Leverage Dynamics without Commitment

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Introduction (1)

Leverage dynamics is at the heart of dynamic corporate finance
- Static trade-off (maximizing firm value) differs from equity’s dynamic optimization
- Challenging, as debt prices interact with future equilibrium leverage policies

Existing literature relies on some ad hoc “commitment” of future debt policies
- Refinance to keep outstanding debt face value constant (Leland 1994 1998)
- Whenever adjusting debt, the firm has to retire the existing debt first, with some transaction costs (Fischer, Heinkel, and Zechner 1989; Goldstein, Leland, Ju 2001)
  - Abrupt adjustment to “target” leverage
- **Empirically counterfactual**: firms actively manage their debt, often incrementally
Introduction (2)
The firm cannot commit to future debt policies
- Otherwise, standard trade-off setting (tax shield vs bankruptcy cost) with stochastic asset growth; no transaction cost
- No commitment at all: say, no covenants
- A more endogenous “friction”, rather than exogenous frictions to adjust leverage

Assumption on seniority and dilution
- Zero recovery ⇒ seniority structure irrelevant. Indirect dilution: issuing more debt hurts default probability
- Positive recovery: pari-passu debt, direct dilution in recovery (not in this presentation)

Leverage may go down via asset growth and debt maturing, but equity never reduces debt voluntarily
- Repurchase debt is never optimal—leverage ratchet effect (Admati DeMarzo Hellwig Pfleiderer, 2018)
- Our setting is more canonical
This Paper (2)

- A general method to solve this class of models
  - A result reminiscent of Coase conjecture

- Closed-form solutions for work-horse log-normal cash-flow setting, and prove uniqueness of this Markov Perfect Equilibrium

- History-dependent leverage dynamics: issue more (less) following good (bad) shocks
  - Leverage dynamics tend to be mean-reverting; no immediate adjustment to leverage “target”

- Dynamic trade-off of equity value ≠ Static trade-off of firm value
  - Two leverage/maturity dynamics drastically different, but both are optimal
  - Lemmon, Roberts, and Zender (2008)
General Model: Environment

Preferences
- Risk-neutral world, with common discount rate $r$

Assets
- Assets in place generate operating income (could allow for jumps):
  \[ dY_t = \mu(Y_t) \, dt + \sigma(Y_t) \, dZ_t \]
- Focus on zero recovery now (debt seniority irrelevant); can be relaxed

Debt contract: aggregate face value $F_t$ (endogenous)
- Each debt with coupon rate $c$, face value 1
- Exponentially retiring (Poisson maturing) with rate $\xi$

Corporate tax: $\pi(Y_t - cF_t)$
Debt Issuance/Repurchase and Default

Evolution of debt

▶ Sell/buyback debt $d \Gamma_t$, so aggregate debt face value evolves as

$$dF_t = \underbrace{-\xi F_t dt}_{\text{contractual debt maturing}} + \underbrace{d \Gamma_t}_{\text{active debt management}}$$

Timing within $[t, t + dt]$ & lack of commitment

▶ Cash flow realizes; either default or pay coupon/principal; announce $d \Gamma_t$; debt price set (and trade); next period

▶ Unable to commit on future $d \Gamma_{t+s}$ for $s > 0$

Equity default at endogenous stopping time $\tau_b$
Equity Value

State variables (Markov Perfect Equilibrium)

- **Exogenous** cash-flows $Y_t$, and **endogenous** debt obligation $F_t$

**Equity’s problem, taking debt prices $p$ as given**

- Equity receives cash-flows (if negative, covered by issuing equity)

$$\begin{align*}
\underbrace{Y_t} - \underbrace{\pi (Y_t - cF_t)} - \underbrace{(c + \xi) F_t} + \underbrace{p_t \cdot d\Gamma_t}
\end{align*}$$

  - cash-flows
  - corporate taxes
  - interest & principal
  - issuance/repurchase

- Endogenous debt price $p_t$ determined later

- Given $Y_t = Y$ and $F_t = F$, equity is solving

$$\max \left\{ \mathbb{E}_t \left\{ \int_t^{\tau_b} e^{-r(s-t)} \left[ Y_s - \pi (Y_s - cF_s) - (c + \xi) F_s + p_s d\Gamma_t \right] ds \right\} \right\}$$

- Controlling 1) debt evolution $dF_t = F_t dt + d\Gamma_t$; and 2) when to default
Debt Price

Debt price

- Competitive risk neutral debt investors price debt rationally
- Given equity default decision $\tau_b$, equilibrium debt price

$$p(Y, F) \equiv \mathbb{E}_t \left\{ \int_t^{\tau_b} e^{-(r+\xi)(s-t)} (c + \xi) \, ds \mid Y_t = Y, F_t = F \right\}$$

Why does commitment matter?

- $p_t$ depends on equilibrium default time $\tau_b$
- $\tau_b$ depends on firm’s future debt policy—the more the future debt, the more likely the default
Discrete-Time (1)

State variables (Markov Perfect Equilibria)

- Exogenous income $Y_t$, and endogenous debt obligation $F_t$

Time line

Default?  
Receive cash flows, pay interest and principal  
Issue or Repurchase Debt

\[
V(Y, F) - (1 - \pi)Y - ((1 - \pi)c + \xi)F \, dt \\
D(Y, F') - F' - (1 - \xi \, dt)F \, D(Y, F')
\]

Dividends $dC_t$
Discrete-Time (1)

State variables (Markov Perfect Equilibria)
- Exogenous income $Y_t$, and endogenous debt obligation $F_t$

Time line

$V(Y, F)$

Default? Receive cash flows, pay interest and principal

$V(Y, F) [(1 - \pi)Y - ((1 - \pi)c + \xi)F]dt$

$D(Y, F') (Y', F')$

Issue or Repurchase Debt

$[F' - (1 - \xi dt)F]D(Y, F')$

Dividends $dC_t$

Simplification (for discrete-time illustration)
- In default, zero debt recovery and zero equity value
Discrete-Time (2)

**Equity Bellman equation**

\[
V(Y, F) = \max_{F'} \left\{ \left( Y - \left( (1 - \pi) c + \xi \right) F \right) dt + p(Y, F') \left( F' - (1 - \xi dt) F \right) \right\}
\]

- **Benefit from today’s consumption**
- **Consumption over** \([t - dt, t] \)**
- **Cost from future debt burden**

**Repayment region** \( \mathbb{R}(Y, F) \equiv \{ V > V \} \) and **debt pricing**

\[
p(Y, F') = e^{-rdt} \mathbb{E}_Y \left[ \mathbb{R}(Y', F') \left( (c + \xi) dt + (1 - \xi dt) p(Y', F'') \right) \right]
\]
Discrete-Time (2)
Equity Bellman equation

\[
V(Y, F) = \max_{F'} \left\{ \begin{aligned}
(Y - ((1 - \pi) c + \zeta) F) \, dt &+ p(Y, F') \left( F' - (1 - \zeta dt) F \right) \\
+ e^{-rdt} \mathbb{E}_Y \left[ \max (V(Y'), V(Y', F')) \right] &+ e^{-rdt} \mathbb{E}_Y \left[ V(Y', F') \right] \\
\text{Benefit from today's consumption} &+ \text{Consumption over } [t-dt, t] &+ \text{Cost from future debt burden}
\end{aligned} \right. 
\]

Repayment region \( R(Y, F) \equiv \{ V > V \} \) and debt pricing

\[
p(Y, F') = e^{-rdt} \mathbb{E}_Y \left[ R(Y', F') \left( (c + \zeta) dt + (1 - \zeta dt) p(Y', F'') \right) \right]
\]

Government MC from future debt burden

\[
- \partial_{F'} \left[ e^{-rdt} \mathbb{E}_Y \left[ \max (V(Y'), V(Y', F')) \right] \right] = e^{-rdt} \mathbb{E}_Y \left[ R(Y', F') \cdot (-V_{F'}(Y', F')) \right] \\
= e^{-rdt} \mathbb{E}_Y \left[ R(Y', F') \left( (1 - \pi) c + \zeta \right) dt + (1 - \zeta dt) p(Y', F') \right] \\
= p(Y, F') - \mathbb{E}_Y \left[ R(Y', F') \right] e^{-rdt} \pi c dt \\
\text{envelope theorem} &+ \text{DTS, source of gain from trade}
\]
Discrete-Time (3)

**Government’s FOC** \((d\Gamma: \text{bond issuance})\)

\[
p(Y, F') + \partial_{F'} p(Y, F') \cdot d\Gamma^* = p(Y, F') - \mathbb{E}_Y \left[ \mathbb{R} (Y', F') \right] e^{-rdt} \pi c dt
\]

- MB from today’s consumption
- MC from future debt burden

\(<0, \text{price impact}\)

**Optimal Policy** \((e^{-rdt} = 1 - rdt + o(dt))\)

\[
d\Gamma^* = \mathbb{E}_Y \left[ \mathbb{R} (Y', F') \right] \cdot \frac{\pi c}{-\partial_{F'} p(Y, F')} dt + o(dt) > 0
\]

▶ Smooth adjustment policy; and never buyback!
Discrete-Time (3)

**Government’s FOC** ($d\Gamma$: bond issuance)

\[
\begin{align*}
p(Y, F') + \partial_{F'} p(Y, F') \cdot d\Gamma^* &= p(Y, F') - E_Y \left[ R(Y', F') \right] e^{-rdt} \pi c dt \\
\text{MB from today’s consumption} &\quad \text{MC from future debt burden}
\end{align*}
\]

**Optimal Policy** ($e^{-rdt} = 1 - rdt + o(dt)$)

\[
\begin{align*}
d\Gamma^* &= \frac{\pi c}{-\partial_{F'} p(Y, F')} \cdot dt + o(dt) > 0
\end{align*}
\]

▶ Smooth adjustment policy; and never buyback!

**Gain from trading over** $[t - dt, t]$ **is bounded by** $O(dt^2)$

\[
\begin{align*}
\left[ p - p - E_Y \left[ R(Y', F') \right] e^{-rdt} \pi c dt \right] \times \frac{d\Gamma^*}{\text{quantity}} &= \frac{E_Y^2 \left[ R(Y', F') \right] \pi^2 c^2}{-\partial_{F'} p} (dt)^2 + o(dt)
\end{align*}
\]

▶ Zero gain from trade; Coase conjecture
Leverage Ratchet and Smooth Equilibrium

Never buyback outstanding debt

▶ **Proposition 1.** In equilibrium \( d\Gamma_t \geq 0 \).
▶ Intuition. Postponing debt buyback—if any—gives shareholders a greater tax subsidy.

**Price impact**

▶ If \( V(Y, F) \) is strictly convex in \( F \) so that there is a strictly positive price impact, then \( \Gamma_t \) is **continuous** in time
▶ Why jump is never optimal? Negative price impact always hurts (see later)

**Focusing on “smooth equilibrium”:** \( d\Gamma_t = G_t dt \)

▶ Equity could adjust debt discretely, but not optimal in such an equilibrium
▶ **Other equilibria with jumps?** see the log-normal case later
Value Equivalence of No-Issuance (1)

- Hamilton-Jacobi-Bellman equation for equity value function $V(Y, F)$

$$rV(Y, F) = \max G \left[ \underbrace{G p(Y, F)}_{\text{issuance/repurchase}} + \underbrace{(G - \xi F) V_F(Y, F)}_{\text{evolution of debt}} \right]$$

$$Y - \pi (Y - cF) - (c + \xi) F + \mu(Y) V_Y(Y, F) + \frac{\sigma^2(Y)}{2} V_{YY}(Y, F)$$

- Objective linear in $G$. Optimal $G \Rightarrow$ First-Order Condition

$$\underbrace{p(Y, F)}_{\text{MB of issuance}} + \underbrace{V_F(Y, F)}_{\text{MC on future value}} = 0$$

- Under FOC, equity indifferent at any $G$ (given equilibrium $p$)
  - Linear control with interior solution (smooth policy $G_t \, dt$)
  - Equity value can be solved by setting $G = 0$ always
Value Equivalence of No-Issuance (2)

- Equity value can be solved by setting $G = 0$ always
  \[
  rV = -\xi FV_F + Y - \pi (Y - cF) - (c + \xi) F + \mu(Y) V_Y + \frac{\sigma^2(Y)}{2} V_{YY}
  \]

- No gain in equilibrium by debt issuance/repurchase
  - Any potential tax shield gain is dissipated by bankruptcy cost caused by future excessive leverage
  - Reminiscent of Coase conjecture; DeMarzo and Urosevic (2006)

- Get equity value $V(Y, F)$ without knowing debt price
Equilibrium Policies

Basic idea

▶ Debt price $p(Y, F)$ must satisfy the valuation equation

\[
p(Y, F) = \mathbb{E}_t \left\{ \int_t^{\tau_b} e^{-(r+\xi)(s-t)} (c + \xi) \, ds \right\}
\]

▶ $V(Y, F)$ gives $-V_F(V, F) = p(Y, F)$ using equity’s FOC

▶ How to make both match? Via debt management $G(Y, F)$
  ▶ ODE for $V_F(V, F)$ (HJB for $V$) does not depend on $G$...
  ▶ while HJB for $p$, which depends on $G$

▶ Recall $-p_F(Y, F) = V_{FF}(Y, F) > 0$, capturing the price impact
Equilibrium Policies

Basic idea

- Debt price $p(Y, F)$ must satisfy the valuation equation

$$p(Y, F) = \mathbb{E}_t \left\{ \int_t^{\tau_b} e^{-(r+\xi)(s-t)} (c + \xi) \, ds \right\}$$

- $V(Y, F)$ gives $-V_F(V, F) = p(Y, F)$ using equity’s FOC

- How to make both match? Via debt management $G(Y, F)$
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  - while HJB for $p$, which depends on $G$

Equilibrium debt issuance policy

$$G^*(Y, F) = \frac{c \cdot \pi'(Y - cF)}{-p_F(Y, F)}$$

- $\pi'(Y - cF) \geq 0$, tax benefit $\Rightarrow$ always issuing debt
- Recall $-p_F(Y, F) = V_{FF}(Y, F) > 0$, capturing the price impact
Strict Optimality in Discrete Time

- Taking the value function at $t + h$ as given, consider equity’s problem at $t$, where time interval $h > 0$
- Denote debt issuance by $\Delta$. Equity is maximizing

$$\max_{\Delta} \quad -(1 - \pi) \cdot \Delta c \cdot h \quad + \Delta \left[ c \cdot h + p(Y, F + \Delta) \right] + V(F + \Delta, Y)$$

- First-order condition w.r.t $\Delta$

$$0 = \pi c \cdot h + p(Y, F + \Delta) + \Delta \cdot p_F(Y, F + \Delta) + V_F(F + \Delta, Y)$$

which implies that

$$\Delta = \frac{\pi c \cdot h + p + V_F}{-p_F} = \frac{\pi c}{-p_F} \cdot h$$

- One can easily check the global optimality
**Proposition 1:** Global optimality of local FOC holds if debt price

\[ p(Y, F) = -V_F(V, F) \]

is non-increasing in debt \( F \).

- Debt price decreasing in \( F \) \( \iff \) Equity value function is convex in \( F \) (option value of default)
  - Buyback, paying a higher price; selling too much hurts price too
What is the impact of debt repurchase on equity value?

Often the intuition is through firm value...

Leverage Ratchet Effect
What is the impact of debt repurchase on equity value?
- Often the intuition is through firm value...

Reducing debt today alleviates future default ⇒ higher firm value
- But does equity benefit strictly from this effect? No. (Do not forget existing debt holders!)
- Equity optimizes default decision ex post already ⇒ zero indirect impact on equity value today (envelope theorem)
Leverage Ratchet Effect

▶ What is the impact of debt repurchase on equity value?
  ▶ Often the intuition is through firm value...

▶ Reducing debt today alleviates future default ⇒ higher firm value
  ▶ But does equity benefit strictly from this effect? No. (Do not forget existing debt holders!)
  ▶ Equity optimizes default decision ex post already ⇒ zero indirect impact on equity value today (envelope theorem)

▶ Tax saving benefit always tempting...leverage ratcheting in ADHP
  ▶ This paper: a more canonical setting
  ▶ Same logic to debt overhang—equity is optimizing investment decisions ex post
Summary of General Model

1. Solve for equity value $V(Y, F)$ by setting $G(Y, F) = 0$

2. Set the equilibrium debt price $p(Y, F) = -V_F(Y, F)$

3. Check the equity holders’ global optimality condition
   - Verifying $p(Y, F)$ is non-increasing in $F$ (or $V(Y, F)$ is convex in $F$)

4. Equilibrium debt issuance $G^*(Y, F) = \frac{\pi'(Y-cF) \cdot c}{-p_F(Y, F)} > 0$
Log-Normal Cash-flows Model

- Scale-invariance, cash-flows $dY_t / Y_t = \mu dt + \sigma dZ_t$
  - The work-horse model of dynamic corporate finance
- One-dimensional state variable: scaled cash-flow $y_t \equiv Y_t / F_t$, Markov Perfect Equilibrium as in Maskin-Tirole (2001)
  - Equity value $V(Y, F) = F \cdot v(y)$, debt price $p(Y, F) = p(y)$; closed-form solutions
  - We prove such equilibrium is unique
- Let $g^*(y_t) \equiv G^*(Y_t, F_t) / F_t$, then

\[
\frac{dy_t}{y_t} = \left( \mu \begin{array}{c} \text{CF growth} \\
\text{debt maturing} \\
\text{debt issuance}
\end{array} \begin{array}{c} \zeta \\
\gamma \\
\gamma
\end{array} - g^*_t \right) dt + \sigma dZ_t
\]

- Debt growth rate $g^*_t - \zeta$; endogenous $g^*_t = \frac{(r+\zeta)\pi c}{c(1-\pi)+\zeta} \frac{1}{\gamma} \left( \frac{y}{y_b} \right)^\gamma > 0$
  - $\gamma$ is a constant depending on parameters
  - Increasing in $y$, i.e., more debt issuance after good fundamental
Net Debt Issuance $g^*(y) - \xi$, Debt Maturity
Two Benchmarks with Commitment

No future debt issuance:
- The firm commits to set $g_t = 0$ always (superscript 0)
- Equity value is the same (so does $y_b$), debt price is higher (by the tax shield)

$$p^0(y) = p(y) + \frac{\pi_c}{r + \zeta} \left(1 - \left(\frac{y}{y_b}\right)^{-\gamma}\right)$$

- Less debt $\Rightarrow$ less likely to default (same $y_b$ but $y$ has a higher drift)

Fixed future debt:
- The firm commits to set $g_t = \zeta$ always; Leland 1998
Model Comparisons: Debt Prices and Credit Spreads

Implication of credit spreads: \( y \to \infty \) i.e. zero current leverage

- \( p^\xi(y) \) and \( p^0(y) \to \frac{c+\xi}{r+\xi} \), with zero credit spread

- \( p(y) \to \frac{c(1-\pi)+\xi}{r+\xi} \), non-zero credit spreads (high future excessive leverage!)
Different from static trade-off setting, it is optimal to set $F_0^* = 0$

Knowing the future temptation of overborrowing.

**Proposition.** Given cash-flow history $\{ Y_s : 0 \leq s \leq t \}$, time-$t$ debt is ($\hat{y}_\zeta$ is a constant depending on parameters)

$$F_t = \frac{1}{\hat{y}_\zeta} \left[ \int_0^t \gamma \zeta Y_s e^{-\gamma \zeta (s-t)} ds \right]^{1/\gamma}$$

Start from $t = 0$ debt grows at the order of $t^{1/\gamma}$

Outstanding debt is average past earnings, with decaying weights $\gamma \zeta$

High mean-reverting speed, or more aggressive in adding leverage given high cash flows, when

Shorter debt maturity (higher $\zeta$)
Optimal Debt Maturity Structure?

- So far the debt maturity structure $\xi$ is taken as a parameter

- Say the firm gets a one-time chance to set $\xi$ optimally for future debt issuance

- **Proposition**: Equity holders are **indifferent** at any $\xi$
  - Why? Because equity value is as if there is **no future debt issuance**...

- This indifference result holds more generally
Long-term vs. Short-term Debt

- Two firms start with zero debt, with different debt maturities (both being optimal)—but have different leverage dynamics/target
  - Lemmon, Roberts, and Zender (2008)
Long-term vs. Short-term Debt

- Two firms start with zero debt, with different debt maturities (both being optimal)—but have different leverage dynamics/target
  - Lemmon, Roberts, and Zender (2008)

- With flexibility of shorter-term debt, the firm borrows more for higher debt tax shield
- But tax shield is a transfer from social perspective—so long-term debt is preferred to minimize bankruptcy cost
Investment

- Special case of log-normal process. Capital $K_t$ evolves as
  \[ \frac{dK_t}{K_t} = (i_t - \delta) \, dt + \sigma \, dZ_t \]
  with quadratic investment cost $\frac{\kappa i_t^2}{2} K_t$, and output $Y_t = AK_t$

- Leverage ratchet effect prevails despite debt overhang considerations

- Equity issues debt more aggressively when controlling investment endogenously, compared to exogenous investment
  - Endogenous investment offers equity more protection later
Conclusion and Future Work

What we have done

▶ A general methodology solving dynamic corporate finance model without commitment
▶ Leverage policy depending on the entire earnings history, new insight on debt maturity and investment
▶ Slow initial adoption of leverage, but leads ultimately to excess

Future extensions

▶ DeMarzo, 2019 AFA presidential address: importance of exclusivity in collateralized borrowing
▶ Modeling sovereign debt and default (DeMarzo, He, and Tourre, 2019)
   ▶ Covenant of no debt issuance once in distress (say for $y < \hat{y}$)
   ▶ Discrete debt issuance (jump to $\hat{y}$) in equilibrium, counter-productive
▶ Internal cash with liquidity-driven default?