# Valuation Uncertainty and Dynamic Entrepreneurial Finance

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# Motivation

*...in early stage investing, valuations are voodoo. As a company gets more established, its valuation gets closer to an actual market value.*

Paul Graham, co-founder – Y Combinator

# Overview

- Borrowing-constrained entrepreneur who must raise external capital to fund fixed operating costs of project
- Flexible, dynamic security issuance
- Entrepreneur faces *Knightian uncertainty* specifically about valuations
    - Uncertainty *declines over time*, in part due to information revealed by security valuations

# Overview

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- Entrepreneur faces *Knightian uncertainty* specifically about valuations
    - Uncertainty *declines over time*, in part due to information revealed by security valuations
- **Main result:** Staged equity financing is optimal.
    - Resembles structure of early-stage entrepreneurial finance (angel investing, VC)

# Static model

# Static model

- Project with random terminal value $Y$
- $Y$ takes $n+1$ possible values $\{0, y_1, ..., y_n\}$ according to probability measure $\mathbf{p}$
- At time zero, agent must issue security with state-contingent payoff $S = (s_0, ..., s_n)$ subject to the limited liability constraint

$$0 \leq s_i \leq y_i$$

in order to finance fixed operating cost $I > 0$.
- Let $\pi(S)$ denote time-zero price of security payoff $S$. Project fails unless $\pi(S) \geq I$

# Static model

- Entrepreneur maximizes expected utility over terminal wealth $W$
- Utility function $u(w)$ increasing, weakly concave, and finite at zero, with absolute risk aversion appropriately bounded
- Entrepreneur can divert excess cash raised for private benefit at rate $\theta \in [0, 1]$

# Security valuations

- For notational simplicity I assume riskless interest rate in this economy is zero
- Represent valuations through risk-neutral probabilities $\mathbf{q}$
- Interpret a specific vector $\mathbf{q}$ as a "valuation model"
- Price of any security with payoff $S$ given by risk-neutral expectation

$$\pi(S; \mathbf{q}) = s \cdot \mathbf{q} \doteq \mathbb{E}^{\mathbf{q}}[S]$$

- For the moment, take $\mathbf{q}$ as known to the entrepreneur (i.e. no valuation uncertainty)

# Entrepreneur's problem (no valuation uncertainty)

- Entrepreneur solves

$$\max_{S} \ \mathbb{E}\left[u\Big((Y - S)\mathbf{1}_{\mathcal{A}} + \theta M\Big)\ \right]$$

subject to the constraints

$$0 \leq s_i \leq y_i \quad \forall i$$
$$M = (\pi(S; \mathbf{q}) - I)$$
$$\mathcal{A} \doteq \{\pi(S; \mathbf{q}) \geq I\}$$

- $M$ denotes excess cash raised, and $\theta \in [0, 1]$ denotes private benefit to entrepreneur

# Optimal security issuance (no uncertainty)

When the entrepreneur is risk neutral and $\mathbf{q}$ is known, optimal security payoff $S$ has following "all-or-nothing" form:

$$s_i = \begin{cases} y_i & \text{if } q_i/p_i > C \\ 0 & \text{if } q_i/p_i < C \\ \text{indeterminate} & \text{if } q_i/p_i = C \end{cases}$$

for all $i \neq 0$. Note that $s_0 = 0$ necessarily due to limited liability constraint

- Optimal security is indeterminate in knife-edge case when $\mathbf{q} \propto \mathbf{p}$ on non-zero states
- More generally when the entrepreneur is risk-averse, any security with monotone (increasing or decreasing) payoff can be optimal

# Uncertainty about security valuations

- Entrepreneur faces uncertainty about how securities will be priced by the market
  - This may represent uncertainty about what investors believe about project, as well as how project risk co-varies aggregate shocks to investor wealth/consumption
- Entrepreneur now considers a *set* of valuation models
- Maximize against "worst-case" valuations in uncertainty set (Gilboa & Schmeidler 1989)

# Uncertainty about security valuations

- Uncertainty is "bounded": Entrepreneur knows:
  - Prices must be consistent with no arbitrage/law of one price
  - Prices must be consistent with interest rate $r = 0$ (i.e. $\pi(1) = 1$)
  - Bounds on time-0 valuation of project

$$\frac{1}{\overline{R}}\mathbb{E}[Y] \leq \pi(Y) \leq \frac{1}{\underline{R}}\mathbb{E}[Y]$$

where $0 < \underline{R} \leq \overline{R}$ with $\overline{R} > 1$.

# Risk-neutral probabilities

- As before, represent prices in terms of vector of risk-neutral probabilities $\mathbf{q}$
- For any security payoff $S$, price assigned by $\mathbf{q}$ is

$$\pi(S; \mathbf{q}) = \mathbb{E}^{\mathbf{q}}[S] = s \cdot \mathbf{q}$$

- Clearly probabilities must be positive $\mathbf{q} > 0$ and sum to one ($\mathbf{q} \cdot \mathbf{1} = 1$)
- Valuation bound implies

$$\overline{R}^{-1} \leq y \cdot \mathbf{q} \leq \underline{R}^{-1}$$

- Denote by $\mathcal{Q}$ the set of risk-neutral probability vectors satisfying these restrictions

# Entrepreneur's problem with uncertainty

- Entrepreneur solves

$$\max_{S} \min_{\mathbf{q} \in \overline{\mathcal{Q}}} \mathbb{E}\left[ u\left( (Y - S)\,\mathbf{1}_{\mathcal{A}} + \theta M \right) \right]$$

subject to the constraints

$$\begin{aligned} 0 \leq s_i \leq y_i \;\; \forall i \\ M = (\pi(S; \mathbf{q}) - I) \\ \mathcal{A} \doteq \{\pi(S; \mathbf{q}) \geq I\}\,. \end{aligned}$$

# Results

- **Result 1:** Optimal security is **equity** ($S = \alpha Y$)
    - **Intuition:** Entrepreneur is uncertain which payoff states are the most important to investors. Non-linear securities give more freedom in constructing hypothetical "worst-case" valuation
- **Result 2:** Entrepreneur issues just enough equity to cover operating cost $I > 0$ under worst-case equity valuation
    - **Intuition:** All capital raised is equity, effectively discounted at worst-case discount rate on equity $\overline{R} > 1$

# Results

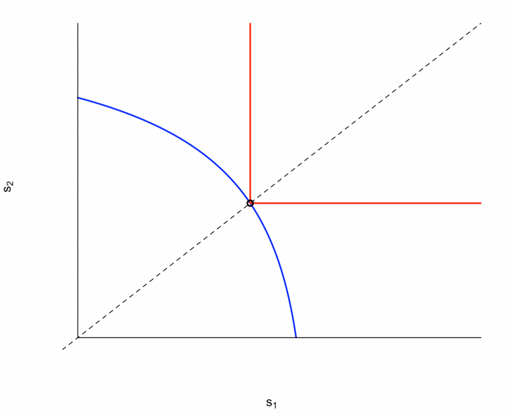


**Figure 1:** Three-state examble. Worst-case iso-price line shown in red. Entrepreneur indifference curve shown in blue. Equity securities correspond do dashed line.

# Alternate formulations of uncertainty

- A natural concern is that the uncertainty here is too “extreme”
- Suppose we reduce the entrepreneur’s uncertainty set by impsing the additional constraint on hypothetical valuation models $\mathbf{q}$

$$c(\mathbf{q}) \leq \kappa$$

  where $c(\cdot)$ is a continuous, convex function

# Alternate formulations of uncertainty

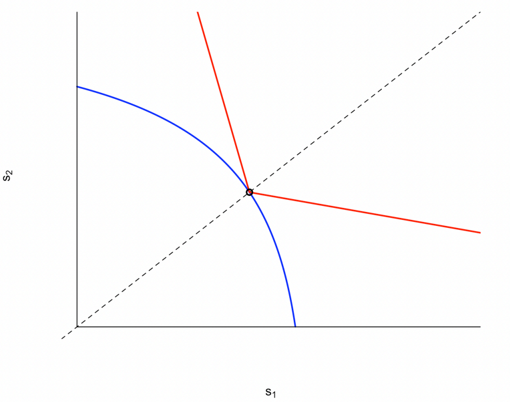


**Figure 2:** Three-state examble. Worst-case iso-price line shown in red. Entrepreneur indifference curve shown in blue. Equity securities correspond do dashed line.

# Price-contingent securities

- When entrepreneur faces uncertainty, entrepreneur may in principle want to issue securities whose terminal payoff conditions on realized security price
- This was implicitly ruled out
- Compare optimal “fixed-fraction” equity security

$$S = I \cdot \frac{Y}{\overline{R}^{-1}\mathbb{E}[Y]}$$

with “fixed-dollar” equity security

$$\widetilde{S} = I \cdot \frac{Y}{\pi(Y)}$$

# Price-contingent securities

- **Result:** In equilibrium, no value to conditioning security payoffs on prices
- However, fixed-dollar equity weakly dominates fixed fraction equity
- Applying an appropriate admissibility criterion (e.g. Brandenburger et al. 2008) yields fixed dollar equity issuance as equilibrium security issuance policy

# Discussion of static model:

- From the static model, we were able to see that (1) equity is uniquely optimal and (2) entrepreneur does not want to raise more capital than necessary under worst-case valuation
- **Limitations:**
  - In static model, no sense in which uncertainty can decline over time (in particular, no learning through prices)
  - Cannot speak to timing of when to raise capital (i.e. raise all capital at once, or gradually over time)
  - Not informative about convertible securities

# Three-period model

# Three-period model

- Next consider a slight generalization to a three-period model with $t \in \{0, 1, 2\}$
- Terminal payoff $Y_2$
- At $t \in \{1, 2\}$ observe signal $i_t \in \{0, ..., n\}$ according to probabilities $\mathbf{p} = (p_0, ..., p_n)$
- Firm "grows" at rate $\mathbf{g}_{i_t}$ for $\mathbf{g} = (g_0, ..., g_n)$ with $g_0 = -1$ (project has failure risk in each period)

$$Y_2 \doteq Y_0(1 + g_{i_1})(1 + g_{i_2})$$

$Y_t \doteq \mathbb{E}_t[Y_T]$.

- Fixed costs $I_0, I_1$ which must be financed by security issuance at time 0 and time 1 (denoted by $S^{(0)}$ and $S^{(1)}$)

# Three-period model

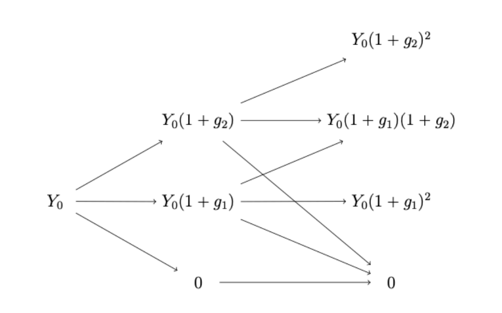


**Figure 3:** Illustration of possible paths of $Y_t$ in 3-period model. Number of non-zero states $n = 2$.

# Risk-neutral probabilities

- Represent prices using one-period risk-neutral transition probabilities $\mathbf{q}^{(0)}, \mathbf{q}^{(1)}(i_1)$
- As before, must represent valid probabilities (positive, sum to one)
- Additional discount rate restriction

$$\overline{R}^{-1} \leq \mathbf{q}^{(t)} \cdot (\mathbf{1}_{n+1} + \mathbf{g}) \leq \underline{R}^{-1}$$

- Denote by $\mathcal{Q}^{(0)}$ the set of pairs $\left\{\mathbf{q}^{(0)}, \mathbf{q}^{(1)}(i_1)\right\}$ satisfying these restrictions

# Learning from prices

- I allow for the entrepreneur to learn from realized prices
- Assume time-zero security with payoff $S^{(0)}$ has realized price $\pi(S^{(0)})$
- Time-1 uncertainty set $\mathcal{Q}^{(1)}$

$$\mathcal{Q}^{(1)} \doteq \Big\{ \mathbf{q} \in \mathbb{R}^{n+1} : \exists \widetilde{\mathbf{q}} \in \overline{\mathcal{Q}}^{(0)} \text{ with } \widetilde{\mathbf{q}}^{(1)}(i_1) = \mathbf{q} \text{ and } \pi^{(0)}(S_2^{(0)}; \widetilde{\mathbf{q}}) = \pi^{(0)}(S_2^{(0)}) \Big\}.$$

- Set $\mathcal{Q}^{(1)}$ is smaller than $\mathcal{Q}^{(0)}$ both due to the passage of time and learning from prices

# Entrepreneur's problem

- Formulate problem of entrepreneur as two-player, zero-sum game between entrepreneur and adversarial agent choosing valuations consistent with time-$t$ uncertainty set $\mathcal{Q}^{(t)}$
- Terminal payoff to entrepreneur:

$$u\left(\left(Y_2 - S_2^{(0)} - S_2^{(1)}\right)\mathbf{1}_{\mathcal{A}} + \theta M_2\right)$$

  where $\mathcal{A}$ denotes set where intertemporal budget constraint is satisfied ($\mathcal{A} = \{M_t \geq 0 \ \forall t \in \{1, 2\}\}$)
- All securities chosen by entrepreneur must satisfy limited liability constraint

# Entrepreneur's problem

**Period 0:**

1. Entrepreneur chooses security payoff $S_0^{(2)}$
2. Adversarial agent chooses $\pi(S_2^{(0)})$ consistent with $\overline{\mathcal{Q}}^{(0)}$

**Period 1:**

1. $i_1$ is realized
2. Entrepreneur chooses security payoff $S_2^{(1)}$
3. Adversarial agent chooses $\pi(S_2^{(1)})$ consistent with $\overline{\mathcal{Q}}^{(1)}$

**Period 2:**

1. $i_2$ is realized
2. Terminal payoff realized

# Observation:

- Right now, security payoffs not allowed to depend on realized prices (even of own security)
    - Can address this by effectively exchanging orders of maximization and minimization

## Results:

- Pay-as-you-go equity issuance is optimal
  - At each period, entrepreneur issues just enough equity to pay cost $I_t$ under worst-case valuation
  - Never wants to over-shoot constraint
- On equilibrium path, no benefit to conditioning payoffs on future realized prices
  - Fixed-dollar equity issuance weakly dominates fixed-fraction equity issuance

# Results:

- Time-0 expected payoff of the entrepreneur has following intuitive form:

$$\overline{R}^2 \left[ \underbrace{\frac{\mathbb{E}_0[Y_2]}{\overline{R}^2}}_{\text{worst-case total value}} - \underbrace{\left( I_0 + \frac{(1-p_0)I_1}{\overline{R}} \right)}_{\text{expected operating costs discounted at } \overline{R}} \right]$$

## Proof (sketch)

- Repeatedly leverage max-min inequality to bound equilibrium payoff of entrepreneur
- Consider static problem where entrepeneur commits at time zero to state-contingent security issuance policy. This gives a lower bound on entrepeneur's payoff.
- Consider fixed $\mathbf{q}^*$ coming from previous problem (may not be unique). Maximizing against this gives upper bound on entrepeneur's equilibrium payoff.
- Observe that lower and upper bounds are equal, and obtained by staged equity issuance.

# What happens if valuation is better than expected?

- Right now equilibrium concept effectively assumes worst-case valuations will be realized
- What happens if at time zero, equity valuation is better than worst-case?

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- Right now equilibrium concept effectively assumes worst-case valuations will be realized
- What happens if at time zero, equity valuation is better than worst-case?
  - Form of time-1 uncertainty set stays constant, with a (possibly) increased value of $\overline{R}$
  - Therefore equity issuance remains optimal

# Generalizations

- Straightforward to generalize to $T$-period model
  - Main results (staged equity issuance is optimal, no benefit to conditioning on prices) go through
- Moral hazard faced by the entrepreneur
  - Incentive compatibility *restricts* space of securities. If staged equity is incentive compatible, then it remains optimal

# Convertible securities

# Convertibles

Two types of convertible securities used in entrepreneurial financing:

- Convertible notes (e.g. SAFEs)
- Convertible debt/preferred stock

Both feature equity-like cash flow rights, but differ in important ways

# Convertible notes

- Early-stage "angel" investors typically invest using equity securities
- Non-trivial minority invest using convertible notes
- Simplest type of convertible note is a *Simple Agreement for Future Equity* (SAFE) which offers conversion to a fixed-dollar amount of equity at first equity funding round
    - Obvious advantages of (1) simplicity and (2) deferring valuation to future (when uncertainty is lower)
- One might naturally expect issuing a SAFE to be at least weakly preferred by the entrepreneur

# Convertible notes

- Assume for concreteness that entrepreneur pre-commits to issue equity at time 1
- At time-0, entrepreneur issues SAFE, which converts to \$$\beta$ of equity at time 1
- State-contingent payoff of SAFE is

$$\beta \frac{Y_2}{\pi^{(1)}(Y_2)} \mathbf{1}(Y_1 > 0)$$

- **Strictly sub-optimal** for entrepreneur!
- One-period on SAFE acts like bet on survival of firm, which introduces non-linearity

# Convertible debt

- Convertible debt or convertible preferred equity contracts are frequently used in late-stage VC rounds
- Differ from equity in how they handle liquidation
- Consider extension where firm has non-zero liquidation value $y_0$, and entrepreneur has non-pecuniary benefit $u_0 < y_0$ of inefficient continuation
- In this case, optimal security features equity-like payoff but allocates control rights over liquidation decision to outside investor

# Discussion and Conclusion

# Discussion and Conclusion

- Optimal choice of financing by entrepreneur who faces (declining) uncertainty about security valuations
- Showed that staged equity financing is optimal
- Valuation uncertainty leads to intuitively “simpler” contracts
- Future exploration: interaction between valuation uncertainty and collateral value, and implications for dynamic debt contracts