}} = -\frac{\frac{1}{k\delta}}{-\frac{\theta^*}{k}} = \frac{1}{\delta \theta^*} \tag{63}$$

Since $l - H\theta_b < 0$, the value of the bank stays constant if and only if the total change in the critical state is zero $\frac{d\theta_b}{d\delta} = 0$, i.e. if and only if depositors' marginal rate of substitution equals the risk-preserving adjustment $k_{\theta_b}(\delta)$. If the total change in critical state is negative, $\frac{d\theta_b}{d\delta} < 0$, bank value increases since $l < H\theta_b$ in (61) that is with $\frac{\partial \theta_b}{\partial k} < 0$ if and only if

$$\frac{\partial k_D}{\partial \delta} \geq \frac{\partial k_{\theta_b}}{\partial \delta} \tag{64}$$

Lemma 7.2. *It holds*

$$k_D'(\delta) = -\frac{\frac{\partial D}{\partial \delta}}{\frac{\partial D}{\partial k}} = -\frac{-\frac{l}{\delta^2}\theta_b + \frac{\partial \theta_b}{\partial \delta}\left(\frac{l}{\delta} - \theta_b k\right)}{\int_{\theta_b}^{1}(1 - n(\theta))\theta\, d\theta + \frac{\partial \theta_b}{\partial k}\left(\frac{l}{\delta} - \theta_b k\right)} \tag{65}$$

If the risk-preserving adjustment and depositors' marginal rate of substitution satisfy (64) the value of the bank increases since the coupon demanded to keep stability constant grows slower than than coupon demanded by depositors for compensation of additional risk. By Lemma 7.2, (64) holds if and only if

$$-\frac{\frac{\partial\theta^*}{\partial\delta}}{\frac{\partial\theta^*}{\partial k}} \leq -\frac{-\frac{l}{\delta^2}\theta_b + \frac{\partial\theta_b}{\partial\delta}\left(\frac{l}{\delta} - \theta_b k\right)}{\int_{\theta_b}^1 (1-n(\theta))\theta\, d\theta + \frac{\partial\theta_b}{\partial k}\left(\frac{l}{\delta} - \theta_b k\right)} \tag{66}$$

Since $\frac{\partial\theta^*}{\partial k} < 0$ and the denominator on the right hand side is positive by $\frac{l}{\delta} - \theta_b k < 0$, by canceling terms we have equivalence to

$$\frac{\frac{\partial\theta^*}{\partial\delta}}{\frac{\partial\theta^*}{\partial k}} \geq -\frac{\frac{l}{\delta^2}\theta_b}{\int_{\theta_b}^1 (1-n(\theta))\theta\, d\theta} \tag{67}$$

where the left hand side equals $-k'_{\theta_b}(\delta)$. Since $n(\theta) = 0$ on $[\theta^* + \varepsilon, 1]$ and $\theta^* + \varepsilon \to \theta^* \to \theta_b$, at the limit $\varepsilon \to 0$,

$$\int_{\theta_b}^1 (1-n(\theta))\theta\, d\theta = \int_{\theta_b}^{\theta^*+\varepsilon} (1-n(\theta))\theta\, d\theta + \int_{\theta^*+\varepsilon}^1 \theta\, d\theta \to \frac{1}{2}(1-\theta_b^2) \tag{68}$$

by Lebesgues dominated convergence Theorem. Plugging in (63), since at the limit, the trigger and the critical state coincide,

$$1 \leq \frac{\frac{l}{\delta}\theta_b^2}{\frac{1}{2}(1-\theta_b^2)} \tag{69}$$

Since $\theta_b \in (0,1)$,

$$\theta_b \geq \sqrt{\frac{\frac{1}{2}}{\frac{l}{\delta} + \frac{1}{2}}} = \sqrt{\frac{\mathbb{E}[\theta]}{\frac{l}{\delta} + \mathbb{E}[\theta]}} =: \theta_M \qquad \square$$

Proof. [Lemma 7.2] From (44)

$$\frac{\partial}{\partial\delta}D = -\frac{l}{\delta^2}\theta_b + \frac{\partial\theta_b}{\partial\delta}\left(-(1-n(\theta_b))\theta_b k\right) + \int_{\theta_b}^{\theta^*+\varepsilon} \frac{\partial n}{\partial\theta^*}\frac{\partial\theta^*}{\partial\delta}(1-\theta k)\, d\theta \tag{70}$$

Since for $\varepsilon \to 0$

$$\int_{\theta_b}^{\theta^*+\varepsilon} \frac{\partial n}{\partial\theta^*}\frac{\partial\theta^*}{\partial\delta}(1-\theta k)\, d\theta = \frac{\partial\theta^*}{\partial\delta}\int_0^{l/\delta} \frac{1}{2\varepsilon}(1-\theta(n,\theta^*)k)\, d\theta \to \frac{l}{\delta}\frac{\partial\theta_b}{\partial\delta}(1-\theta^* k) \tag{71}$$

We have with $n(\theta_b) = \frac{l}{\delta}$ and $\theta^* \to \theta_b$

$$\frac{\partial}{\partial \delta} D = -\frac{l}{\delta^2}\theta_b + \frac{\partial \theta_b}{\partial \delta}\left(\frac{l}{\delta} - \theta_b k\right) \tag{72}$$

Similarly,

$$\frac{\partial}{\partial k} D \to \frac{\partial \theta_b}{\partial k}\left(\frac{l}{\delta} - \theta_b k\right) + \int_{\theta_b}^{1} (1 - n(\theta))\theta \, d\theta \tag{73}$$

Further, since $\frac{\partial \theta_b}{\partial k} < 0$ and by definition of the lower dominance region $\theta_b k > 1 > l/\delta$, $\frac{\partial}{\partial k} D > 0$ and thus also $\frac{\partial}{\partial k} EU = \frac{\partial}{\partial k}(D/\delta) > 0$. □

Proof. [Proposition 3.4] The change of bank value for a change in debt ratio is given by (61) where now the total change in critical state $\theta_b(\delta, k^u(\delta))$ derives from a direct change in debt and the indirect change via interior, unconstrained maximizer of equity value k^u given depositors' constraint is slack

$$\frac{d\theta_b}{d\delta} = \frac{\partial \theta_b}{\partial \delta} + \frac{\partial \theta_b}{\partial k}\frac{\partial k^u}{\partial \delta} \tag{74}$$

Since k^u is implicitly defined by

$$\frac{\partial}{\partial k} EV(\delta, k^u(\delta)) = 0 \tag{75}$$

with $l < H\theta_b$ and $\frac{\partial \theta_b}{\partial k} < 0$, by (61) bank value deteriorates in debt if $\frac{d\theta_b}{d\delta} > 0$ or alternatively

$$\frac{\partial k^u}{\partial \delta} < -\frac{\frac{\partial \theta_b}{\partial \delta}}{\frac{\partial \theta_b}{\partial k}} \tag{76}$$

where the right hand side is the change in coupon necessary for an incremental change in debt to maintain the bankruptcy state at a constant level. By definition (75), as debt increases, k^u has to change in a way that marginal equity value remains zero, i.e. such that the total derivative satisfies $\frac{d}{d\delta}\frac{\partial}{\partial k} EV(\delta, k^u(\delta)) = 0$. The total derivative equals

$$\frac{d}{d\delta}\frac{\partial}{\partial k} EV(\delta, k^u(\delta)) = \frac{\partial}{\partial \delta}\frac{\partial}{\partial k} EV(\delta, k^u(\delta)) + \frac{\partial^2}{\partial k^2} EV(\delta, k^u(\delta))\frac{\partial k^u(\delta)}{\partial \delta} \tag{77}$$

setting equal to zero yields

$$\frac{\partial k^u(\delta)}{\partial \delta} = -\frac{\frac{\partial}{\partial \delta}\frac{\partial}{\partial k} EV(\delta, k^u(\delta))}{\frac{\partial^2}{\partial k^2} EV(\delta, k^u(\delta))} \tag{78}$$

Calculating the cross-derivative, with $n(\theta_b) = \frac{l}{\delta}$, applying (51), $\frac{\partial}{\partial\delta}\frac{\partial\theta_b}{\partial k} = -\frac{1}{k}\frac{\partial\theta^*}{\partial\delta}$ and $-\frac{1}{k}\theta_b = \frac{\partial\theta^*}{\partial k}$, at the limit yields

$$\frac{\partial}{\partial\delta}\frac{\partial}{\partial k}EV = \frac{1}{1-\delta}\Big[-2\frac{\partial\theta^*}{\partial\delta}\frac{\partial\theta_b}{\partial k}\,(H-\delta k) + \frac{\partial\theta_b}{\partial k}\theta_b k + \frac{\partial\theta_b}{\partial\delta}\theta_b\,\delta - \int_{\theta_b}^{1}\theta(1-n(\theta))d\theta\,\Big] + \frac{1}{1-\delta}\frac{\partial}{\partial k}EV \tag{79}$$

With our results from above, inequality (76) is equivalent to

$$-\frac{\Big[-2\frac{\partial\theta^*}{\partial\delta}\frac{\partial\theta_b}{\partial k}\,(H-\delta k) + \frac{\partial\theta_b}{\partial k}\theta_b k + \frac{\partial\theta_b}{\partial\delta}\theta_b\,\delta - \int_{\theta_b}^{1}\theta(1-n(\theta))d\theta\,\Big] + \frac{\partial}{\partial k}EV}{\Big[-\Big(\frac{\partial^2\theta_b}{\partial k^2}\theta_b + \big(\frac{\partial\theta_b}{\partial k}\big)^2\Big)(H-k\delta) + 2\delta\,\frac{\partial\theta_b}{\partial k}\theta_b\Big]} < -\frac{\frac{\partial\theta_b}{\partial\delta}}{\frac{\partial\theta_b}{\partial k}} \tag{80}$$

Lemma 7.3. *Inequality (80) is at the limit equivalent to*

$$\theta_b^2 - \theta_b\,\frac{2H}{k(2H-k)}\frac{1-\delta}{\delta} - \frac{k}{(2H-k)} < 0 \tag{81}$$

Setting the left hand side equal to zero yields the solution

$$\theta_{b,N1/N2} = \frac{H}{k(2H-k)}\frac{1-\delta}{\delta} \pm \sqrt{\left(\frac{H}{k(2H-k)}\frac{1-\delta}{\delta}\right)^2 + \frac{k}{(2H-k)}} \tag{82}$$

Since $\frac{k}{(2H-k)} > 0$, the smaller solution $\theta_{b,N1}$ undercuts zero. Since θ_b can only take values in $[0,1]$, the critical states which satisfy (81) and for which bank value deteriorates in debt, lie in the interval $[0, \theta_{b,N2}]$ with

$$\theta_{b,N2} = \frac{H}{k(2H-k)}\frac{1-\delta}{\delta} + \sqrt{\left(\frac{H}{k(2H-k)}\frac{1-\delta}{\delta}\right)^2 + \frac{k}{(2H-k)}} \tag{83}$$

where $\theta_{b,N2}$ monotonically decreases in δ. For $\theta_b > \theta_{b,N2}$, the equilibrium change in coupon is such that bank value improves in debt. For $\theta_b = \theta_{b,N2}$, k_u changes such that the critical state remains constant. Last apply $E[\theta] = 1/2$. □

Proof. [Lemma 4.1]

In the run-proof case the payoff indifference equation becomes

$$0 = \frac{1}{2\varepsilon} \int_{\theta^*-\varepsilon}^{\theta^*+\varepsilon} (\theta\, k - 1)\; d\theta \tag{84}$$

which yields the solution $\theta^* = \frac{1}{k} = \underline{\theta}$ which decreases in k. □

Proof. [Lemma 7.3]

With $\frac{\partial^2\theta_b}{\partial k^2} = -\frac{2}{k}\frac{\partial\theta^*}{\partial k}$ and since $-\frac{\partial\theta^*}{\partial k} > 0$, multiplication on both sides of (80) with $-\frac{\partial\theta^*}{\partial k}$ yields

$$\frac{\Big[- 2\frac{\partial\theta^*}{\partial\delta}\frac{\partial\theta_b}{\partial k}\, (H - \delta k) + \frac{\partial\theta_b}{\partial k}\theta_b k + \frac{\partial\theta_b}{\partial\delta}\theta_b\, \delta - \int_{\theta_b}^{1} \theta(1-n(\theta))d\theta \Big] + \frac{\partial}{\partial k}EV}{\big[\big(\frac{2}{k}\theta_b - \big(\frac{\partial\theta_b}{\partial k}\big)\big)\, (H - k\delta) + 2\delta\, \theta_b\big]} < \frac{\partial\theta_b}{\partial\delta} \tag{85}$$

since at the limit, $\frac{\partial\theta^*}{\partial k} = \frac{\partial\theta_b}{\partial k}$. Further, since the denominator on the left hand side is positive, multiplication and plugging in $\frac{\partial}{\partial k}EV$ from (52), $k'_{\theta_b}(\delta) = -\frac{\frac{\partial\theta_b}{\partial\delta}}{\frac{\partial\theta_b}{\partial k}} = \frac{1}{\delta\theta_b}$ and the cross derivatives of the trigger yields

$$- \theta_b^2\delta k - (\frac{k\delta}{1-\delta}) \int_{\theta_b}^{1} \theta(1-n(\theta))d\theta\; + \frac{\delta}{1-\delta}\theta_b^2(H - k\delta) < \theta_b\, \frac{H}{k} \tag{86}$$

As $\varepsilon \to 0$, since $n(\theta) = 0$ on $[\theta^* + \varepsilon, 1]$, we have

$$\int_{\theta_b}^{1} \theta(1-n(\theta))d\theta \to \frac{1}{2}(1-\theta_b^2) \tag{87}$$

Plugging in and sorting terms, since $H > k$ the inequality follows.

□

Proof. [Proposition 4.1] Assume the bank is run-proof, $\delta < l$. All depositors only withdraw if they infer from their signals that the true state lies in the lower dominance region, that is we have $\theta_b = \underline{\theta} > \theta_e$. The lender of last resort assists with liquidity for all states above critical state $\theta_b = \underline{\theta}$. Since the bank is run-proof a fraction of the asset remains invested independently of how many depositors withdraw if the bank decides to do so. Assume $\theta \geq \underline{\theta}$, then all depositors roll over. If the bank liquidates the asset, she realizes t_2 equity value $(l - \delta k)/(1 - \delta) > 0$ while if she continues she realizes $\theta(H - \delta k)/(1 - \delta)$ since for states above the critical state LOLR pays liquidity assistance so the bank does not need to liquidate assets. She therefore voluntarily liquidates if and only if

$$\theta \leq \frac{l-\delta k}{H-\delta k} \tag{88}$$

But $\frac{l-\delta k}{H-\delta k} < \frac{l}{H} < \frac{1}{k} = \underline{\theta}$ by Lemma 3.3, since $H > l$. Thus for states $\theta \geq \underline{\theta}$ the bank does not liquidate voluntarily. Now assume $\theta < \underline{\theta}$. Then all depositors withdraw, $n = 1$ in t_1 and no liquidity assistance is paid. The total continuation value of investment is $\theta H(1-\delta/l)$ since the bank needs to liquidate fraction δ/l of the asset to repay withdrawing depositors. If she liquidates the entire asset, she realizes $l-\delta$. Thus, she voluntarily liquidates assets if and only if

$$\theta \leq \theta_v = \frac{l-\delta}{H(1-\delta/l)} = \frac{l}{H} = \theta_e \tag{89}$$

that is the bank liquidates exactly for states below the efficiency cut-off. Expected equity value for $1-\delta$ units invested in the run-proof bank is thus

$$EV = (l-\delta)\,\theta_e + \int_{\theta_e}^{\underline{\theta}-\varepsilon} \theta\, H(1-\frac{\delta}{l})\, d\theta + \int_{\underline{\theta}-\varepsilon}^{\underline{\theta}} \theta \left(H(1-\frac{\delta n(\theta)}{l}) - (1-n(\theta))\delta k) \right) d\theta \tag{90}$$

$$+ \int_{\underline{\theta}}^{\underline{\theta}+\varepsilon} \theta(H - \delta n(\theta) - (1-n(\theta))\delta k)\; d\theta + \int_{\underline{\theta}+\varepsilon}^{1} \theta(H-\delta k)\, d\theta \tag{91}$$

where $\delta n/l < 1$ by $\delta < l$, while value of debt equals

$$D = 1\cdot(\underline{\theta}-\varepsilon)\delta + \int_{\underline{\theta}-\varepsilon}^{\underline{\theta}+\varepsilon} \delta n(\theta) + (1-n(\theta))\delta k\theta\, d\theta + \int_{\underline{\theta}+\varepsilon}^{1} \theta\delta k\, d\theta \tag{92}$$

LOLR intervenes for states above $\underline{\theta}$ while assistance is needed only for states below $\underline{\theta}+\varepsilon$, otherwise all depositors roll over. Thus, the value of liquidity assistance is

$$L = \int_{\underline{\theta}}^{\underline{\theta}+\varepsilon} -\delta n(\theta) + \theta\delta n(\theta)\; d\theta$$

The value of the bank is the sum of equity value, value of debt and liquidity assistance, canceling out all transfers and considering $n = 0$ for $\theta < \underline{\theta} - \varepsilon$

$$V^{rp} = l\,\theta_e + \int_{\theta_e}^{1} \theta\, H\; d\theta + \delta \int_{\theta_e}^{\underline{\theta}} n(\theta,\theta^*)(1-\theta H/l))\; d\theta \tag{93}$$

The states θ_e and $\underline{\theta}$ are constant in debt ratio. Only the last term is negative due to inefficient liquidation and depends on debt via the integrand and the trigger in the function $n(\theta, \theta^*)$. The trigger increases in debt and function n increases in the trigger such that the last term and thus value of the run-proof bank monotonically decrease in debt ratio. At the limit, all depositors withdraw $n(\theta) = 1$ for states in the lower dominance region $\theta < \underline{\theta}$ and the formula follows. □

Lemma 7.4. *Consider a run-proof bank. If LOLR intervenes with liquidity assistance for all states, still the bank's risk class varies in debt ratio.*

Proof. [Lemma 7.4] This policy can be justified, since a run-proof bank is in fact solvent for all states at the interim period. Then, value of the run-proof bank will take a different form. Still, an inefficiency whose size depends on the debt ratio will persist since the bank will not liquidate voluntarily at the efficient state. Thus again, her risk-class will vary in her debt ratio. Assume $\theta < \underline{\theta}$. Then all depositors withdraw in t_1. Equity value from voluntary liquidation is $l - \delta$ and equity value from continuation is $\theta(H - \delta)$ since the LOLR intervenes. Thus, the bank liquidates voluntarily for states

$$\theta < \theta_v := \frac{l - \delta}{H - \delta} < \frac{l}{H} = \theta_e \tag{94}$$

and the cutoff below she liquidates voluntarily depends on her debt ratio. Thus, an inefficiency remains and again her risk-class depends on her debt ratio. □

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