Leverage Dynamics without Commitment

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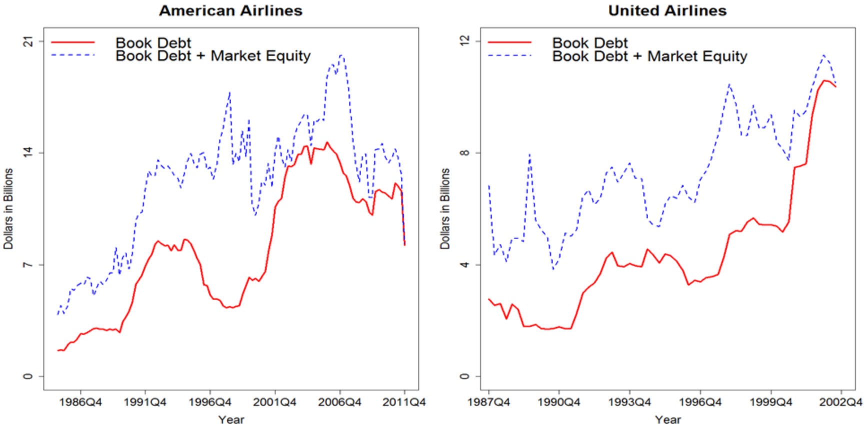
Introduction (1)

- Leverage dynamics is at the heart of dynamic corporate finance
  - Static trade-off (maximizing firm value) differs from equity's dynamic optimization
  - Challenging, as debt prices interact with future equilibrium leverage polices
- Existing literature relies on some ad hoc "commitment" of future debt policies
  - Refinance to keep outstanding debt face value constant (Leland 1994 1998)
  - Whenever adjusting debt, the firm has to retire the existing debt first, with some transaction costs (Fischer, Heinkel, and Zechner 1989; Goldstein, Leland, Ju 2001)
    - Abrupt adjustment to "target" leverage
  - Empirically counterfactual: firms actively manage their debt, often incrementally

Introduction (2)



This Paper (1)

- **The firm cannot commit to future debt policies**
  - Otherwise, standard trade-off setting (tax shield vs bankruptcy cost) with stochastic asset growth; no transaction cost
  - No commitment at all: say, no covenants
  - A more endogenous "friction", rather than exogenous frictions to adjust leverage
- Assumption on seniority and dilution
  - Zero recovery $\Rightarrow$ seniority structure irrelvant. **Indirect** dilution: issuing more debt hurts default probability
  - Positive recovery: pari-passu debt, **direct** dilution in recovery (not in this presentation)
- Leverage may go down via asset growth and debt maturing, but equity never reduces debt voluntarily
  - Repurchase debt is never optimal—**leverage ratchet effect** (Admati DeMarzo Hellwig Pfleiderer, 2018)
  - Our setting is more canonical

This Paper (2)

- A general method to solve this class of models
  - A result reminiscent of Coase conjecture
- Closed-form solutions for work-horse log-normal cash-flow setting, and prove uniqueness of this Markov Perfect Equilibrium
- History-dependent leverage dynamics: issue more (less) following good (bad) shocks
  - Leverage dynamics tend to be mean-reverting; no immediate adjustment to leverage "target"
- Dynamic trade-off of equity value $\neq$ Static trade-off of firm value
  - Two leverage/maturity dynamics drastically different, but both are optimal
  - Lemmon, Roberts, and Zender (2008)

General Model: Environment

Preferences

- Risk-neutral world, with common discount rate r

Assets

- Assets in place generate operating income (could allow for jumps):

$$dY_t = \mu(Y_t)\,dt + \sigma(Y_t)\,dZ_t$$

- Focus on zero recovery now (debt seniority irrelevant); can be relaxed

Debt contract: aggregate face value F_t (endogenous)

- Each debt with coupon rate c, face value 1
- Exponentially retiring (Poisson maturing) with rate ξ

Corporate tax: $\pi(Y_t - cF_t)$

Debt Issuance/Repurchase and Default

Evolution of debt

- Sell/buyback debt $d\Gamma_t$, so aggregate debt face value evolves as

$$dF_t = \underbrace{-\xi F_t dt}_{\text{contractual debt maturing}} + \underbrace{d\Gamma_t}_{\text{active debt managment}}$$

Timing within $[t, t + dt]$ & lack of commitment

- Cash flow realizes; either default or pay coupon/principal; announce $d\Gamma_t$; debt price set (and trade); next period
- Unable to commit on future $d\Gamma_{t+s}$ for $s > 0$

Equity default at endogenous stopping time τ_b

Equity Value

State variables (Markov Perfect Equilibrium)

- Exogenous cash-flows Y_t, and endogenous debt obligation F_t

Equity's problem, taking debt prices p as given

- Equity receives cash-flows (if negative, covered by issuing equity)

$$\underbrace{Y_t}_{\text{cash-flows}} - \underbrace{\pi\left(Y_t - cF_t\right)}_{\text{corproate taxes}} - \underbrace{\left(c+\xi\right)F_t}_{\text{interest \& principal}} + \underbrace{p_t \cdot d\Gamma_t}_{\text{issuance/repurchase}}$$

  - Endogenous debt price p_t determined later
- Given $Y_t = Y$ and $F_t = F$, equity is solving

$$\max_{\{d\Gamma_t\},\tau_b} \mathbb{E}_t \left\{ \int_t^{\tau_b} e^{-r(s-t)} \left[Y_s - \pi\left(Y_s - cF_s\right) - \left(c+\xi\right)F_s + p_s d\Gamma_t \right] ds \right\}$$

  - Controlling 1) debt evolution $dF_t = F_t dt + d\Gamma_t$; and 2) when to default

Debt Price

Debt price

- Competitive risk neutral debt investors price debt rationally
- Given equity default decision τ_b, equilibrium debt price

$$p\left(Y,F\right)\equiv\mathbb{E}_t\left\{\int_t^{\tau_b}e^{-(r+\xi)(s-t)}\left(c+\xi\right)ds\,\middle|\,Y_t=Y,F_t=F\right\}$$

Why does commitment matter?

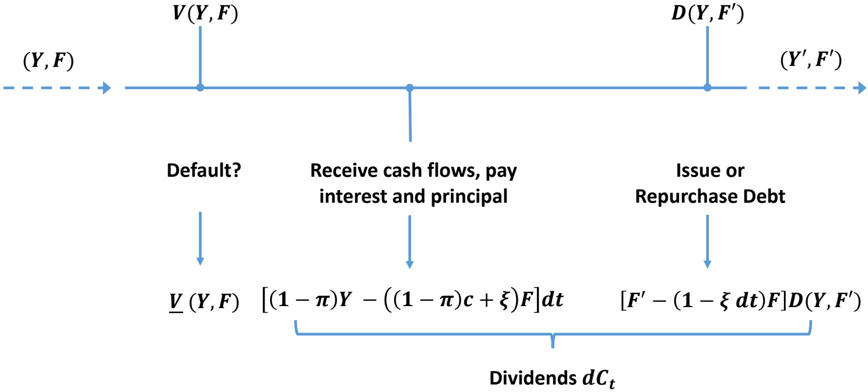
- p_t depends on equilibrium default time τ_b
- τ_b depends on firm's future debt policy—the more the future debt, the more likely the default

Discrete-Time (1)

State variables (Markov Perfect Equilibria)

- Exogenous income Y_t, and endogenous debt obligation F_t

Time line



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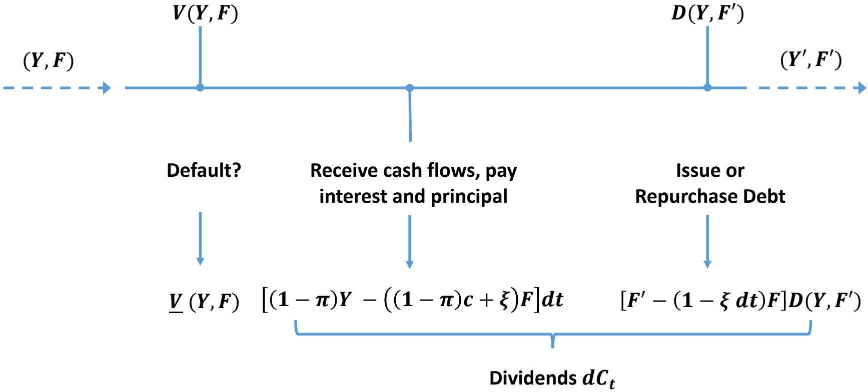

Simplification (for discrete-time illustration)

- In default, zero debt recovery and zero equity value

Discrete-Time (2)

Equity Bellman equation

$$V(Y,F)=\max_{F'}\left\{\begin{array}{c}\underbrace{(Y-((1-\pi)c+\xi)F)\,dt+\overbrace{p(Y,F')\,(F'-(1-\xi dt)F)}^{\text{Benefit from today's consumption}}}_{\text{Consumption over }[t-dt,t]}\\ +\underbrace{e^{-rdt}\mathbb{E}_Y\left[\max\left(\underline{V}(Y'),V(Y',F')\right)\right]}_{\text{Cost from future debt burden}}\end{array}\right\}$$

Repayment region $\mathbb{R}(Y,F)\equiv\{V>\underline{V}\}$ **and debt pricing**

$$p(Y,F')=e^{-rdt}\mathbb{E}_Y\left[\mathbb{R}(Y',F')\left((c+\xi)\,dt+(1-\xi dt)\,p(Y',F'')\right)\right]$$

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Government MC from future debt burden

$$-\partial_{F'}\left[e^{-rdt}\mathbb{E}_Y\left[\max\left(\underline{V}(Y'),\, V(Y',F')\right)\right]\right] = e^{-rdt}\mathbb{E}_Y\left[\mathbb{R}(Y',F')\cdot\overbrace{\left(-V_{F'}(Y',F')\right)}^{\text{envelope theorem}}\right]$$

$$= e^{-rdt}\mathbb{E}_Y\left[\mathbb{R}(Y',F')\left[((1-\pi)c+\xi)\,dt + (1-\xi dt)\,p(Y', F\right.\right.$$

$$= p(Y,F') - \underbrace{\mathbb{E}_Y\left[\mathbb{R}(Y',F')\right]e^{-rdt}\pi c dt}_{\text{DTS, source of gain from trade}}$$

Discrete-Time (3)

Government's FOC ($d\Gamma$: bond issuance)

$$\underbrace{p\left(Y,F'\right)+\overbrace{\partial_{F'}p\left(Y,F'\right)}^{<0\text{, price impact}}\cdot d\Gamma^*}_{\text{MB from today's consumption}}=\underbrace{p\left(Y,F'\right)-\mathbb{E}_Y\left[\mathbb{R}\left(Y',F'\right)\right]e^{-rdt}\pi cdt}_{\text{MC from future debt burden}}$$

Optimal Policy $\left(e^{-rdt}=1-rdt+o\left(dt\right)\right)$

$$d\Gamma^*=\overbrace{\mathbb{E}_Y\left[\mathbb{R}\left(Y',F'\right)\right]}^{\leq 1\text{, vanishing in continuous-time}}\cdot\frac{\pi c}{-\partial_{F'}p\left(Y,F'\right)}dt+o\left(dt\right)>0$$

- Smooth adjustment policy; and never buyback!

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Gain from trading over $\left[t-dt,t\right]$ **is bounded by** $O\left(dt^2\right)$

$$\left[\underbrace{p}_{\text{price}}-\underbrace{p-\mathbb{E}_Y\left[\mathbb{R}\left(Y',F'\right)\right]e^{-rdt}\pi c dt}_{\text{MC}}\right]\times\underbrace{d\Gamma^*}_{\text{quantity}}=\frac{\mathbb{E}_Y^2\left[\mathbb{R}\left(Y',F'\right)\right]\pi^2c^2}{-\partial_F p}\left(dt\right)^2+o\left(\right.$$

- Zero gain from trade; Coase conjecture

Leverage Ratchet and Smooth Equilibiurm

Never buyback outstanding debt

- **Proposition 1**. In equilirium $d\Gamma_t \geq 0$.
- Intuition. Postponing debt buyback—if any—gives shareholders a greater tax subsidy.

Price impact

- If $V(Y, F)$ is strictly convex in F so that there is a strictly positive price impact, then Γ_t is **continuous** in time
- Why jump is never optimal? Negative price impact always hurts (see later)

Focusing on "smooth equilibrium": $d\mathit{\Gamma}_t = G_t dt$

- Equity could adjust debt discretely, but not optimal in such an equilibrium
- Other equilibria with jumps? see the log-normal case later

Value Equivalence of No-Issuance (1)

- Hamilton-Jacobi-Bellman equation for equity value function $V(Y, F)$

$$
\begin{aligned}
rV(Y,F) &= \max_{G} \left[\underbrace{G\,p(Y,F)}_{\text{issuance/repurchase}} + \underbrace{(G - \xi F)\,V_F(Y,F)}_{\text{evolution of debt}} \right] \\
& Y - \pi(Y - cF) - (c + \xi)F + \mu(Y)\,V_Y(Y,F) + \frac{\sigma^2(Y)}{2} V_{YY}(Y,F)
\end{aligned}
$$

- Objective linear in G. Optimal $G \Rightarrow$ First-Order Condition

$$
\underbrace{p(Y,F)}_{\text{MB of issuance}} + \underbrace{V_F(Y,F)}_{\text{MC on future value}} = 0
$$

- Under FOC, equity indifferent at any G (given equilibrium p)
  - Linear control with interior solution (smooth policy $G_t dt$)
- Equity value can be solved by setting $G = 0$ always

Value Equivalence of No-Issuance (2)

- Equity value can be solved by setting $G = 0$ always

$$rV = -\xi F V_F + Y - \pi (Y - cF) - (c + \xi) F + \mu (Y) V_Y + \frac{\sigma^2 (Y)}{2} V_{YY}$$

- No gain in equilibrium by debt issuance/repurchase
  - Any potential tax shield gain is dissipated by bankruptcy cost caused by future excessive leverage
  - Reminiscent of Coase conjecture; DeMarzo and Urosevic (2006)
- **Get equity value $V(Y, F)$ without knowing debt price**

Equilibrium Policies

Basic idea

- Debt price $p(Y, F)$ must satisfy the valuation equation

$$p(Y, F) = \mathbb{E}_t \left\{ \int_t^{\tau_b} e^{-(r+\xi)(s-t)} (c + \xi) \, ds \right\}$$

- $V(Y, F)$ gives $-V_F(V, F) = p(Y, F)$ using equity's FOC
- How to make both match? Via debt management $G(Y, F)$
  - ODE for $V_F(V, F)$ (HJB for V) does not depend on G...
  - while HJB for p, which depends on G

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Equilibrium debt issuance policy

$$G^*(Y,F) = \frac{c \cdot \pi'(Y-cF)}{-p_F(Y,F)}$$

- $\pi'(Y-cF) \geq 0$, tax benefit $\Rightarrow$ always issuing debt
- Recall $-p_F(Y,F) = V_{FF}(Y,F) > 0$, capturing the price impact

Strict Optimality in Discrete Time

- Taking the value function at $t+h$ as given, consider equity's problem at t, where time interval $h>0$
- Denote debt issuance by Δ. Equity is maximizing

$$\max_{\Delta} \underbrace{-(1-\pi)\cdot\Delta c\cdot h}_{\text{after-tax interest payment}} + \underbrace{\Delta\left[c\cdot h+p\left(Y,F+\Delta\right)\right]}_{\text{new debt proceeds}} + \underbrace{V\left(F+\Delta,Y\right)}_{\text{future equity Value}}$$

- First-order condition w.r.t Δ

$$0=\pi c\cdot h+p\left(Y,F+\Delta\right)+\Delta\cdot p_F\left(Y,F+\Delta\right)+V_F\left(F+\Delta,Y\right)$$

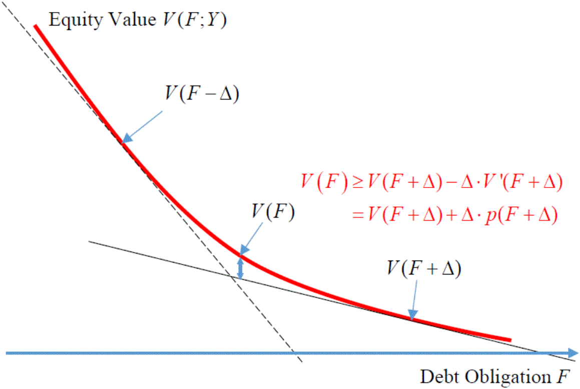
which implies that

$$\Delta=\frac{\underbrace{\pi c\cdot h}_{\text{tax benefit}}+\underbrace{p+V_F}_{\text{FOC}=0}}{-p_F}=\frac{\pi c}{-p_F}\cdot h$$

- One can easily check the global optimality

Sufficiency of Local FOC

Proposition 1: Global optimality of local FOC holds if debt price $p(Y, F) = -V_F(V, F)$ is non-increasing in debt F



- Debt price decreasing in $F \Leftrightarrow$ Equity value function is convex in F (option value of default)
  - Buyback, paying a higher price; selling too much hurts price too

Leverage Ratchet Effect

- What is the impact of debt repurchase on **equity value**?
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- Reducing debt today alleviates future default $\Rightarrow$ higher firm value
  - But does equity benefit strictly from this effect? **No.** (Do not forget existing debt holders!)
  - Equity optimizes default decision ex post already $\Rightarrow$ zero indirect impact on equity value today (**envelope theorem**)

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- Tax saving benefit always tempting...leverage ratcheting in ADHP
  - This paper: a more canonical setting
  - Same logic to debt overhang—equity is optimizing investment decisions ex post

Summary of General Model

1. Solve for equity value $V(Y, F)$ by setting $G(Y, F) = 0$
2. Set the equilibrium debt price $p(Y, F) = -V_F(Y, F)$
3. Check the equity holders' global optimality condition
   - Verifying $p(Y, F)$ is non-increasing in F (or $V(Y, F)$ is convex in F)
4. Equilibrium debt issuance $G^*(Y, F) = \frac{\pi'(Y-cF)\cdot c}{-p_F(Y,F)} > 0$

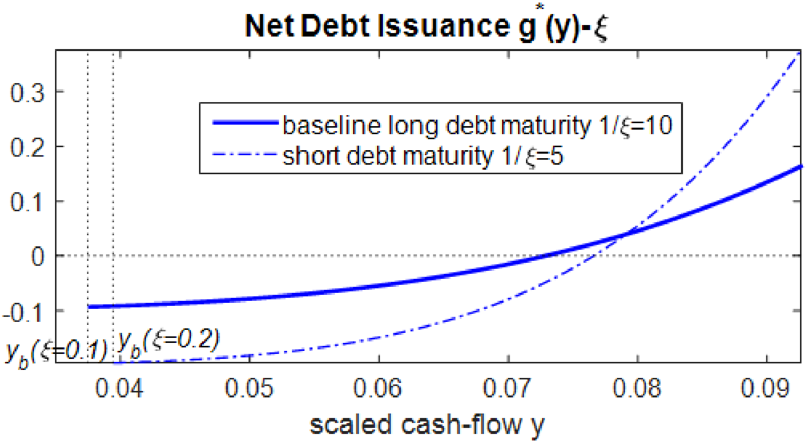
Log-Normal Cash-flows Model

- Scale-invariance, cash-flows $dY_t / Y_t = \mu dt + \sigma dZ_t$
  - The work-horse model of dynamic corporate finance
- One-dimensional state variable: scaled cash-flow $y_t \equiv Y_t / F_t$, Markov Perfect Equilibrium as in Maskin-Tirole (2001)
  - Equity value $V(Y, F) = F \cdot v(y)$, debt price $p(Y, F) = p(y)$; closed-form solutions
  - We prove such equilibrium is unique
- Let $g^*(y_t) \equiv G^*(Y_t, F_t) / F_t$, then

$$\frac{dy_t}{y_t} = \left(\underbrace{\mu}_{\text{CF growth}} + \underbrace{\xi}_{\text{debt maturing}} - \underbrace{g_t^*}_{\text{debt issuance}} \right) dt + \underbrace{\sigma dZ_t}_{\text{CF shocks}}$$

  - Debt growth rate $g_t^* - \xi$; endogenous $g_t^* = \frac{(r+\xi)\pi c}{c(1-\pi)+\xi} \frac{1}{\gamma} \left(\frac{y}{y_b} \right)^{\gamma} > 0$
    - γ is a constant depending on parameters
  - Increasing in y, i.e., more debt issuance after good fundamental

Net Debt Issuance $g^*(y) - \xi$, Debt Maturity



Two Benchmarks with Commitment

No future debt issuance:

- The firm commits to set $g_t = 0$ always (superscript 0)
- Equity value is the same (so does y_b), debt price is higher (by the tax shield)

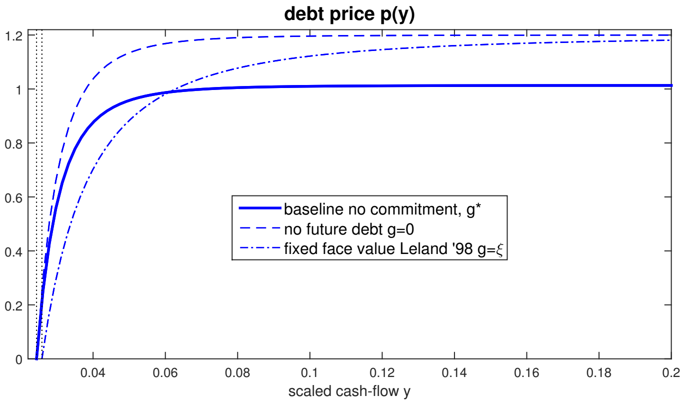
$$p^0(y) = p(y) + \frac{\pi c}{r + \xi}\left(1 - \left(\frac{y}{y_b}\right)^{-\gamma}\right)$$

- Less debt $\Rightarrow$ less likely to default (same y_b but y has a higher drift)

Fixed future debt:

- The firm commits to set $g_t = \xi$ always; Leland 1998

Model Comparisons: Debt Prices and Credit Spreads



Implication of credit spreads: $y \to \infty$ i.e. zero **current** leverage

- $p^{\xi}(y)$ and $p^{0}(y) \to \frac{c+\xi}{r+\xi}$, with zero credit spread
- $p(y) \to \frac{c(1-\pi)+\xi}{r+\xi}$, **non-zero credit spreads (high future excessive leverage!)**

Equilibrium Debt Dynamics

- Different from static trade-off setting, it is optimal to set $F_0^* = 0$
  - Knowing the future temptation of overborrowing....
- **Proposition.** Given cash-flow history $\{Y_s : 0 \leq s \leq t\}$, time-$t$ debt is ($\hat{y}_\xi$ is a constant depending on parameters)

$$F_t = \frac{1}{\hat{y}_\xi} \left[\int_0^t \gamma \xi Y_s^\gamma e^{-\gamma \xi (s-t)} ds \right]^{1/\gamma}$$

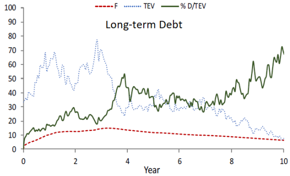
  - Start from $t = 0$ debt grows at the order of $t^{1/\gamma}$
  - Outstanding debt is average past earnings, with decaying weights $\gamma\xi$
- High mean-reverting speed, or more aggressive in adding leverage given high cash flows, when
  - Shorter debt maturity (higher ξ)

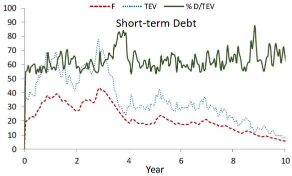
Optimal Debt Maturity Structure?

- So far the debt maturity structure ξ is taken as a parameter
- Say the firm gets a one-time chance to set ξ optimally for future debt issuance
- **Proposition**: Equity holders are **indifferent** at any ξ
  - Why? Because equity value is as if there is **no future debt issuance**...
- This indifference result holds more generally

Long-term vs. Short-term Debt

- Two firms start with zero debt, with different debt maturities (both being optimal)—but have different leverage dynamics/target
  - Lemmon, Roberts, and Zender (2008)





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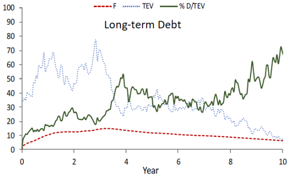

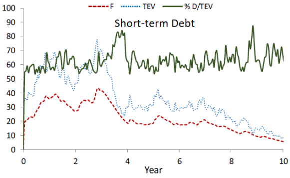

- With flexibility of shorter-term debt, the firm borrows more for higher debt tax shield
- But tax shield is a transfer from social perspective—so long-term debt is preferred to minimize bankruptcy cost

Investment

- Special case of log-normal process. Capital K_t evolves as

$$\frac{dK_t}{K_t} = (i_t - \delta)\, dt + \sigma dZ_t$$

  with quadratic investment cost $\frac{\kappa i_t^2}{2} K_t$, and output $Y_t = AK_t$

- Leverage ratchet effect prevails despite debt overhang considerations
- Equity issues debt more aggressively when controlling investment endogenously, compared to exogenous investment
  - Endogenous investment offers equity more protection later

Conclusion and Future Work

What we have done

- A general methodology solving dynamic corporate finance model without commitment
- Leverage policy depending on the entire earnings history, new insight on debt maturity and investment
- Slow initial adoption of leverage, but leads ultimately to excess

Future extensions

- DeMarzo, 2019 AFA presidential address: importance of exclusivity in collateralized borrowing
- Modeling sovereign debt and default (DeMarzo, He, and Tourre, 2019)
  - Covenant of no debt issuance once in distress (say for $y < \hat{y}$)
  - Discrete debt issuance (jump to $\hat{y}$) in equilibrium, counter-productive
- Internal cash with liquidity-driven default?