WEB APPENDIX FOR:

"The Analytic Theory of a Monetary Shock"*

Fernando Alvarez
University of Chicago and NBER

Francesco Lippi
LUISS University and EIEF

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Abstract

This document contains all the proofs of the paper "The Analytic Theory of a Monetary Shock". The document also contains two applications of the method developed in the paper for Multiproduct firms and for a general random fixed cost problem. Finally the document contains two extensions concerning problems with drift and problems with asymmetries in the return function.

JEL Classification Numbers: E3, E5

Key Words: Menu costs, Impulse response, Dominant Eigenvalue, Selection.

*Appendix to be posted online.

Contents

A Proofs of propositions in the body of the paper

Proof. (of Proposition 1). Using the definitions of $\mathcal{H}$, $\mathcal{G}$ and τ we have the following recursion:

$$\mathcal{H}(f)(x,t) = \mathcal{G}(f)(x,t) + \mathbb{E}\left[1_{\{t>\tau\}}\mathcal{H}(f)(x^*, t-\tau)\big|x\right] \text{ for all } x \in [\underline{x}, \bar{x}] \text{ and for all } t > 0.$$

Let us begin by defining the following object: $D(x,t) \equiv \mathbb{E}\left[1_{\{t>\tau\}}\mathcal{H}(f)(x^*, t-\tau)\big|x\right]$. We first consider case (i) and show that $D(x,t) = 0$ for all x and all t. This follows since $\mathcal{H}(f)(x^*, s) = 0$ for all s. This in turn follows because f is antisymmetric, thus we have $\mathbb{E}\left[f(x(t))\,|x(\tau) = x^*\right] = 0$, which follows immediately by the symmetry of the distribution $g(x,t)$ and the antisymmetric property of f. It follows that $\mathbb{E}\left[1_{\{t\geq\tau\}}f(x(t))\,|x(0) = x\right] = 0$. Hence, since $\mathcal{H} = \mathcal{G}$, this implies that $G(t) = H(t)$ for any $p(\cdot, t)$.

Now we turn to case (ii). We note that $D(x,t)$ is symmetric in x around $x^* = (\underline{x} + \bar{x})/2$. This follows since the law of motion of x is symmetric so $g(x,t)$ is symmetric around x^*. This in turn implies that the probability of hitting either barrier at time s, starting with $x(0) = x$, is symmetric in x, which directly implies the symmetry of $D(x,t)$. Now we use that $D(x,t)$ is symmetric and that

$$H(t,f,p) - G(t,f,p) = \int_{\underline{x}}^{\bar{x}} D(x,t)\left(p(x,0) - \bar{p}(x)\right)\, dx \quad .$$

Since $D(x,t)$ is symmetric and $p(x,0) - \bar{p}(x)$ is antisymmetric we have that the right hand side is zero so that $H(t) = G(t)$. □

Proof. (of Proposition 2). We analyze the eigenvalue-eigenfunctions problem defined equation (12) for $\{\lambda_j, \gamma_j\}$, which can be rewritten as

$$\lambda_j \gamma_j(x) = \frac{\sigma^2}{2}\gamma''(x) - V(x)\gamma_j(x) \text{ where } V(x) \equiv \xi(x) + \tfrac{1}{2}\tfrac{\mu^2}{\sigma^2} \tag{A.1}$$

As a matter of notation we refer to the bounded domain case when $-\infty < \underline{x} < \bar{x} < +\infty$, and to the unbounded domain case when $-\infty = \underline{x} < \bar{x} = +\infty$. We use results from Section 3 of Chapter 4 of Zettl (2010) for the bounded domain case, and from Section 2.3 of Chapter 2 of Berezinn and Shubin (1991) for the unbounded domain case. Both references use a different notation from each other, which also differs from the one we use. Relative to the notation in chapter 4 of Zettl (2010) our boundary condition of the o.d.e for the eigenvalue-eigenfunction pair in the bounded domain case corresponds to the "Separated self-adjoint BC", our $\sigma^2 > 0$ corresponds to a constant and positive function $p > 0$, our function V corresponds to the function q, and the function w can be taken to be identically one. The notation in Chapter 2 of Berezinn and Shubin (1991), corresponds to the case where we divide both sides of equation (A.1) by $\sigma^2/2$. Equivalently, we can assume that $\sigma^2/2 = 1$, in which case our potential $V(x)$ corresponds to $v(x)$ in the notation of Chapter 2 of Berezinn and Shubin (1991). Relative to notation convention in both references our eigenvalues corresponds to minus theirs, since the term with the product of the eigenvalue times the eigenfunction is on the other side of the inequality.

- (E_1) The existence of a countably many eigenvalues follows from the spectral theorem for compact self-adjoint operators. where in our case the operator on an arbitrary function f is defined as $L(f)(x) = \frac{\sigma^2}{2}f''(x) - V(x)f(x)$ for $x \in [\underline{x}, \bar{x}]$ and $f(\bar{x}) = f(\underline{x}) = 0$.

  In particular, for the bounded domain case it follows from Theorem 4.3.1 in Zettl (2010). For the unbounded domain case it follows from Theorem 3.1 part 1 in Berezinn and Shubin (1991).

- (E_2) That the eigenvalues are all real follows immediately because the operator L defined above is Hermitian or self-adjoint. That L is self-adjoint follows by direct computation, using integration by parts, and the boundary conditions. This is a standard result for Sturm-Liouville equations.

  That the eigenvalues are strictly ordered and that they diverge follows form Theorem 4.3.1 parts 4 and 6 in Theorem 4.3.1 in Zettl (2010) in the bounded domain case. That the eigenvalues are ordered, and that they diverge follows form Theorem 3.1 Berezinn and Shubin (1991) in the unbounded domain case.

  That the eigenvalues are non-repeated, i.e. that each eigenvalue is associated with only one linearly independent eigenfunction follows from Proposition 3.3 in Berezinn and Shubin (1991) in the unbounded domain case and from part 6 of Theorem 4.3.1 in Zettl (2010).

  The the eigenvalues are negative follows from $\sigma^2 > 0$ and $V \geq 0$. To see why, take $\lambda_j \gamma_j = L(\gamma_j)$ multiply it by γ_j and integrate it between $\underline{x}$ and $\bar{x}$. Integrating by parts, and using the boundary conditions we obtain $\lambda_j|\gamma_j|^2 = -\frac{\sigma^2}{2}\int_{\underline{x}}^{\bar{x}} \gamma_j'(x)^2 dx - \int_{\underline{x}}^{\bar{x}} V(x)\gamma_j(x)^2 dx < 0$ since $|\gamma_j| = 1$.

- (E_3) That the eigenfunctions $\{\gamma_j\}_{j=1}^{\infty}$ form a complete orthonormal base in L^2 follow from Theorem 2.27 in Al-Gwaiz (2008) for the bounded domain case, and from Theorem 3.1 Berezinn and Shubin (1991) in the unbounded domain case. Equivalently, for any $g \in L^2$ we have $||g - \sum_{j=1}^{\infty}\langle g, \gamma_j\rangle_2 \, \gamma_j||_2 = 0$ where for any $g, h \in L^2$ we define the standard L^2 norm an inner product as $\langle g, h\rangle_2 \equiv \int_{\underline{x}}^{\bar{x}} g(x)h(x)dx$ and where $||g||_2^2 \equiv \langle g, g\rangle$.

  Next we extend the result to show that $\{\varphi_j\}_{j=1}^{\infty}$ form an orthonormal base for L_w^2. Take any $f \in L_w^2$, or equivalently:

$$\langle f, f\rangle = \int_{\underline{x}}^{\bar{x}} (f(x))^2 \, e^{\frac{2\mu}{\sigma^2}x} dx = \int_{\underline{x}}^{\bar{x}} \left(f(x)e^{\frac{\mu}{\sigma^2}x}\right)^2 dx = \int_{\underline{x}}^{\bar{x}} (g(x))^2 \, dx = \langle g, g\rangle_2$$

  where we define $g(x) \equiv f(x)e^{\frac{\mu}{\sigma^2}x}$, and hence $g \in L^2$. By the result above, we have

$$\begin{aligned} 0 &= \int_{\underline{x}}^{\bar{x}} \left(g(x) - \sum_{j=1}^{\infty}\langle g, \gamma_j\rangle_2 \, \gamma_j(x) \right)^2 dx \\ &= \int_{\underline{x}}^{\bar{x}} \left(g(x)w(x)^{-\frac{1}{2}} - \sum_{j=1}^{\infty}\langle g, \gamma_j\rangle_2 \, \gamma_j(x)w(x)^{-\frac{1}{2}} \right)^2 w(x)dx \\ &= \int_{\underline{x}}^{\bar{x}} \left(g(x)e^{-\frac{\mu}{\sigma^2}x} - \sum_{j=1}^{\infty}\langle g, \gamma_j\rangle_2 \, \gamma_j(x)e^{-\frac{\mu}{\sigma^2}x} \right)^2 w(x)dx \\ &= \int_{\underline{x}}^{\bar{x}} \left(f(x) - \sum_{j=1}^{\infty}\langle g, \gamma_j\rangle_2\varphi_j(x) \right)^2 w(x)dx = ||f - \sum_{j=1}^{\infty}\langle g, \gamma_j\rangle_2\varphi_j||^2 \end{aligned}$$

  Finally, notice that

$$\begin{aligned} \langle g, \gamma_j\rangle_2 &= \int_{\underline{x}}^{\bar{x}} g(x)\gamma_j(x)dx = \int_{\underline{x}}^{\bar{x}} g(x)w(x)^{-\frac{1}{2}}\gamma_j(x)w(x)^{-\frac{1}{2}}w(x)dx \\ &= \int_{\underline{x}}^{\bar{x}} f(x)\varphi_j(x)w(x)dx = \langle f, \varphi_j\rangle \end{aligned}$$

  Thus we have shown that for ant arbitrary $f \in L_w^2$ we have $0 = ||f - \sum_{j=1}^{\infty}\langle f, \varphi_j\rangle\varphi_j||$.

- (E_4) That the eigenfunctions can be indexed by the number of zeros follow from part 6 of Theorem 4.3.1 in Zettl (2010) for the bounded domain case, and from Theorem 3.5 in the Berezinn and Shubin (1991) unbounded domain case.

- (E_5) The parity of the eigenfunctions follows immediately from the assumption of symmetry of the problem. Under the symmetry assumption, let's normalize the values of $x^* = 0$ so that $\underline{x} = -\bar{x}$ and $V(-x) = \xi(-x) = \xi(x) = V(x)$. In this case, one can easily check that γ_j will be of the form $\gamma_j(x) = c_j\gamma_j(-x)$ for some non-zero constant

c_j solve the o.d.e and Dirichlet boundary condition. Since there is only one linearly independent eigenfunction for each eigenvalue, this is the form of the eigenfunctions. It will be symmetric or antisymmetric depending on the sign of c_j. Since γ_j has exactly $j-1$ zeros, then $c_j > 0$ for $j = 1, 3, \ldots$ and $c_j < 0$ for $j = 2, 4, \ldots,$.

□

Proof. (of Theorem 1). The result follows by Proposition 2, the definition of the projection coefficients $\langle \varphi_j, f \rangle$ and $\langle \varphi_j, \hat{P}/w \rangle$ and the definition of the response function in equation (7). In particular, let's start with the definition of IRF G in equation (7) as an integral of $\mathcal{G}$ where $\mathcal{G}$ is the conditional expectation given by equation (7). Thus, fixing f, the function $\mathcal{G}$ when viewed as a function of (x,t) must satisfy the following Kolmogorov Backward p.d.e. with boundary conditions:

$$\partial_t \mathcal{G}(f)(x,t) = \mu \partial_x \mathcal{G}(f)(x,t) + \frac{\sigma^2}{2} \partial_{xx} \mathcal{G}(f)(x,t) \tag{A.2}$$
$$- \xi(x) \mathcal{G}(f)(x,t) \text{ for all } x \in [\underline{x}, \bar{x}], \text{ and } t > 0$$
$$0 = \mathcal{G}(f)(\bar{x},t) = \mathcal{G}(f)(\underline{x},t) = 0, \text{ for all } t > 0 \tag{A.3}$$
$$f(x) = \mathcal{G}(f)(x,0) \text{ for all } x \in [\underline{x}, \bar{x}]\,. \tag{A.4}$$

We postulate that this equation has as solution:

$$\mathcal{G}(f)(x,t) = \sum_{j=1}^{\infty} e^{\lambda_j t} \langle f, \varphi_j \rangle \varphi_j(x) \text{ for all } x \in [\underline{x}, \bar{x}] \text{ and } t \geq 0 \tag{A.5}$$

To see that equation (A.5) is the solution, first we check that for each j the function $e^{\lambda_j t}\varphi_j(x)$ satisfies the p.d.e. in equation (A.2), and the Dirichlet boundary condition in equation (A.3). Substituting this guess, and dividing both sides by $e^{\lambda_j t}$, this function solves the p.d.e. and Dirichlet boundary condition if φ_j satisfy the following o.d.e.

$$\lambda_j \varphi_j(x) = \mu \partial_x \varphi_j'(x) + \frac{\sigma^2}{2} \varphi_j''(x) - \xi(x)\varphi_j(x) \text{ for all } x \in [\underline{x}, \bar{x}] \tag{A.6}$$

and $\varphi_j(\bar{x}) = \varphi_j(\underline{x}) = 0$. Using that $\varphi_j(x) = \gamma_j(x) e^{-\frac{\mu}{\sigma^2}x}$ and that the pair $\{\lambda_j, \gamma_j\}$ satisfies the o.d.e. and Dirichlet boundary condition in equation (12), a direct computation of the derivatives of φ_j shows that the pair $\{\lambda_j, \varphi_j\}$ satisfies equation (A.6) and its Dirichlet boundary condition. Since the p.d.e in equation (A.2) is linear, any linear combination of $e^{\lambda_j t}\varphi_j(x)$ satisfies it too. The coefficients in the linear combination in equation (A.5) are chosen to solve the space boundary condition equation (A.4) at $t = 0$. This can be done since, as shown in Proposition 2, the set of eigenfunctions $\{\varphi_j\}_{j=1}^{\infty}$ form an orthonormal base of L_w^2. Finally, using the definition of the IRF G, we replace $\mathcal{G}$ by equation (A.5) and integrate each of the terms of the infinite sum with respect to $\hat{P}$, use the definition of $\langle \cdot, \cdot \rangle$ to reinterpret the integrals, and that $\hat{P}$ is a CDF with finitely many mass points, to obtain the desired expression.

□

Proof. (of Corollary 2) Straightforward differentiation of the density function $\bar{p}(x)$ gives

$$\bar{p}'(x) = \begin{cases} -\frac{\theta^2\left[-e^{-\theta x}-e^{2\theta\bar{x}}e^{\theta x}\right]}{2\left[1-2e^{\theta\bar{x}}+e^{2\theta\bar{x}}\right]} & \text{if } x\in[-\bar{x},0] \\ -\frac{\theta^2\left[e^{\theta x}+e^{2\theta\bar{x}}e^{-\theta x}\right]}{2\left[1-2e^{\theta\bar{x}}+e^{2\theta\bar{x}}\right]} & \text{if } x\in[0,\bar{x}] \end{cases}$$

where $\theta \equiv \bar{x}^2\zeta/\sigma^2$.

The linear projection of $\bar{p}'(x)$ onto φ_j gives the projection coefficients. Let us compute: $\int_{-\bar{x}}^{\bar{x}} \bar{p}'(x)\varphi_j(x)dx = 2\int_{-\bar{x}}^{0} \bar{p}'(x)\varphi_j(x)dx$ for $j = 2,4,6,\ldots$. The function $\bar{p}'$ is antisymmetric and φ_j is antisymmetric for j even, with respect to $x = 0$. For $j = 1,3,5,...$ this integral is zero, since φ_j is symmetric, see equation (13). For $j = 2,4,\ldots$ we thus have:

$$\begin{aligned} \langle\varphi_j,\bar{p}'\rangle = 2\int_{-\bar{x}}^{0}\bar{p}'(x)\varphi_j(x)dx &= \frac{\theta^2}{[1-2e^{\theta\bar{x}}+e^{2\theta\bar{x}}]}\int_{-\bar{x}}^{0}\left[e^{-\theta x}+e^{2\theta\bar{x}}e^{\theta x}\right]\frac{1}{\sqrt{\bar{x}}}\sin\left(\frac{(x+\bar{x})}{2\bar{x}}j\pi\right)dx \\ &= \frac{e^{\bar{x}\theta}4\theta^2\bar{x}}{\sqrt{\bar{x}}\,[1-2e^{\theta\bar{x}}+e^{2\theta\bar{x}}]}\frac{\left[j\pi\left(1-\cosh\left(\bar{x}\theta\right)(-1)^{j/2}\right)\right]}{4\theta^2\bar{x}^2+j^2\pi^2} \\ &= \frac{8\phi\, e^{\sqrt{2\phi}}}{\bar{x}^{3/2}\left[1-2e^{\sqrt{2\phi}}+e^{2\sqrt{2\phi}}\right]}\frac{\left[j\pi\left(1-\cosh\left(\sqrt{2\phi}\right)(-1)^{j/2}\right)\right]}{8\phi+\pi^2j^2} \\ &= \frac{j\pi}{4\bar{x}^{3/2}}\;\frac{(-2)}{\left(1+\frac{j^2\pi^2}{8\phi}\right)}\;\frac{1-\cosh\left(\sqrt{2\phi}\right)(-1)^{j/2}}{1-\cosh\left(\sqrt{2\phi}\right)} \end{aligned}$$

where we used that $\theta\bar{x} = \sqrt{2\phi}$ and that $\cosh(x) = (1+e^x)/(2e^x)$. Combining it with the expression for $\langle\varphi_j, f\rangle$ in equation (16) gives the desired result. $\square$

Proof. (of Proposition 3) Let us define the centered even k-th moment for the variable x: $M_k(t,\delta) \equiv \mathbb{E}_\delta\left(x(t)-\mathbb{E}(x(t))\right)^k$, where $k = 2,4....$ and the subscript δ denotes that probabilities are those of an impulse response following a marginal shock δ to the invariant distribution of gaps at zero inflation.

The objective is to show that the $\frac{\partial}{\partial\delta}M_k(t,\delta)\Big|_{\delta=0} = 0$ for all t, i.e. that a marginal shock δ has no first-order effect on the even centered moments at every t. The proof follows two steps. First, to show that the impulse response of any *even* moment is flat at zero. Second, to show that the impulse response of any *centered* moment is well approximated, up to second order terms, by the impulse response of the corresponding non-centered moment.

The first step is readily established since a marginal shock triggers an antisymmetric displaced distribution $\hat{p}(x,0) = \bar{p}'(x)\delta$, whose projection coefficients on all even-indexed eigenfunctions $j = 2,4,...$ are zero (since such eigenfunctions are symmetric). Note next that even (non-centered) moments $k = 2,4,...$ are symmetric by definition, which immediately implies that their projection coefficients on all odd-indexed eigenfunctions $j = 1,3,...$ are zero. It follows that none of the eigenfunctions will have a non-zero coefficient. This proves the first step.

To prove the second step write in terms of the non-centered moments

$$M_k(t,\delta) = B_0\mathbb{E}_\delta\left(x(t)^k\right)+B_1\mathbb{E}_\delta\left(x(t)^{k-1}\right)\mathbb{E}_\delta\left(x(t)\right)+....+B_{k-1}\mathbb{E}_\delta\left(x(t)\right)\left(\mathbb{E}_\delta\left(x(t)\right)\right)^{k-1}+B_k\left(\mathbb{E}_\delta\left(x(t)\right)\right)^k$$

where the B_j are the binomial coefficients. Next, lets us replace each of the moments with its first order expansion in δ, namely let $\mathbb{E}_\delta\left(x(t)^k\right) = a_k\delta + o(\delta)$ where a_k is moment-k first derivative. We get

$$M_k(t,\delta) = B_0(a_k\delta + o(\delta)) + B_1(a_{k-1}\delta + o(\delta))(a_1\delta + o(\delta)) + ... + B_k(a_1\delta + o(\delta))^k$$

It is apparent that the only first order term in δ is a_k, i.e. the coefficient of the non-centered moment. This concludes the proof. □

Proof. (of Proposition 4) The equation for eigenvalue-eigenfunction pair $\{\lambda_j, \varphi_j\}$ for the case where ξ is quadratic, i.e. when $\xi(x) = \xi_0 + \frac{1}{2}\xi_2 x^2$ and where $-\underline{x} = \bar{x} = +\infty$, is, after a change in variables, identical to the one dimensional time independent Schrodinger equation for the eigenstate Ψ_j. This equation is typically written as:

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\Psi_j(x) + \frac{1}{2}m\omega^2 x^2 \Psi_j(x) = E_j\Psi_j(x) \text{ for } x \in \mathbb{R} \tag{A.7}$$

where Ψ_j is the j^{th} eigenstate, $\hbar$ the Planck constant, E_j the energy of the eigenstate, ω is the natural frequency, and m is the mass of the particle. As can be seen in Chapter 2, Section 3 of Griffiths (2015) the solution for the energy levels and for the eigenstates are:

$$E_j = \left(j+\tfrac{1}{2}\right)\hbar\omega \text{ and } \Psi_j(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\frac{1}{\sqrt{2^j j!}}e^{-\eta^2\frac{x^2}{2}}H_j(\eta x) \text{ for all } x \text{ and } j = 0,1,2,\ldots \tag{A.8}$$

where $\eta = (\frac{m\omega}{\hbar})^{1/2}$ and where H_j is the physicist j^{th} Hermite's polynomial. Note that in equation (A.8) we are following the convention, common in Physics, of labeling the state with the smaller energy as $j = 0$. Thus, Φ_j corresponds to our φ_{j+1}, E_j corresponds to our $-\lambda_{j+1} - \xi_0$, $m\omega^2$ corresponds to ξ_2 and $\frac{\hbar^2}{m}$ corresponds to our σ^2. So we can set $m = 1$, $\sigma = \hbar$ and $\omega = \sqrt{\xi_2}$.

Proof. (of Proposition 5) We first show that $\frac{\partial}{\partial\mu}Y(t; f, \mu, a)\Big|_{\mu=0,a=0} = 0$ as in equation (28) holds. To simplify the notation we omit a in the expression in this part of the proof. The proof proceeds in several steps. First we analyze properties of the decision rules (optimal thresholds) as a function of μ. Second, we analyze the direct and indirect (i.e. via the decision rules) implications of μ for the transition probabilities of the state at a given horizon t. Third, we establish a symmetry property of the impulse $\hat{p}(\cdot;\delta,\mu)$ as a function of (μ,δ). Fourth, we use the properties of the transition probabilities and decision rules to derive an antisymmetric property of H, viewed as joint function of (δ,μ) for any fixed t. Fifth, we use this antisymmetric property to obtain a zero cross derivative of H, which implies the desired result.

1) We write the boundaries of the inaction range and the optimal return point as functions of μ. They satisfy

$$x^*(\mu) = -x^*(-\mu)\,,\; \bar{x}(\mu) = -\underline{x}(-\mu) \text{ and } \underline{x}(\mu) = -\bar{x}(-\mu)\,.$$

This can be shown using a guess and verify strategy together with the corresponding guess of the value function $v(x, \mu) = v(-x, -\mu)$.

2) We define $P_t(y|x; \mu)$ to be the transition function for the state starting at $x(0) = x$ to $x(t) = y$, where the state evolves as follows. For $0 < s < t$, then $dx(s) = \mu ds + \sigma dW(s)$ as long as $x(s) \in (\underline{x}(\mu), \bar{x}(\mu))$ and the free adjustment opportunity has not arrived at time s. On the other hand, if $x(s)$ hits either $\bar{x}(\mu)$ or $\underline{x}(\mu)$, or the free adjustment opportunity arrives, then $x_+(s) = x^*(\mu)$, i.e. the firm is re-injected at the optimal return point. Using the properties of the decision rules, and the symmetry of the innovations in BM we have :

$$P_t\,(y|x; \mu) = P_t\,(-y| - x; -\mu)$$

To see why, write $y = x^*(\mu) + \Delta_y$ and $x = x^*(\mu) + \Delta_x$, so that for (y', x') given by $y' = -y$ and likewise $x' = -x$ we have $P_t\,(y|x; \mu) = P_t\,(y'|x' - \mu)$. But $y' = x^*(-\mu) - \Delta_y = -x^*(\mu) - \Delta_y = -y$ and likewise $x' = -x'$, establishing the required result.

3) Recall that $\hat{p}(\cdot; \delta, \mu) = \bar{p}(x + \delta; \mu) - \bar{p}(x; \mu)$. Using the properties of the decision rules and of the Kolmogorov forward equation for the steady state density $\bar{p}$, we get that $\bar{p}(x, \mu) = \bar{p}(-x, -\mu)$ is symmetric, which can be proved by a guess and verified strategy.

4) Using P_t we can write the impulse response as:

$$H\,(t; f, \hat{p}(\cdot, \delta, \mu), \mu) = \int \int f(y)\, P_t(y|x; \mu)\, \hat{p}(x, \mu, \delta)\, dy\, dx$$

Recall that we define

$$Y(r; f, \mu) = \frac{\partial}{\partial \delta} H\,(t; f, \hat{p}(\cdot, \delta, \mu), \mu)\Big|_{\delta=0}$$

and thus

$$\frac{\partial}{\partial \mu} Y(r; f, \mu) = \frac{\partial^2}{\partial \delta \partial \mu} H\,(t; f, \hat{p}(\cdot, \delta, \mu), \mu)\Big|_{\delta=0, \mu=0}$$

We will show below that:

$$H\,(t; f, \hat{p}(\cdot, \delta, \mu), \mu) = -H\,(t; f, \hat{p}(\cdot, -\delta, -\mu), -\mu) \tag{A.9}$$

for all μ, δ. Using this, we have:

$$\frac{\partial^2}{\partial \mu \partial \delta} H\,(t; f, \hat{p}(\cdot, \delta, \mu), \mu) = -\frac{\partial^2}{\partial \mu \partial \delta} H\,(t; f, \hat{p}(\cdot, -\delta, -\mu), -\mu)$$

which, evaluated at $(\mu, \delta) = (0, 0)$ gives

$$\frac{\partial^2}{\partial \mu \partial \delta} H\,(t; f, \hat{p}(\cdot, 0, 0), 0) = -\frac{\partial^2}{\partial \mu \partial \delta} H\,(t; f, \hat{p}(\cdot, 0, 0), 0)$$

so the cross derivative has to be zero, establishing the desired results.

To finish the proof we show equation (A.9) holds. We have

$$
\begin{aligned}
& H\left(t; f, \hat{p}(\cdot,\delta,\mu),\mu\right) = \int\int f(y)\, P_t(y|x;\mu)\, \hat{p}(x,\mu,\delta)\, dy\, dx \\
&= -\int\int f(-y)\, P_t(y|x;\mu)\, \hat{p}(x,\mu,\delta)\, dy\, dx \\
&= -\int\int f(-y)\, P_t(-y|-x;-\mu)\, \hat{p}(x,\mu,\delta)\, dy\, dx \\
&= -\int\int f(-y)\, P_t(-y|-x;-\mu)\, \left[\bar{p}(x+\delta,\mu) - \bar{p}(x,\mu)\right]\, dy\, dx \\
&= -\int\int f(-y)\, P_t(-y|-x;-\mu)\, \left[\bar{p}(-x-\delta,-\mu) - \bar{p}(-x,-\mu)\right]\, dy\, dx \\
&= -\int\int f(-y)\, P_t(-y|-x;-\mu)\, \hat{p}(-x,-\mu,-\delta)\, dy\, dx \\
&= -\int\int f(y')\, P_t(y'|x';-\mu)\, \hat{p}(x',-\mu,-\delta)\, dy'\, dx' \\
&= -H\left(t; f, \hat{p}(\cdot,-\delta,-\mu),-\mu\right)
\end{aligned}
$$

where we use the definition of H, that f is antisymmetric, that P_t is symmetric (as shown above), the definition of $\hat{p}$, the symmetry of $\bar{p}$ (as shown above), the definition of $\hat{p}$ again, a change of variables of integration, and again the definition of H. This finishes the proof.

The proof that $\frac{\partial}{\partial a} Y(t; f, \mu, a)\Big|_{\mu=0,a=0} = 0$ is almost identical to the previous one, step by step replacing μ by a. □

B Generalized Random Fixed-Cost Model

In this appendix we write down the problem that the firm solves, which give rise to the decision rule described by threshold $\bar{x}$, a function $\xi : [-\bar{x}, \bar{x}] \to \mathbb{R}$ and the volatility σ^2. Recall that the Calvo-plus model supplements the traditional Calvo model with the possibility that the firm can change its price by paying a fixed menu cost at any time. The generalization allows the firm to draw a fixed menu cost ψ from a distribution with CDF W at random times – arriving at a Poisson rate $\kappa > 0$. The menu costs drawn by the firm can be zero or strictly positive. If the cost is zero the firm changes its price to the ideal one (i.e. it "closes its price gap"), just like in Calvo. If the firm draws a strictly positive cost, it will either ignore it or change its price depending on the value of the "price gap" relative to the realization of the fixed cost. In particular, the optimal decision rule will be characterized by a threshold rule that gives the maximum adjustment cost that the firm is willing to pay for adjustment. For all fixed costs smaller than the threshold the firm changes its price, while for larger costs it keeps the price unchanged.

We also allow the firm to have a price change at any time by paying a large fixed cost, which we denote by $\Psi > 0$ and refer to as the "deterministic fixed cost". We let $\bar{x}$ the threshold so that if $|x| \geq \bar{x}$, the firm will pay the deterministic fixed cost Ψ and adjust its price. If $\Psi = \infty$, then the firm has no such alternative. We can write the value function of the firm, $v(x)$, as:

$$rv(x) = \min\left\{Bx^2 + \frac{\sigma^2}{2}v''(x) + \kappa\int_0^{\Psi} \min\left\{\psi + \min_{x'} v(x') - v(x)\,,\, 0\right\}dW(\psi)\,,\, r\left(\Psi + \min_{x'} v(x')\right)\right\}$$

If $\Psi = \infty$ then $\bar{x} = \infty$, and thus there is no second branch in the Bellman equation. The term $\min_{x'} v(x')$ is the value right after adjustment, and given the symmetry of the return function $x^* = 0$ or $v(0) = \min_{x'} v(x')$. Thus we can simply write that for all x

$$rv(x) = \min\left\{Bx^2 + \frac{\sigma^2}{2}v''(x) + \kappa\int_0^{\Psi} \min\{\psi + v(0) - v(x)\,,\, 0\}dW(\psi)\,,\, r\left(\Psi + v(0)\right)\right\}$$

It is easy to verify that v is increasing in $|x|$. We also have the following smooth pasting and optimal return point conditions:

$$v'(-\bar{x}) = v'(\bar{x}) = v'(0) = 0 \tag{A.10}$$

We are now ready to define the *generalized hazard function* corresponding to this model, $\xi : (-\bar{x}, \bar{x}) \to \mathbb{R}_+$, which gives the probability (per unit of time) that a firm with $x \in (-\bar{x}, \bar{x})$ will change its price. And conditional on changing its price, the price change is $-x$, i.e. it closes its gap. The function ξ is defined by the optimal decision rule, or the value function, as well the Poisson arrival rate $\kappa > 0$ and the distribution of fixed cost W is:

$$\xi(x) = \kappa\, W\left(v\left(x\right) - v\left(0\right)\right) \text{ for all } x \in (-\bar{x}, \bar{x}) \quad . \tag{A.11}$$

The function ξ is symmetric around $x = 0$ and weakly increasing in $|x|$, inheriting these properties from $v(x)$. It is continuous at x if W is continuous at $\psi = v\left(x\right) - v\left(0\right)$, and bounded above by κ. While the function ξ is not defined at $x = \pm\bar{x}$, we abuse notation and let $\xi(\bar{x}) = \lim_{x\to\bar{x}} \xi(x) = \kappa\, W\left(\Psi\right) = \kappa$.

In Alvarez, Lippi, and Oskolkov (2020) we show that for every function $\xi : (-\bar{x}, \bar{x}) \to \mathbb{R}_+$ that is piecewise continuous, positive, symmetric around $x = 0$, increasing in $|x|$, and bounded above, there is a distribution of cost with CDF W that rationalize it.

B.1 Example: Quadratic Hazard Model

We let $S(t)$ the survival function and its corresponding hazard rate as function of the duration of the price changes $h(t)$ for a price spell for the case of a quadratic function $\xi(x) = \xi_0 + \xi_2 x^2/2$, with $\bar{x} = -\underline{x} = \infty$. We have:

Proposition 6. The survival function $S(t)$ and the hazard rate $h(t)$ are:

$$S(t) = \sum_{n=0}^{\infty} (-1)^n \sqrt{2/\pi}\, \frac{\Gamma\left(\frac{1}{2}+n\right)}{n!}\, e^{\lambda_{2n} t} \text{ for all } t \geq 0 \tag{A.12}$$

$$h(t) = -\sum_{n=0}^{\infty} \lambda_{2n} \mathcal{H}_n(t) \text{ for all } t > 0 \text{ where} \tag{A.13}$$

$$\mathcal{H}_n(t) \equiv \frac{(-1)^n \frac{\Gamma\left(\frac{1}{2}+n\right)}{n!} e^{\lambda_{2n} t}}{\sum_{m=0}^{\infty} (-1)^m \frac{\Gamma\left(\frac{1}{2}+m\right)}{m!} e^{\lambda_{2m} t}} \text{ for all } n = 0, 1, 2, \ldots \tag{A.14}$$

Moreover, let $S_0(t)$ the survival function for $\xi_0 = 0$, so using the identity for competing risk durations we have $S(t) = e^{-\xi_0 t} S_0(t)$. For the case of $\xi_0 = 0$ we have

$$S_0(t) = \sqrt{\operatorname{sech}\left(t\sqrt{\sigma^2 \xi_2}\right)} \tag{A.15}$$

$$N = \frac{\sqrt{\sigma^2 \xi_2}}{4\sqrt{2/\pi}\, \Gamma\left(\frac{5}{4}\right)^2} \text{ and } \lambda_j = -N\left(j - \frac{1}{2}\right) 4\sqrt{2/\pi}\, \Gamma\left(\tfrac{5}{4}\right)^2 \tag{A.16}$$

The expression for $S(t)$ follows directly from the general expression of the survival function in Theorem 1 using $f(x) = 1$ and a degenerate initial condition concentrated at $x = 0$, and the expressions for the eigenvalues and eigenfunction for the quadratic case in Proposition 4. The expression for $h(t)$ follows from differentiating $S(t)$. The expression for S_0 can be verified by comparing the series expansion of $S(t)$ when $\xi_0 = 0$. Alternatively, one can use the Laplace transform of the square of the integral of a standard Brownian W –see Example 1 Kac (1949) page 11– and any constant u gives

$$\mathbb{E}\left[e^{-u \int_0^t (W(s))^2 ds}\right] = \sqrt{\operatorname{sech}\left(t\sqrt{2u}\right)} \tag{A.17}$$

and setting $u = \sigma^2 \xi_2 / 2$ we obtain equation (A.15). The expression for N in the case of $\xi_0 = 0$ uses that the expected duration of price spells, or its reciprocal, the expected number of price changes for the case satsify

$$\frac{1}{N} = \int_0^\infty S(t) dt = \int_0^\infty \sqrt{\operatorname{sech}\left(\sqrt{\sigma^2 \xi_2}\, t\right)} dt = 4\sqrt{\frac{2}{\pi\, \sigma^2 \xi_2}}\, \Gamma\left(\tfrac{5}{4}\right)^2$$

Using this expression into the general expression of the eigenvalues, we eliminate *eta* to obtain the desired expression.

B.2 Example: Absolute Value Generalized Hazard Function

In this appendix we characterize the odd (antisymmetric) eigenvalues and eigenfunctions for the absolute value $\xi(x) = A|x|$. The eigenfunctions are given by displaced Airy functions and the eigenvalues are the zeros of the Airy functions $Ai(\cdot)$. We give formulas and numerical implementations for the eigenvalues λ_k in equation (A.19), the antisymmetric eigenfunctions $\varphi_j(\cdot)$ in equation (A.20), the invariant distribution $\bar{p}(\cdot)$ in equation (A.21), the expected number of price changes N in equation (A.22), and the projections $\langle \varphi_j(\cdot), -x\rangle$ in equation (A.23) and $\langle \varphi_j(\cdot), \bar{p}'(\cdot)\rangle$ in equation (A.24).

Let's start with the equation we wish to solve:

$$[Ax + \lambda_k]\,\varphi_k(x) = \frac{\sigma^2}{2}\varphi_k''(x) \text{ for } x \geq 0, \varphi_k(x) = -\varphi_k(-x) \text{ and } k = 2, 4, 6, \ldots$$

First let $z = bx$ for some $b >$ or $x = z/b$ and define $\tilde{\varphi}_k(z) = \varphi_k(z/b)$ so that $\tilde{\varphi}_k''(z) = \varphi_k''(z)/b^2$ or $\varphi_k''(z) = b^2\tilde{\varphi}_k''(z)$ and thus

$$\tilde{\varphi}_k(z)\left[z\frac{A}{b} + \lambda_k\right] = \frac{\sigma^2}{2}b^2\tilde{\varphi}_k''(z) \text{ or } \tilde{\varphi}_k(z)\left[z\frac{A}{b^3\sigma^2/2} + \frac{\lambda_k}{b^2\sigma^2/2}\right] = \tilde{\varphi}_k''(z)$$

Set b

$$b \equiv \left(\frac{2A}{\sigma^2}\right)^{1/3} \text{ thus } \tilde{\varphi}_k(z)\left[z + \tilde{\lambda}_k\right] = \tilde{\varphi}_k''(z) \text{ where } \tilde{\lambda}_k \equiv \lambda_k\frac{\left(\frac{\sigma^2/2}{A}\right)^{2/3}}{\sigma^2/2}$$

The Airy function $Ai(z)$ solves $Ai(z)z = Ai''(z)$ and has $Ai(z) \to 0$ as $z \to +\infty$. Moreover it has infinitely many negative zeros, denoted by $0 > a_1 > a_2 > \ldots$. Thus the solution for $\tilde{\varphi}_{2k+1}$ is:

$$\tilde{\varphi}_{2k+2}(z) = Ai(z + a_k) \text{ for all } z > 0 \text{ and } \tilde{\lambda}_{2k+2} = a_{k+1} \text{ for } k = 0, 1, \ldots$$

While there are no closed expression for the zeros of the Airy functions, there are excellent approximations, which can be used to find numerically exact values of them. For instance:

$$a_k = -\frac{1}{4}(m^2+20)^{1/3} + \bar{E}_k\frac{457}{(m^3(m^2+40)^{1/6}}, \text{ where } m = (12k-3)\pi \text{ for } k = 1, 2, \ldots$$

where $\bar{E}_k$ is an approximation error that is less than one in absolute value. See, Theorem 10 in Krasikov (2014). Thus, our analytical approximation to the odd eigenvalues is:

$$\lambda_{2k+2} = a_{k+1}\frac{\sigma^2/2}{\left(\frac{\sigma^2/2}{A}\right)^{2/3}} \text{ for } k = 0, 1, 2, \ldots \tag{A.18}$$

$$\approx -\frac{1}{4}\left([12(k+1)-3\pi]^2 + 20\right)^{1/3}\frac{\sigma^2/2}{\left(\frac{\sigma^2/2}{A}\right)^{2/3}} \tag{A.19}$$

and the antisymmetric eigenfunctions are:

$$\varphi_{2k+2}(x) = \beta Ai\left(a_{k+1} + \left(\frac{2A}{\sigma^2}\right)^{1/3}x\right) \text{ for } x \geq 0 \text{ for } k = 0, 1, 2, \ldots$$

where β is a normalizing constant. We will need to normalize the eigenfunctions by: $2\int_0^\infty \varphi_k(x)^2 dx =$

1. For this we note that for any c:

$$\int_c^\infty [Ai(z)]^2\, dz = -c[Ai(c)]^2 + [Ai'(c)]^2$$

Thus

$$\begin{aligned}1 &= 2\int_0^\infty \varphi_{2k+2}(x)^2 dx = 2\beta^2 \int_0^\infty Ai(a_{k+1}+bx)^2 dx = \frac{2\beta^2}{b}\int_0^\infty Ai(a_{k+1}+bx)^2 dbx \\ &= 2\beta^2 \frac{1}{b}\int_0^\infty Ai(a_{k+1}+z)^2 dz = \frac{2\beta^2}{b}\int_{a_{k+1}}^\infty Ai(s)^2 ds = \frac{2\beta^2}{b}\left[-a_{k+1}[Ai(a_{k+1})]^2 + [Ai'(a_{k+1})]^2\right] \\ &= \frac{2\beta^2}{b}[Ai'(a_{k+1})]^2\end{aligned}$$

Thus the normalized eigenfunctions are given by

$$\varphi_{2k+1}(x) = \frac{\sqrt{\left(\frac{2A}{\sigma^2}\right)^{1/3}}}{\sqrt{2}\,|Ai'(a_{k+1})|} Ai\left(a_{k+1} + \left(\frac{2A}{\sigma^2}\right)^{1/3} x\right) \text{ for } x \geq 0 \text{ for } k = 0,1,2,\ldots \quad \text{(A.20)}$$

Likewise, for the invariant distribution $\bar{p}$, satisfying $\bar{p}(x)Ax = \sigma^2/2\bar{p}(x)$ for $x > 0$. We define again $z = bx$ and $\tilde{p}(z) = \bar{p}(z/b)$ so that $\tilde{p}(z)Az/b = \sigma^2/2\tilde{p}(z)b^2$ and setting again $b = (2A/\sigma^2)^{1/3}$ we get: $\tilde{p}(z)z = \tilde{p}''(z)$, which is solved by the Airy function. Thus:

$$\bar{p}(x) = Ai\left((2A/\sigma^2)^{1/3}x\right)/\alpha \text{ with } \alpha = 2\int_0^\infty Ai\left((2A/\sigma^2)^{1/3}x\right) dx$$

We can use that:

$$\int_0^\infty Ai(z)dz = 1/3 \text{ and thus } \alpha = \frac{2}{b}\int_0^\infty Ai\left(bx\right) dbx = \frac{2}{3b}$$

to obtain

$$\bar{p}(x) = \frac{3\,(2A/\sigma^2)^{1/3}}{2} Ai\left((2A/\sigma^2)^{1/3}x\right) \text{ for } x \geq 0 \quad \text{(A.21)}$$

Note that $N = -\sigma^2\bar{p}'(0)$ and thus

$$N = -\sigma^2\bar{p}'(x)|_{x=0} = -\sigma^2\frac{3\,(2A/\sigma^2)^{2/3}}{2} Ai'\left((2A/\sigma^2)^{1/3}x\right)|_{x=0}$$

using that $Ai'(0) = -1/\left(3^{1/3}\Gamma(1/3)\right)$ we have:

$$N = -\sigma^2\frac{3\,(2A/\sigma^2)^{2/3}}{2} Ai'\left(0\right)$$

using that $Ai'(0) = 1/\left(3^{1/3}\Gamma(1/3)\right)$ we have:

$$N = \frac{\sigma^2}{2}\left(\frac{2A}{\sigma^2}\right)^{2/3}\frac{3}{3^{1/3}\Gamma(1/3)} \tag{A.22}$$

Let φ_{2k+2} be an antisymmetric eigenfunction. Then the projections for the IRF are:

$$\langle\varphi_{2k+2}, -x\rangle = -\frac{\sqrt{b}}{b^2\sqrt{2}|Ai'(a_{k+1})|}2\int_0^\infty zAi(a_{k+1}+z)dz \tag{A.23}$$

$$\langle\varphi_{2k+2}, \bar{p}'\rangle = \frac{\sqrt{b}}{\sqrt{2}|Ai'(a_{k+1})|}2b\frac{3}{2}\int_0^\infty Ai'(z)Ai(a_{k+1}+z)dz \tag{A.24}$$

Note that the product $\langle\varphi_{2k+2}, -x\rangle\langle\varphi_{2k+2}, \bar{p}'\rangle$ is independent of b.

C Monetary propagation with volatility shocks

This section discusses the effect that changes to the volatility of shocks exert on the propagation of monetary shocks. The issue matters to e.g. the effectiveness of monetary policy in recessions vs boom, when the state of the economy is assumed to feature, respectively, high vs low volatility of shocks as in Vavra (2014). Our method provides a sharp analytic answer to this question.

For concreteness we illustrate the problem by using the pure menu cost model (without Calvo adjustment i.e. $\zeta = 0$ so that $\phi = \ell = 0$), whose output response to a small monetary shock was given in equation (18). We conduct a comparative static exercise to analyze how the propagation is affected by an innovation of the "volatility shocks", namely a permanent change in the common value of the idiosyncratic volatility σ.[1]

We start with a steady state for the model with idiosyncratic volatility σ. We characterize the effect of a small monetary shock, $\delta > 0$, which occurs $s \geq 0$ periods after a change in idiosyncratic volatility from σ to $\tilde{\sigma}$, so that $\tilde{\sigma} = \left(1 + \frac{d\sigma}{\sigma}\right)\sigma$. In particular we let $Y(t; s, d\sigma/\sigma)\delta$ denote the output's IRF $t \geq 0$ periods after an unexpected monetary shock of size δ starting with a cross sectional distribution that has evolved s periods since the change in σ.[2]

While we characterize Y for all $t > 0$ and $s \geq 0$, two interesting cases are worthwhile to mention separately: the short-run and the long-run effect of volatility. The short-run effect, defined as $Y(t; 0, d\sigma/\sigma)$ or $s = 0$, consists of considering a simultaneous permanent change of both σ (to $\tilde{\sigma}$) and $\delta > 0$. After the shock the forward looking firm's decision rules adjusts immediately to the new volatility $\tilde{\sigma}$, while the initial distribution of price gaps corresponds to the stationary distribution obtained under the old decision rule. The long-run effect, denoted by $Y(t; \infty, d\sigma/\sigma)$ or $s \to \infty$, is equivalent to computing the effect of a monetary shock δ for a new steady state with volatility $\tilde{\sigma}$. We refer to this as the *long-run* effect since it is the

[1] For simplicity and clarity of the results we consider here once and for all shocks to volatility. It is simple to modify the setup to consider a two-state Markov switching volatility process and to solve the associated firm's decision rules.

[2] We will keep using the notation of Y as the output's IRF per unit of monetary shock, and then omit the δ in the expressions below.

effect of an unanticipated monetary shock once the distribution of price gaps has achieved its new invariant distribution. In this case the firm's decision rule corresponds to the new volatility $\tilde{\sigma}$ and the economy is described by the new invariant distribution of price gaps.

The general case characterizes an IRF whose coefficients are indexed by the parameter $0 < s < \infty$. The key feature of this case is that the monetary shock δ occurs s periods after the volatility shock, thus displacing a cross-section distribution of price gaps that is in a transition towards the new invariant distribution. Our analytic method allows us to exactly compute the evolution of this distribution and hence the effect of a monetary shock.

The next proposition uses the notation introduced above, where $Y(t;0,0)$ denotes the impulse before any change in volatility occurs, which we use as a benchmark. Also, the difference $Y\left(t;s,\frac{d\sigma}{\sigma}\right) - Y\left(t;\infty,\frac{d\sigma}{\sigma}\right)$ is the correction to the long run effect of a volatility shock $d\sigma/\sigma$ due to a finite duration s.

Proposition 7. Let $Y\left(t;s,\frac{d\sigma}{\sigma}\right)$ denote the time-t value of the output marginal impulse response that occurs s-periods after a volatility increase from σ to $\tilde{\sigma} = \left(1+\frac{d\sigma}{\sigma}\right)\sigma$. The *long run* effect ($s \to \infty$) of the volatility shock $\frac{d\sigma}{\sigma}$ on the impulse response of output to a monetary shock is:

$$Y\left(t;\infty,\frac{d\sigma}{\sigma}\right) = Y\left(t\left(1+\frac{d\sigma}{\sigma}\right);0,0\right) \quad \text{for all } t \geq 0\,. \tag{A.25}$$

The *short run* effect ($s \to 0$) of the volatility shock $\frac{d\sigma}{\sigma}$ on the impulse response of output to a monetary shock is:

$$Y\left(t;0,\frac{d\sigma}{\sigma}\right) = \left(1+\frac{d\sigma}{\sigma}\right)\; Y\left(t\left(1+\frac{d\sigma}{\sigma}\right);0,0\right) \quad \text{for all } t \geq 0 \quad . \tag{A.26}$$

The deviation from the long run response as a function of s is given by:

$$Y\left(t;s,\frac{d\sigma}{\sigma}\right) - Y\left(t;\infty,\frac{d\sigma}{\sigma}\right) = \sum_{k=1}^{\infty} e^{\lambda_{2k}t} b_{2k}[f]\, b_{2k}\left[\hat{p}'(\cdot,s)\right] \quad \text{for all } t,s \geq 0\,. \tag{A.27}$$

where $\hat{p}'(\cdot,s)$ is the initial condition (i.e. a displaced cross section) at the time of the monetary shock, s periods after the change in volatility, whose projection coefficients are given by:

$$b_{2k}\left[\hat{p}'(\cdot,s)\right] = \frac{d\sigma}{\sigma}\frac{1}{\bar{x}^{\frac{3}{2}}} \sum_{j=1,3,5,\ldots}^{\infty} e^{\lambda_j s}\left(2\frac{4(-1)^{\frac{j+3}{2}} - j\pi}{(j\,\pi)^2}\right)\left(\frac{4kj}{(4k^2-j^2))}\right) \quad , \quad k=1,2,3,\ldots \tag{A.28}$$

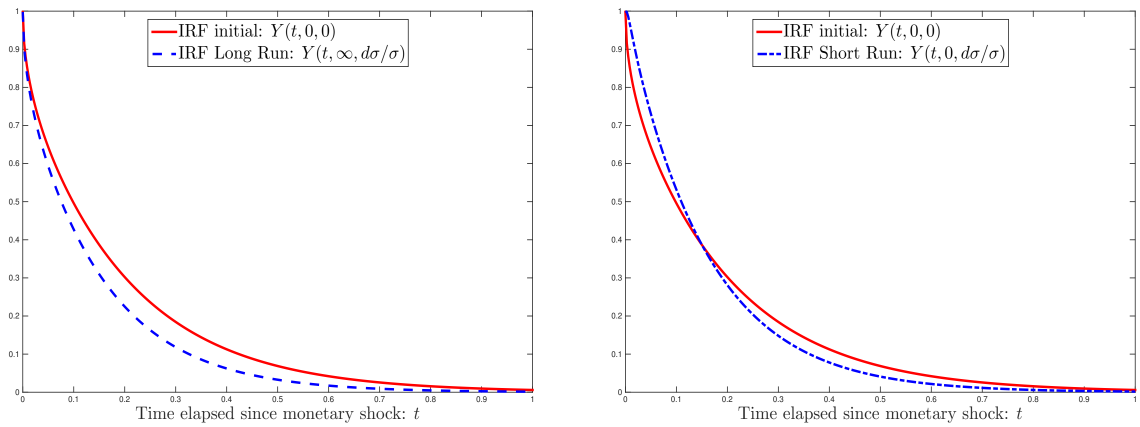
and where $b_{2k}[f] = 2\bar{x}^{3/2}/(k\pi)$ as in equation (16).

A few comments are in order.

(i) Figure 1 illustrates the difference between the short run and long run effect of an increase in volatility on the output's response to a monetary shock. The left panel compares the IRF with no change in volatility, $Y(t;0,0)$ to the one where the volatility increase has occurred $s \to \infty$ periods ago, i.e. $Y(t;\infty,d\sigma/\sigma)$ the long run effect. The right panel compares the IRF with no change in volatility, $Y(t;0,0)$ to the one where the volatility increase has occurred at the same time as the monetary shock $s=0$ periods ago, i.e. $Y(t;0,d\sigma/\sigma)$ the short run effect.

Figure 1: Short-run and long-run IRF vs. IRF before volatility increases

Long run $(s \to \infty)$ Short run $(s = 0)$



Note: $N = 1$ (one price adjustment per unit of time, on average) and $d\sigma/\sigma = 0.1$.

(ii) For this proposition we use the form of the decision rules for the threshold $\bar{x}$, which as the discount rate goes to zero is $\bar{x} = \left(6\frac{\psi}{B}\sigma^2\right)^{\frac{1}{4}}$ where ψ is the fixed cost –as fraction of the frictionless profit and B is the curvature of the profit function around the frictionless profit. This implies that the elasticity of $\bar{x}$ to σ is 1/2. This elasticity is the so called "option value" effect on the optimal decision rules.

(iii) The rescaling of time in $Y\left(t\left(1+\frac{d\sigma}{\sigma}\right);0,0\right)$ in the expressions for the long and short run effect of volatility reflects the change in the eigenvalues, which depend on the value of N, the implied average number of price changes per unit of time, as $\lambda_j = -N(\pi j)^2/8$ (see equation (21) for $\zeta = 0$). Recall that $N = (\sigma/\bar{x})^2$, and hence all the eigenvalues change proportionally with σ.

(iv) For the case of the impact effect and in which $\tilde{\sigma} > \sigma$, the invariant distribution just before the monetary shock is narrower than the range of inaction that corresponds to the new wider barriers. This explains the extra multiplicative term level $\left(1+\frac{d\sigma}{\sigma}\right)$ in the impact effect in equation (A.26): since firms have price gaps that are discretely away from the inaction bands, then prices react more slowly, generating the extra effect on output. The logic for the case where $\tilde{\sigma} < \sigma$ is similar.

(v) In equation (A.27) we use only the even terms for the projections, i.e. the index for the projection $b_{2k}[\cdot]$ runs on $2k$ because f is antisymmetric. For these coefficients, as was the case without volatility shocks, the eigenvalues that control the effect of the horizon t in the IRF are the even ones, i.e. $\lambda_2, \lambda_4, \ldots$, starting with the leading one λ_2.

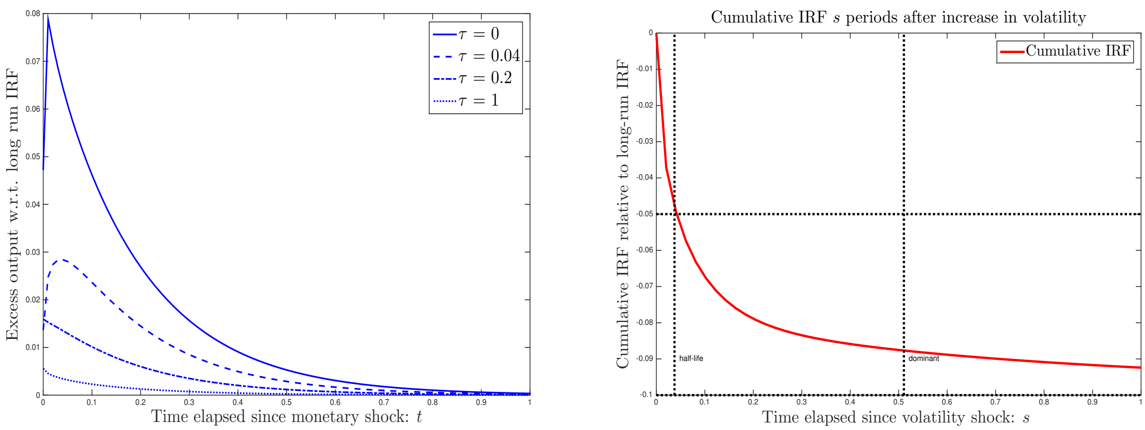
(vi) The expressions in equation (A.27) and equation (A.28) show that what governs the difference between the long run and the short run volatility effects are the odd eigenvalues, i.e. $\lambda_1, \lambda_3, \ldots$, since these are the only elements where s affect the expressions. In particular, λ_1 is the dominant eigenvalue.

(vii) We note that the expression for the correction term in equation (A.27) involves no parameter for the model with the exception of N, which enters only in the eigenvalues

$\lambda_j = -N(j\pi)^2/8$. This gives a meaning to the units of t and s, which are measured relative to the (new) steady state duration of price changes $1/N$. This remark is needed to interpret the time units in the horizontal axes of both panels of Figure 2.

Figure 2: The propagation of monetary shocks as s grows

$Y\,(t; s, d\sigma/\sigma) - Y\,(t; \infty, d\sigma/\sigma)$ Change in Cumulated output: $\mathcal{C}(s, d\sigma/\sigma)$



Note: $N = 1$ (one price adjustment per unit of time, on average) and $d\sigma/\sigma = 0.1$.

(vii) To illustrate the general case of $0 < s < \infty$ in Figure 2 we display two plots. First, the left panel of Figure 2 plots equation (A.27), evaluated at 4 values of s. It is apparent that as s becomes bigger monetary policy becomes less effective and gradually converges to the long run value. This can be seen by comparing the correction for any given t across the four values of s. Second, the right panel, plots the cumulated IRF of a monetary shock s periods after the volatility shock, relative to the cumulative IRF of a monetary shock when there is no volatility shock. In particular we plot:

$$\mathcal{C}(s, d\sigma/\sigma) \equiv \frac{\int_0^\infty Y(t, s, d\sigma/\sigma)dt}{\int_0^\infty Y(t, 0, 0)dt} - 1$$

We use the cumulated IRF to obtain a simple one-dimensional summary of this effect across all times t. Notice the following properties of $\mathcal{C}$: for all s we have $\mathcal{C}\,(s, d\sigma/\sigma) = (d\sigma/\sigma)\mathcal{C}\,(s, 1)$, since it is based on a derivative, and for extreme values of s we have $\mathcal{C}\,(\infty, d\sigma/\sigma) = -d\sigma/\sigma$, and $\mathcal{C}\,(0, d\sigma/\sigma) = 0$. From Figure 2 it is clear that the transition to the higher volatility occurs very fast, a cumulative effect of $\mathcal{C}$ half as large as half of the one in $s \to \infty$ will occur when $s_{1/2} \approx 0.05/N$, a half-life indicated by a vertical bar in the right panel. More precisely, $s_{1/2}$ is defined as $\mathcal{C}(s_{1/2}, d\sigma/\sigma) = -(1/2)(d\sigma/\sigma)$. This effect is much faster than the half-life corresponding to the dominant eigenvalue $\lambda_1 = -N\pi^2/8$, which is given by $t_{1/2} \equiv -8\log(0.5)/(N\pi^2) \approx 0.56/N$, and it is indicated by another vertical bar in the right panel. The ratio of the two times is very large: $t_{1/2}/s_{1/2} \approx 12$, and it is independent of any parameter of the model.[3] From this comparison we conclude that for this particular

[3]The vertical distance on the correction between $s = 0$ and $s \to \infty$ plotted on the right panel is $d\sigma/\sigma$,

model using exclusively the dominant eigenvalue λ_1 to approximate the time it takes for the distribution to converge after the change in volatility will be misleading. Summarizing, in the Golosov-Lucas model the short run effect of the volatility change is only relevant when the monetary shock occurs almost immediately after the volatility change.

C.1 Proofs for uncertainty shocks

Proof. (of Proposition 7) First we consider case (i), i.e. the *long run* effect of a volatility shock $\frac{d\sigma}{\sigma}$, so that $\tilde{\sigma} = \left(1 + \frac{d\sigma}{\sigma}\right)\sigma$ on the impulse response of output to a monetary shock. We note that the expression for $Y(t)$ for the Golosov Lucas model does not feature $\bar{x}$, which is a function of σ (see equation (18)). Indeed the only place where σ enters in the expression for $Y(t)$ is in the eigenvalues (the parameter $N(j\pi)^2/8$ in equation (18)). Since, $N = \sigma^2/\bar{x}^2$ and $\bar{x} = \left(6\frac{\psi}{B}\sigma^2\right)^{\frac{1}{4}}$, then $d\log\bar{x} = 1/2 d\log\sigma$ and $d\log N = 2\left(d\log\bar{x} - d\log\sigma\right)$, hence $d\log N = d\log\sigma$. Substituting this into the eigenvalue $\lambda_j = -\tilde{N}(j\pi)^2/8 = -(1+\frac{d\sigma}{\sigma})N(j\pi)^2/8$ were N is the average number of price changes before the volatility shock. Using the expression for the impulse response in terms of the post-shock objects we have:

$$Y\left(t;\infty,\frac{d\sigma}{\sigma}\right) = \sum_{j=1}^{\infty}\langle\varphi_j,f\rangle\langle\varphi_j,\bar{p}'\rangle e^{-\left(1+\frac{d\sigma}{\sigma}\right)N\frac{(j\pi)^2}{8}t} = Y\left(t\left(1+\frac{d\sigma}{\sigma}\right);0,0\right)$$

and we obtain the desired result.

Now we consider short run effect, i.e. the *impact* effect of a volatility shock $\frac{d\sigma}{\sigma}$, so that $\tilde{\sigma} = \left(1 + \frac{d\sigma}{\sigma}\right)\sigma$ on the impulse response of output to a monetary shock. As in the previous case the eigenvalues can be written as functions of the shock and the old value of the expected number of price changes. Also as the previous case we have $f(x) = -x$. The difference is on the initial distribution $p(x,0)$. The initial condition is given by $p(x,0) = \bar{p}(x+\delta;\bar{x}(\sigma))$ where we write $\bar{x}(\sigma)$ to indicate that the distribution depends on σ. Indeed, since we are using the expression for $Y\left(t,0,\frac{d\sigma}{\sigma}\right)$ in terms of the value of $\bar{x}$ that corresponds to the post-shock value of σ, we need to consider the effect on $\bar{x}$ of a decrease of σ in the proportion $d\sigma/\sigma$. To do this we take a second order expansion of $p(x,0) = \bar{p}(x+\delta;\bar{x}(\sigma))$ with respect to δ and σ evaluated at $\delta = 0$ and $d\sigma = 0$.

$$\begin{aligned}
p(x;0) \equiv \bar{p}(x+\delta;\bar{x}(\sigma)) &= \bar{p}(x) + \frac{\partial}{\partial\delta}\bar{p}(x+\delta;\bar{x}(\sigma))\big|_{\delta=0}\delta - \frac{\partial}{\partial\bar{x}}\bar{p}(x+\delta;\bar{x}(\sigma))\big|_{\delta=0}\frac{\partial\bar{x}(\sigma)}{\partial\sigma}d\sigma \\
&+ \frac{1}{2}\frac{\partial^2}{\partial\delta^2}\bar{p}(x+\delta;\bar{x}(\sigma))\big|_{\delta=0}\delta^2 \\
&+ \frac{1}{2}\frac{\partial^2}{\partial\bar{x}^2}\bar{p}(x+\delta;\bar{x}(\sigma))\big|_{\delta=0}\left(\frac{\partial\bar{x}(\sigma)}{\partial\sigma}\right)^2 d\sigma^2 + \frac{1}{2}\frac{\partial}{\partial\bar{x}}\bar{p}(x+\delta;\bar{x}(\sigma))\big|_{\delta=0}\frac{\partial^2\bar{x}(\sigma)}{\partial\sigma^2}d\sigma^2 \\
&- \frac{\partial^2}{\partial\bar{x}\partial\delta}\bar{p}(x+\delta;\bar{x}(\sigma))\big|_{\delta=0}\frac{\partial\bar{x}(\sigma)}{\partial\sigma}d\sigma\delta + o\left(||(\delta,d\sigma)||^2\right)
\end{aligned}$$

for $x \in [\underline{x},\bar{x}]$ and $x \neq 0$. Recall that the invariant distribution for this model is the triangular

which is 0.1 for this example. For other values, the vertical axis scales proportionally.

density $\bar{p}(x) = 1/\bar{x} - |x|/\bar{x}^2$ for $x \in (-\bar{x}, \bar{x})$. Using this functional form we have:

$$\frac{\partial}{\partial \delta}\bar{p}(\delta + x; \bar{x}) = \begin{cases} +\frac{1}{\bar{x}^2} & \text{if } x \in [-\bar{x}, 0) \\ -\frac{1}{\bar{x}^2} & \text{if } x \in (0, \bar{x}] \end{cases} \quad , \quad \frac{\partial}{\partial \bar{x}}\bar{p}(\delta + x; \bar{x})\big|_{\delta=0} = \begin{cases} -\frac{x}{\bar{x}^2}\frac{2}{\bar{x}} & \text{if } x \in [-\bar{x}, 0) \\ +\frac{x}{\bar{x}^2}\frac{2}{\bar{x}} & \text{if } x \in (0, \bar{x}] \end{cases}$$

$$\frac{\partial^2}{\partial \delta \partial \bar{x}}\bar{p}(x + \delta; \bar{x}) = \begin{cases} -\frac{1}{\bar{x}^2}\frac{2}{\bar{x}} & \text{if } x \in [-\bar{x}, 0) \\ +\frac{1}{\bar{x}^2}\frac{2}{\bar{x}} & \text{if } x \in (0, \bar{x}] \end{cases} \quad , \quad \frac{\partial^2}{\partial \bar{x}^2}\bar{p}(\delta + x; \bar{x}) = \begin{cases} +\frac{x}{\bar{x}^2}\frac{6}{\bar{x}^2} & \text{if } x \in [-\bar{x}, 0) \\ -\frac{x}{\bar{x}^2}\frac{6}{\bar{x}^2} & \text{if } x \in (0, \bar{x}] \end{cases}$$

Notice that the first order derivatives with respect to δ as well as the cross partial derivative are antisymmetric functions of x around $x = 0$, while the derivatives with respect to $\bar{x}$ are symmetric functions of x. Finally we have $\frac{\partial^2}{\partial \delta^2}\bar{p}(x + \delta; \bar{x}) = 0$.

Now we use the expansion and compute the impulse response coefficients $\beta_j \equiv \langle \varphi_j, f \rangle \langle \varphi_j, p(\cdot, 0) \rangle$. The first order term for $d\sigma$ is zero because f is antisymmetric (so that $\langle \varphi_j, f \rangle = 0$ for $j = 2, 4, 6, ...$) and the first derivative with respect to $\bar{x}$ is symmetric (so that $\langle \varphi_j, p(\cdot, 0) \rangle = 0$ for $j = 1, 3, 5, ...$) hence the $\beta_j = 0$ for $j = 1, 2, 3, 4, ...$ Likewise the second order terms for $d\sigma^2$ are zero since f is antisymmetric and the first and second derivative with respect to $\bar{x}$ are symmetric. The second order term δ^2 is zero because the second derivative with respect to δ is zero. This leaves us with two non-zero terms. The first order term on δ, which is the term for the IRF with respect to a monetary shock, and the second order term corresponding to the cross-derivative. For the cross-partial term we note that, using that $\bar{x}$ has elasticity $1/2$ with respect to σ, we can write

$$\begin{aligned} -\frac{\partial^2}{\partial \delta \partial \bar{x}}\bar{p}(x + \delta; \bar{x})\frac{\partial \bar{x}(\sigma)}{\partial \sigma} d\sigma \delta &= -\frac{\partial^2}{\partial \delta \partial \bar{x}}\bar{p}(x + \delta; \bar{x})\bar{x}(\sigma)\left[\frac{\partial \bar{x}(\sigma)}{\partial \sigma}\frac{\sigma}{\bar{x}(\sigma)}\right]\frac{d\sigma}{\sigma}\delta \\ &= -\frac{2}{\bar{x}(\sigma)}\frac{\partial}{\partial \delta}\bar{p}(\delta + x; \bar{x})\bar{x}(\sigma)\frac{1}{2}\frac{d\sigma}{\sigma}\delta \quad = -\frac{\partial}{\partial \delta}\bar{p}(\delta + x; \bar{x})\frac{d\sigma}{\sigma}\delta \end{aligned}$$

Thus we have that each β_j term is given by the sum of the (non-zero) terms corresponding to the first order term on δ and the second-order term corresponding to the cross-derivative:

$$\langle \varphi_j, f \rangle \langle \varphi_j, \bar{p}'(\cdot) \rangle \delta + \langle \varphi_j, f \rangle \langle \varphi_j, \bar{p}'(\cdot) \rangle \delta \frac{d\sigma}{\sigma} = \langle \varphi_j, f \rangle \langle \varphi_j, \bar{p}'(\cdot) \rangle \delta \left(1 + \frac{d\sigma}{\sigma}\right)$$

This gives the projection coefficients for the short run impact that appear in equation (A.26) in the proposition. In particular it shows that the coefficients for the short run are equal to the ones for the long-run multiplied by the factor $(1 + d\sigma/\sigma)$.

Finally, we consider the case of a monetary shock that occurs s periods after the volatility shock. We proceed in three steps.

Step 1: find initial signed measure $\hat{p}(x, p)$. For a small σ shock, the signed measure $\hat{p}(x, 0)$ right after the uncertainty shock is given by

$$\hat{p}(x, 0) \equiv \bar{p}(x; \bar{x}(\sigma)) - \bar{p}(x; \bar{x}(\tilde{\sigma})) = \bar{p}(x; \bar{x}(\sigma)) - \bar{p}(x; \bar{x}(\sigma + d\sigma))$$

which is by the difference between the original invariant distribution and the new long run distribution. We now take an expansion around the original invariant distribution and write

$$\hat{p}(x,0) = \bar{p}(x;\bar{x}(\sigma)) - \bar{p}(x;\bar{x}(\sigma)) - \bar{p}_{\bar{x}}(x;\bar{x}(\sigma))\bar{x}'d\sigma + o(d\sigma)$$

where $\bar{p}_{\bar{x}}$ is the derivative of the density function with respect to $\bar{x}$. For the pure menu cost model we have $\bar{p}(x,\bar{x}) = 1/\bar{x} - (1/\bar{x}^2)|x|$ for $x \in (\bar{x},\bar{x})$ so we have:

$$\bar{p}_{\bar{x}}(x,\bar{x}) = \frac{1}{\bar{x}^2}\left(-1 + \frac{2|x|}{\bar{x}}\right) \quad \text{and} \quad \frac{\partial}{\partial\sigma}\bar{x}(\sigma) = \frac{1}{2}\frac{\bar{x}}{\sigma}$$

where we use that $\bar{x}(\sigma) = (6\psi/B\sigma^2)^{1/4}$. Replacing into the expression for $\hat{p}$ we have

$$\hat{p}(x,0) = -\bar{p}_{\bar{x}}(x;\bar{x}(\sigma))\bar{x}'d\sigma + o(d\sigma) = \frac{1}{\bar{x}}\left(\frac{|x|}{\bar{x}} - \frac{1}{2}\right)\frac{d\sigma}{\sigma} + o(d\sigma) \text{ for } \; x \in (-\bar{x},\bar{x})$$

Step 2: find signed measure after s periods $\hat{p}(x,s)$. The function $\hat{p}(x,s)$ describes the evolution of this signed measure s periods after the uncertainty shock. We use our characterization of the density of transition function $\sum_j \exp(\lambda_j s)\,\phi_j(x_s)\phi(x_0)$ between time $t = 0$ and $t = s$ with eigenfunctions φ_j and eigenvalues λ_j with $\bar{x}(\tilde{\sigma})$ and $N = \tilde{\sigma}^2/\bar{x}(\tilde{\sigma})^2$ to construct the evolution of $\hat{p}(x,s)$. We represent the signed measure (deviation from the invariant distribution) as follows

$$\hat{p}(x,s) = \frac{d\sigma}{\sigma}\sum_{j=1,3,5,...} e^{\lambda_j s}\varphi_j(x)\left\langle\varphi_j,\hat{p}(\cdot,0)\right\rangle \tag{A.29}$$

where the projection coefficients $\langle\varphi_j,\hat{p}\rangle = 0$ for $j = 2,4,6,...$ since the function $\hat{p}(x,0)$ is a symmetric function while the even-indexed φ_j functions are antisymmetric. The non zero coefficients are

$$\langle\varphi_j,\hat{p}(\cdot,0)\rangle = \int_{-\bar{x}}^{\bar{x}}\varphi_j(x)\hat{p}(x,0)dx = 2\int_0^{\bar{x}}\frac{1}{\bar{x}}\left(\frac{x}{\bar{x}} - \frac{1}{2}\right)\varphi_j(x)dx \quad \text{for} \quad j = 1,3,5,...$$

Direct calculation gives

$$\langle\varphi_j,\hat{p}(\cdot,0)\rangle = \frac{2}{\bar{x}^{1/2}}\,\frac{4(-1)^{\frac{j+3}{2}} - j\pi}{(j\,\pi)^2} \quad \text{for} \quad j = 1,3,5,... \tag{A.30}$$

Step 3: Find excess impulse response $Y(t;s,\frac{d\sigma}{\sigma}) - Y(t;\infty,\frac{d\sigma}{\sigma})$. The cross section distribution right after the monetary shock is

$$p(x+\delta,s;\tilde{\sigma}) = \bar{p}(x+\delta;\tilde{\sigma}) + \hat{p}(x+\delta,\tau)$$

the first term is the invariant distribution (under the new variance $\tilde{\sigma}$) which will settle in the long run, the second term is the deviation between the current cross-section distribution and the invariant, discussed above. Then

$$p(x+\delta, s; \tilde{\sigma}) - \bar{p}(x; \tilde{\sigma}) \approx \delta\left(\bar{p}'(x; \tilde{\sigma}) + \hat{p}'(x, s)\right)$$

We let $Y(t; \infty, \frac{d\sigma}{\sigma}) = \sum_{k=1}^{\infty} e^{\lambda_{2k} t}\, b_{2k}[f] b_{2k}[\bar{p}'(\cdot; \tilde{\sigma})]$ be the long run output response to a monetary shock, after the initial uncertainty shock has settled down (i.e. for $s \to \infty$). Our main proposition implies that impulse response to a monetary shock s periods after the uncertainty shock is

$$Y\left(t; s, \frac{d\sigma}{\sigma}\right) = \sum_{k=1}^{\infty} e^{\lambda_{2k} t}\, b_{2k}[f] b_{2k}[\hat{p}'(\cdot, s)] \; + Y\left(t; \infty, \frac{d\sigma}{\sigma}\right) \tag{A.31}$$

Note that the above summation only uses even-indexed eigenfunctions since the function of interest for the output $f(x) = -x$ is antisymmetric, we know that all $b_{2k+1}[f] = 0$ for $k = 1, 2, 3, ...$

Now we turn to the computation of $b_{2k}[\hat{p}'(\cdot, \tau)]$, given by$b_{2k}[\hat{p}'(\cdot, \tau)] \equiv \int_{-\bar{x}}^{\bar{x}} \varphi_{2k}(x) \hat{p}'(x, s) dx$. Note that from equation (2) we can write $\hat{p}'(\cdot, \tau)$ as

$$\hat{p}'(x, s) = \frac{d\sigma}{\sigma} \sum_{j=1,3,5,\ldots} e^{\lambda_j s} \varphi_j'(x) \left\langle \varphi_j, \hat{p}(\cdot, 0)\right\rangle$$

Using equation (A.29) and the form of the eigenfunctions:

$$b_{2k}[\hat{p}'(\cdot, s)] = \int_{-\bar{x}}^{\bar{x}} \varphi_{2k}(x) \hat{p}'(x, s) dx = \frac{d\sigma}{\sigma} \sum_{j=1,3,5,\ldots} e^{\lambda_j s} \langle \varphi_j, \hat{p}(\cdot, 0)\rangle b_{2k}[\varphi_j'] \quad , \quad k = 1, 2, 3, \ldots$$

Direct computation for $k = 1, 2, 3, \ldots$ and $j = 1, 3, 5, \ldots$ gives

$$\begin{aligned} b_{2k}[\varphi_j'] &\equiv \int_{-\bar{x}}^{\bar{x}} \varphi_{2k}(x) \varphi_j'(x) dx \\ &= \frac{j\pi}{2\bar{x}} \int_{-\bar{x}}^{\bar{x}} \sin\left(k\pi\left(\frac{x+\bar{x}}{\bar{x}}\right)\right) \cos\left(j\pi\left(\frac{x+\bar{x}}{2\bar{x}}\right)\right) dx = \frac{4kj}{\bar{x}(4k^2 - j^2))} \end{aligned}$$

and hence

$$b_{2k}[\hat{p}'(\cdot, s)] = \frac{d\sigma}{\sigma} \frac{1}{\bar{x}^{3/2}} \sum_{j=1,3,5,\ldots}^{\infty} e^{\lambda_j s} \left(2 \frac{4(-1)^{\frac{j+3}{2}} - j\pi}{(j\,\pi)^2}\right) \left(\frac{4kj}{(4k^2 - j^2))}\right) \quad , \quad k = 1, 2, 3, \ldots$$

□

D Sticky Price Multiproduct firms

Multiproduct models consider a firm that produces n different products and that faces increasing returns in the price adjustment: if a pays a fixed cost it can adjust simultaneously the n prices. Variations on this model have been studied by Midrigan (2011) and Bhattarai and Schoenle (2014). These models are appealing because they match several empirical regularities: synchronization among price changes within a store and the coexistence of both small and large price changes. Their economic analysis is of interest because in an economy populated by multiproduct firms the monetary shocks have more persistent real effects. In Alvarez and Lippi (2014) we derived results for impulse responses to this multidimensional setup and explore the sense in which such a model is realistic. Here we show that the characterization of the selection effect, as the difference between the survival function and the output IRF holds in this model, with the number of products n serving as the parameter that control selection. We also show that in this case a single eigenvalue gives a poor characterization of the output IRF.

In the multiproduct model the price gap is given by a vector of n price gaps, each of them given by an independently standard BM's $(p_1, p_2, \ldots p_n)$, driftless and with innovation variance σ^2. We are interested only on two functions of this vector, the sum of its squares and it sum:

$$y = \sum_{i=1}^{n} p_i^2 \text{ and } z = \sum_{i=1}^{n} p_i$$

It is interesting to notice that while the original state is n dimensional, (y, z) can be described as a two dimensional diffusion –see Alvarez and Lippi (2014) and Appendix D.1 for details.

We are interested in the sum of its squares y because in Alvarez and Lippi (2014) under the assumption of symmetric demand the optimal decision rule is to adjust the firm time that y hits a critical value $\bar{y}$. We are interested in z, the sum of the price gaps, because this give the contribution of firm to the deviation of the price level relative to the steady state value, and hence $-z$ is proportional to its contribution to output. Note that the domain of (y, z) is $0 \leq y \leq \bar{y}$ and $-\sqrt{ny} \leq z \leq \sqrt{n\, y}$. In Alvarez and Lippi (2014) we show that the expected number of adjustments per unit of time is given by $N = \frac{n\sigma^2}{\bar{y}}$ and also give a characterization of $\bar{y}$ in terms of the parameters for the firm's problem. For the purpose in this paper we find it convenient to rewrite the state as (x, w) defined as

$$x = \sqrt{y} \text{ and } w = \frac{z}{\sqrt{ny}}.$$

In Lemma 1 in Appendix D.1 we analyze the behavior of the $(x, w) \in [0, \bar{x}] \times [-1, 1]$ process with $\bar{x} \equiv \sqrt{\bar{y}}$. Clearly we can recover (y, z) from (x, w). For instance, $z = w\sqrt{n x}$. Yet with this change on variables, even though the original problem is n dimensional, we define a two dimensional process for which we can analytically find its associated eigenfunctions and eigenvalues for the operator:

$$\mathcal{G}(f)\left((x, w), t\right) = \mathbb{E}\left[f\left(x(t), w(t)\right)\, 1_{y \geq \bar{y}} \,\middle|\, (x(0), w(0)) = (x, w) \right]$$

where $f : [0, \bar{x}] \times [-1, 1] \to \mathbb{R}$. The relevant p.d.e. is defined and its solution via eigenfunctions and eigenvalues, is characterized in Proposition 10 in Appendix D.1. Moreover the eigenfunctions and eigenvalues are indexed by a countably double infinity indices $\{m, k\}$.

Eigenfunctions. The eigenfunctions φ have a multiplicative nature, so $\varphi_{m,k}(x, w) = h_m(w) g_{m,k}(x)$ where for each number of products n then h_m and $g_{m,k}$ are known analytic functions indexed by k and by (k, m) respectively. Indeed h_m are scaled Gegenbauer polynomials, and $g_{m,k}$ are scaled Bessel functions –see Proposition D.1 for the exact expressions and definition.[4]

Eigenvalues. For each n the eigenvalues can be also indexed by a countably double-infinity $\{\lambda_{m,k}\}$. As in the baseline case, the eigenvalues are proportional to N, the expected number of price changes per unit of time:

$$\lambda_{m,k} = -N \, \frac{\left(j_{\frac{n}{2}-1+m,k}\right)^2}{2\,n} \quad \text{for } m = 0, 1, \ldots, \text{ and } k = 1, 2, \ldots$$

$j_{\nu,k}$ denote the ordered zeros of the Bessel function of the first kind $J_\nu(\cdot)$ with index ν.

The second sub-index k in the root of the Bessel function denote their ordering, so $k = 1$ is the smallest positive root. Also fixing k the roots $j_{m+\frac{n}{2}-1,k}$ are increasing in m. Thus, the *dominant* eigenvalue is given by $\lambda_{0,1}$. We will argue below that the smallest (in absolute value) eigenvalue that is featured in the (marginal) output IRF is $\lambda_{1,1}$. A very accurate approximation of the eigenvalues consists on using the first three leading terms in its expansion, as is given by: $j_{\nu,k} \approx \nu + \nu^{1/3} 2^{-1/3} a_k + (3/20)(a_k)^2 2^{1/3} \nu^{-1/3}$ where a_k are the zeros of the Airy function. [5] Using this approximation into the expression for the eigenvalues, one can see that keeping fixed N, the absolute value both $\lambda_{0,1}$ and $\lambda_{1,1}$ go to infinity, and that the difference between the two decreases and converges to $N/2$. Figure 3 displays the difference between these two eigenvalues.

Impulse response. As before, we want to compute $G(t)$, the conditional expectation of $f : [0, \bar{x}] \times [-1, 1] \to \mathbb{R}$ for (x, w) following equation (A.32)-equation (A.33), integrated with respect to $p(w, x; 0)$. We are interested in functions $f : [0, \bar{x}] \times [-1, 1]$ that can be written as:

$$f(x, w) = \sum_{m=0}^{\infty} \sum_{k=1}^{\infty} b_{m,k}[f] \varphi_{m,k}(x, w)$$

Using the same logic as in the one dimensional case:[6]

$$G(t) = \sum_{k=1}^{\infty} \sum_{m=0}^{\infty} e^{\lambda_{m,k}\, t}\, b_{m,k}[f]\, b_{m,k}\left[p(\cdot, 0)/\omega\right] \, \langle \varphi_{m,k}, \varphi_{m,k} \rangle$$

where the term $\langle \varphi_{m,k}, \varphi_{m,k} \rangle$ appears because the have, as it is customary in this case, use an orthogonal, but not orthonormal base, and where $\omega(w, x)$ is a weighing function appropriately defined – see Appendix D.1. So that $b_{m,k}\left[p(\cdot, 0)/\omega\right]$ are the projections of the ratio of the functions $p(\cdot, 0)$ and ω.

Functions of interest. We analyze two important functions of interest f. The first one a constant, $f(w, x) = 1$ which is used to compute the measure of firms that have not adjusted, or the survival function $S(t)$. The second one is the one that gives the average price gap

[4] The Gegenbauer polynomials are orthogonal to each other, and so are the Bessel functions when using an appropriately weighted inner product, as defined in Appendix D.1.

[5] In our case, we are interested in $k = 1$ which is about $a_1 = -2.33811$. See Figure 4 in the APP where we plots both eigenvalues, as well as its approximation for several n.

[6] See Appendix D.1 for a derivation

among the n product of the firm, i.e. $f(w,z) = -z/n = -wx/\sqrt{n}$. This is, as before, the negative of the average across the n products of the price gaps. This is the function f used for the impulse response of output to a monetary shock. An important property of the Gegenbauer polynomials is that the $m = 0$ equals a constant, for $m = 1$ is proportional to w, and in general for m odd are antisymmetric on w and symmetric for even m. Thus for $f = 1$ we can use just the Gegenbauer polynomial with $m = 0$ and all the Bessel functions corresponding to $m = 0$ and $k \geq 1$. Instead for $f(w,x) = wx/sqrtn = z$ we can use just the Gegenbauer polynomial with $m = 1$ and all the Bessel functions corresponding to $m = 0$ and $k \geq 1$.

Initial shifted distribution for a small shock. We have derived the invariant distribution of (z,y) in Alvarez and Lippi (2014). Using the change in variables (y,z) to $y = x^2$ and $z = \sqrt{yn}\,w =$ we can define the steady state density as $\bar{p}(w,x) = \bar{h}(w)\bar{g}(x)$ – see Appendix D.1 for the expressions. We perturb this density with a shock of size δ in each of the n price gaps. We want to subtract δ to each component of $(p_1, ..., p_n)$. This means that the density for each $x = ||p||$ just after the shock becomes the density of $x(\delta) = ||(p_1 + \delta, \ldots, p_n + \delta)||$ just before. Likewise the density corresponding to each w becomes th one for $w(\delta) = (z + n\delta)/(\sqrt{n}\,x(\delta))$. We consider the initial condition given by density $p_0(w,x;\delta) = \bar{h}(w(\delta))\bar{g}(x(\delta))$. We will use the first order terms, which are appropriate for the case of a small shock δ. The expressions can be found in Appendix D.1.

Interpretation of dominant eigenvalue, and irrelevance for the marginal IRF. We are now ready to generalize our interpretation of the dominant eigenvalue (as well as those corresponding to symmetric functions of z), as well as its irrelevance for the marginal output IRF.

Proposition 8. The coefficient of the marginal impulse response of output for a monetary shock are a function of the $\{\lambda_{1,k}, \varphi_{1,k}\}_{k=1}^{\infty}$ eigenvalue-eigenfunctions pairs, so that:

$$Y(t) = \sum_{k=1}^{\infty} \beta_{1,k}\, e^{\lambda_{1,k}\, t} \quad \text{and} \quad -\lambda_{1,1} = \lim_{t\to\infty} \frac{\log|Y(t)|}{t}$$

where $\beta_{1,k} = b_{1,k}\,[wx/\sqrt{n}]\; b_{1,k}\,[\bar{p}'(w,x)]$. In particular, the dominant eigenvalue $\lambda_{0,1}$ does not characterize the limiting behavior of the impulse response. Instead the survival function for price changes $S(t)$, can be written in terms of $\{\lambda_{0,k}, \varphi_{0,k}\}_{k=1}^{\infty}$, and hence the asymptotic hazard rate is equal to the dominant eigenvalue $\lambda_{0,1}$, i.e.

$$S(t) = \sum_{k=1}^{\infty} \beta_{0,k}\, e^{\lambda_{0,k}\, t} \quad \text{and} \quad -\lambda_{0,1} = \lim_{t\to\infty} \frac{\log S(t)}{t}$$

where $\beta_{0,k} = b_{0,k}\,[1]\; b_{0,k}\,[\delta_0]$ where δ_0 is the Dirac delta function for $(p_1, \ldots, p_n)$ transformed to the (x,w) coordinates. Recall that $0 > \lambda_{0,1} > \lambda_{1,1}$.

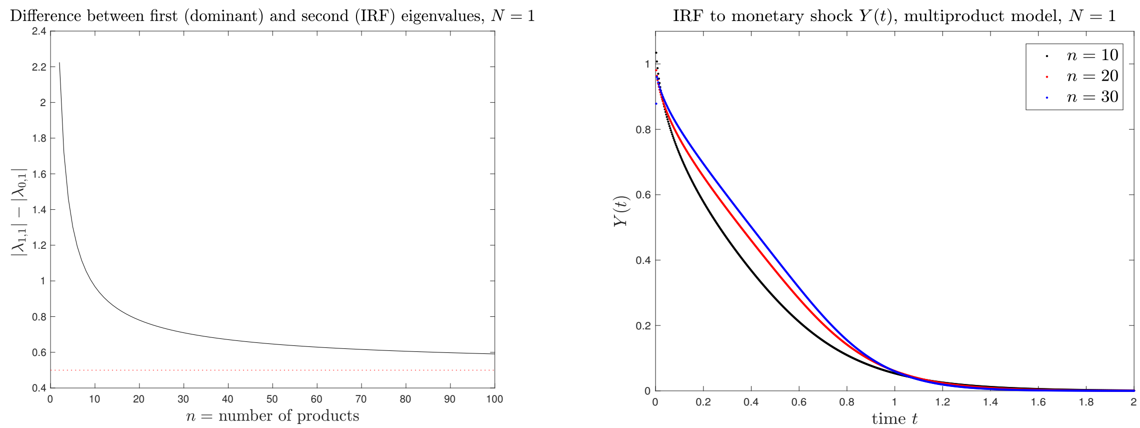
Given the importance of the difference between the eigenvalues $\lambda_{1,k}$ and $\lambda_{0,k}$ we show that for a fixed k they both increase with n, but it difference decreases to asymptote to $1/2$.

Proposition 9. Fixing $k \geq 1$, the k^{th} eigenvalue for the IRF $Y(\cdot)$ given by $\lambda_{1,k}$ and the k^{th} eigenvalue for the survival function $S(\cdot)$ given by $\lambda_{1,k}$ both increase with the number of products n, diverging towards $-\infty$ as $n \to \infty$. The difference $\lambda_{0,k} - \lambda_{1,k} > 0$ decreases with n, converging to $1/2$ as $n \to \infty$.

Figure 3: Shock propagation in Multiproduct models

Selection effect

Impulse response



Keeping fixed $N = 1$ for all n

Figure 3 illustrates Proposition 9 for the case of $k = 1$, i.e. the eigenvalue that dominates the long run behaviour of the survival and IRF functions. Proposition 9 extends the result for all k. Increasing the number of products n in the multi product model decreases the selection effect at the time of a price change. As n goes to infinity, the eigenvalues that control the duration of the price changes (S) and those that control the marginal output IRF (Y) converge. This result shows that the characterization of selection effect in terms of dynamics controlled by two different types of eigenvalues is present not only in the Calvo$^+$ model, but also in this setup.

In Appendix D.1 we include Proposition 11 which gives a closed form solution for $\bar{p}'(w, x; 0)$ and for the coefficients for $b_{1,k}$ of the output impulse response function. All these expressions depends only of the number of products n. Instead we include a figure of the impulse responses for three values of n. It is clear both the output IRF and the survival function cannot be well described using one eigenfunction-eigenvalue for large n. For instance, as $n \to \infty$ the output's IRF Y becomes a linearly declining function until it hits zero at $t = 1/N$, and the survival function S is zero until it becomes infinite at $t = 1/N$.

D.1 Details of the multiproduct model

Law of motion for y, z.

$$dy = \sigma^2 n\, dt + 2\sigma\sqrt{y}\, dW^a$$
$$dz = \sigma\sqrt{n}\left[\frac{z}{\sqrt{ny}}\, dW^a + \sqrt{1-\left(\frac{z}{\sqrt{ny}}\right)^2}\, dW^b\right]$$

where W^a, W^b are independent standard BM's.

LEMMA 1. Define

$$x = \sqrt{y} \text{ and } w = \frac{z}{\sqrt{ny}}$$

so that the domain is $0 \le x \le \bar{x} \equiv \sqrt{\bar{y}}$ and $-1 \le w \le 1$. They satisfy:

$$dx = \sigma^2 \frac{n-1}{2\,x} dt + \sigma dW^a \tag{A.32}$$
$$dw = \frac{w}{x^2}\left(\frac{1-n}{2}\right) dt + \frac{\sqrt{1-w^2}}{x} dW^b \tag{A.33}$$

We look for a solution to the eigenvalue-eigenfunction problem (λ, φ) given by equation (A.32) and equation (A.33). They must satisfy

$$\lambda\varphi(w,x) = \varphi_x(w,x)\sigma^2\left(\frac{n-1}{2x}\right) + \varphi_w(w,x)\frac{w}{x^2}\left(\frac{1-n}{2}\right) + \frac{1}{2}\varphi_{ww}(w,x)\frac{(1-w^2)}{x^2} + \frac{1}{2}\sigma^2\varphi_{xx}(w,x)$$

for all $(x,w) \in [0,\bar{x}] \times [-1,1]$, with $\varphi(\bar{x}, w) = 0$, all w and φ^2 integrable.

PROPOSITION 10. The eigenfunctions-eigenvalues of (w,x) satisfying equation (A.32)-equation (A.33) denoted by $\{\varphi_{m,k}(\cdot), \lambda_{m,k}\}$ for $k = 1, 2, \ldots$ and $m = 0, 1, \ldots$ are given by:

$$\varphi_{m,k}(x,w) = h_m(w)\, g_{m,k}(x) \text{ where}$$
$$h_m(w) = C_m^{\frac{n}{2}-1}(w) \text{ for } m = 0,1,2,\ldots \text{ and}$$
$$g_{m,k}(x) = x^{1-n/2} J_{\frac{n}{2}-1+m}\left(j_{\frac{n}{2}-1+m,k}\,\frac{x}{\bar{x}}\right) \text{ for } k = 1,2,\ldots \text{ and}$$
$$\lambda_{m,k} = -N\,\frac{\left(j_{\frac{n}{2}-1+m,k}\right)^2}{2\,n} \text{ for } m = 0,1,\ldots, \text{ and } k = 1,2,\ldots$$

where $C_m^{\frac{n}{2}-1}(\cdot)$ denote the Gegenbauer polynomials, and where $J_{\frac{n}{2}-1+m}(\cdot)$ denote the Bessel function of the first kind, $j_{\nu,k}$ denote the ordered zeros of the Bessel function of the first kind $J_\nu(\cdot)$ with index ν.

Note that the expressions for the eigenfunctions are only valid only for $n > 2$. For $n = 2$ the expression take a different special form, which we skip to save space. The expressions for the eigenvalues are valid for $n \geq 2$.

We remind the reader how the Gegenbauer polynomial and Bessel function, which form an orthogonal base, are defined. The Gegenbauer polynomial $C_m^{\frac{n}{2}-1}(w)$ is given by:

$$C_m^{\frac{n}{2}-1}(w) = \sum_{k=0}^{\lfloor m/2 \rfloor} (-1)^k \frac{\Gamma(m-k+\frac{n}{2}-1)}{\Gamma(\frac{n}{2}-1)k!(m-2k)!}(2w)^{m-2k} \tag{A.34}$$

For a fixed n, the polynomials are orthogonal on with respect to the weighting function $(1-w^2)^{\frac{n}{2}-1-\frac{1}{2}}$ so that:[7]

$$\int_{-1}^{1} C_m^{(\frac{n}{2}-1)}(w)\, C_j^{(\frac{n}{2}-1)}(w)\,(1-w^2)^{\frac{n}{2}-1-\frac{1}{2}}\,dw = 0 \text{ for } m \neq j \tag{A.35}$$

and for $m = j$ we get

$$\int_{-1}^{1} \left[C_m^{(\frac{n}{2}-1)}(w)\right]^2 (1-w^2)^{\frac{n}{2}-1-\frac{1}{2}}\,dw = \frac{\pi\, 2^{1-2(\frac{n}{2}-1)}\Gamma(m+2(\frac{n}{2}-1))}{m!(m+\frac{n}{2}-1)[\Gamma(\frac{n}{2}-1)]^2} \tag{A.36}$$

The Bessel function of the first kind is given by :

$$J_\nu(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(k+\nu+1)}\left(\frac{x}{2}\right)^{2k+\nu} \tag{A.37}$$

For a given ν, the following functions are orthogonal, using the weighting function x^{n-1} so that:[8]

$$\begin{aligned}
&\int_0^{\bar{x}} \left[x^{1-\frac{n}{2}} J_\nu\left(j_{\nu,k}\frac{x}{\bar{x}}\right)\right] \left[x^{1-\frac{n}{2}} J_\nu\left(j_{\nu,s}\frac{x}{\bar{x}}\right)\right] x^{n-1}\,dx \\
&= \int_0^{\bar{x}} J_\nu\left(j_{\nu,k}\frac{x}{\bar{x}}\right) J_\nu\left(j_{\nu,s}\frac{x}{\bar{x}}\right) x\,dx = 0 \text{ if } k \neq s \in \{1,2,3,\ldots\} \text{ and} \\
&\int_0^{\bar{x}} \left[x^{1-\frac{n}{2}} J_\nu\left(j_{\nu,k}\frac{x}{\bar{x}}\right)\right]^2 x^{n-1}\,dx = \bar{x}^2 \int_0^{\bar{x}} \frac{x}{\bar{x}} \left[J_\nu\left(j_{\nu,k}\frac{x}{\bar{x}}\right)\right]^2 \frac{dx}{\bar{x}} \qquad \text{(A.38)} \\
&= \frac{1}{2}\left(\bar{x}\, J_{\nu+1}\left(j_{\nu,k}\right)\right)^2 \text{ for all } k \in \{1,2,3\ldots\} \qquad \text{(A.39)}
\end{aligned}$$

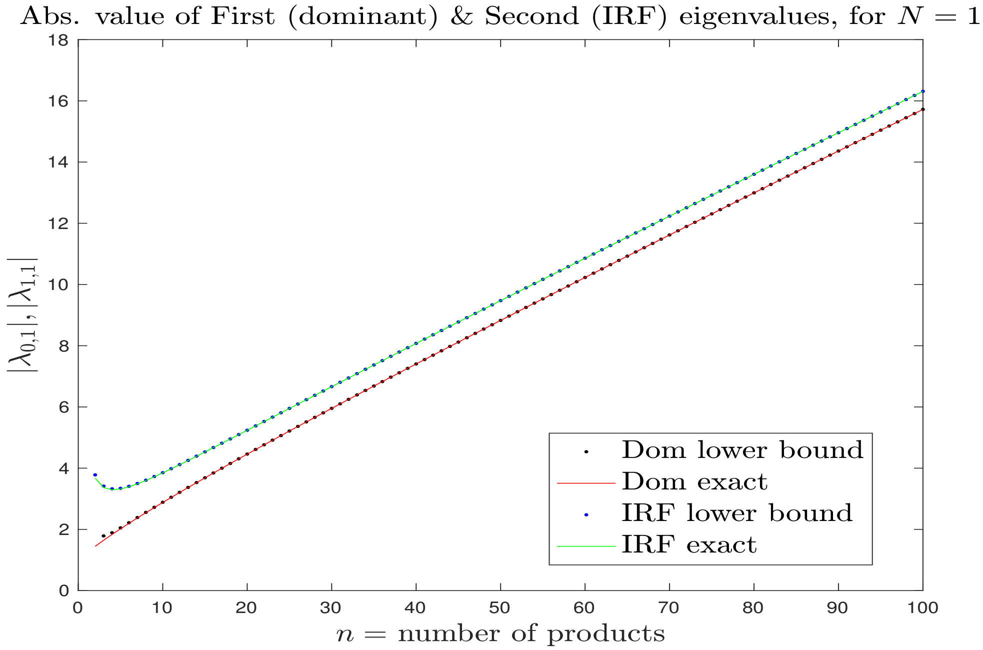
where $j_{\nu,k}$ and $j_{\nu,s}$ are two zeros of $J_\nu(\cdot)$.

---

[7] By this we mean that we define the inner product between functions a, b from $[-1,1]$ to $\mathbb{R}$ as : $\langle a,b\rangle = \int_{-1}^{1} a(w)b(w)\left(1-w^2\right)^{\frac{n}{2}-1-\frac{1}{2}} dw$.

[8] By this we mean that we define the inner product between functions a, b from $[0,\bar{x}]$ to $\mathbb{R}$ as: $\langle a,b\rangle = \int_0^{\bar{x}} a(x)b(x)x^{n-1}dx$.

Figure 4: Eigenvalues for multiproduct model



Kepping fixed $N = 1$ for all n

Derivation of IRF. Thus we have

$$G(t) \equiv \int_0^{\bar{x}} \int_{-1}^{1} \mathcal{G}(f)(x, w, t)\, p(x, w; 0) dw\, dx$$

As in Section 3, we can write this expected value as:

$$\begin{aligned} Y(t) &= \int_0^{\bar{x}} \int_{-1}^{1} \mathcal{G} \left(\sum_{k=1}^{\infty} \sum_{m=0}^{\infty} b_{m,k}[f]\, \varphi_{k,m} \right) (x, w, t)\ p(x, w; 0) dw\, dx \\ &= \int_0^{\bar{x}} \int_{-1}^{1} \sum_{k=1}^{\infty} \sum_{m=0}^{\infty} b_{m,k}[f]\, \mathcal{G}\left(\varphi_{k,m}\right) (x, w, t)\ p(x, w; 0) dw\, dx \\ &= \int_0^{\bar{x}} \int_{-1}^{1} \sum_{k=1}^{\infty} \sum_{m=0}^{\infty} b_{m,k}[f]\, e^{\lambda_{m,k}\, t}\, \varphi_{m,k}(x, w)\, p(x, w; 0) dw\, dx \\ &= \sum_{k=1}^{\infty} \sum_{m=0}^{\infty} e^{\lambda_{m,k}\, t}\, b_{m,k}[f] \int_0^{\bar{x}} \int_{-1}^{1} \varphi_{m,k}(x, w)\, p(x, w; 0) dw\, dx \end{aligned}$$

Then we get:

$$G(t) = \sum_{k=1}^{\infty} \sum_{m=0}^{\infty} e^{\lambda_{m,k}\, t}\, b_{m,k}[f]\, b_{m,k}\left[p(\cdot, 0)/\omega\right]\, \langle \varphi_{m,k}, \varphi_{m,k} \rangle$$

Inner product. We let $\omega(w,x) = x^{1-n}\,(1-w^2)^{\frac{n-3}{2}}$. The inner product of functions a, b from $[0,\bar{x}] \times [-1,1]$ to $\mathbb{R}$ is defied as

$$\langle a, b\rangle = \int_0^{\bar{x}} \int_{-1}^{1} a(x,w)\, b(x,w)\, x^{1-n}\,(1-w^2)^{\frac{n-3}{2}}\, dw\, dx$$

The term $\langle \varphi_{m,k}, \varphi_{m,k}\rangle$ is given by the product of equation (A.36) and equation (A.39) found above. Indeed since the polynomials are orthogonal we have:

$$\begin{aligned} b_{m,k}[f] = \frac{\langle f, \varphi_{m,k}\rangle}{\langle \varphi_{m,k}, \varphi_{m,k}\rangle} &= \frac{\int_0^{\bar{x}} \left[\int_{-1}^{1} f(x,w) h_m(w) (1-w^2)^{\frac{n-3}{2}} dw\right] g_{m,k}(x)\, x^{n-1}\, dx}{\left[\int_{-1}^{1} (h_m(w))^2\,(1-w^2)^{\frac{n-3}{2}} dw\right]\left[\int_0^{\bar{x}} (g_{m,k}(x))^2\, x^{n-1}\, dx\right]} \\ &= \frac{\int_0^{\bar{x}} \left[\int_{-1}^{1} f(x,w) C_m^{\frac{n}{2}-1}(w) (1-w^2)^{\frac{n-3}{2}} dw\right] J_{m+\frac{n}{2}-1}\left(j_{m+\frac{n}{2}-1,k}\,\frac{x}{\bar{x}}\right) x^{\frac{n}{2}}\, dx}{\left[\int_{-1}^{1} \left(C_m^{\frac{n}{2}-1}(w)\right)^2 (1-w^2)^{\frac{n-3}{2}} dw\right]\left[\int_0^{\bar{x}} \left(J_{m+\frac{n}{2}-1}\left(j_{m+\frac{n}{2}-1,k}\,\frac{x}{\bar{x}}\right)\right)^2 x\, dx\right]} \end{aligned}$$

Invariant Distribution. After the change in variables we have:

$$\bar{h}(w) = \frac{1}{Beta\left(\frac{n-1}{2},\frac{1}{2}\right)}\ \left(1-w^2\right)^{(n-3)/2} \quad \text{for} \quad w \in (-1\,,\,1) \tag{A.40}$$

$$\bar{g}(x) = x\ (\bar{x})^{-n}\left(\frac{2n}{n-2}\right)\left[\bar{x}^{n-2} - x^{n-2}\right] \qquad \text{for } x \in [0,\bar{x}] \tag{A.41}$$

Initial distribution after a small monetary shock.

$$\begin{aligned} &p(w,x;0) = \bar{h}(w(\delta))\bar{g}(x(\delta)) = \bar{h}(w)\bar{g}(x) + \bar{p}'(w,x;0)\delta + o(\delta) \text{ with} \\ &\bar{p}'(w,x;0) = \bar{g}(x)\bar{h}'(w)w'(0) + \bar{h}(w)\bar{g}'(x)x'(0) \end{aligned}$$

where:

$$\begin{aligned} &\frac{\partial}{\partial\delta} x(\delta)|_{\delta=0} = x'(0) = \sqrt{n}\, w \quad \text{and} \quad \frac{\partial}{\partial\delta}\bar{h}(w(\delta))|_{\delta=0} = \bar{h}'(w)w'(0) \\ &\frac{\partial}{\partial\delta} w(\delta)|_{\delta=0} = w'(0) = \frac{\sqrt{n}\,(1-w^2)}{x} \quad \text{and} \quad \frac{\partial}{\partial\delta}\bar{g}(x(\delta))|_{\delta=0} = \bar{g}'(x)x'(0) \end{aligned}$$

Proposition 11. The expressions for $\bar{p}'(x,w;0)$ and the coefficients $b_{1,k}(n)$ for the impulse response of output are given by:

$$\begin{aligned} \bar{p}'(w,x;0) &= \bar{g}(x)\bar{h}'(w)w'(0) + \bar{h}(w)\bar{g}'(x)x'(0) \\ &= \frac{w\,(1-w^2)^{(n-3)/2}}{Beta\left(\frac{n-1}{2},\frac{1}{2}\right)}\sqrt{n}\left(\frac{2n}{n-2}\right)\frac{[(4-n)\bar{x}^{n-2} - (4+n)x^{n-2}]}{\bar{x}^n} \end{aligned}$$

and the coefficients for the impulse response $b_{1,k}(n) = b_{1,k}[f]\, b_{1,k}[\bar{p}'(\cdot,0)/\omega]\, \langle\varphi_{1,k},\varphi_{1,k}\rangle$ are given by

$$b_{1,k}(n) = -\frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n}{2}+1\right)} \frac{2n}{(n-2)\, j_{\frac{n}{2},k}\, J_{\frac{n}{2}+1}\left(j_{\frac{n}{2},k}\right)} \left[(4-n)\left(\frac{2^{1-\frac{n}{2}}}{\Gamma(\frac{n}{2})\left(j_{\frac{n}{2},k}\right)^{2-\frac{n}{2}}} - \frac{J_{\frac{n}{2}-1}\left(j_{\frac{n}{2},k}\right)}{j_{\frac{n}{2},k}}\right)\right.$$
$$\left. -(4+n)2^{-1-\frac{n}{2}}(j_{\frac{n}{2},k})^{\frac{n}{2}}\Gamma\left(\frac{n}{2}\right) {}_1\tilde{F}_2\left(\frac{n}{2}; 1+\frac{n}{2}, 1+\frac{n}{2}; -\frac{(j_{\frac{n}{2},k})^2}{4}\right)\right]$$

where ${}_1\tilde{F}_2(a_1; b_1, b_2; z)$ is the regularized generalized hypergeometric function, i.e. it is defined as ${}_1\tilde{F}_2(a_1; b_1, b_2; z) = {}_1F_2(a_1; b_1, b_2; z)/\left(\Gamma(b_1)\Gamma(b_2)\right)$ where ${}_1F_2$ is the generalized hypergeometric function and $j_{\frac{n}{2},k}$ is the k^{th} ordered zero of the Bessel function $J_{\frac{n}{2}}(\cdot)$.

Note that, as our notation emphasizes, the coefficients $b_j(n)$ depends only on the number of products.

D.2 Proofs for the Multiproduct model of Appendix D.1

Proof. (of Proposition 8) First take $f(w,x) = w\,x/\sqrt{n} = \frac{1}{n}\sum_{i=1}^{n} p_i$. But note that the Gegenbauer polynomial of degree 1 is

$$C_1^{\frac{n}{2}-1}(w) = \sum_{k=0}^{\lfloor 1/2 \rfloor} (-1)^k \frac{\Gamma(1-k+\frac{n}{2}-1)}{\Gamma(\frac{n}{2}-1)k!(1-2k)!}(2w)^{1-2k} = \frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n}{2}-1)}(2w) = (n-2)w$$

Thus for $f(w,x) = wx/\sqrt{n}$ we can simply write:

$$f(x,w) = \sum_{k=1}^{\infty} b_{1,k}[f]\varphi_{1,k}(x,w)$$

since

$$b_{m,k}[f] = \frac{1}{\sqrt{n}(n-2)} \frac{\left[\int_{-1}^{1} C_1^{\frac{n}{2}-1}(w) C_m^{\frac{n}{2}-1}(w)(1-w^2)^{\frac{n-3}{2}}dw\right]\left[\int_0^{\bar{x}} x\, J_{m+\frac{n}{2}-1}\left(j_{m+\frac{n}{2}-1,k}\frac{x}{\bar{x}}\right) x^{\frac{n}{2}} dx\right]}{\left[\int_{-1}^{1}\left(C_m^{\frac{n}{2}-1}(w)\right)^2 (1-w^2)^{\frac{n-3}{2}}dw\right]\left[\int_0^{\bar{x}}\left(J_{m+\frac{n}{2}-1}\left(j_{m+\frac{n}{2}-1,k}\frac{x}{\bar{x}}\right)\right)^2 x\, dx\right]}$$

and thus $b_{m,k}[f] = 0$ for all $m \neq 1$, since the polynomials are orthogonal, and

$$b_{1,k}[f] = \frac{1}{\sqrt{n}(n-2)} \frac{\int_0^{\bar{x}} x J_{\frac{n}{2}}\left(j_{\frac{n}{2},k}\frac{x}{\bar{x}}\right) x^{\frac{n}{2}} dx}{\int_0^{\bar{x}}\left(J_{\frac{n}{2}}\left(j_{\frac{n}{2},k}\frac{x}{\bar{x}}\right)\right)^2 x\, dx} \text{ for all } k \geq 1$$

Now we argue that fixing x the function $p(w,x;0)$ is odd (antisymmetric) viewed as a function of w. This is because $\bar{h}$ is even and $x'(0)$ is odd, so $\bar{h}'(w)x'(0)$ is odd. Also $\bar{h}'$ is odd and $w'(0)$ is even, hence $\bar{h}'(w)w'(0)$ is odd. Hence $p(w,x;0)$ is not orthogonal to the $C_1^{\frac{n}{2}-1}(\cdot)$. Thus $b_{1,k}[\bar{p}'] \neq 0$.

Finally, to represent the survival function, take $f(w,x) = 1$. Note that this also coincides with a Gegenbauer polynomial for $m = 0$, i.e. $C_0^{\frac{n}{2}-1}(w) = 1$. Thus:

$$f(x,w) = \sum_{k=1}^{\infty} b_{0,k}[f]\varphi_{1,k}(x,w)$$

since

$$b_{m,k}[f] = \frac{\left[\int_{-1}^{1} C_0^{\frac{n}{2}-1}(w) C_m^{\frac{n}{2}-1}(w)(1-w^2)^{\frac{n-3}{2}} dw\right]\left[\int_0^{\bar{x}} J_{m+\frac{n}{2}-1}\left(j_{m+\frac{n}{2}-1,k}\frac{x}{\bar{x}}\right) x^{\frac{n}{2}} dx\right]}{\left[\int_{-1}^{1}\left(C_m^{\frac{n}{2}-1}(w)\right)^2 (1-w^2)^{\frac{n-3}{2}} dw\right]\left[\int_0^{\bar{x}}\left(J_{m+\frac{n}{2}-1}\left(j_{m+\frac{n}{2}-1,k}\frac{x}{\bar{x}}\right)\right)^2 x\, dx\right]}$$

and since the Gegenbauer polynomials are orthogonal, and thus $b_{m,k}[f] = 0$ for all $m > 0$, and

$$b_{0,k}[f] = \frac{\int_0^{\bar{x}} J_{\frac{n}{2}-1}\left(j_{\frac{n}{2}-1,k}\,\frac{x}{\bar{x}}\right) x^{\frac{n}{2}} dx}{\left[\int_0^{\bar{x}}\left(J_{\frac{n}{2}-1}\left(j_{\frac{n}{2}-1,k}\,\frac{x}{\bar{x}}\right)\right)^2 x\, dx\right]} \text{ for all } k \geq 1$$

□

Proof. (of Proposition 9) Recall that for each $k \geq 1$:

$$\lambda_{1,k} = -N\frac{\left(j_{\frac{n}{2},k}\right)^2}{2n} \text{ and } \lambda_{0,k} = -N\frac{\left(j_{\frac{n}{2}-1,k}\right)^2}{2n}$$

and use $\nu = n/2$ in the first case and $\nu = n/2 - 1$ in the second. It is well know that $j_{\nu,k}$ is strictly increasing in both variables –see Elbert (2001). From here we know that $|\lambda_{1,k}| - |\lambda_{0,k}| > 0$ for all n and k. Also in Elbert (2001) we see that $\frac{\partial}{\partial v} j_{v,k} < 0$ for $\nu > -k$ and $k \geq 1/2$. Thus, the difference between $|\lambda_{1,k}| - |\lambda_{0,k}|$ is decreasing in n.

From Qu and Wong (1999) we have the lower and upper bound for the zeros of the Bessel function $J_\nu(\cdot)$:

$$\nu + \nu^{1/3}2^{-1/3}|a_k| \leq j_{\nu,k} \leq \nu + \nu^{1/3}2^{-1/3}|a_k| + \frac{3}{20}|a_k|^2 2^{1/3}\nu^{-1/3}$$

where a_k is the k^{th} zero of the Airy function. Thus, as $n \to \infty$ then $v \to \infty$ and thus both $\lambda_{1,k}$ and $\lambda_{0,k}$ diverge towards $-\infty$. From the same bounds we see that as $n \to \infty$, the difference $\lambda_{0,k} - \lambda_{1,k} \to 1/2$. □

Proof. (of Lemma 1) Using Ito's lemma we have: $dx = (1/2)y^{-1/2}dy - (1/2)(1/4)y^{-3/2}dy^2$ which gives

$$dx = \frac{n-1}{2\,x}dt + dW^a$$

and $w = f(y,z) = z/\sqrt{ny}$. We have:

$$dw = f_y dy + f_z dz + \frac{1}{2}f_{yy}(dy)^2 + \frac{1}{2}f_{zz}(dz)^2 + f_{yz}dydz$$

where $f = (z/\sqrt{n})\, y^{-1/2}$, and thus

$$
\begin{aligned}
f_y &= -\frac{z}{2\sqrt{n}} y^{-3/2} \\
f_z &= \frac{1}{\sqrt{n}} y^{-1/2} \\
f_{yy} &= \frac{3z}{4\sqrt{n}} y^{-5/2} \\
f_{zz} &= 0 \\
f_{yz} &= -\frac{1}{2\sqrt{n}} y^{-3/2}
\end{aligned}
$$

We thus have:

$$
\begin{aligned}
dw = &-\frac{z}{2\sqrt{n}} y^{-3/2}\left(ndt + 2\sqrt{y}\, dW^a\right) \\
&+ \frac{1}{\sqrt{n}} y^{-1/2}\sqrt{n}\left(\frac{z}{\sqrt{ny}}\, dW^a + \sqrt{1 - \left(\frac{z}{\sqrt{ny}}\right)^2}\, dW^b\right) \\
&+ \frac{1}{2}\frac{3z}{4\sqrt{n}} y^{-5/2} 4y dt - \frac{1}{2\sqrt{n}} y^{-3/2} 2z dt
\end{aligned}
$$

which we can rearrange as:

$$
\begin{aligned}
dw = &\frac{z}{\sqrt{n}} y^{-3/2}\left(\frac{1-n}{2}\right) dt \\
&+ \left(\frac{1}{\sqrt{n}} y^{-1/2}\sqrt{n}\frac{z}{\sqrt{n}\sqrt{y}} - \frac{z}{2\sqrt{n}} y^{-3/2} 2\sqrt{y}\right) dW^a \\
&+ \frac{1}{\sqrt{n}} y^{-1/2}\sqrt{n}\sqrt{1 - \left(\frac{z}{\sqrt{n}\sqrt{y}}\right)^2}\, dW^b
\end{aligned}
$$

or

$$
\begin{aligned}
dw &= \frac{w}{x^2}\left(\frac{1-n}{2}\right) dt + \left(\frac{z}{\sqrt{ny}} - \frac{z}{\sqrt{ny}}\right) dW^a + \frac{1}{x}\sqrt{1 - (w)^2}\, dW^b \\
&= \frac{w}{x^2}\left(\frac{1-n}{2}\right) dt + \frac{1}{x}\sqrt{1 - w^2}\, dW^b
\end{aligned}
$$

□

Proof. (of Proposition 10) We try a multiplicative solution of the form:

$$
\varphi(w, x) = h(w)\, g(x)
$$

To simplify the proof we set $\sigma^2 = 1$. Thus

$$\lambda h(w)g(x) = h(w)g'(x)\left(\frac{n-1}{2x}\right) + h'(w)g(x)\frac{w}{x^2}\left(\frac{1-n}{2}\right) + \frac{1}{2}h''(w)g(x)\frac{(1-w^2)}{x^2} + \frac{1}{2}h(w)g''(x)$$

Dividing by $h(w)$ in both sides we have:

$$\lambda g(x) = g'(x)\left(\frac{n-1}{2x}\right) + \frac{h'(w)w}{h(w)}\frac{g(x)}{x^2}\left(\frac{1-n}{2}\right) + \frac{1}{2}\frac{h''(w)\,(1-w^2)}{h(w)}\frac{g(x)}{x^2} + \frac{1}{2}g''(x)$$

Collecting terms:

$$\lambda g(x) = \frac{g(x)}{x^2}\left[\frac{h'(w)w}{h(w)}\left(\frac{1-n}{2}\right) + \frac{1}{2}\frac{h''(w)\,(1-w^2)}{h(w)}\right] + g'(x)\left[\frac{n-1}{2x}\right] + \frac{1}{2}g''(x)$$

Which suggests to try the following separating variable:

$$\mu = \frac{h'(w)w}{h(w)}\left(\frac{1-n}{2}\right) + \frac{1}{2}\frac{h''(w)\,(1-w^2)}{h(w)}$$

or

$$0 = -2\mu h(w) + h'(w)w\,(1-n) + h''(w)\left(1-w^2\right)$$

The solution of this equation is given by the Gegenbauer polynomials $C_m^{\alpha}(w)$. The Gegenbauer polynomials are the solution to the following o.d.e.:

$$(1-w^2)h(w)'' - (2\alpha+1)wh'(w) + m(m+2\alpha)h(w) = 0 \text{ for } w \in [-1,1]$$

for integer $m \geq 0$. Matching coefficients we have:[9]

$$-2\mu = m(m+2\alpha) \text{ and } -(2\alpha+1) = (1-n)$$

which gives

$$\alpha = \frac{n}{2} - 1 \text{ and } \mu = -\frac{m}{2}(m+n-2)$$

[9]See https://en.wikipedia.org/wiki/Gegenbauer_polynomials, which is bases on Abramowitz, Milton; Stegun, Irene Ann, eds. (1983) [June 1964], Chapter 22, Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables. Applied Mathematics Series. 55, Dover Publications.

Then given $\mu = -(m/2)(m+n-2)$ the o.d.e. for g is:

$$\lambda g(x) = \frac{g(x)}{x^2}\mu + g'(x)\left[\frac{n-1}{2x}\right] + \frac{1}{2}g''(x)$$

or

$$0 = g(x)\left(\mu - x^2\lambda\right) + g'(x)\left[\frac{n-1}{2}\right]x + \frac{1}{2}g''(x)x^2$$

or

$$0 = g(x)\left(2\mu - x^2 2\lambda\right) + g'(x)x\,(n-1) + g''(x)x^2$$

with boundary condition $g(\bar{x}) = 0$. The solution of this o.d.e., which does not explode at $x = 0$ is given by a Bessel function of the first kind. This is because the following o.d.e.:

$$g(x)(c + bx^2) + g'(x)x\,a + g''(x)x^2 = 0$$

has solution:[10]

$$g(x) = x^{(1-a)/2} J_\nu\left(\sqrt{b}\,x\right) \text{ where } \nu = \frac{1}{2}\sqrt{(1-a)^2 - 4c}$$

where $J_\nu(\cdot)$ is the Bessel function of the first kind. Matching coefficients we have:

$$a = n-1,\; b = -2\lambda,\; c = 2\mu \text{ and}$$
$$\nu = \frac{1}{2}\sqrt{(n-2)^2 - 8\mu} = \frac{1}{2}\sqrt{(n-2)^2 + 8(m/2)(m+n-2)} = \frac{n}{2} - 1 + m$$

We argue that $\nu = n/2 - 1 + m$ to see that note we have

$$4\nu^2 = (n-2)^2 + 4m(m+n-2) \text{ and}$$
$$4\nu^2 = 4\left(\frac{n-2+2m}{2}\right)^2 = (n-2)^2 + 4m(n-2) + 4m^2$$

which verifies the equality. So we have:

$$g(x) = x^{1-n/2} J_{\frac{n}{2}-1+m}\left(\sqrt{-2\lambda}\,x\right)$$

We still have to determine the eigenvalue λ. For this we use the boundary condition $g(\bar{x}) = 0$ and that $J_\nu(\cdot)$ has infinitely strictly orderer positive zeros, denoted by $j_{\nu,k}$ for $k = 1, 2, \ldots$ so that $J_\nu(j_{\nu,k}) = 0$. Thus fixing μ, i.e. $m \geq 0$, we have:

$$0 = g(\bar{x}) = (\bar{x})^{1-n/2} J_{\frac{n}{2}-1+m}\left(\sqrt{-2\lambda}\,\bar{x}\right)$$

[10]See http://eqworld.ipmnet.ru/en/solutions/ode/ode0215.pdf which uses Polyanin, A. D. and Zaitsev, V. F., Handbook of Exact Solutions for Ordinary Differential Equations, 2nd Edition, Chapman & Hall/CRC, Boca Raton, 2003.

so that:

$$0 = (\bar{x})^{1-n/2} J_{\frac{n}{2}-1+m}\left(\sqrt{-2\,\lambda_{m,k}}\,\bar{x}\right)$$

Hence

$$j_{\frac{n}{2}-1+m,k} = \sqrt{-2\,\lambda_{m,k}}\,\bar{x} \quad \text{or} \quad \lambda_{m,k} = -\frac{\left(j_{\frac{n}{2}-1+m,k}\right)^2}{2\,\bar{x}^2}$$

Collecting the terms for h, g and λ we obtain the desired result.

Since $\sigma^2 \neq 1$ changes the units of time, we need only to adjust the eigenvalue by its value, so that

$$\lambda_{m,k} = -\sigma^2 \frac{\left(j_{\frac{n}{2}-1+m,k}\right)^2}{2\,\bar{x}^2}$$

Using that $N = n\sigma^2/\bar{x}^2$ we get

$$\lambda_{m,k} = -\frac{n\sigma^2}{\bar{x}^2}\frac{\left(j_{\frac{n}{2}-1+m,k}\right)^2}{2n} = N\frac{\left(j_{\frac{n}{2}-1+m,k}\right)^2}{2n}$$

□

Proof. (of Proposition 11) We start with the projections for $z/n = f(w,x) = wx/\sqrt{n}$. We are looking for:

$$\begin{aligned} f(x,w) = wx/\sqrt{n} \sim \sum_{k=1}^{\infty} b_{1,k}[f]\varphi_{1,k}(x,w) &= \sum_{k=1}^{\infty} b_{1,k}[f] h_1(w) g_{1,k}(x) \\ &= \sum_{k=1}^{\infty} b_{1,k}[f]\, C_1^{\frac{n}{2}-1(w)}\, J_{\frac{n}{2}}\left(j_{\frac{n}{2},k}\frac{x}{\bar{x}}\right)\, x^{1-\frac{n}{2}} \\ &= w\, x^{1-\frac{n}{2}}\,(n-2)\sum_{k=1}^{\infty} b_{1,k}[f] J_{\frac{n}{2}}\left(j_{\frac{n}{2},k}\frac{x}{\bar{x}}\right) \end{aligned}$$

We can replace the expression we obtain below for $b_{1,k}[f]$ to get:

$$\begin{aligned} \sum_{k=1}^{\infty} b_{1,k}[f]\varphi_{1,k}(x,w) &= w\, x^{1-\frac{n}{2}}\,(n-2)\sum_{k=1}^{\infty}\frac{2\,\bar{x}^{\frac{n}{2}}}{\sqrt{n}(n-2)\, j_{\frac{n}{2},k}\, J_{\frac{n}{2}+1}\left(j_{\frac{n}{2},k}\right)} J_{\frac{n}{2}}\left(j_{\frac{n}{2},k}\frac{x}{\bar{x}}\right) \\ &= \frac{w\,x}{\sqrt{n}}\sum_{k=1}^{\infty}\frac{2\,(x/\bar{x})^{-\frac{n}{2}}\, J_{\frac{n}{2}}\left(j_{\frac{n}{2},k}\frac{x}{\bar{x}}\right)}{j_{\frac{n}{2},k}\, J_{\frac{n}{2}+1}\left(j_{\frac{n}{2},k}\right)} \end{aligned}$$

To get the coefficients we start by computing

$$\int_0^{\bar{x}} J_{\frac{n}{2}}\left(j_{\frac{n}{2},k}\frac{x}{\bar{x}}\right) x^{\frac{n}{2}+1} dx = \left(\frac{\bar{x}}{j_{\frac{n}{2},k}}\right)^{\frac{n}{2}+2}\int_0^{\bar{x}}\left(j_{\frac{n}{2},k}\frac{x}{\bar{x}}\right)^{\frac{n}{2}+1} J_{\frac{n}{2}}\left(j_{\frac{n}{2},k}\frac{x}{\bar{x}}\right) j_{\frac{n}{2},k}\frac{dx}{\bar{x}}$$

$$= \left(\frac{\bar{x}}{j_{\frac{n}{2},k}}\right)^{\frac{n}{2}+2}\int_0^{j_{\frac{n}{2},k}} (z)^{\frac{n}{2}+1} J_{\frac{n}{2}}(z)\, dz$$

Using that

$$\int_a^b z^{\nu+1} J_\nu(z) dz = z^{\nu+1} J_{\nu+1}(z)\Big|_a^b$$

then

$$\int_0^{\bar{x}} J_{\frac{n}{2}}\left(j_{\frac{n}{2},k}\frac{x}{\bar{x}}\right) x^{\frac{n}{2}+1} dx = \left(\frac{\bar{x}}{j_{\frac{n}{2},k}}\right)^{\frac{n}{2}+2} \int_0^{j_{\frac{n}{2},k}} (z)^{\frac{n}{2}+1} J_{\frac{n}{2}}(z)\, dz$$
$$= \left(\frac{\bar{x}}{j_{\frac{n}{2},k}}\right)^{\frac{n}{2}+2} \left(j_{\frac{n}{2},k}\right)^{\frac{n}{2}+1} J_{\frac{n}{2}+1}\left(j_{\frac{n}{2},k}\right)$$

Using that

$$\bar{x}^2 \int_0^{\bar{x}} \frac{x}{\bar{x}} \left[J_\nu\left(j_{\nu,k}\frac{x}{\bar{x}}\right)\right]^2 \frac{dx}{\bar{x}} = \frac{1}{2}\left(\bar{x}\, J_{\nu+1}\left(j_{\nu,k}\right)\right)^2 \text{ for all } k \in \{1, 2, 3 \ldots\}$$

we have

$$\int_0^{\bar{x}} \left(J_{\frac{n}{2}}\left(j_{\frac{n}{2},k}\frac{x}{\bar{x}}\right)\right)^2 x\, dx = \frac{1}{2}\left(\bar{x}\, J_{\frac{n}{2}+1}\left(j_{\frac{n}{2},k}\right)\right)^2$$

Thus:

$$b_{1,k}[f] = \frac{2}{\sqrt{n}(n-2)} \frac{\left(\frac{\bar{x}}{j_{\frac{n}{2},k}}\right)^{\frac{n}{2}+2} \left(j_{\frac{n}{2},k}\right)^{\frac{n}{2}+1} J_{\frac{n}{2}+1}\left(j_{\frac{n}{2},k}\right)}{\left(\bar{x}\, J_{\frac{n}{2}+1}\left(j_{\frac{n}{2},k}\right)\right)^2}$$
$$= \frac{2\,\bar{x}^{\frac{n}{2}}}{\sqrt{n}(n-2)\, j_{\frac{n}{2},k}\, J_{\frac{n}{2}+1}\left(j_{\frac{n}{2},k}\right)} \text{ for all } k \geq 1$$

Now we turn to compute: $b_{1,k}[\bar{p}'(\cdot,0)]\langle\varphi 1, k, \varphi 1, k\rangle$. We start deriving an explicit expression for $\bar{p}'(\cdot,0)$. We have

$$\bar{h}(w) = \frac{1}{Beta\left(\frac{n-1}{2}, \frac{1}{2}\right)} \left(1-w^2\right)^{(n-3)/2} \quad \text{for} \quad w \in (-1\,,\,1)$$
$$\bar{g}(x) = x\ (\bar{x})^{-n} \left(\frac{2n}{n-2}\right) \left[\bar{x}^{n-2} - x^{n-2}\right] \qquad \text{for } x \in [0, \bar{x}]$$

$$p(w,x;0) = \bar{h}(w(\delta))\bar{g}(x(\delta)) = \bar{h}(w)\bar{g}(x) + \bar{p}'(w,x;0)\delta + o(\delta) \text{ with}$$
$$\bar{p}'(w,x;0) = \bar{g}(x)\bar{h}'(w)w'(0) + \bar{h}(w)\bar{g}'(x)x'(0)$$

where:

$$\frac{\partial}{\partial \delta} x(\delta)|_{\delta=0} = x'(0) = \sqrt{n}\, w \quad \text{and} \quad \frac{\partial}{\partial \delta} \bar{h}(w(\delta))|_{\delta=0} = \bar{h}'(w) w'(0)$$
$$\frac{\partial}{\partial \delta} w(\delta)|_{\delta=0} = w'(0) = \frac{\sqrt{n}\,(1-w^2)}{x} \quad \text{and} \quad \frac{\partial}{\partial \delta} \bar{g}(x(\delta))|_{\delta=0} = \bar{g}'(x) x'(0)$$

so:

$$\begin{aligned}
\bar{p}'(w,z;0) &= \bar{g}(x)\bar{h}'(w)w'(0) + \bar{h}(w)\bar{g}'(x)x'(0) \\
&= -\left(\bar{x}\right)^{-n} \left(\frac{2n}{n-2}\right) \left[\bar{x}^{n-2} - x^{n-2}\right] \frac{(n-3)w\left(1-w^2\right)^{(n-3)/2}}{Beta\left(\frac{n-1}{2}, \frac{1}{2}\right)} \sqrt{n} \\
&+ \frac{w\left(1-w^2\right)^{(n-3)/2}}{Beta\left(\frac{n-1}{2}, \frac{1}{2}\right)}\, \bar{x}^{-n} \left[\left(\frac{2n}{n-2}\right)\left[\bar{x}^{n-2} - x^{n-2}\right] - 2nx^{n-2}\right] \sqrt{n} \\
&= \frac{w\left(1-w^2\right)^{(n-3)/2}}{Beta\left(\frac{n-1}{2}, \frac{1}{2}\right)} \frac{\sqrt{n}}{\bar{x}^n} \left(\frac{2n}{n-2}\right) \left[(4-n)\left(\bar{x}^{n-2} - x^{n-2}\right) - 2nx^{n-2}\right] \\
&= \frac{w\left(1-w^2\right)^{(n-3)/2}}{Beta\left(\frac{n-1}{2}, \frac{1}{2}\right)} \sqrt{n} \left(\frac{2n}{n-2}\right) \frac{\left[(4-n)\bar{x}^{n-2} - (4+n)x^{n-2}\right]}{\bar{x}^n}
\end{aligned}$$

We want to compute:

$$b_{1,k}[\bar{p}'(\cdot,0)/\omega]\langle \varphi_{1,k}, \varphi_{1,k}\rangle = \int_0^{\bar{x}} \int_{-1}^{1} \bar{p}'(x,w;0) h_{1,k}(x) g_{m,k}(w)\, dw dx$$

So we split the integral in the product of two terms. The first term involves the integral over w given by:

$$\begin{aligned}
&\int_{-1}^{1} (n-2)w\, \frac{w\left(1-w^2\right)^{(n-3)/2}}{Beta\left(\frac{n-1}{2}, \frac{1}{2}\right)} \sqrt{n} \left(\frac{2n}{n-2}\right) dw \\
&= \frac{2n\sqrt{n}}{Beta\left(\frac{n-1}{2}, \frac{1}{2}\right)} \int_{-1}^{1} w^2 \left(1-w^2\right)^{(n-3)/2} dw = \frac{2n\sqrt{n}}{Beta\left(\frac{n-1}{2}, \frac{1}{2}\right)} \frac{\sqrt{\pi}\,\Gamma(\frac{n-1}{2})}{2\,\Gamma\left(\frac{n}{2}+1\right)} \\
&= \frac{n\sqrt{n}\Gamma\left(\frac{n-1}{2}+\frac{1}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{1}{2}\right)} \frac{\sqrt{\pi}\,\Gamma(\frac{n-1}{2})}{\Gamma\left(\frac{n}{2}+1\right)} = n\sqrt{n}\, \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)} \frac{\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n}{2}+1\right)} = n\sqrt{n}\, \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n}{2}+1\right)}
\end{aligned}$$

where we use that $C_1^{\frac{n}{2}-1}(w) = (n-2)w$, and properties of the $Beta$ and Γ functions.

The second term involves the integral over x and is given by:

$$\begin{aligned}
&\frac{1}{\bar{x}^n}\int_0^{\bar{x}} \left[(4-n)\bar{x}^{n-2}-(4+n)x^{n-2}\right] J_{n/2}\left(j_{\frac{n}{2},k}\frac{x}{\bar{x}}\right) x^{1-\frac{n}{2}}\,dx \\
&= \frac{\bar{x}^{1-\frac{n}{2}}\bar{x}^{n-2}}{\bar{x}^n}\int_0^{\bar{x}} \left[(4-n)-(4+n)\left(\frac{x}{\bar{x}}\right)^{n-2}\right] J_{n/2}\left(j_{\frac{n}{2},k}\frac{x}{\bar{x}}\right)\left(\frac{x}{\bar{x}}\right)^{1-\frac{n}{2}}\,dx \\
&= \bar{x}^{-\frac{n}{2}}\int_0^{\bar{x}} \left[(4-n)-(4+n)\left(\frac{x}{\bar{x}}\right)^{n-2}\right] J_{n/2}\left(j_{\frac{n}{2},k}\frac{x}{\bar{x}}\right)\left(\frac{x}{\bar{x}}\right)^{1-\frac{n}{2}}\frac{dx}{\bar{x}} \\
&= \frac{\bar{x}^{-\frac{n}{2}}}{\left(j_{\frac{n}{2},k}\right)^{2-\frac{n}{2}}}(4-n)\int_0^{\bar{x}} J_{n/2}\left(j_{\frac{n}{2},k}\frac{x}{\bar{x}}\right)\left(\frac{j_{\frac{n}{2},k}x}{\bar{x}}\right)^{1-\frac{n}{2}}\frac{j_{\frac{n}{2},k}dx}{\bar{x}} \\
&- \frac{\bar{x}^{-\frac{n}{2}}}{\left(j_{\frac{n}{2},k}\right)^{\frac{n}{2}}}(4+n)\int_0^{\bar{x}} \left(\frac{j_{\frac{n}{2},k}x}{\bar{x}}\right)^{n-2} J_{n/2}\left(j_{\frac{n}{2},k}\frac{x}{\bar{x}}\right)\left(\frac{j_{\frac{n}{2},k}x}{\bar{x}}\right)^{1-\frac{n}{2}}\frac{j_{\frac{n}{2},k}dx}{\bar{x}} \\
&= \frac{\bar{x}^{-\frac{n}{2}}}{\left(j_{\frac{n}{2},k}\right)^{2-\frac{n}{2}}}(4-n)\int_0^{j_{\frac{n}{2},k}} z^{1-\frac{n}{2}}J_{\frac{n}{2}}(z)\,dz \\
&- \frac{\bar{x}^{-\frac{n}{2}}}{\left(j_{\frac{n}{2},k}\right)^{\frac{n}{2}}}(4+n)\int_0^{j_{\frac{n}{2},k}} z^{\frac{n}{2}-1}J_{\frac{n}{2}}(z)\,dz
\end{aligned}$$

To find an expression for this integrals note that:

$$\int_0^a z^{1-\frac{n}{2}}J_{\frac{n}{2}}(z)\,dz = -\frac{2^{1-n/2}\left(-1+{}_0F_1\left(n/2,-a^2/4\right)\right)}{\Gamma(n/2)} = \frac{2^{1-\frac{n}{2}}}{\Gamma(\frac{n}{2})} - a^{1-\frac{n}{2}}J_{\frac{n}{2}-1}(a)$$

and

$$\int_0^a z^{\frac{n}{2}-1}J_{\frac{n}{2}}(z)\,dz = 2^{-1-\frac{n}{2}}a^n\Gamma\left(\frac{n}{2}\right){}_1\tilde{F}_2\left(\frac{n}{2};1+\frac{n}{2},1+\frac{n}{2};-\frac{a^2}{4}\right)$$

where ${}_1\tilde{F}_2(a_1;b_1,b_2;z)$ is the regularized generalized hypergeometric function, i.e. it is defined as ${}_1\tilde{F}_2(a_1;b_1,b_2;z) = {}_1F_2(a_1;b_1,b_2;z)/\left(\Gamma(b_1)\Gamma(b_2)\right)$ where ${}_1F_2$ is the generalized hypergeometric function. Thus

$$\begin{aligned}
&\frac{1}{\bar{x}^n}\int_0^{\bar{x}} \left[(4-n)\bar{x}^{n-2}-(4+n)x^{n-2}\right] J_{n/2}\left(j_{\frac{n}{2},k}\frac{x}{\bar{x}}\right) x^{1-\frac{n}{2}}\,dx \\
&= \bar{x}^{-\frac{n}{2}}\left[\frac{(4-n)}{\left(j_{\frac{n}{2},k}\right)^{2-\frac{n}{2}}}\left(\frac{2^{1-\frac{n}{2}}}{\Gamma(\frac{n}{2})} - (j_{\frac{n}{2},k})^{1-\frac{n}{2}}J_{\frac{n}{2}-1}\left(j_{\frac{n}{2},k}\right)\right)\right. \\
&\left. -\frac{(4+n)}{\left(j_{\frac{n}{2},k}\right)^{\frac{n}{2}}}2^{-1-\frac{n}{2}}(j_{\frac{n}{2},k})^n\Gamma\left(\frac{n}{2}\right){}_1\tilde{F}_2\left(\frac{n}{2};1+\frac{n}{2},1+\frac{n}{2};-\frac{(j_{\frac{n}{2},k})^2}{4}\right)\right]
\end{aligned}$$

Thus we have:

$$
\begin{aligned}
&b_{1,k}[f]b_{1,k}[\bar{p}'(\cdot,0)]\langle\varphi_{1,k},\varphi_{1,k}\rangle \\
&= n\sqrt{n}\,\frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n}{2}+1\right)}\frac{2\,\bar{x}^{\frac{n}{2}}}{\sqrt{n}(n-2)\,j_{\frac{n}{2},k}\,J_{\frac{n}{2}+1}\left(j_{\frac{n}{2},k}\right)} \\
&\bar{x}^{-\frac{n}{2}}\left[\frac{(4-n)}{\left(j_{\frac{n}{2},k}\right)^{2-\frac{n}{2}}}\left(\frac{2^{1-\frac{n}{2}}}{\Gamma(\frac{n}{2})}-(j_{\frac{n}{2},k})^{1-\frac{n}{2}}J_{\frac{n}{2}-1}\left(j_{\frac{n}{2},k}\right)\right)\right. \\
&\left.-\frac{(4+n)}{\left(j_{\frac{n}{2},k}\right)^{\frac{n}{2}}}2^{-1-\frac{n}{2}}(j_{\frac{n}{2},k})^{n}\Gamma\left(\frac{n}{2}\right){}_1\tilde{F}_2\left(\frac{n}{2};1+\frac{n}{2},1+\frac{n}{2};-\frac{(j_{\frac{n}{2},k})^2}{4}\right)\right] \\
&= \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n}{2}+1\right)}\frac{2n}{(n-2)\,j_{\frac{n}{2},k}\,J_{\frac{n}{2}+1}\left(j_{\frac{n}{2},k}\right)} \\
&\left[\frac{(4-n)}{\left(j_{\frac{n}{2},k}\right)^{2-\frac{n}{2}}}\left(\frac{2^{1-\frac{n}{2}}}{\Gamma(\frac{n}{2})}-(j_{\frac{n}{2},k})^{1-\frac{n}{2}}J_{\frac{n}{2}-1}\left(j_{\frac{n}{2},k}\right)\right)\right. \\
&\left.-\frac{(4+n)}{\left(j_{\frac{n}{2},k}\right)^{\frac{n}{2}}}2^{-1-\frac{n}{2}}(j_{\frac{n}{2},k})^{n}\Gamma\left(\frac{n}{2}\right){}_1\tilde{F}_2\left(\frac{n}{2};1+\frac{n}{2},1+\frac{n}{2};-\frac{(j_{\frac{n}{2},k})^2}{4}\right)\right] \\
&= \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n}{2}+1\right)}\frac{2n}{(n-2)\,j_{\frac{n}{2},k}\,J_{\frac{n}{2}+1}\left(j_{\frac{n}{2},k}\right)} \\
&\left[(4-n)\left(\frac{2^{1-\frac{n}{2}}}{\Gamma(\frac{n}{2})\left(j_{\frac{n}{2},k}\right)^{2-\frac{n}{2}}}-\frac{J_{\frac{n}{2}-1}\left(j_{\frac{n}{2},k}\right)}{j_{\frac{n}{2},k}}\right)\right. \\
&\left.-(4+n)2^{-1-\frac{n}{2}}(j_{\frac{n}{2},k})^{\frac{n}{2}}\Gamma\left(\frac{n}{2}\right){}_1\tilde{F}_2\left(\frac{n}{2};1+\frac{n}{2},1+\frac{n}{2};-\frac{(j_{\frac{n}{2},k})^2}{4}\right)\right]
\end{aligned}
$$

□

E Asymmetric Problems (Dealing with Reinjection)

In this section we study the impulse response function for problems where the symmetry assumptions of Proposition 1 do not hold. In such a case the computation of the impulse response function requires keeping track of firms after their first adjustment, so that the impulse response function $H(t)$ cannot be computed by means of the simpler operator $G(t)$. The solution to this problem is to compute the law of motion of the cross-sectional distribution using the Kolmogorov forward equation, keeping track of the reinjections that occur after the adjustments.

The nature of reinjection in our set up differs from the one in Gabaix et al. (2016) and hence we cannot use their results for the ergodic case. The added complexity of our case originates because the exit points (of our pricing problem) are not independent of where x is, as in the case of poisson adjustments. Rather, prices are changed when either barrier $\underline{x}$ or $\bar{x}$ is hit, and then the measure of products whose prices are changed are all reinjected at single value, the optimal return point x^*.

The set-up consists of an unregulated BM $dx = \mu dt + \sigma dB$, which returns to a single point x^* the first time that x hits either of the barriers $\underline{x}$ or $\bar{x}$ or that a Poisson counter with intensity ζ changes. As implied by Proposition 1 we cannot ignore the reinjections at x^* if either $x^* \neq (\bar{x} + \underline{x})/2$ or $\mu \neq 0$. For simplicity consider the case with no drift, so we set $\mu = 0$. We can use the method in Appendix **??** to modify the result accordingly. This set up can be used to study the problem of a firm with a non-symmetric period return function in an economy without inflation ($\mu = 0$). In this case the optimal decision rule implies $x^* \neq (\bar{x} + \underline{x})/2$, i.e. the reinjection point (after adjustment) is not located in the middle of the inaction region. Note that the number of price adjustment per unit of time is given by $N = \frac{\sigma^2}{(\bar{x}-\underline{x})^2\alpha(1-\alpha)}$.

Let $\hat{p}(x)$ denote the initial condition for the density of firms relative the invariant distribution $\bar{p}(x)$, i.e. $\hat{p}(x) = p(x) - \bar{p}(x)$ for some density p, where $\bar{p}$ is the asymmetric (steady state) tent map. Notice that to analyze small shocks δ, i.e. an initial condition $p(x) = \bar{p}(x + \delta)$ the signed measure is $\hat{p}(x) = \delta \bar{p}'(x)$ by a simple Taylor expansion and mass preservation requires that $\int_{\underline{x}}^{\bar{x}} \hat{p}(x)dx = 0$, so that

$$\hat{p}(x)/\delta = \begin{cases} \frac{2}{(\bar{x}-\underline{x})^2\alpha} & \text{if } x \in [\underline{x}, x^*) \\ \frac{-2}{(\bar{x}-\underline{x})^2(1-\alpha)} & \text{if } x \in (x^*, \bar{x}]\,. \end{cases} \tag{A.42}$$

We define the Kolmogorov forward operator $\mathcal{H}^*(\hat{p}) : [\underline{x}, \bar{x}] \times \mathbb{R}_+ \to \mathbb{R}$ for the process with reinjection, where $\mathcal{H}^*(\hat{p})(x, t)$ denotes the cross-sectional density of the firms t periods after the shock. In this case we have that for all $x \in [\underline{x}, x^*) \cup (x^*, \bar{x}]$ and for all $t > 0$:

$$\partial_t \mathcal{H}^*(\hat{p})(x, t) = \frac{\sigma^2}{2} \partial_{xx} \mathcal{H}^*(\hat{p})(x, t) - \zeta \mathcal{H}^*(\hat{p})(x, t) \tag{A.43}$$

with boundary conditions:

$$\mathcal{H}^*(\hat{p})(\underline{x}, t) = \mathcal{H}^*(\hat{p})(\bar{x}, t) = 0 \quad , \quad \lim_{x \uparrow x^*} \mathcal{H}^*(\hat{p})(x, t) = \lim_{x \downarrow x^*} \mathcal{H}^*(\hat{p})(x, t) \tag{A.44}$$

$$\partial_x^- \mathcal{H}^*(\hat{p})(\bar{x}, t) - \partial_x^+ \mathcal{H}^*(\hat{p})(x^*, t) + \partial_x^- \mathcal{H}^*(\hat{p})(x^*, t) - \partial_x^+ \mathcal{H}^*(\hat{p})(\underline{x}, t) = \frac{2\zeta}{\sigma^2} \tag{A.45}$$

$$\mathcal{H}^*(\hat{p})(x, 0) = \hat{p}(x) \text{ for all } x \in [\underline{x}, \bar{x}] \tag{A.46}$$

The p.d.e. in equation (A.43) is standard, we just note that it does not need to hold at the reinjection point x^*. The boundary conditions in equation (A.44) are also standard, given that $\underline{x}$ and $\bar{x}$ are exit points, and that with $\sigma^2 > 0$, the density must be continuous everywhere. The condition in equation (A.45) ensures that the measure is preserved, or equivalently that there is no change in total mass across time: $\int_{\underline{x}}^{\bar{x}} \mathcal{H}^*(\hat{p})(x, t)dx = \int_{\underline{x}}^{\bar{x}} \hat{p}(x)dx$ for all t. This is a small extension of Proposition 1 in Caballero (1993).

We can use $\mathcal{H}^*$ to compute the Impulse response function defined above as follows:

$$H(t, f, \hat{p}) = \int_{\underline{x}}^{\bar{x}} f(x)\, \mathcal{H}^*(\hat{p})(x, t)\, dx\,. \tag{A.47}$$

If $\zeta > 0$ and $\bar{x} \to \infty$ as well as $\underline{x} = \infty$, we will have the pure Calvo case, and we can use the ideas in Gabaix et al. (2016), and thus the case without reinjection and with reinjection are quite similar. To highlight the difference, we consider the opposite case, and set $\zeta = 0$ and use an eigenvalue decomposition of $\mathcal{H}^*$.

Proposition 12. Assume that $\zeta = 0$ and that α is not a rational number. The orthonormal eigenfunctions of $\mathcal{H}^*$ are:

$$\varphi_j^m(x) = \sqrt{\frac{2}{(\bar{x} - \underline{x})}} \sin\left(\left[\frac{x - \underline{x}}{\bar{x} - \underline{x}}\right] 2\pi j\right) \text{ if } x \in [\underline{x}, \bar{x}] \tag{A.48}$$

$$\varphi_j^l(x) = \sqrt{\frac{2}{(x^* - \underline{x})}} \sin\left(\left[\frac{x - \underline{x}}{x^* - \underline{x}}\right] 2\pi j\right) \text{ if } x \in [\underline{x}, x^*] \text{ and } 0 \text{ otherwise} \tag{A.49}$$

$$\varphi_j^h(x) = \sqrt{\frac{2}{(\bar{x} - x^*)}} \sin\left(\left[\frac{x - x^*}{\bar{x} - x^*}\right] 2\pi j\right) \text{ if } x \in [x^*, \bar{x}] \text{ and } 0 \text{ otherwise} \quad , \tag{A.50}$$

with corresponding eigenvalues:

$$\lambda_j^m = -\frac{\sigma^2}{2}\frac{(2\pi j)^2}{(\bar{x} - \underline{x})^2} \ , \ \lambda_j^l = -\frac{\sigma^2}{2}\frac{(2\pi j)^2}{(x^* - \underline{x})^2}, \text{ and } \lambda_j^h = -\frac{\sigma^2}{2}\frac{(2\pi j)^2}{(\bar{x} - x^*)^2} \ , \tag{A.51}$$

for all $j = 1, 2, \ldots$. The eigenfunctions in the set $\{\varphi_j^m\}_{j=1}^{\infty}$ are orthogonal to each other, and so are those in the set $\{\varphi_j^l, \varphi_j^h\}_{j=1}^{\infty}$. The eigenfunctions $\{\varphi_j^m, \varphi_j^l, \varphi_j^h\}_{j=1}^{\infty}$ span the set of functions $g : [\underline{x}, \bar{x}] \to \mathbb{R}$, piecewise differentiable, with countably many discontinuities, and with $\int_{\underline{x}}^{\bar{x}} g(x)dx = 0$.

Proof. (of Proposition 12) The proof consists on checking that the functions described are the only ones that satisfy the sufficient conditions equation (A.43), equation (A.44), equation (A.45) and equation (A.46) for $\mathcal{H}^*(\varphi_j^k(x)e^{\lambda_j^k t})$ for $k \in l, m, j$ and $j = 1, 2, \ldots$.

First, let's consider the case of eigenvalues $\lambda \neq 0$. In this case the only non-constant real function that satisfy the o.d.e.: $\lambda\varphi(x) = \partial_{xx}\varphi(x)\sigma^2/2$ for $\lambda < 0$ is $\varphi(x) = \sin(\phi + \omega x)$ for some ϕ and for $\lambda = -\frac{\sigma^2}{2}\omega^2$. The cases below use this characterization when the function is not constant, to determine the values of ϕ and ω.

Second, consider the case of functions that are differentiable in the entire domain $[\underline{x}, \bar{x}]$. The continuity at $x = x^*$ is satisfied immediately. In this case, the o.d.e. : $\lambda\varphi(x) = \partial_{xx}\varphi(x)\sigma^2/2$, with boundaries equation (A.44) and equation (A.45) is satisfied only by $\varphi(x) = \varphi_j^m(x)$ for all $j = 1, 2, ...$ This gives the particular value of ϕ and ω, for $j = 1, 2, \ldots$, and hence no other differentiable function different from zero satisfy all the conditions.

Third, consider the case of functions $\varphi(x)$ which are constant in an interval of strictly positive length included in $[\underline{x}, x^*]$. Then $\varphi(x) = 0$, to satisfy the boundary condition equation (A.44) at $x = \underline{x}$. Then $\varphi(x) = 0$ for all $x \in [\underline{x}, x^*)$, since φ can only be non-differentiable at $x = x^*$. Then, for $x \in (x^*, \bar{x}]$ it has to be differentiable, non-identically equal to zero, satisfy $\varphi(x^*) = 0$ so that it is continuous at $x = x^*$, and also $\varphi(\bar{x}) = 0$, to satisfy the boundary condition equation (A.44) at $x = \bar{x}$. Finally, to satisfy the measure preserving condition equation (A.45), it has to be of the form of $\varphi_j^l(x)$ for $j = 1, 2, \ldots$.

Fourth, consider the case of functions $\varphi(x)$ which are constant in an interval of strictly positive length included in $[x^*, \bar{x}]$. Following the same steps as in the previous case, we obtain that $\varphi(x) = \varphi^h_j(x)$ for $j = 1, 2, \ldots$ for this case.

For the fifth and remaining case, we consider the case of functions $\varphi(x)$ which are non-constant for all intervals included in $[\underline{x}, \bar{]}$, and that $\varphi(x)$ is not differentiable at $x = x^*$. To satisfy the o.d.e. in each segment $[\underline{x}, x^*)$ and $(x^*, \bar{x}]$ then we must have $\varphi(x) = \sin\left(\underline{\phi} + \underline{\omega}x\right)$ and $\varphi(x) = \sin\left(\bar{\phi} + \bar{\omega}x\right)$ in each of the respective segments. Since the eigenvalue has to be the same for all segments, then we have that $\underline{\omega} = \bar{\omega} \equiv \omega$. The eigenfunction φ must be measure preserving, so that

$$0 = \cos\left(\underline{\phi} + \omega x^*\right) - \cos\left(\underline{\phi} + \omega \underline{x}\right) + \cos\left(\bar{\phi} + \omega \bar{x}\right) - \cos\left(\bar{\phi} + \omega x^*\right)$$

To satisfy the boundary conditions equation (A.44) we require $\sin\left(\underline{\phi} + \omega \underline{x}\right) = \sin\left(\bar{\phi} + \omega \bar{x}\right) = 0$. Thus $\cos\left(\underline{\phi} + \omega \underline{x}\right) = \pm 1$ and $\cos\left(\bar{\phi} + \omega \bar{x}\right) = \pm 1$. Hence, we have that:

$$\text{either } 0 = \cos\left(\underline{\phi} + \omega x^*\right) - \cos\left(\bar{\phi} + \omega x^*\right) \;\text{ or }\; \pm 2 = \cos\left(\underline{\phi} + \omega x^*\right) - \cos\left(\bar{\phi} + \omega x^*\right)$$

In the first case we have:

$$0 = \cos\left(\underline{\phi} + \omega x^*\right) - \cos\left(\bar{\phi} + \omega x^*\right) \;\text{ and } 0 = \sin\left(\underline{\phi} + \omega x^*\right) - \sin\left(\bar{\phi} + \omega x^*\right)$$

so the function is differentiable at $x = x^*$, which is a contradiction. So we must have the second case, and because eigenfunctions are defined up to sign, must have:

$$2 = \cos\left(\underline{\phi} + \omega x^*\right) - \cos\left(\bar{\phi} + \omega x^*\right) \;\text{ and } 0 = \sin\left(\underline{\phi} + \omega x^*\right) - \sin\left(\bar{\phi} + \omega x^*\right)$$

Using the properties of cos it must be the case that $\sin\left(\underline{\phi} + \omega x^*\right) = \sin\left(\bar{\phi} + \omega x^*\right) = 0$. Then, φ must be zero in the extremes of each of the two following segment $[\underline{x}, x^*]$ and $[x^*, \bar{x}]$. This requires that $[\underline{x}, x^*]$ and $[x^*, \bar{x}]$ be an in an multiple integer of each other, since in each of the segments φ is a sine function with the same frequency ω which is zero at the two extremes. But this violate that $[\underline{x}, x^*]/[x^*, \bar{x}]$ is not rational.

Now we show that the eigenfunctions span the densities for the signed measures. It suffices to show that if $g : [\underline{x}, \bar{x}] \to \mathbb{R}$ is in the domain of $\mathcal{H}^*$, and $\langle g, \varphi^m_j \rangle = \langle g, \varphi^l_j \rangle = \langle g, \varphi^h_j \rangle = 0$ for all $j = 1, 2, \ldots$, then it must be that $g = 0$.

As a way of contradiction, suppose we have a function $g \neq 0$ that g is orthogonal to all the eigenfunctions. Given that the eigenfunctions can span antisymmetric functions defined in different domains as explained above, it must be that g is a symmetric function as defined in $[\underline{x}, \bar{x}]$, so that it is orthogonal to $\{\varphi^m_j\}_{j=1}^\infty$, and also a symmetric in the following restricted domains $[\underline{x}, x^*]$ and $[x^*, \bar{x}]$, so that g is also orthogonal each of eigenfunctions $\{\varphi^l_j\}_{j=1}^\infty$ and $\{\varphi^h_j\}_{j=1}^\infty$ when defined in the restricted domains.

Now, without loss of generality, assume that $x^* < (\underline{x} + \bar{x})/2$. Below we sketch a proof that for a function g to be even (or symmetric) in these three domains, it must be the case that $[\bar{x} - x^*]$ is an integer multiple of $[x^* - \underline{x}]$, which contradicts the assumption that $[x^* - \underline{x}]/[\bar{x} - x^*]$ is not a rational number.

Let $L = x^* - \underline{x}$. To arrive to this conclusion we first notice that since g must be symmetric in the entire domain $[\underline{x}, \bar{x}]$, then it must be the case that g has identical symmetric shape in the segment $[\underline{x}, \underline{x}+L]$ than in the segment $[\bar{x}-L, \bar{x}]$. Then using that g is symmetric in the restricted domain $[x^*, \bar{x}]$, it must be that it also has the same symmetric shape in the interval $[x^*, x^*+L]$ than in both intervals $[\underline{x}, \underline{x}+L]$ and $[\bar{x}-L, \bar{x}]$. If it is the case that $x^*+L = \bar{x}-L$, then $[\bar{x}-x^*]$ is an integer multiple of $[x^*-\underline{x}]$, and find a contradiction. If this is not the case, i.e. if $x^*+L < \bar{x}-L$, we use the g is symmetric in the entire domain, to say that again g must take the same symmetric shape in the interval $[\bar{x}-2L, \bar{x}-L]$. Now either $x*+L = \bar{x}-2L$, which gives a contradiction, or we continue using the symmetry of g in either the entire domain $[\underline{x}, \bar{x}]$ or in the restricted domain $[x^*, \bar{x}]$ until we get that $[\bar{x}-x^*]$ is an integer multiple of $[x^*-\underline{x}]$, which is a contradiction with our assumption. Formally, this can be set up as an induction step, but it requires to develop enough notation, which we skip to shorten. □

By defining $\mathcal{H}^*$ for initial conditions given by the differences of a density relative to the density of the invariant distribution, we are excluding the zero eigenvalue and its corresponding eigenfunction, the invariant distribution $\bar{p}$ from the its representation. From the proposition we see what are the first two non-zero eigenvalues.

$$\lambda_1 = -\frac{\sigma^2}{2}\left(\frac{2\pi}{\bar{x}-\underline{x}}\right)^2 > \lambda_2 = -\frac{\sigma^2}{2}\left(\frac{2\pi}{\max\left\{(\bar{x}-x^*),(x^*-\underline{x})\right\}}\right)^2 \tag{A.52}$$

Notice that the difference between the first and the second eigenvalues depends on the asymmetry of the bands.

The proposition also proves that the eigenfunctions φ_j^k with $k = \{l, h, m\}$ form a base, so that projecting the initial condition onto them is possible. It is however more involved than in the symmetric case since the eigenfunctions are not all orthogonal with each other, e.g. $\langle \varphi_j^m, \varphi_j^h \rangle \neq 0$. Note however that $\{\varphi_j^m, \lambda_j^m\}$ coincide withe the antisymmetric eigenfunction and eigenvalues for the case without reinjection and $\mu = \zeta = 0$. Because of this, any piecewise differentiable function $g : [\underline{x}, \bar{x}] :\to \mathbb{R}$ that is antisymmetric around $(\underline{x}+\bar{x})/2$ can be represented, in a L^2 sense, as a Fourier series using $\{\varphi_j^m\}$.

In spite of the lack of orthogonality, the general logic for constructing the impulse response function is the same. Given the projection of the initial condition on the eigenfunctions

$$\hat{p}(x,0) = \sum_k \sum_{j=1}^{\infty} a_j^k \varphi_j^k(x)$$

where $k = \{l, h, m\}$. We use the linearity of $\mathcal{H}^*$ to write the operator in equation (A.47) as

$$\mathcal{H}^*(\hat{p})(x,t) = \mathcal{H}^*\left(\sum_k \sum_{j=1}^{\infty} a_j^k \varphi_j^k\right)(x,t) = \sum_k \sum_{j=1}^{\infty} a_j^k \mathcal{H}^*(\varphi_j^k)(x,t) = \sum_k \sum_{j=1}^{\infty} a_j^k e^{\lambda_j^k t} \varphi_j^k(x)$$

where the last equality uses that the $\varphi_j^k(x)$ are eigenfunctions. Thus, given the a_j^k coefficients (whose computation is discussed below), we can write the impulse response in equation (A.47) as

$$H(t, f, \hat{p}) = \sum_{\{k=l,h,m\}} \sum_{j=1}^{\infty} e^{\lambda_j^k t}\, a_j^k \int_{\underline{x}}^{\bar{x}} f(x)\, \varphi_j^k(x)\, dx\,.$$

or, computing the inner products $b_j^k[f] = \int_{\underline{x}}^{\bar{x}} f(x)\, \varphi_j^k(x)\, dx$

$$H(t, f, \hat{p}) = \sum_{k=\{l,h,m\}} \sum_{j=1}^{\infty} e^{\lambda_j^k t}\, a_j^k\, b_j^k[f]\,. \tag{A.53}$$

A straightforward numerical approach to finding the projection coefficients a_j^k requires running a simple linear regression of $\hat{p}(x, 0)$ on the basis $\{\varphi_j^h(x), \varphi_j^l(x), \varphi_j^m(x)\}_{j=1}^{J}$ (up to some order frequency J). For the output impulse response, given the function of interest $f(x) = -x$, the projection coefficients are also readily computed $b_j^k[f] = \int_{\underline{x}}^{\bar{x}} f(x)\varphi_j^k(x)dx$ for $k = \{m, l, h\}$, and $j = 1, 2, 3....$ which gives

$$b_j^m[f] = \frac{(\bar{x} - \underline{x})^{3/2}}{\sqrt{2}\pi j} \quad , \quad b_j^l[f] = \frac{(x^* - \underline{x})^{3/2}}{\sqrt{2}\pi j} \quad , \quad b_j^h[f] = \frac{(\bar{x} - x^*)^{3/2}}{\sqrt{2}\pi j} \quad . \tag{A.54}$$

Figure 5: Response to monetary shock for asymmetric problem

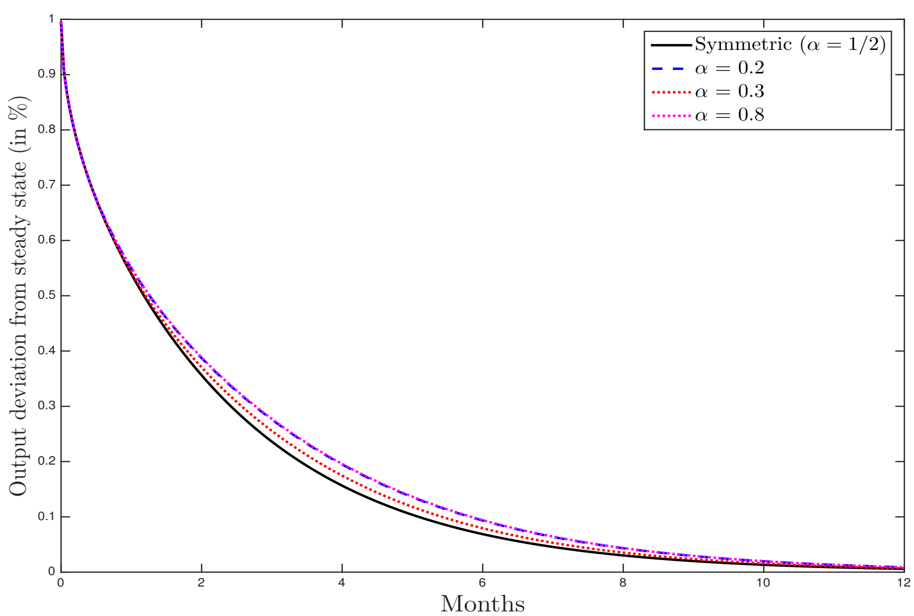


Figure 5 displays some impulse response generated by asymmetric problems where $\alpha \neq 1/2$ and contrasts them to the one produced by the symmetric problem where $\alpha = 1/2$. Two remarks are in order: first, modest degrees of asymmetry do not have a major effect on the impulse response: the impulse response function for $\alpha = 0.4$ would be barely distinguishable from the symmetric impulse response. Second, once quantitatively large asymmetries are considered, such as the small values of α considered in the figure, the impulse response becomes more persistent than the symmetric one. The presence of the asymmetry makes the convergence to the mean x value of the invariant distribution slower; this is intuitive since for symmetric problems the mean of the distribution is obtained right after the first adjustment, while this is not anymore true.

Irrelevance of the sign of the reinjection point. We discuss here the irrelevance of whether the optimal return point x^* is to the left of the interval's midpoint, as when $x^* < 0$, or to the right of it hence with $x^* > 0$. Formally we consider two problems: the first one has $\alpha = 1/2 - z$ where $z \in (0, 1/2)$ and the second problem has $\tilde{\alpha} = 1/2 + z$. We will show that, somewhat surprisingly to us, the sign of the optimal return point x^* is irrelevant for the impulse response which is the same one for the problem with α and for the one with $\tilde{\alpha}$. We have the following result

Proposition 13. Consider the inaction region for x defined by the interval $(-\bar{x}, \bar{x})$, let $z \in (0, 1/2)$ be a non-rational number. Consider a problem with reinjection point $\alpha = 1/2 - z$ and another problem with reinjection point $\tilde{\alpha} = 1/2 + z$. Then the impulse response function is the same for both problems.

Proof. of Proposition 13. Consider the first problem with $\alpha < 1/2$. Normalize (WLOG) the interval width to $2\bar{x} = 1$ and rewrite the initial condition as $\hat{p} = \hat{p}^s + \hat{p}^a$, respectively the symmetric and antisymmetric component as

$$\hat{p}^s(x) = \begin{cases} \frac{1-2\alpha}{\alpha(1-\alpha)} \\ \frac{-1}{(1-\alpha)} \end{cases} , \quad \hat{p}^a(x) = \begin{cases} \frac{1}{\alpha(1-\alpha)} & \text{for} \quad x \in (-\bar{x}, -z) \cup (z, \bar{x}) \\ 0 & \text{for } x \in (-z, z) \end{cases}$$

Notice that for $\alpha = 1/2 - z$ the slope of the antisymmetric part is either zero or $\frac{1}{1/4 - z^2}$. The same obtains for $\tilde{\alpha} = 1/2 + z$. Thus the asymmetric component of the initial condition $\hat{p}^a(x)$ is the same for α and for $\tilde{\alpha}$. The symmetric component $\hat{p}^s(x)$ is as follows

$$\hat{p}^s(x, \alpha) = \begin{cases} \frac{2z}{1/4 - z^2} \\ \frac{-1}{1/2 + z} \end{cases} , \quad \hat{p}^s(x, \tilde{\alpha}) = \begin{cases} \frac{-2z}{1/4 - z^2} & \text{for} \quad x \in (-\bar{x}, -z) \cup (z, \bar{x}) \\ \frac{1}{1/2 + z} & \text{for } x \in (-z, z) \end{cases}$$

which reveals that the symmetric component of the initial condition for the problem with $\tilde{\alpha}$ is given by -1 times the symmetric component of the initial condition for the problem with α.

Now let's consider the consequences for the output impulse response as defined in equation (A.47). For the problem with α we use the decomposition $\hat{p} = \hat{p}^s + \hat{p}^a$ and the linearity of $\mathcal{H}^*$ to write the IRF as

$$H_\alpha(t, f, \hat{p}(\alpha)) = H_\alpha(t, f, \hat{p}^a(\alpha)) + H_\alpha(t, f, \hat{p}^s(\alpha))$$

where we use the subscript to emphasize that this is the impulse response for the problem with reinjection point α. Using the properties for the initial condition associated to the problem with $\tilde{\alpha}$ discussed above we can the write its impulse response as

$$H_{\tilde{\alpha}}(t, f, \hat{p}(\tilde{\alpha})) = H_{\tilde{\alpha}}(t, f, \hat{p}^a(\alpha)) + H_{\tilde{\alpha}}(t, f, -\hat{p}^s(\alpha))$$

where we used that $\hat{p}^a(\alpha) = \hat{p}^a(\tilde{\alpha})$ and that $\hat{p}^s(\alpha) = -\hat{p}^s(\tilde{\alpha})$.

It is immediate to see that $H_\alpha(t, f, \hat{p}^a(\alpha)) = H_{\tilde{\alpha}}(t, f, \hat{p}^a(\alpha))$, i.e. that the IRF component triggered by the asymmetric part of the initial condition, is the same in both problems. This follows since $\hat{p}^a(\alpha) = \hat{p}^a(\tilde{\alpha})$ and because both problems share the same identical base for asymmetric functions, given by the eigenfunctions φ_j^m.

Finally, we argue that $H_\alpha(t, f, \hat{p}^s(\alpha)) = H_{\tilde{\alpha}}(t, f, -\hat{p}^s(\alpha))$. To see this notice that the symmetric part of the impulse response function is obtained by projecting the initial condition on the orthogonalized symmetric eigenfunctions v_j^k, where $k = \{l, h\}$, produced by e.g. the Gram-Schmidt algorithm. The key is to notice that the symmetrized eigenfunction for the problem with $\tilde{\alpha}$, equals -1 times the eigenfunctions for the problem with α, formally $v_j^k(\alpha) = -v_j^k(\tilde{\alpha})$. Inspection if the eigenfunctions φ_j^h and φ_j^l reveals that, for all $x \in (-\bar{x}, \bar{x})$ they obey $\varphi_1^h(x; x^* = -z) = -\varphi_1^l(-x; x^* = z)$. It therefore follows that $H_\alpha(t, f, \hat{p}(\alpha)) = H_{\tilde{\alpha}}(t, f, \hat{p}(\tilde{\alpha}))$. □

Figure 5 illustrates the results of the proposition by showing that the impulse response for $\alpha = 0.2$ coincides with the one for $\alpha = 0.8$. An important implication of this property is that the derivative of the impulse response function with respect to α evaluated at $alpha = 1/2$ must be zero, which explains why small deviations from the symmetric benchmark produce results that are essentially almost indistinguishable from those produced by the symmetric case. Overall this result suggests that the symmetric benchmark is an accurate approximation of problems with modest degrees of asymmetry.

F Additional results on the Calvo plus model

Next we discuss whether it is possible to approximate the impulse response function in a parsimonious way, a question that is naturally related to the shape of the impulse response. A natural candidate would be to analyze the impulse response associated to the leading eigenvalue as defined in Section 4.2, namely the largest eigenvalue associated with non-zero projection coefficient b_j in equation (14), for a case in which the IRF is close to exponential. We analyze this question by focusing on a small monetary shock that causes a marginal displacement of the invariant distribution. We assume a symmetric problem and present results for the baseline Calvo$^+$ model as well as for a model with price plans.

The next proposition gives a characterization of the ratio between the true area under the output impulse response and the approximate one, computed using only the leading eigenvalue:

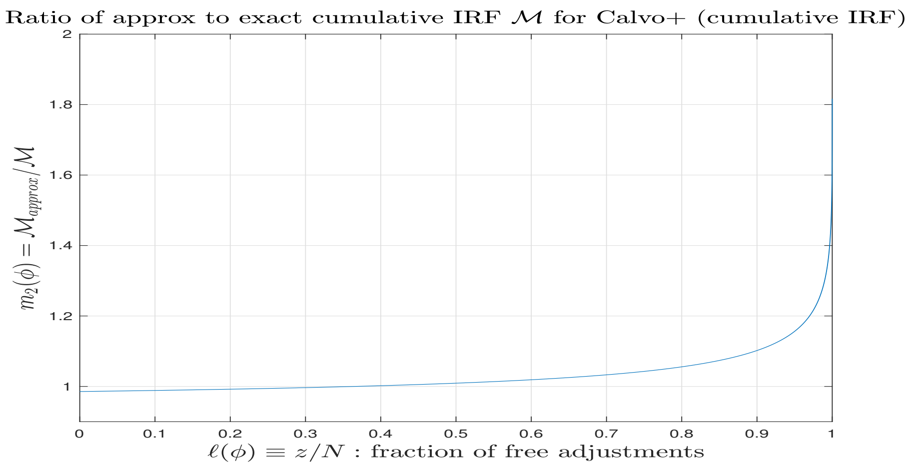
Proposition 14. Consider the marginal impulse response for output, so that $f(x) = -x$ and $\hat{p} = \delta \bar{p}'$. Define the ratio of the approximate cumulative impulse response based on the leading eigenvalue relative to the area under the impulse response as:

$$m_2(\phi) = \frac{\beta_2(\phi)/\lambda_2(\phi)}{\sum_{j=1}^{\infty} \beta_j(\phi)/\lambda_j(\phi)} = 2 \frac{\left[1 + \cosh\left(\sqrt{2\phi}\right)\right]}{\left[\cosh(\sqrt{2\phi}) - 1 - \phi\right]\left[1 + \frac{\pi^2}{2\phi}\right]^2}$$

We note that $m_2(0) = \frac{16}{\pi^4} 6 \approx 0.98$, $m_2'(\phi) > 0$ and $m_2(\phi) \to 2$ as $\phi \to \infty$.

Proposition 14 shows that the leading eigenvalue provides an accurate approximation of the total cumulative IRF for most variants of the calvo-plus model. At values of $\ell \approx 0.7$ the approximate function is close to 95% of the true effect. Figure 6 shows that accuracy degenerates as the model converges towards a pure Calvo model $\ell \to 1$.

Figure 6: Calvo-plus model



We can also use the expression for the coefficients of the impulse response to show that the slope of Y at $t = 0$ is minus infinity. This is intuitive since, after the shock, there are firms that are just on the boundary of the inaction region where they will increase prices, but there are no firms at the boundary at which they want to decrease prices.

Proposition 15. The derivative of the IRF with respect to t at $t = 0$ is given by:

$$\frac{\partial}{\partial t} Y(t)\Big|_{t=0} = -\infty \quad \text{for} \quad 0 \leq \phi < \infty.$$

Note that when $\phi \to \infty$, so we get the pure Calvo model, then the impulse response becomes $Y(t) = \exp(-Nt)$, and thus $Y'(0)$ is finite.

F.1 Proofs

Proof. (of Proposition 14) Rewriting the expression for m_2:

$$\begin{aligned}
m_2(\phi) &= \frac{\beta_2(\phi)/\lambda_2(\phi)}{\sum_{j=1}^{\infty} \beta_j(\phi)/\lambda_j(\phi)} = \frac{\beta_2(\phi)/\lambda_2(\phi)}{Kurt(\phi)/(6\,N)} \\
&= \frac{\left[\frac{1+\cosh\left(\sqrt{2\phi}\right)}{\cosh(\sqrt{2\phi})-1}\right]\left[\frac{8(2\phi)}{4(2\phi)+4\pi^2}\right]}{N\,\ell\left(\sqrt{2\phi}\right)\left[1+\frac{\pi^2}{2\phi}\right]} \frac{N\left(\exp(\sqrt{2\phi})+\exp(-\sqrt{2\phi})-2\right)^2}{(\exp(\sqrt{2\phi})+\exp(-\sqrt{2\phi}))(\exp(\sqrt{2\phi})+\exp(-\sqrt{2\phi})-2-2\phi)} \\
&= 2\frac{\left[1+\cosh\left(\sqrt{2\phi}\right)\right]}{\left[\cosh(\sqrt{2\phi})-1-\phi\right]\left[1+\frac{\pi^2}{2\phi}\right]^2}
\end{aligned}$$

where the first line follows from the definition, and the first equality from the sufficient statistic result in Alvarez, Le Bihan, and Lippi (2016). The second line uses the expression for β_2, λ_2 derived above, as well as the expression for the Kurtosis derived in Alvarez, Le Bihan, and Lippi (2016). The third line uses the expression for ℓ. The remaining lines are simplifications. □

Proof. (of Proposition 15) First use Proposition 2 to write

$$\frac{\partial}{\partial t} Y(t)|_{t=0} = \lim_{M\to\infty} \sum_{j=1}^{M} \beta_j(\phi)\lambda_j(\phi) = \lim_{M\to\infty} \sum_{i=0}^{M} [\beta_{2+4i}\lambda_{2+4i} + \beta_{4+4i}\lambda_{4+4i}]$$

Using the coefficients for β_j in Proposition 2 and the expression for the eigenvalues in equation (21) we write

$$\frac{\partial}{\partial t} Y(t)|_{t=0} = -N\ell(\phi) \lim_{M\to\infty} \sum_{i=0}^{M} 2\left(\left[\frac{1+\cosh\left(\sqrt{2\phi}\right)}{\cosh(\sqrt{2\phi})-1}\right] - 1\right) = -2N\ell(\phi) \lim_{M\to\infty} M\left(\left[\frac{1+\cosh\left(\sqrt{2\phi}\right)}{\cosh(\sqrt{2\phi})-1}\right] - 1\right)$$

which diverges towards minus infinity for any $0 \leq \phi < \infty$. □

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