Uniqueness of MFG Equilibrium

Fernando Alvarez

University of Chicago

MFR, August 1^{st} 2022

- Decision Maker Problem:
  - Forward looking, standard Dynamic Programming Problem
  - State x, given optimal decision, markov process
  - Flow benefits depend on time varying cross-sectional density $m(t)$
  - Value function $u(x, t)$ solve HJB eqn, optimal decision $a^*(x, t)$.
- Distribution of agents across state at tine t is $m(x, t)$
  - Law of motion for distribution if all agents follow $a^*(x, t)$.
  - Density $m(x, t)$ evolves according Kolmogorov Forward (KFE)
  - Initial distribution $m_0(x)$ given.
- Equilibrium $u(x, t)$ and $m(x, t)$ fixed point.

- HJB eqn for u given path $\{m(x,t)\}$ of cross sectional distribution and terminal value function $u_T(x, m(T))$, for all x & $t \in [0, T]$:

$$u(x,t)$$
$$= \max_{\{a\}} \mathbb{E}\left[\int_t^T e^{-\rho(s-t)} \left[F(x(s), m(s)) + G(x(s), a(s))\right] ds | x(t) = x\right]$$
$$+ \mathbb{E}\left[e^{-\rho(T-t)} u_T(x(T), m(T)) | x(t) = x\right]$$

subject to:

$$dx(s) = \mu(x(s), a(s))dt + \sigma^2(x(s))dW(s) \text{ for all } s \in [t, T]$$

- Optimal decision rule $a^*(x,t)$
- Cross sectional density $\{m(x,t)\}$ given:
  - decision rules $a^*(x,t)$ and
  - initial condition $m_0(x)$.

- Period return function $F(x, m(t)) + G(x, a)$.
- $F(x, \cdot)$ depends on the ENTIRE distribution.
- Example: $F(x, \bar{m}_1, \bar{m}_2, \ldots, \bar{m}_K)$ where $\bar{m}_i \equiv \int x^i m(x) dx$
- $F(x, m(t))$ effect of equilibrium prices, as function of aggregates.
- Interpret effect of m in $F(x, m)$: interpretation in static game
    - Strategic Substitutes: F is higher if x “different“ to others in m
    - Strategic Complements: F is higher if x “similar“ to others in m

Other elements of the MFG

- discount rate $\rho \geq 0$.
- Effect of action on period return $G(x, a)$.
- Effect of action on law of motion: drift control $\mu(x, a)$.
- Volatility $\sigma(x)$. (Can also consider controlling it.)
- Terminal time T and value $u_T(x, m(T))$.

What is state of decision maker?

- In principle, infinite dimensional problem, depends on m
- But, we do NOT include m as argument on $u(x, t)$.
- Instead, this is captures by including t.
- This simplifies the problem <u>enormously</u>:

  infinite dimension state vs introducing one more dimension.

What is new/useful with MFGs?

- Same definition as most equilibria in macro.
- Case considered here, no aggregate random shock:
  equilibrium from arbitrary initial condition, similar to an IRF
- Advantage: analytical characterization.
- Helps on numerical computation:
  - finite difference nicely matches with the structure.
  - See Ben Moll notes on duality of KFE and HJB eqns.
- Existence result – but this can be done in other ways.
- Uniqueness: big advantage of MFGs, this note.
- Perturbations: in infinite dimensions, not here.

1. Write Hamilton-Jacobi-Bellman (HJB) eqn in continous time as p.d.e. for $u(x, t)$
2. Introduce Hamiltonean and its properties
3. Write law of motion for m, Kolmogorov forward Eqn (KFE) as a p.d.e. for $m(x, t)$.
4. Use Hamiltonean to “couple" p.d.e.’s for u and m.
5. Introduce Monotonicity definition and relationship uniqueness in static Nash. Slope of Best response.
6. Examples of Monotone F.
7. Prove the if F is monotone, MFG has only one equilibrium.

1. Hamilton-Jacobi-Bellman (HJB), see Note II

- Given path $m(\cdot, t)$ then u must solve:

$$\begin{aligned}\rho u(x,t) &= F(x,m(t)) + \max_a G(x,a) + \mu(x,a)u_x(x,t) \\ &\quad + \tfrac{1}{2}\sigma^2(x)u_{xx}(x,t) + u_t(x,t) \\ &= F(x,m(t)) + H(u_x(x,t),x) + \tfrac{1}{2}\sigma^2(x)u_{xx}(x,t) + u_t(x,t)\end{aligned}$$

for all x and $t \in [0,T]$ and $u(x,T) = u_T(x, m(T))$.

- The Hamiltonean is defined as, for any p:

$$H(p,x) = \max_a G(x,a) + p\mu(x,a)$$

- If $dx = \mu(x,a)dt + \sigma(x)dW$, then for any function $f(x,t)$:

$$\begin{aligned}&\lim_{\Delta \downarrow 0} \frac{1}{\Delta}\mathbb{E}\left[f(x(t+\Delta), t+\Delta) - f(x,t) | x(t) = x\right] \\ &\quad = \mu(x,a)f_x(x,t) + \tfrac{1}{2}\sigma^2(x)f_{xx}(x,t) + f_t(x,t)\end{aligned}$$

2. Hamiltonean (See Note II)

- Definition:

$$H(p,x) = \max_a G(x,a) + p\mu(x,a)$$

- F.o.c. and s.o.c:

$$0 = G_a(x, a^*(x,p)) + p\mu_a(x, a^*(x,p))$$
$$0 \geq G_{aa}(x, a^*(x,p)) + p\mu_{aa}(x, a^*(x,p))$$

- Envelope:

$$H_p(x,p) = \mu(x, a^*(x,p))$$

- Convexity of H, and assumption on uniform bound

$$H_{pp}(x,p) = \mu_a(x,a^*)a_p^* = \frac{\mu_a^2(x,a^*)}{-(G_{aa}(x,a^*) + p\mu_{aa}(x,a^*))} \geq \underline{H}_{pp} > 0$$

3. Law of motion of m, Kolmogorov Forward Equation

- Suppose that x evolves with

$$dx(t) = \tilde{\mu}(x(t), t)dt + \sigma(x(t))dW(t)$$

- With given initial condition $m(x, 0) = m_0(x)$
- Then, for each t and x we have

$$m_t(x, t) = -\left(\tilde{\mu}(x, t)\, m(x, t)\right)_x + \left(\tfrac{1}{2}\sigma^2(x)\, m(x, t)\right)_{xx}$$

- Why the negative sign? See Note III
- Why the first and second derivative of the product ? See Note III
- Use optimal policy and envelope:

$$\tilde{\mu}(x, t) = \mu(x, a^*(x, t)) = H_p(u_x(x, t), x)$$

- replacing into KFE:

$$m_t(x, t) = -\left(H_p(u_x(x, t), x)\, m(x, t)\right)_x + \left(\tfrac{1}{2}\sigma^2(x)\, m(x, t)\right)_{xx}$$

4. MFG: HJB and KFE coupled by Hamiltonean

- An eqbm of MFG is a pair u, m that solves:
- pair of (non-lineal) p.d.e.:

$$\rho u(x,t) = F(x, m(t)) + H(u_x(x,t), x) + \frac{1}{2}\sigma^2(x) u_{xx}(x,t) + u_t(x,t)$$

$$m_t(x,t) = -\left(H_p(u_x(x,t), x)\, m(x,t)\right)_x + \left(\frac{1}{2}\sigma^2(x)\, m(x,t)\right)_{xx}$$

for all x and all $t \in [0, T]$

- terminal and initial condition:

$$u(x, T) = u_T(x, m(T)) \text{ and } m(x, 0) = m_0(x)$$

for all x.

5. Monotonicity

- Def.: A function F satisfies monotonicity if for any two m^1, m^2:

$$\int \left[F(x, m^1) - F(x, m^2)\right] \left(m^1(x) - m^2(x)\right) dx \leq 0$$

  with strict inequality if $m^1 \neq m^2$.

- Sufficient condition for uniqueness of static Nash.

- Static Nash Def.: m^* is an equilibrium if for all $x \in$ support m^*:

$$F(x, m^*) \geq F(x', m^*) \text{ for all } x'$$

  i.e. if m^* puts positive mass on x, then x is optimal against m^*.

- Simple result (4 lines, see Note IV):

  Monotonicity of $F \implies$ Unique Static Nash

5. Monotonicity, Static Nash degenerate case

- Def.: A function F satisfies monotonicity if for any two m^1, m^2:

$$\int \left[F(x, m^1) - F(x, m^2)\right] \left(m^1(x) - m^2(x)\right) dx \leq 0, \quad < \text{ if } m^1 \neq m^2$$

- Specialize to the case where m degnerate, prob. 1 on single X.
- Best response $B(X)$ for m, concentrated on X.

$$B(X) = \arg\max_x F(x, X) \quad \text{abusing notation using } F$$

- Static Nash is defined as $x^* = B(x^*)$
- F is Monotone $\implies B'(X) \leq 1$. (simple proof, see Note IV)
- Simple (graphical) result: If $B'(X) < 1$, then equilibrium is unique
  - $B'(X) < 0$: strategic substitutability
  - $B'(X) > 0$: strategic complementarity

6. Examples of Monotone F

- Def.: A function F satisfies monotonicity if for any two m^1, m^2:

$$\int \left[F(x, m^1) - F(x, m^2)\right] \left(m^1(x) - m^2(x)\right) dx \leq 0, \quad < \text{ if } m^1 \neq m^2$$

- If F_1 monotone, F_a depends on x only, & F_b depends only on m:

  Then $F = F_1 + F_a + F_b$ is monotone.

- If F_1, F_2 are monotone, then $F = \alpha_1 F_1 + \alpha_2 F_2$ is monotone.

- Monotone if $\theta > 0$:

$$F(x, m) = -\left(x + \theta \int y\, m(y)\, dy\right)^2$$

$$F(x, m) = \theta \int (x - y)^2\, m(y)\, dy$$

$$F(x, m) = -\theta \int K(x, y) m(y) dy \text{ where } K \text{ is almost positive definite}$$

7. Monotonicity of $F, u_T \implies$ MFG unique eqbm

- Result by Larsry-Lions. Add assumption $\underline{H}_{pp} > 0$.

- Proof by contradiction. Assume (u^1, m^1) and (u^2, m^2) are two different eqbm.

- define $\hat{u}(x,t) \equiv u^1(x,t) - u^2(x,t)$, $\hat{m}(x,t) \equiv m^1(x,t) - m^2(x,t)$ and $I(t) \equiv \int \hat{u}(x,t)\hat{m}(x,t)dx$

- Show that $I(t) = \int_0^t e^{\rho(t-\tau)} S(\tau) d\tau$ where

$$S(\tau) \geq \underline{H}_{pp} \int \left(m^1(x,\tau) + m^2(x,\tau)\right) \left(u_x^1(x,\tau) - u_x^2(x,\tau)\right)^2 dx$$

- By monotonicity $I(T) \leq 0$, since $u^i(x, m^i(T)) = u_T(x, m^i(T))$.

- Then $S(t) = 0$ all t and hence $m^1 = m^2$ and $u^1 = u^2$.

- $I(t) = \int \hat{u}(x,0)\hat{m}^(x,0)dx = \int \hat{u}(x,0)\, 0\, dx$ since $m^i(x,0) = m_0(x)$

- Time differetiating I:

$$\frac{d}{dt}I(t) = \frac{d}{dt}\int \hat{u}(x,t)\hat{m}(x,t)dx$$
$$= \int \hat{u}_t(x,t)\hat{m}(x,t)dx + \int \hat{u}(x,t)\hat{m}_t(x,t)dx$$

- Replace u_t^i from HJB eqn. for $i = 1,2$

- Replace m_t^i from KFE eqn. for $i = 1,2$

- Integration by parts w.r.t. x to eliminate second derivatives

- Use monotonicity of F to sign term with difference in $F's$.

- Use convexity of $H(p,x)$ w.r.t. p, second order expansion.

- Use $\hat{u}(x,t) \equiv u^1(x,t) - u^2(x,t)$, $\hat{m}(x,t) \equiv m^1(x,t) - m^2(x,t)$ and replace

$$u_t^i(x,t) = -F(x, m^i(t)) - H(u_x^i(x,t), x) - \tfrac{1}{2}\sigma^2(x) u_{xx}^i(x,t) + \rho u^i(x,t)$$
$$m_t^i(x,t) = -\left(H_p u_x^i(x,t), x)\, m^i(x,t)\right)_x + \left(\tfrac{1}{2}\sigma^2(x)\, m^i(x,t)\right)_{xx}$$

- into $\dot{I}(t) \equiv \frac{d}{dt} I(t) = \int \hat{u}_t(x,t)\hat{m}(x,t)dx + \int \hat{u}(x,t)\hat{m}_t(x,t)dx$ or

$$\dot{I}(t) = \int \hat{u}_t \hat{m} + \int \hat{m}_t \hat{u} = A + B$$

$$A = \int \hat{u}_t \hat{m} = -\int \left(F(m^1) - F(m^2)\right) \hat{m} - \int \left(H(u_x^1) - H(u_x^2)\right) \hat{m}$$
$$- \int \frac{\sigma^2}{2} \hat{u}_{xx} \hat{m} + \rho \int \hat{u}\hat{m}$$

$$B = \int \hat{m}_t \hat{u} = -\int \left(H_p(u_x^1) m^1 - H_p(u_x^2) m^2\right)_x \hat{u} + \int \left(\frac{\sigma^2}{2} \hat{m}\right)_{xx} \hat{u}$$

- Integration by parts twice on the third term in A gives

$$-\int \frac{\sigma^2}{2}\hat{u}_{xx}\hat{m} = -\int \left(\frac{\sigma^2}{2}\hat{m}\right)_{xx}\hat{u}$$

- Using monotonicity of F we have the first term in A:

$$-\int \left(F(m^1) - F(m^2)\right)\hat{m} = -\int \left(F(m^1) - F(m^2)\right)\left(m^1 - m^2\right) \geq 0$$

- Integrate by parts the first term in B to obtain:

$$-\int \left(H_p(u_x^1)m^1 - H_p(u_x^2)m^2\right)_x \hat{u} = \int \left(H_p(u_x^1)m^1 - H_p(u_x^2)m^2\right)\hat{u}_x$$

- Use the previous equality and two inequalities to write

$$\dot{I}(t) - \rho I(t) = S(t)$$

where

$$S(t) \geq -\int \left(H(u_x^1) - H(u_x^2)\right) \hat{m} + \int \left(H_p(u_x^1)m^1 - H_p(u_x^2)m^2\right) \hat{u}_x$$

- Use a second order expansion of $H(u_x^i)$ to write:

$$\begin{aligned} H(p) &= H(p^0) + H_p(p^0)(p - p^0) + \tfrac{1}{2}H_{pp}(p - p^0)^2 \\ &\geq H(p^0) + H_p(p^0)(p - p^0) + \tfrac{1}{2}\underline{H}_{pp}(p - p^0)^2 \end{aligned}$$

and set $p = u_x^i$ and $p^0 = u_x^j$ to write:

$$\begin{aligned} H(u_x^1) - H(u_x^2) &\geq H_p(u_x^2)(u_x^1 - u_x^2) + \tfrac{1}{2}\underline{H}_{pp}(u_x^1 - u_x^2)^2 \\ H(u_x^2) - H(u_x^1) &\geq H_p(u_x^1)(u_x^2 - u_x^1) + \tfrac{1}{2}\underline{H}_{pp}(u_x^1 - u_x^2)^2 \end{aligned}$$

- We develop the terms and use def of $\hat{u}$:

$$\int \left(H_p(u_x^1)m^1 - H_p(u_x^2)m^2\right)\hat{u}_x$$
$$= H_p(u_x^1)m^1(u_x^1 - u_x^2) + H_p(u_x^2)m^2(u_x^2 - u_x^1)$$

- Use def of $\hat{m}$, and inequalities based on convexity:

$$-\int \left(H(u_x^1) - H(u_x^2)\right)\hat{m}$$
$$= \left(H(u_x^2) - H(u_x^1)\right)m^1 + \left(H(u_x^1) - H(u_x^2)\right)m^2$$
$$\geq H_p(u_x^1)(u_x^2 - u_x^1)m^1 + \tfrac{1}{2}H_{pp}(u_x^1 - u_x^2)^2 m^1$$
$$+ H_p(u_x^2)(u_x^1 - u_x^2)m^2 + \tfrac{1}{2}H_{pp}(u_x^1 - u_x^2)^2 m^2$$

- Combining both terms:

$$-\int \left(H(u_x^1) - H(u_x^2)\right)\hat{m} + \int \left(H_p(u_x^1)m^1 - H_p(u_x^2)m^2\right)\hat{u}_x$$
$$\geq \tfrac{1}{2}\underline{H}_{pp}\int (u_x^1 - u_x^2)^2\left(m^1 + m^2\right)$$

- Thus we have for $I(t) \equiv \int \hat{u}(x,t)\hat{m}(x,t)dx$:
- $I(t) = \int_0^t e^{\rho(t-\tau)} S(\tau) d\tau$ with $I(T) \leq 0$, where

$$S(\tau) \geq \underline{H}_{pp} \int \left(m^1(x,\tau) + m^2(x,\tau)\right) \left(u_x^1(x,\tau) - u_x^2(x,\tau)\right)^2 dx \geq 0$$

- Then it must be that $S(t) = 0$ all t.
- Hence, $u_x^1 = u_x^2$ at all points with positive probability.
- But $\tilde{\mu}^i = H_p(u_x^i)$ for $i = 1, 2$ which using KFE $\implies m^1 = m^2$.
- Thus it must be that $u^1 = u^2$ in levels too, using HJB.
- This concludes the proof.

Extensions

- Infinite Horizon $T = \infty$.

  If $|u_T|$ is bounded, then it follows by taking limits.

- Volatiliy, instead of drift control.
  - Redefine HJB, Hamiltonean, and KFE.
  - Essentially the same proof.
- Case where x is a vector
  - Using divergence in KFE
  - Essentially the same proof.
- Case where period return is not additive
  - If $J(x, m, a)$ instead of $F(x, m) + G(x, a)$.
  - More complex version of Monotoncity

Volatility Control

- An eqbm of MFG is a pair u, m that solves:
- pair of (non-lineal) p.d.e.:

$$\rho u(x,t) = F(x, m(t)) + H(u_{xx}(x,t), x) + \mu(x) u_x(x,t) + u_t(x,t)$$

$$m_t(x,t) = -\left(m(x,t)\mu(x)\right)_x + \left(H_p(u_{xx}(x,t), x)\ m(x,t)\right)_{xx}$$

for all x and all $t \in [0, T]$

- terminal and initial condition:

$$u(x,T) = u_T(x, m(T)) \text{ and } m(x,0) = m_0(x)$$

for all x.

- Hamiltonean:

$$H(p,x) = \max_a G(x,a) + p\frac{\sigma^2(x,a)}{2}$$

$$H_p(p,x) = \frac{\sigma^2(x, a^*(x,t))}{2}$$

Vector case $x = (x^1, x^2, \ldots, x^n)$

- pair of (non-lineal) p.d.e.:

$$\rho u(x,t) = F(x, m(t)) + H(\vec{u}_x(x,t), x) + \frac{1}{2}\sum_{i=1}^{n} \sigma_i^2(x) u_{x_i x_i}(x,t) + u_t(x,t)$$

$$m_t(x,t) = -\sum_{i=1}^{n} (H_{p_i}(u_x(x,t), x) m(x,t))_{x_i} + \frac{1}{2}\sum_{i=1}^{n} \left(\sigma_i^2(x)\, m(x,t)\right)_{x_i x_i}$$

  for all x and all $t \in [0, T]$

- terminal and initial condition:

$$u(x,T) = u_T(x, m(T)) \text{ and } m(x,0) = m_0(x)$$

  for all x.

- Hamiltonean:

$$H(\vec{p}, x) = \max_a G(x,a) + \sum_{i=1}^{n} p_i\, \mu^i(x,a)$$

$$H_{p_i}(\vec{p}, x) = \mu^i(x, a^*(x,t))$$

Summary

- Example of usefulness of using structure of a problem.
- Result:
  - sufficient conditions for uniqueness of static Nash

    $\Downarrow$
  - sufficient conditions uniqueness of MFG.
- Interpretation of sufficient conditions static Nash:

  \- Bound on strategic complementarity
- Same duality HJB-KFE and Hamiltonean can be used for:
  - ☐ computations
  - ☐ analytic characterizations
  - ☐ perturbations (linearizations)