Valuation Uncertainty and Dynamic Entrepreneurial Finance

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...in early stage investing, valuations are voodoo. As a company gets more established, its valuation gets closer to an actual market value.

Paul Graham, co-founder – Y Combinator
Overview

- Borrowing-constrained entrepreneur who must raise external capital to fund fixed operating costs of project
- Flexible, dynamic security issuance
- Entrepreneur faces *Knightian uncertainty* specifically about valuations
  - Uncertainty *declines over time*, in part due to information revealed by security valuations
Overview

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- Flexible, dynamic security issuance
- Entrepreneur faces *Knightian uncertainty* specifically about valuations
  - Uncertainty *declines over time*, in part due to information revealed by security valuations
- **Main result:** Staged equity financing is optimal.
  - Resembles structure of early-stage entrepreneurial finance (angel investing, VC)
Static model
Static model

- Project with random terminal value $Y$
- $Y$ takes $n + 1$ possible values $\{0, y_1, \ldots, y_n\}$ according to probability measure $p$
- At time zero, agent must issue security with state-contingent payoff $S = (s_0, \ldots, s_n)$ subject to the limited liability constraint

$$0 \leq s_i \leq y_i$$

in order to finance fixed operating cost $I > 0$.
- Let $\pi(S)$ denote time-zero price of security payoff $S$. Project fails unless $\pi(S) \geq I$
Static model

- Entrepreneur maximizes expected utility over terminal wealth $W$
- Utility function $u(w)$ increasing, weakly concave, and finite at zero, with absolute risk aversion appropriately bounded
- Entrepreneur can divert excess cash raised for private benefit at rate $\theta \in [0, 1]$
For notational simplicity I assume riskless interest rate in this economy is zero.

- Represent valuations through risk-neutral probabilities \( q \).
- Interpret a specific vector \( q \) as a “valuation model”.
- Price of any security with payoff \( S \) given by risk-neutral expectation

\[
\pi(S; q) = s \cdot q = \mathbb{E}^q[S]
\]

- For the moment, take \( q \) as known to the entrepreneur (i.e. no valuation uncertainty).
Entrepreneur’s problem (no valuation uncertainty)

- Entrepreneur solves

$$\max_S \mathbb{E}\left[u\left( (Y - S)1_A + \theta M \right) \right]$$

subject to the constraints

$$0 \leq s_i \leq y_i \quad \forall i$$

$$M = (\pi(S; q) - l)$$

$$A \doteq \{\pi(S; q) \geq l\}$$

- $M$ denotes excess cash raised, and $\theta \in [0, 1]$ denotes private benefit to entrepreneur
Optimal security issue (no uncertainty)

When the entrepreneur is risk neutral and $q$ is known, optimal security payoff $S$ has following “all-or-nothing” form:

$$s_i = \begin{cases} 
  y_i & \text{if } q_i/p_i > C \\
  0 & \text{if } q_i/p_i < C \\
  \text{indeterminate} & \text{if } q_i/p_i = C 
\end{cases}$$

for all $i \neq 0$. Note that $s_0 = 0$ necessarily due to limited liability constraint.

- Optimal security is indeterminate in knife-edge case when $q \propto p$ on non-zero states
- More generally when the entrepreneur is risk-averse, any security with monotone (increasing or decreasing) payoff can be optimal
• Entrepreneur faces uncertainty about how securities will be priced by the market
  ▪ This may represent uncertainty about what investors believe about project, as well as how project risk co-varies aggregate shocks to investor wealth/consumption

• Entrepreneur now considers a set of valuation models

• Maximize against “worst-case” valuations in uncertainty set (Gilboa & Schmeidler 1989)
Uncertainty about security valuations

- Uncertainty is “bounded”: Entrepreneur knows:
  - Prices must be consistent with no arbitrage/law of one price
  - Prices must be consistent with interest rate $r = 0$ (i.e. $\pi(1) = 1$)
  - Bounds on time-0 valuation of project

$$\frac{1}{R}E[Y] \leq \pi(Y) \leq \frac{1}{\overline{R}} E[Y]$$

where $0 < R \leq \overline{R}$ with $\overline{R} > 1$. 
Risk-neutral probabilities

- As before, represent prices in terms of vector of risk-neutral probabilities \( q \)
- For any security payoff \( S \), price assigned by \( q \) is
  \[
  \pi(S; q) = \mathbb{E}^q[S] = s \cdot q
  \]
- Clearly probabilities must be positive \( q > 0 \) and sum to one \( (q \cdot 1 = 1) \)
- Valuation bound implies
  \[
  \overline{R}^{-1} \leq y \cdot q \leq R^{-1}
  \]
- Denote by \( Q \) the set of risk-neutral probability vectors satisfying these restrictions
Entrepreneur’s problem with uncertainty

- Entrepreneur solves

\[
\max_S \min_{q \in \mathcal{Q}} \mathbb{E} \left[ u \left( (Y - S) \mathbf{1}_A + \theta M \right) \right]
\]

subject to the constraints

\[
0 \leq s_i \leq y_i \quad \forall i
\]

\[
M = (\pi(S; q) - l)
\]

\[
A \triangleq \{ \pi(S; q) \geq l \}.
\]
Results

- **Result 1**: Optimal security is **equity** \((S = a Y)\)
  - **Intuition**: Entrepreneur is uncertain which payoff states are the most important to investors. Non-linear securities give more freedom in constructing hypothetical “worst-case” valuation

- **Result 2**: Entrepreneur issues just enough equity to cover operating cost \(I > 0\) under worst-case equity valuation
  - **Intuition**: All capital raised is equity, effectively discounted at worst-case discount rate on equity \(\overline{R} > 1\)
Figure 1: Three-state example. Worst-case iso-price line shown in red. Entrepreneur indifference curve shown in blue. Equity securities correspond do dashed line.
A natural concern is that the uncertainty here is too “extreme”

Suppose we reduce the entrepreneur’s uncertainty set by imposing the additional constraint on hypothetical valuation models $q$

$$c(q) \leq \kappa$$

where $c(\cdot)$ is a continuous, convex function.
Figure 2: Three-state examble. Worst-case iso-price line shown in red. Entrepreneur indifference curve shown in blue. Equity securities correspond do dashed line.
Price-contingent securities

- When entrepreneur faces uncertainty, entrepreneur may in principle want to issue securities whose terminal payoff conditions on realized security price.
- This was implicitly ruled out.
- Compare optimal “fixed-fraction” equity security

\[
S = I \cdot \frac{Y}{\bar{R}^{-1} \mathbb{E}[Y]}
\]

with “fixed-dollar” equity security

\[
\tilde{S} = I \cdot \frac{Y}{\pi(Y)}
\]
Price-contingent securities

- **Result:** In equilibrium, no value to conditioning security payoffs on prices
- However, fixed-dollar equity weakly dominates fixed fraction equity
- Applying an appropriate admissibility criterion (e.g. Brandenburger et al. 2008) yields fixed dollar equity issuance as equilibrium security issuance policy
Discussion of static model:

- From the static model, we were able to see that (1) equity is uniquely optimal and (2) entrepreneur does not want to raise more capital than necessary under worst-case valuation

- **Limitations:**
  - In static model, no sense in which uncertainty can decline over time (in particular, no learning through prices)
  - Cannot speak to timing of when to raise capital (i.e. raise all capital at once, or gradually over time)
  - Not informative about convertible securities
Three-period model
Next consider a slight generalization to a three-period model with \( t \in \{0, 1, 2\} \).

- Terminal payoff \( Y_2 \)
- At \( t \in \{1, 2\} \) observe signal \( i_t \in \{0, \ldots, n\} \) according to probabilities \( p = (p_0, \ldots, p_n) \)
- Firm “grows” at rate \( g_{it} \) for \( g = (g_0, \ldots, g_n) \) with \( g_0 = -1 \) (project has failure risk in each period)

\[
Y_2 = Y_0(1 + g_{i_1})(1 + g_{i_2})
\]

\[
Y_t = \mathbb{E}_t[Y_T].
\]

- Fixed costs \( l_0, l_1 \) which must be financed by security issuance at time 0 and time 1 (denoted by \( S^{(0)} \) and \( S^{(1)} \))
**Three-period model**

*Figure 3:* Illustration of possible paths of $Y_t$ in 3-period model. Number of non-zero states $n = 2$. 

![Diagram of possible paths of $Y_t$ in 3-period model.](image)
Risk-neutral probabilities

- Represent prices using one-period risk-neutral transition probabilities $q^{(0)}$, $q^{(1)}(i_1)$
- As before, must represent valid probabilities (positive, sum to one)
- Additional discount rate restriction

$$\overline{R}^{-1} \leq q^{(t)} \cdot (1_{n+1} + g) \leq R^{-1}$$

- Denote by $Q^{(0)}$ the set of pairs $\{q^{(0)}, q^{(1)}(i_1)\}$ satisfying these restrictions
Learning from prices

- I allow for the entrepreneur to learn from realized prices
- Assume time-zero security with payoff $S^{(0)}$ has realized price $\pi(S^{(0)})$
- Time-1 uncertainty set $Q^{(1)}$

$$Q^{(1)} = \left\{ q \in \mathbb{R}^{n+1} : \exists \tilde{q} \in \overline{Q}^{(0)} \quad \text{with} \quad \tilde{q}^{(1)}(i_1) = q \right\}$$

and $\pi^{(0)}(S^{(0)}; \tilde{q}) = \pi^{(0)}(S^{(0)})$

- Set $Q^{(1)}$ is smaller than $Q^{(0)}$ both due to the passage of time and learning from prices
Entrepreneur’s problem

- Formulate problem of entrepreneur as two-player, zero-sum game between entrepreneur and adversarial agent choosing valuations consistent with time-\( t \) uncertainty set \( Q(t) \)

- Terminal payoff to entrepreneur:

\[
u \left( \left( Y_2 - S_2^{(0)} - S_2^{(1)} \right) 1_A + \theta M_2 \right)\]

where \( A \) denotes set where intertemporal budget constraint is satisfied (\( A = \{ M_t \geq 0 \ \forall t \in \{1, 2\} \} \))

- All securities chosen by entrepreneur must satisfy limited liability constraint
Entrepreneur’s problem

**Period 0:**

1. Entrepreneur chooses security payoff $S_0^{(2)}$
2. Adversarial agent chooses $\pi(S_2^{(0)})$ consistent with $Q^{(0)}$

**Period 1:**

1. $i_1$ is realized
2. Entrepreneur chooses security payoff $S_2^{(1)}$
3. Adversarial agent chooses $\pi(S_2^{(1)})$ consistent with $Q^{(1)}$

**Period 2:**

1. $i_2$ is realized
2. Terminal payoff realized
Observation:

- Right now, security payoffs not allowed to depend on realized prices (even of own security)
  - Can address this by effectively exchanging orders of maximization and minimization
Results:

- Pay-as-you-go equity issuance is optimal
  - At each period, entrepreneur issues just enough equity to pay cost $I_t$ under worst-case valuation
  - Never wants to over-shoot constraint
- On equilibrium path, no benefit to conditioning payoffs on future realized prices
  - Fixed-dollar equity issuance weakly dominates fixed-fraction equity issuance
Results:

- Time-0 expected payoff of the entrepreneur has following intuitive form:

\[
\bar{R}^2 \left[ \frac{\mathbb{E}_0[Y_2]}{\bar{R}^2} \right] - \left( l_0 + \left( 1 - p_0 \right) l_1 \right)
\]

worst-case total value expected operating costs discounted at $\bar{R}$
Proof (sketch)

- Repeatedly leverage max-min inequality to bound equilibrium payoff of entrepreneur.
- Consider static problem where entrepreneur commits at time zero to state-contingent security issuance policy. This gives a lower bound on entrepreneur’s payoff.
- Consider fixed $q^*$ coming from previous problem (may not be unique). Maximizing against this gives upper bound on entrepreneur’s equilibrium payoff.
- Observe that lower and upper bounds are equal, and obtained by staged equity issuance.
What happens if valuation is better than expected?

- Right now equilibrium concept effectively assumes worst-case valuations will be realized.
- What happens if at time zero, equity valuation is better than worst-case?
What happens if valuation is better than expected?

- Right now equilibrium concept effectively assumes worst-case valuations will be realized
- What happens if at time zero, equity valuation is better than worst-case?
  - Form of time-1 uncertainty set stays constant, with a (possibly) increased value of $\bar{R}$
  - Therefore equity issuance remains optimal
Generalizations

- Straightforward to generalize to $T$-period model
  - Main results (staged equity issuance is optimal, no benefit to conditioning on prices) go through
- Moral hazard faced by the entrepreneur
  - Incentive compatibility restricts space of securities. If staged equity is incentive compatible, then it remains optimal
Convertible securities
Two types of convertible securities used in entrepreneurial financing:

- Convertible notes (e.g. SAFEs)
- Convertible debt/preferred stock

Both feature equity-like cash flow rights, but differ in important ways
Convertible notes

- Early-stage “angel” investors typically invest using equity securities
- Non-trivial minority invest using convertible notes
- Simplest type of convertible note is a *Simple Agreement for Future Equity* (SAFE) which offers conversion to a fixed-dollar amount of equity at first equity funding round
  - Obvious advantages of (1) simplicity and (2) deferring valuation to future (when uncertainty is lower)
- One might naturally expect issuing a SAFE to be at least weakly preferred by the entrepreneur
Assume for concreteness that entrepreneur pre-commits to issue equity at time 1

At time-0, entrepreneur issues SAFE, which converts to $\beta$ of equity at time 1

State-contingent payoff of SAFE is

$$\beta \frac{Y_2}{\pi^{(1)}(Y_2)} \mathbf{1}(Y_1 > 0)$$

Strictly sub-optimal for entrepreneur!

One-period on SAFE acts like bet on survival of firm, which introduces non-linearity
Convertible debt

- Convertible debt or convertible preferred equity contracts are frequently used in late-stage VC rounds.
- Differ from equity in how they handle liquidation.
- Consider extension where firm has non-zero liquidation value $y_0$, and entrepreneur has non-pecuniary benefit $u_0 < y_0$ of inefficient continuation.
- In this case, optimal security features equity-like payoff but allocates control rights over liquidation decision to outside investor.
Discussion and Conclusion
Discussion and Conclusion

- Optimal choice of financing by entrepreneur who faces (declining) uncertainty about security valuations
- Showed that staged equity financing is optimal
- Valuation uncertainty leads to intuitively “simpler” contracts
- Future exploration: interaction between valuation uncertainty and collateral value, and implications for dynamic debt contracts