Internet Appendix
"Intermediary Balance Sheets and the Treasury Yield Curve"

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A Dealers and Levered Clients

In Section 2, we developed net long and net short curves from the perspective of a securities dealer, yields at which the dealer would be willing to either net long or net short Treasury bonds. In this section, we extend our model to consider the perspective of a levered Treasury investor (e.g. a hedge fund) financed by a security dealer of the kind considered in that section. The main result is that levered clients will have the same net long and net short curves as the dealer that finances them. That is, the net long curve represents a yield at which the levered client would be willing to buy the Treasury bond, irrespective of its beliefs about the stochastic process driving Treasury yields, and a symmetric result holds for the net short curve. This result occurs in spite of the fact that the levered client is not itself directly affected by balance sheet constraints.

This result is important from a general equilibrium perspective. Dealers are never on net long or short a large quantity of Treasury bonds during our sample, relative to the overall Treasury supply. Dealers moved from a net short of roughly 100 hundred billion in 2005 to a net long of 200 hundred billion in 2020. The overall supply of Treasury securities rose from 4 trillion to 22 trillion over the same period.

However, dealers intermediate repo and reverse-repo for their levered clients in much greater quantities– on the order of trillions each day. In this section we will argue that the recipients of much of this financing will act like dealers, and subsequently provide some suggestive evidence on this point.

Consider the following trading strategy for the dealer: finance a client's purchase of a Treasury using bilateral repo, use the resulting collateral to raise financing in the tri-party repo market, and reduce CIP activity so that the trade is balance sheet neutral. In a competitive market, the profits of such a strategy are zero:

$$\underbrace{(1-h)(e^{\frac{1}{12}r_t^{bi}}-e^{\frac{1}{12}r_t^{tri}})}_{\text{Lending/Borrowing Spread}}-\underbrace{(1-h)(e^{\frac{1}{12}r_t^{syn}}-e^{\frac{1}{12}r_t^{ois}})}_{\text{Forgone CIP profits}}=0. \tag{A1}$$

That is, the dealer must be indifferent between matched book repo lending and taking advantage of CIP arbitrage, as both activities use balance sheet. Note that this is expressed per dollar of Treasury collateral, and that we have assumed the same haircut in both markets.

Let's now consider the perspective of a levered client who can purchase a Treasury bond, financed by this intermediary, and can trade derivatives with the securities dealer. Because the levered client can trade derivatives with the dealer, the projection of its stochastic discount factor onto the space of derivative returns must agree with the same projection for the dealer's SDF. Equivalently, the risk-neutral measure Q is shared (within this space) by the levered clients and the dealer.

We will also assume that the levered client can engage in risk-free unsecured borrowing[1] from the unsecured dealer at the synthetic lending rate. The dealer is unwilling to lend at a rate lower than this, as otherwise it would be better off engaging in CIP arbitrage.

Under these assumptions, the levered client considers buying an n-month Treasury and then selling one month later:

$$(1-h)\cdot \underbrace{e^{-ny_{n,t}}e^{r_t^{bi}}}_{\text{secured financing}} + h\cdot \underbrace{e^{-ny_{n,t}}e^{r_t^{syn}}}_{\text{unsecured financing}} \geq E_t^{\mathbb{Q}}[e^{-(n-1)y_{n-1,t+1}}]. \tag{A2}$$

Substituting in (A1), this condition becomes identical to (7). It follows immediately that levered clients must be willing to go net long if the yield reaches the net long curve.

Essentially identical logic applies to the net short curve: the dealers indifference between matched book repo (in the net short case, intermediating between security lenders and short-sellers) and CIP arbitrage converts the levered client's indifference condition to the dealer's indifference condition.

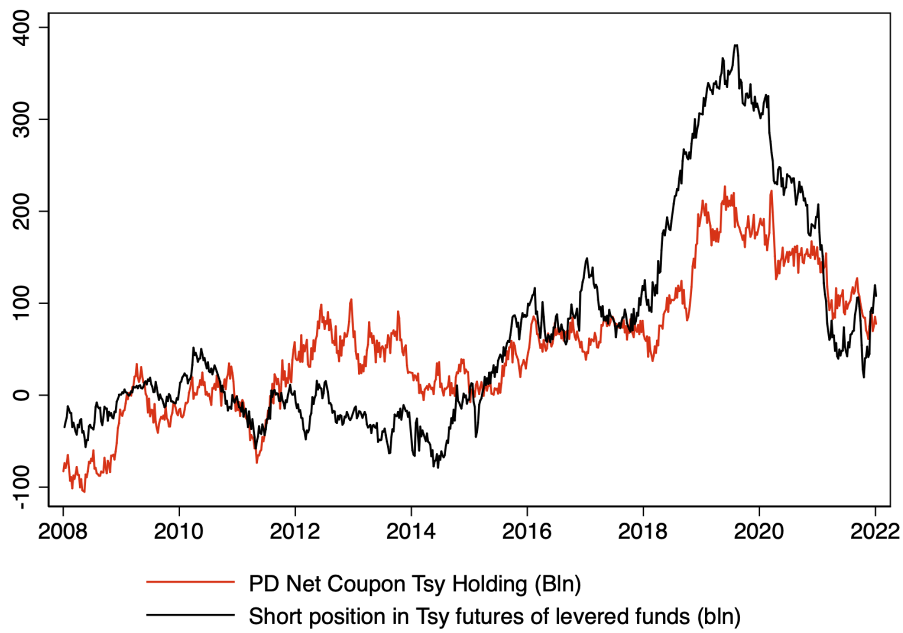
We conclude that levered clients who are dependent on dealers for financing will act as if they face the same balance sheet costs that dealers face, even if they are not themselves directly regulated. As a result, balance sheets costs will influence a substantial segment of the Treasury market, even though dealers are on their own hold a relatively small quantity of Treasury bonds on net.

Below, we provide evidence consistent with this perspective. While the Treasury positions of

[1] It is probably better to think of this as secured borrowing using non-Treasury securities that the dealer cannot itself finance in a repo market.

levered investors are not publicly available, we can infer the holdings of investors that engage in Treasury cash-future trades from Treasury futures positions. Figure A1 plots the primary dealer net coupon holdings and levered funds' short positions in Treasury futures contracts published by the CTFC. For relative value hedge funds that arbitrage Treasury cash-futures basis, a short position in Treasury futures corresponds to a long position in the cash Treasury bonds. We see that primary dealer positions and the levered funds' short Treasury futures position are strongly positively correlated, which is consistent with our result that dealers and levered investors take similar positions and can be considered as a consolidated intermediary.

Figure A1: Primary Dealer Treasury Holdings and Implied Treasury Holding of Levered Investors



Notes: This figure plots the primary dealer's net position in coupon-bearing Treasury securities from Primary Dealer Statistics published by the Federal Reserve Bank of New York, and the short position in the Treasury futures market by levered funds from the Commitments of Traders Report published by the Commodity Futures Trading Commission.

B Partial Equilibrium Arbitrage Bounds

In this appendix section, we construct the net short and net long curves described in the main text as arbitrage bounds under weaker assumptions than those employed in the main text. In particular, in the main text we assumed that zero-cost, zero-balance sheet trades are weakly unattractive under a common SDF (i.e., a version of the no-arbitrage assumption). That assumption leads to $y_{n-1,t+1} \leq y^b_{n-1,t+1}$ with probability one. Here, we instead assume that there could be profitable zero-cost, zero-balance sheet trading strategies under the intermediary's stochastic discount factor. Then we consider the question of whether this intermediary is willing to go net long or net short a Treasury bond, irrespective of the intermediary's preferences or beliefs about the stochastic process driving Treasury yields.

We will assume that this intermediary's SDF prices derivatives, and that the intermediary believes with probability one that $x_{1,t} \geq r^{ois}_t \geq x_{2,t}$, where $x_{1,t}$ and $x_{2,t}$ are defined as in the main text. We discuss the role of this assumption below.

B.1 The Net Long Curve

Consider first the trade in which the intermediary buys a zero-coupon seven-month Treasury bond, and then sells it in one month, at which time the Treasury bond becomes a zero-coupon T-bill. The intermediary can finance this purchase with tri-party repo, up to the standard two percent haircut h, and finance the remainder with unsecured debt. This trade, in combination with a reduction in CIP activity, is a balance-sheet neutral, zero-financing trade. The intermediary is therefore willing to get net long if this strategy is weakly appealing under the SDF that prices derivatives. Let $\mathbb{Q}$ denote the risk-neutral measure associated with this SDF. We assume that r^{ois}_t is the log risk-free rate associated with this SDF.[2]

Let $y^b_{7,t}$ denote a yield at which this trade is attractive to the dealer, and define $y^b_{6,t} = y^{bill}_t$. The

[2] That is, we assume the one-month OIS swap rate is the intermediary's unsecured borrowing rate. This assumption is consistent with the empirical observation that the one-month OIS rate closely tracks other unsecured rates, for example the one-month highly rated financial commercial paper rate. It is also consistent with the industry practice of using the OIS curve to discount derivative cashflows. Lastly, it is consistent with the observation that the unsecured borrowing rate is the appropriate discount rate for off-balance-sheet cashflows, under our generalized no-arbitrage assumption.

dealer will be indifferent between employing and not employing this trading strategy if

$$\underbrace{e^{-\frac{7}{12}y^{b}_{7,t}}}_{\text{Purchase price}}\Big(\underbrace{he^{\frac{1}{12}r^{ois}_t}}_{\text{Unsecured Financing}}+\underbrace{(1-h)e^{\frac{1}{12}r^{tri}_t}}_{\text{Repo Financing}}+\underbrace{(e^{\frac{1}{12}r^{syn}_t}-e^{\frac{1}{12}r^{ois}_t})}_{\text{Forgone CIP profits}}\Big)=\underbrace{E^{\mathbb{Q}}_t[e^{-\frac{6}{12}y^{bill}_{t+1}}]}_{\text{Sale price}}.$$

Cheap financing ($r^{tri}_t < r^{ois}_t$) makes the trade attractive and hence decreases the required yield, while the opportunity cost of using balance sheet ($r^{syn}_t > r^{ois}_t$) has the opposite effect. We assume that $x_{1,t} \geq r^{ois}_t$, which is consistent with the post-GFC data and implies that

$$e^{\frac{1}{12}r^{syn}_{t+j}}-e^{\frac{1}{12}r^{ois}_{t+j}} \geq (1-h)(e^{\frac{1}{12}r^{ois}_{t+j}}-e^{\frac{1}{12}r^{tri}_{t+j}}).$$

This assumption states that the balance sheet cost exceeds the financing advantage. It can be justified on the grounds that, if it did not hold, it would be efficient for dealers to purchase Treasury bills from money market funds, financed by repo loans from those same money market funds. This would lead to large dealer balance sheets, causing the leverage constraint to tighten, and hence cannot be part of an equilibrium.

Let us now define a yield curve , $y^{l}_{n,t}$, such that the dealer will be certainly be willing to purchase an n-month Treasury bond, regardless of her preferences or beliefs, if its yield exceeds this value. This will be the net long curve. We will conjecture and verify that the curve defined recursively by

$$e^{-\frac{n}{12}y^{l}_{n,t}}((1-h)(e^{\frac{1}{12}r^{tri}_t}-e^{\frac{1}{12}r^{ois}_t})+e^{\frac{1}{12}r^{syn}_t}))=E^{\mathbb{Q}}_t[e^{-\frac{n-1}{12}y^{l}_{n-1,t+1}}]$$

has this property. That is, the net long curve is defined by the discount rate $e^{x_{1,t}}=(1-h)(e^{\frac{1}{12}r^{tri}_t}-e^{\frac{1}{12}r^{ois}_t})+e^{\frac{1}{12}r^{syn}_t}$, as in the main text.

Fix some $n>7$ and suppose $y^{l}_{m,t}$ is defined by this recursion for all $m\in\{6,\dots,n-1\}$. Consider a trading strategy that purchases the bond, finances the trade with repo and unsecured borrowing, offsets the balance sheet cost by reducing CIP activity, and unwinds at the first moment at which the bond yield becomes weakly lower than $y^{l}_{m,t}$. Let τ denote the months elapsed and let $y_{n-\tau,t+\tau} \leq y^{l}_{n-\tau,t+\tau}$ be the bond price at which the trade is unwound. According to the strategy, we have $y_{m,t} \geq y^{l}_{m,t}$ for all $m\in\{6,\dots,n-1\}$. Further, $\tau \leq n-6$ is guaranteed because by assumption, the intermediary always unwinds the trade once the bond has six-month remaining maturity.

The intermediary will be willing to engage in this strategy provided that

$$\underbrace{e^{-\frac{n}{12}y_{n,t}}}_{\text{purchase price}} + E_t^{\mathbb{Q}}[\sum_{j=0}^{\tau-1} \underbrace{e^{-\sum_{k=0}^{j} r_{t+k}^{ois}}}_{\text{discount rate}} \underbrace{e^{-\frac{n-j}{12}y_{n-j,t+j}}}_{\text{interim bond price}} (\underbrace{(1-h)(e^{\frac{1}{12}r_{t+j}^{tri}} - e^{\frac{1}{12}r_{t+j}^{ois}})}_{\text{repo financing benefits}} + \underbrace{(e^{\frac{1}{12}r_{t+j}^{syn}} - e^{\frac{1}{12}r_{t+j}^{ois}})}_{\text{forgone CIP profits}})]$$
$$\leq E_t^{\mathbb{Q}}[\underbrace{e^{-\sum_{k=0}^{\tau-1} r_{t+k}^{ois}}}_{\text{discount rate}} \underbrace{e^{-\frac{n-\tau}{12}y_{n-\tau,t+\tau}}}_{\text{sale price}}]$$

Since derivatives are priced by the intermediary, hedging does not affect the profit in the above trade. We could add a hedging component to this equation, so that certain future fluctuations in the financing rate are fixed at the beginning of the trade, as illustrated by Figure 4. We omit this extra zero-cost component for simplicity.

However, this strategy cannot be fully hedged by interest rates swaps. First, the time τ at which the bond yield falls below $y_{m,t}^{l}$ is uncertain, as is the ultimate sale price. Second, the interim price of the bond before τ affects the size of the trade that needs to be financed, and consequently both the benefit of cheap financing via tri-party repo and the opportunity cost of the balance sheet. The effects of intermediate bond prices occur because the intermediary uses short term, as opposed to term, financing, and because the assets are marked to market. Thus, even if it were possible to perfectly hedge all of the relevant interest rates, the attractiveness of this trade would depend in part on the intermediary's beliefs about the stochastic process driving bond yields.

However, the worse case scenario for the sale price is that it is exactly equal to the unwinding threshold, $y_{n-\tau,t+\tau} = y_{n-\tau,t+\tau}^{l}$. Under the assumption that $x_{1,t} \geq r_t^{ois}$, the worse case scenario for the intermediate bond yields is that they are as low as possible (i.e. $y_{n-j,t+j} = y_{n-j,t+j}^{l}$), which is to say that the trading strategy uses up the maximum possible balance sheet capacity. Consequently, the intermediary will definitely be willing to buy the bond if, for all possible stopping times τ ,

$$e^{-\frac{n}{12}y_{n,t}} \leq -E_t^{\mathbb{Q}}[\sum_{j=0}^{\tau-1} e^{-\frac{n-j}{12}y_{n-j,t+j}^{l}} e^{-\sum_{k=0}^{j} r_{t+k}^{ois}}((1-h)(e^{\frac{1}{12}r_{t+j}^{tri}} - e^{\frac{1}{12}r_{t+j}^{ois}}) + (e^{\frac{1}{12}r_{t+j}^{syn}} - e^{\frac{1}{12}r_{t+j}^{ois}}))]$$
$$+E_t^{\mathbb{Q}}[e^{-\sum_{k=0}^{\tau-1} r_{t+k}^{ois}} e^{-\frac{n-\tau}{12}y_{n-\tau,t+\tau}^{l}}]$$

and this is in fact the tightest possible bound. Rewriting the definition of net long curve, we obtain

$$e^{-\Sigma_{k=0}^{j-1} r^{ois}_{t+k}} e^{-\frac{n-j}{12} y^l_{n-j,t+j}} = -e^{-\Sigma_{k=0}^{j} r^{ois}_{t+k}} e^{-\frac{n}{12} y^l_{n-j,t+j}} \left((1-h)(e^{\frac{1}{12} r^{tri}_{t+j}} - e^{\frac{1}{12} r^{ois}_{t+j}}\right) \\ + e^{-\Sigma_{k=0}^{j} r^{ois}_{t+k}} E^{\mathbb{Q}}_{t+j}[e^{-\frac{n-j-1}{12} y^l_{n-j-1,t+j+1}}]$$

for any j, and thus it also holds for any bounded stopping time τ. By the definition of the net long curve, this inequality is equivalent to

$$e^{-\frac{n}{12} y_{n,t}} \leq e^{-\frac{n}{12} y^l_{n,t}}$$

Thus, the intermediary will be willing to buy the bond, regardless of the nature of the intermediary's preferences and beliefs about the bond price process, if $y_{n,t} \geq y^l_{n,t}$.

We conclude that the intermediary's demand for a zero-coupon bond should be high if its yield exceeds the net long curve yield. This demand is limited only by the intermediary's leverage constraint: at some point, the intermediary will have switched entirely to doing the Treasury arbitrage as opposed to other arbitrages, at which point $r^{syn}_t - r^{ois}_t$ is no longer a valid measure of the opportunity cost of balance sheet. We therefore predict that if a bond's yield exceeds the buy yield, the intermediary's demand should be substantial.

B.2 The Net Short Curve

We next develop parallel logic for the case of short-selling. In this case, we assume that the intermediary borrows the security from a securities lender in exchange for cash equal to the market value of the security, and receives a log interest rate $r^{sec}_t < r^{tri}_t$ on the cash lent.

The intermediary will be willing to short a seven-month bond at yield $y_{7,t}$ if

$$\underbrace{e^{-\frac{7}{12} y_{7,t}}}_{\text{Sale price}} \underbrace{e^{\frac{1}{12} r^{sec}_t}}_{\text{Gross return on cash in sec. lending}} \geq \underbrace{E^{\mathbb{Q}}_t[e^{-\frac{6}{12} y^{bill}_{t+1}}]}_{\text{Repurchase price}} + e^{-\frac{7}{12} y_{7,t}} \underbrace{(e^{\frac{1}{12} r^{syn}_t} - e^{\frac{1}{12} r^{ois}_t})}_{\text{Forgone CIP profits}}. \tag{B-1}$$

Note that the sign of the forgone CIP profits has changed, relative to the analogous equation for the net long curve, reflecting the fact that both buying and short-selling increase the size of the balance sheet. In equation (B-1), moving the right-hand-side OIS term to the left and dividing both sides by $\exp(\frac{1}{12}r_t^{ois})$, we obtain

$$e^{-\frac{7}{12}y_{7,t}} \geq e^{-\frac{7}{12}y_{7,t}} e^{-\frac{1}{12}r_t^{ois}} \left(e^{\frac{1}{12}r_t^{syn}} - e^{\frac{1}{12}r_t^{sec}}\right) + e^{-\frac{1}{12}r_t^{ois}} E_t^{\mathbb{Q}}[e^{-\frac{6}{12}y_{t+1}^{bill}}] \tag{B-2}$$

Under the assumption that yields are weakly positive, $y_{7,t} \geq 0$, the intermediately is definitely willing to short if

$$e^{-\frac{7}{12}y_{7,t}} \geq e^{-\frac{7}{12}y_{7,t}^s} \equiv e^{-\frac{1}{12}r_t^{ois}} \left(e^{\frac{1}{12}r_t^{syn}} - e^{\frac{1}{12}r_t^{sec}}\right) + e^{-\frac{1}{12}r_t^{ois}} E_t^{\mathbb{Q}}[e^{-\frac{6}{12}y_{t+1}^{bill}}]. \tag{B-3}$$

Following the same spirit, let us define $y_{n,t}^s$ recursively for $n \geq 8$ as

$$e^{-\frac{n}{12}y_{n,t}^s} = e^{-\frac{1}{12}r_t^{ois}} \left(e^{\frac{1}{12}r_t^{syn}} - e^{\frac{1}{12}r_t^{sec}} + E_t^{\mathbb{Q}}[e^{-\frac{n-1}{12}y_{n-1,t+1}^s}]\right), \tag{B-4}$$

which can be interpreted as the pricing equation for a bond with a monthly coupon of $e^{\frac{1}{12}r_t^{syn}} - e^{\frac{1}{12}r_t^{sec}}$, discounted using the OIS curve.

As above, fix some $n > 7$ and suppose $y_{m,t}^s$ is defined as above. Consider a trading strategy that short-sells the bond, borrows the bond from a securities lender, offsets the balance sheet cost by reducing CIP activity, and unwinds at the first moment at which the bond yield becomes weakly higher than $y_{m,t}^s$. Let τ denote this time and let $y_{n-\tau,t+\tau} \geq y_{n-\tau,t+\tau}^s$ be the bond price at which the trade is unwound. According to the strategy, we have $y_{m,t} \leq y_{m,t}^s$ for all $m \in \{6, 7, \cdots, n-1\}$. Further, $\tau \leq n-6$ is guaranteed by the assumption that dealers always unwinds the trade once the bond has six-month remaining maturity. The intermediary will be willing to engage in this strategy provided it is profitable,

$$e^{-\frac{n}{12}y_{n,t}} \geq E_t^{\mathbb{Q}}[\sum_{j=0}^{\tau-1} e^{-\frac{n-j}{12}y_{n-j,t+j}} e^{-\sum_{k=0}^{j} r_{t+k}^{ois}} (e^{\frac{1}{12}r_{t+j}^{syn}} - e^{\frac{1}{12}r_{t+j}^{sec}})] + E_t^{\mathbb{Q}}[e^{-\sum_{k=0}^{\tau-1} r_{t+k}^{ois}} e^{-\frac{n-\tau}{12}y_{n-\tau,t+\tau}}]. \tag{B-5}$$

Note that, because $r_t^{sec} < r_t^{syn}$, the worst-case scenario is the one that makes intermediate bond prices as high as possible. Unlike the net long curve, the fact that $y_{n-j,t+j} < y_{n-j,t+j}^s$ is of no help is generating a bound. In this case, we instead assume a lower bound on yields, $y_{m,t} \geq 0$, motivated the possibility of substitution to cash. In the worst-case scenario, the pricing condition becomes

$$e^{-\frac{n}{12}y_{n,t}} \geq E_t^{\mathbb{Q}}[\sum_{j=0}^{\tau-1} e^{-\Sigma_{k=0}^{j} r_{t+k}^{ois}}(e^{\frac{1}{12}r_{t+j}^{syn}} - e^{\frac{1}{12}r_{t+j}^{sec}})] + E_t^{\mathbb{Q}}[e^{-\Sigma_{k=0}^{\tau-1} r_{t+k}^{ois}} e^{-\frac{n-\tau}{12}y_{n-\tau,t+\tau}^{s}}]. \tag{B-6}$$

For all stopping times τ (bounded above by $n-6$), this is equivalent to

$$e^{-\frac{n}{12}y_{n,t}} \geq e^{-\frac{n}{12}y_{n,t}^{s}}, \tag{B-7}$$

which is to say that the intermediary will be willing to short-sell if yields are below $y_{n,t}^{s}$, irrespective of intermediary's preferences or beliefs about future bond prices.[3]

Finally, we will illustrate that to a first-order approximation, the net-short curve in this appendix is the same as the net-short curve (23) in the main text. Ignoring the covariance terms, the net-short curve in this appendix is

$$1 + \frac{1}{12}r_t^{ois} - \frac{n}{12}y_{n,t}^{s} \approx \frac{1}{12}r_t^{syn} - \frac{1}{12}r_t^{sec} + E_t^{\mathbb{Q}}[1 - \frac{n-1}{12}y_{n-1,t+1}^{s}]$$

$$ny_{n,t}^{s} \approx r_t^{sec} - (r_t^{syn} - r_t^{ois}) - (n-1)E_t^{\mathbb{Q}}[y_{n-1,t+1}^{s}]$$

$$ny_{n,t}^{s} \approx E_t^{\mathbb{Q}}\left[\sum_{j=0}^{n-7}(r_t^{sec} - (r_t^{syn} - r_t^{ois})) + \frac{6}{12}y_{t+n-6}^{bill}\right]$$

It is straightforward to show that equation (23) in the main text also leads to the same linear approximation.

C Data

C.1 Data Sources

We obtain the Treasury term structure from Bloomberg, for maturities 0.25, 0.5, 1, 3, 5, 10, 15, 20, and 30 years, all at daily frequency. The T-bills are from the ticker "GB", representing actively traded T-bill yields, and the non-bills are from the ticker "C082", representing the widely-used

[3] Subject to the caveat that the intermediary must believe in the zero lower bound. Our formulas can be readily generated to other (non-zero) lower bounds, at the expense of additional notation.

Bloomberg fair value Treasury yield curve.

We obtain OIS term structure denominated in USD from Bloomberg for maturities 0.25, 0.5, 1, 3, 5, 10, 15, 20, and 30 years, all at daily frequency. The ticker is "USSO" and data are from Nov 1996 to Dec 2021.

We construct the synthetic dollar lending rate from Euro (EUR). For this purpose, we obtain OIS term structure for EUR for maturities 1, 3, 5, 10, 15, 20, and 30 years. The EUR OIS data are from Aug 2009 to Dec 2021.

Then we obtain the above-one-year maturity LIBOR basis at a daily frequency for EUR-USD from Bloomberg. The EUR-USD LIBOR basis covers Nov 1999 to Dec 2021, and includes the following maturities (in years): 1, 3, 5, 10, 15, 20, 30. EUR 3-month LIBOR basis is from Bloomberg, and they are at daily frequency from Jan 2000 to Dec 2021.

To construct the OIS basis, we also collect EUR inter-bank interest-rate swap (IRS) term structure from Bloomberg at daily frequency. EUR IRS data are from Sep 1999 to Dec 2021. Then we construct the OIS basis for each maturity as

$$\text{EUR-USD OIS basis} = \text{EUR-USD LIBOR basis} + (\text{USD OIS - USD IRS}) - (\text{EUR OIS - EUR IRS})$$

where each term has the same maturity.

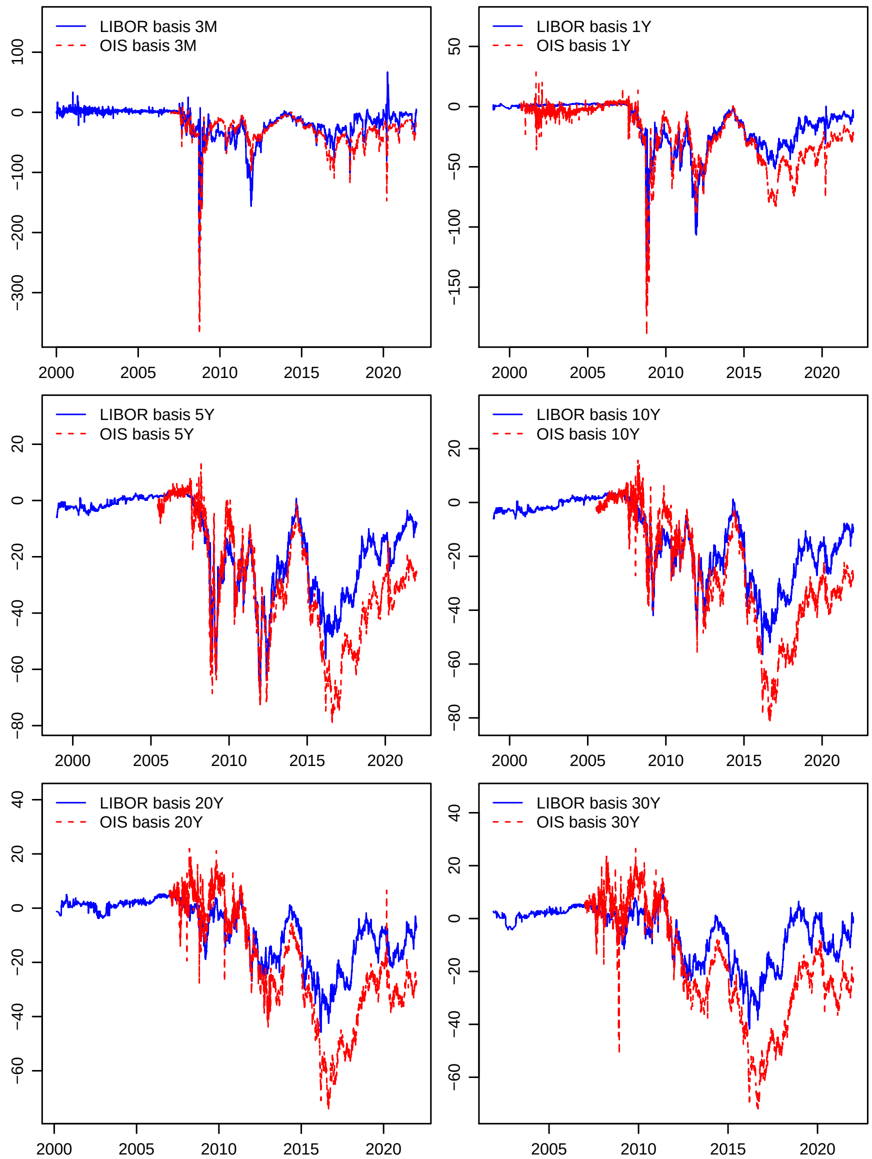
Due to data limitation, we use the “hybrid OIS basis”, defined as follows:

- Whenever OIS data are available, we construct the OIS basis from the LIBOR basis and LIBOR-OIS basis swap.
- When OIS data are not available (only happens before 2008), we use the LIBOR basis instead

This approach is essentially OIS basis throughout the whole sample period because OIS basis and LIBOR basis are almost the same before the global financial crisis. A comparison between the OIS basis and the LIBOR basis is shown in Figure A2.

On the quantity side, dealer net holdings are Treasury securities are based on the primary dealer statistics published by the Federal Reserve Bank of New York.

Figure A2: Comparison of LIBOR EUR Basis and OIS EUR Basis

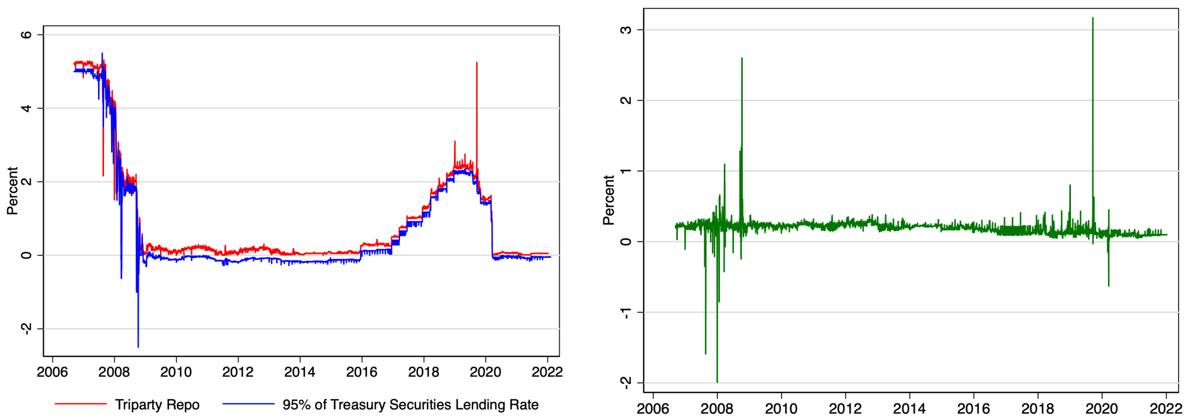


Notes: This figure illustrates the LIBOR EUR-USD basis and the OIS EUR-USD basis. The cross-currency basis is defined as the dollar rate minus the synthetic rate, which is exactly the opposite to the CIP violations we used in the model.

C.2 Treasury Securities Lending Rebate Rates

We use data from Market Securities Finance to calculate the rebate rate on the cash collateral when the dealer is borrowing Treasury bonds from a security lender. Figure A3 shows that the 95 percentile of all Treasury securities lending rebate rate is consistently below the triparty repo rate. The spread between triparty and rebate rate is about 20 basis points on average and quite stable throughout our sample.

Figure A3: Comparison Between Securities Lending Rebate and Triparty Repo Rates



(a) 95 pct of Rebate vs. Triparty

(b) Spread between 95 pct of Rebate and Triparty

Notes: Panel (a) plots the yield spread between the 10-year Treasury bond and the 3-month Treasury bill (in blue), and the primary dealers' net holdings of Treasury bonds. Panel (b) plots the relationship between the two variables post-2009 in a scatter plot.

D Functional Forms and Parameters for Figures

This appendix section describes the functional form and parameter assumptions used to generate Figures 14, 15, 16, 17, and 18. These functional form and parametric assumptions are for illustrative purposes only and do not represent a calibration of the model.

We assume a constant elasticity functional form for the Treasury demand curves. Note that both of these demand curves are functions of the hedged bond log risk premium $\pi_{n,H}$ (the expected excess log return using r^{syn} as the risk-free rate). We assume that

$$D_H(\pi_{n,H}) = D_{H,0}\exp(\eta_H \pi_{n,H}) \tag{D-1}$$

where $D_{H,0} > 0$ represents the demand at zero risk premium. The parameter $\eta_H > 0$ is the semi-elasticity of bond demand to the log risk premium.

We similarly assume that

$$D_U(\pi_{n,U}) = D_{U,0}\exp(\eta_U \pi_{n,U}), \tag{D-2}$$

where $\pi_{n,U}$ is the log risk premium with respect to Treasury bills, with $D_{U,0} > 0$ and $\eta_U > 0$. Note that $\pi_{n,H}$ and $\pi_{n,U}$ are log risk premia; an η_U or η_H of 50 implies a roughly $1.35x$ change in demand given a 1% excess return.

For the synthetic demand curve, we assume that

$$D^{syn}(x) = D_0^{syn} x^{-\xi}, \tag{D-3}$$

where $x = r^{syn} - r^{ois}$ is the spread in basis points and $\xi > 0$ is the elasticity of demand to the spread. This functional form imposes an Inada-type condition that ensures that demand is large as the spread becomes close to zero.

Note that these functional forms satisfy Assumption 1, irrespective of the parameters employed.

We use three sets of parameters to generate the figures used in the main text. The illustrative parameters are chosen to generate clear graphs, and in particular have the property that the regime can change given modest changes in term premium or bond supply. The pre-GFC parameter set perturbs this parameter set using a smaller Treasury supply, larger dealer balance sheet capacity, and larger repo-bill spread. The post-GFC parameter set uses instead a large bond supply, comparatively tight dealer balance sheet, and zero repo-bill spread.

The parameters are chosen under the assumption of an annual holding period and that the bond is a two-year bond ($n = 2$). The table below lists the sets of parameters we employ. Note that Figures 14 and 15 plot dealer indifference curves for different levels of $y_{\mathbb{Q}}$, holding all else constant. Likewise, Figures 16, 17, and 18 have the OIS term premium on the x-axis, which is equivalent to $y_{\mathbb{Q}}$ (holding $y_{\mathbb{P}}$ constant). For this reason, we do not list $y_{\mathbb{Q}}$ in the set of parameters below.

Table A1: Parameters for Figures

Parameter	Illustrative Value	Pre-GFC	Post-GFC
S^{bond}	10.5	9.5	14.5
$\bar{q}$	2	7	2
y^{bill} (bps)	95	65	95
$y_{\mathbb{P}}$ (bps)	95	65	95
r^{ois} (bps)	100		
r^{tri} (bps)	95		
h	2%		
r^{sec} (bps)	75		
D_0^{syn}	4		
ξ	1		
$D_{U,0}$	9.5		
$D_{H,0}$	0.5		
$\eta_U = \eta_H$	50		

E Details of the Term Structure Model

The term structure model consists $\mathbb{P}$ and $\mathbb{Q}$ dynamics

$$z_{t+1} = k_{0,z}^{\mathbb{P}} + K_{1,z}^{\mathbb{P}} \cdot z_t + (\Sigma_z)^{1/2} \varepsilon_{z,t+1}^{\mathbb{P}}, \varepsilon_{z,t+1}^{\mathbb{P}} \sim N(0, I_N),$$

$$z_{t+1} = k_{0,z}^{\mathbb{Q}} + K_{1,z}^{\mathbb{Q}} \cdot z_t + (\Sigma_z)^{1/2} \varepsilon_{z,t+1}^{\mathbb{Q}}, \varepsilon_{z,t+1}^{\mathbb{Q}} \sim N(0, I_N)$$

The state variable vector z_t is 5-by-1, include the first three PCs of OIS term structure (i_t^{PC1}, i_t^{PC2}, and i_t^{PC3}) and the first two PCs of the cross-currency basis term structure ($r_t^{cip,PC1}$ and

$r_t^{cip,PC2}$),

$$z_t = \begin{bmatrix} i_t^{PC1} \\ i_t^{PC2} \\ i_t^{PC3} \\ r_t^{cip,PC1} \\ r_t^{cip,PC2} \end{bmatrix}.$$

The monthly OIS rate and the monthly synthetic rate are both affine functions of the state vector,

$$\frac{1}{12} r_t^{ois} = \delta_0 + (\delta_1)^T z_t,$$
$$\frac{1}{12} r_t^{syn} = \hat{\delta}_0 + (\hat{\delta}_1)^T z_t,$$

For pricing Treasury securities, we also need the state vector $x_t = (x_{1,t}, x_{2,t}, x_{3,t})$, constructed from the data as

$$x_{1,t} = \ln((1-h)(e^{\frac{1}{12} r_t^{tri}} - e^{\frac{1}{12} r_t^{ois}}) + e^{\frac{1}{12} r_t^{syn}})$$
$$x_{2,t} = \ln(e^{\frac{1}{12} r_t^{sec}} + e^{\frac{1}{12} r_t^{ois}} - e^{\frac{1}{12} r_t^{syn}})$$
$$x_{3,t} = \frac{1}{12} y_t^{bill}$$

To operationalize the term structure model and reduce dimensionality, we assume that the vector x_t is affine in the state vector z_t,

$$x_t = \begin{bmatrix} x_{1,t} \\ x_{2,t} \\ x_{3,t} \end{bmatrix} = \gamma_0 + \Gamma_1 z_t + (\Sigma_x)^{\frac{1}{2}} \varepsilon_{x,t}, \varepsilon_{x,t} \sim N(0, I_3).$$

All state variables $x_{k,t}, k \in \{1,2,3\}$ represent yields at the monthly frequency. However, due to the lack of data, we use overnight tri-party rate and overnight security lending rate as proxies for the monthly counterparts. Furthermore, the one-month CIP basis is subject to a quarter-end effect, where the one-month CIP basis spikes at the end of each quarter due to capital regulation, as documented by Du et al. (2018b). To avoid such effect, we instead use the three-month CIP basis to construct the synthetic rate. The underlying assumption is that the rate difference due to maturity difference between one month and three months is negligible.

From our estimations, variance matrix Σ_x is close to zero (the maximum eigen value of Σ_x is about 7×10^{-5}, and much smaller than the maximum eigenvalue of Σ_z which is 4×10^{-3}). To simplify expositions, we set $\Sigma_z = 0$ and limit the actual state space to be five-dimensional. Thus, we will proceed with

$$x_t = \gamma_0 + \Gamma_1 z_t$$

In what follows, we first show the derivations of the OIS term structure and the basis term structure. Then we provide details on how the model generates dealer net long and net short curves. Next, we discuss the conversions between zero-coupon yields and par yields. Finally, we discuss how to estimate the model.

E.1 OIS Term Structure

The zero-coupon OIS term structure is the "risk-free rate" term structure in our model. Denote the swap rate as $i_{n,t}$. The swap exchanges floating payment pegged to the short-term OIS rate $i_t (\equiv r_t^{ois})$ to the fixed swap rate $i_{n,t}$. By construction, the floating leg and the fixed leg should have the same present value. Thus,

$$\exp(n i_{n,t}) E_t^{\mathbb{Q}}[\exp(\sum_{k=1}^{n} -i_{t+k-1})] = 1$$

Conjecture

$$n i_{n,t} = A_n + B_n z_t$$

Then we have

$$\begin{aligned}
\exp(-n \cdot i_{n,t}) = \exp(-A_n - B_n z_t) &= E_t^{\mathbb{Q}}[\exp(-\sum_{k=1}^{n} i_{t+k-1})] \\
&= E_t^{\mathbb{Q}}[E_{t+1}^{\mathbb{Q}}[\exp(-\sum_{k=1}^{n-1} i_{(t+1)+k-1})] \exp(-i_t)] \\
&= E_t^{\mathbb{Q}}[\exp(-A_{n-1} - B_{n-1} z_{t+1} - \delta_0 - \delta_1 z_t)] \\
&= E_t^{\mathbb{Q}}[\exp(-A_{n-1} - B_{n-1} k_{0,z}^{\mathbb{Q}} - B_{n-1} K_{1,z}^{\mathbb{Q}} z_{t+1} + \frac{1}{2} B_{n-1} \Sigma_z (B_{n-1})^T - \delta_0 - \delta_1 z_t)]
\end{aligned}$$

which implies

$$
\begin{aligned}
A_n &= \delta_0 + A_{n-1} + B_{n-1} k_{0,z}^{\mathbb{Q}} - \frac{1}{2} B_{n-1} \Sigma_z (B_{n-1})^T \\
B_n &= \delta_1 + B_{n-1} K_{1,z}^{\mathbb{Q}}
\end{aligned}
$$

for all $n \geq 1$. The starting values are $A_0 = B_0 = 0$.

E.2 Synthetic-Rate Term Structure

We denote the synthetic rate as $i_{n,t} + r_{n,t}^{cip} \equiv r_{n,t}^{syn}$, i.e., composed of both OIS rate and the cross-currency basis. Conjecture that the cumulative synthetic rate is affine in the state vector,

$$
n(r_{n,t}^{cip} + i_{n,t}) = A_n^{syn} + B_n^{syn} z_t
$$

Then we have

$$
\begin{aligned}
&\exp(-n(r_{n,t}^{cip} + i_{n,t})) = \exp(-A_n^{syn} - B_n^{syn} z_t) \\
&= E_t^{\mathbb{Q}}[\exp(\sum_{k=1}^{n} (-r_{t+k-1}^{cip} - i_{t+k-1}))] \\
&= E_t^{\mathbb{Q}}[E_{t+1}^{\mathbb{Q}}[\exp(\sum_{k=1}^{n-1} (-r_{(t+1)+k-1}^{cip} - i_{(t+1)+k-1}))] \exp(-r_t^{cip} - i_t)] \\
&= E_t^{\mathbb{Q}}[\exp(-A_{n-1}^{syn} - B_{n-1}^{syn} z_{t+1} - (\delta_0 + \hat{\delta}_0) - (\delta_1 + \hat{\delta}_1) z_t)] \\
&= E_t^{\mathbb{Q}}[\exp(-A_{n-1}^{syn} - B_{n-1}^{syn} k_{0,z}^{\mathbb{Q}} - B_{n-1}^{syn} K_{1,z}^{\mathbb{Q}} z_{t+1} + \frac{1}{2} B_{n-1}^{syn} \Sigma_z (B_{n-1}^{syn})^T - (\delta_0 + \hat{\delta}_0) - (\delta_1 + \hat{\delta}_1) z_t)]
\end{aligned}
$$

The above equation is the present value of a CIP strategy that earns the CIP deviations, and the values is the same as the long-term CIP discounted at the long-term discount rate. Then we obtain the following iteration:

$$
\begin{aligned}
A_n^{syn} &= \delta_0 + \hat{\delta}_0 + A_{n-1}^{syn} + B_{n-1}^{syn} k_{0,z}^{\mathbb{Q}} - \frac{1}{2} B_{n-1}^{syn} \Sigma_z (B_{n-1}^{syn})^T \\
B_n^{syn} &= \delta_1 + \hat{\delta}_1 + B_{n-1}^{syn} K_{1,z}^{\mathbb{Q}}
\end{aligned}
$$

with the starting values $A_0^{syn} = B_0^{syn} = 0$.

E.3 Treasury Net Long Curve

Next, we derive the iteration steps for the Treasury net long curve.

$$e^{-\frac{n}{12}y^l_{n,t}}e^{x_{1,t}} = E^Q_t[e^{-\frac{n-1}{12}y^l_{n-1,t+1}}]$$

We use $\iota_1 = (1,0,0)$ to denote the indicator vector of the first element, so $x_{1,t} = \iota_1 x_t = \iota_1(\gamma_0 + \Gamma_1 z_t)$. Conjecture that the cumulative yield is affine in the state vector,

$$\frac{n}{12}y^l_{n,t} = A^l_n + B^l_n z_t$$

For all $n \geq 7$, the iteration is

$$\begin{aligned}
\exp(-\left(A^l_n + B^l_n z_t\right)) &= E^Q_t[e^{-\frac{n-1}{12}y^l_{n-1,t+1} - \iota_1(\gamma_0 + \Gamma_1 z_t)}] \\
&= E^Q_t[\exp(-\left(A^l_{n-1} + B^l_{n-1}z_{t+1} + \iota_1(\gamma_0 + \Gamma_1 z_t)\right))] \\
&= E^Q_t[\exp(-\left(A^l_{n-1} + B^l_{n-1}(k^{\mathbb{Q}}_{0,z} + K^{\mathbb{Q}}_{1,z} \cdot z_t + (\Sigma_z)^{1/2}\varepsilon^{\mathbb{Q}}_{z,t+1}) + \iota_1(\gamma_0 + \Gamma_1 z_t)\right))] \\
&= E^Q_t[\exp(-\left(A^l_{n-1} + B^l_{n-1}k^{\mathbb{Q}}_{0,z} + \iota_1\gamma_0 - \frac{1}{2}B^l_{n-1}\Sigma_z(B^l_{n-1})' + (B^l_{n-1}K^{\mathbb{Q}}_{1,z} + \iota_1\Gamma_1)\cdot z_t\right))]
\end{aligned}$$

which implies the iteration equation

$$\begin{aligned}
A^l_n &= \iota_1\gamma_0 + A^l_{n-1} + B^l_{n-1}k^{\mathbb{Q}}_{0,z} - \frac{1}{2}B^l_{n-1}\Sigma_z(B^l_{n-1})^T \\
B^l_n &= \iota_1\Gamma_1 + B^l_{n-1}K^{\mathbb{Q}}_{1,z}
\end{aligned}$$

At $n = 6$, we have

$$\frac{6}{12}y^l_{6,t} = \frac{6}{12}y^{bill}_t = 6x_{3,t} = 6\iota_3(\gamma_0 + \Gamma_1 z_t)$$

with initial values

$$A^l_6 = 6\iota_3\gamma_0, \quad B^l_6 = 6\iota_3\Gamma_1$$

E.4 Treasury Net Short Curve

Next, we derive the iteration steps for the Treasury net short curve.

$$e^{-\frac{n}{12}y^s_{n,t}}e^{x_{2,t}} = E^Q_t[e^{-\frac{n-1}{12}y^s_{n-1,t+1}}]$$

Similar arguments as in the last section will lead to cumulative yield

$$\frac{n}{12}y^s_{n,t} = A^s_n + B^s_n z_t$$

where

$$A^s_n = \iota_2\gamma_0 + A^s_{n-1} + B^s_{n-1}k^{\mathbb{Q}}_{0,z} - \frac{1}{2}B^s_{n-1}\Sigma_z(B^s_{n-1})^T$$
$$B^s_n = \iota_2\Gamma_1 + B^s_{n-1}K^{\mathbb{Q}}_{1,z}$$

At $n = 6$, we have

$$\frac{6}{12}y^s_{6,t} = \frac{6}{12}y^{bill}_t = 6x_{3,t} = 6\iota_3(\gamma_0 + \Gamma_1 z_t)$$

with initial values

$$A^s_6 = 6\iota_3\gamma_0, \quad B^s_6 = 6\iota_3\Gamma_1$$

E.5 Par Curve and Zero Curve Conversion

In our term structure model, all the yields are zero-coupon yields. In the data, on the other hand, yields are par yields. The ideal way to resolve the mismatch is asking the model to convert all zero-coupon yields into par yields. However, the model is solved thousands of times when we estimate it, and the extra conversion significantly slows the estimation process. Thus, we do the following:

- We convert the OIS term structure and the CIP basis term structure into zero-coupon yields for model estimation purpose.
- Once we finish estimating the model, then we generate the net long and net short zero-coupon curves, and convert them into par yields.

For the par-to-zero conversion, we follow the standard Svensson (1994) method that fits the whole yield curve with a parsimonious functional form and infer the zero yields.

For the zero-to-par conversion, we directly use the definition. We want to transform the annualized zero-coupon yields $r_{n,t}$ into annualized par yields $r^{par}_{n,t}$ with coupon payment every 6 months. Then for a coupon-bond of maturity n, the pricing relationship is

$$q^{par}_{n,t} = \frac{r^{par}_{n,t}}{2}\left(e^{-\frac{6}{12}r_{t,6}} + e^{-\frac{12}{12}r_{t,12}} + \cdots + e^{-\frac{n}{12}r_{n,t}}\right) + e^{-\frac{n}{12}r_{n,t}} \tag{E-1}$$

For a bond at the par, the price is $q^{par}_{n,t} = 1$, indicating the par yield as

$$r^{par}_{n,t} = 2 \times \frac{1 - e^{-\frac{n}{12}r_{n,t}}}{e^{-\frac{6}{12}r_{t,6}} + e^{-\frac{12}{12}r_{t,12}} + \cdots + e^{-\frac{n}{12}r_{n,t}}} \tag{E-2}$$

E.6 Model Estimation

We estimate the model to fit the OIS and basis term structure. Then we use regression-implied coefficients γ_0 and Γ_1 to obtain the model-implied net long and net short curves. Denote the observed OIS yield of maturity n at time t as

$$\hat{i}_{n,t} = i_{n,t} + e^{ois}_{n,t}, \quad e^{ois}_t \sim \mathcal{N}(0, \Sigma_{ois})$$

and the observed basis as

$$\hat{r}^{cip}_{n,t} = r^{cip}_{n,t} + e^{basis}_{n,t}, \quad e^{basis}_t \sim \mathcal{N}(0, \Sigma_{basis})$$

We denote the stacked OIS yields (across different maturities) as $\hat{i}_t$, and the stacked basis rates as $\hat{r}^{cip}_t$. For the estimation step, the set of parameters is $\Theta = \{k^{\mathbb{Q}}_{0,z}, K^{\mathbb{Q}}_{1,z}, k^{\mathbb{P}}_{0,z}, K^{\mathbb{P}}_{1,z}, \Sigma_z, \Sigma_{ois}, \Sigma_{basis}, \delta_0, \delta_1, \hat{\delta}_0, \hat{\delta}_1\}$. The objective of the estimation is to maximize the log likelihood that the observed yields are generated by the model,

$$\mathscr{L}(\{\hat{i}_t, \hat{r}^{cip}_t, z_t\}_{t\in\text{data}}; \Theta)$$

Denote the log likelihood an N-variable normal variable Z with mean μ and variance matrix Σ as $\mathscr{G}(Z,\mu,\Sigma)$. Then the objective function is

$$\mathscr{L}(\{\hat{i}_t,\hat{r}_t^{cip},z_t\}_{t\in\text{data}};\Theta)=$$

$$\sum_{t\in\text{data}}\left(\underbrace{\mathscr{G}(z_t-k^{\mathbb{P}}_{0,z}-K^{\mathbb{P}}_{1,z}\cdot z_{t-1},0,\Sigma_z)}_{\text{state variable physical dynamics}}+\underbrace{\mathscr{G}(\hat{i}_t-i_t,0,\Sigma_{ois})}_{\text{OIS fitting}}+\underbrace{\mathscr{G}(\hat{r}_t^{cip}-r_t^{cip},0,\Sigma_{basis})}_{\text{basis fitting}}\right)$$

The whole estimation problem is thus

$$\max_{k^{\mathbb{Q}}_{0,z},K^{\mathbb{Q}}_{1,z},k^{\mathbb{P}}_{0,z},K^{\mathbb{P}}_{1,z},\Sigma_z,\Sigma_{ois},\Sigma_{basis},\delta_0,\delta_1,\hat{\delta}_0,\hat{\delta}_1}\mathscr{L}(\{\hat{i}_t,\hat{r}_t^{cip},z_t\}_{t\in\text{data}};\Theta) \tag{E-3}$$

To reduce dimensionality, we assume that the covariance matrices for observation errors are in the form of $\Sigma_{ois}=\sigma_{ois}I$ and $\Sigma_{basis}=\sigma_{basis}I$.

Compared to the classical term structure estimation problem, the key challenge of this problem is that we need to estimate two inter-linked term structures simultaneously. However, the canonical form transformation in Joslin et al. (2011) only applies to one term structure. To resolve the challenge and at the same time taking advantage of the canonical form, we design the following two-step procedure that applies the canonical form to each individual term struccture as initialization (the initial values for this high-dimensional optimization problem are quite important):

1. Divide the state-space into two blocks, an OIS block, $z_t^{ois}=(z_{1,t},z_{2,t},z_{3,t})$, and a basis block $z_t^{basis}=(z_{4,t},z_{5,t})$. Similarly, we denote the associated sub-group risk-neutral dynamic parameters as $k^{\mathbb{Q}}_{0,z^{ois}},K^{\mathbb{Q}}_{1,z^{ois}}$ and $k^{\mathbb{Q}}_{0,z^{basis}},K^{\mathbb{Q}}_{1,z^{basis}}$. Denote the sub-group physical dynamic parameters as $k^{\mathbb{P}}_{0,z^{ois}},K^{\mathbb{P}}_{1,z^{ois}}$ and $k^{\mathbb{P}}_{0,z^{basis}},K^{\mathbb{P}}_{1,z^{basis}}$. Also divide the observations into the OIS group and basis group. Then apply the standard canonical form estimation procedure to two models separately,

$$\mathscr{L}(\{\hat{i}_t,z_t\}^{ois}_{t\in\text{data}};k^{\mathbb{Q}}_{0,z^{ois}},K^{\mathbb{Q}}_{1,z^{ois}},k^{\mathbb{P}}_{0,z^{ois}},K^{\mathbb{P}}_{1,z^{ois}},\Sigma_z^{ois},\Sigma_{ois},\delta_0^{ois},\delta_1^{ois})$$

$$\mathscr{L}(\{\hat{i}_t,z_t\}^{basis}_{t\in\text{data}};k^{\mathbb{Q}}_{0,z^{basis}},K^{\mathbb{Q}}_{1,z^{basis}},k^{\mathbb{P}}_{0,z^{basis}},K^{\mathbb{P}}_{1,z^{basis}},\Sigma_z^{basis},\Sigma_{basis},\delta_0^{basis},\delta_1^{basis})$$

where the short rate in the first estimation is $\delta_0^{ois}+\delta_1^{ois}*z_t^{ois}$, and the short rate in the second estimation is $\delta_0^{basis}+\delta_1^{basis}*z_t^{basis}$ is a two-dimensional vector that loads on z_t^{basis}. The covariance matrix Σ_z^{ois} is 3×3 and Σ_z^{basis} is 2×2. After estimating the above dynamics, we construct an initialization of the original problem as

$$k_{0,z}^{\mathbb{Q}} = \begin{pmatrix} k_{0,z^{ois}}^{\mathbb{Q}} \\ k_{0,z^{basis}}^{\mathbb{Q}} \end{pmatrix}, \quad K_{1,z}^{\mathbb{Q}} = \begin{pmatrix} K_{1,z^{ois}}^{\mathbb{Q}} & \\ & K_{1,z^{basis}}^{\mathbb{Q}} \end{pmatrix}, \quad \Sigma_z = \begin{pmatrix} \Sigma_z^{ois} & \\ & \Sigma_z^{basis} \end{pmatrix}$$

$$\delta_0 = \delta_0^{ois}, \quad \delta_1 = \begin{pmatrix} \delta_1^{ois} \\ 0 \\ 0 \end{pmatrix}, \quad \hat{\delta}_0 = \delta_0^{ois} + \delta_0^{basis}, \quad \hat{\delta}_1 = \begin{pmatrix} \delta_1^{ois} \\ \delta_1^{basis} \end{pmatrix}$$

We initialize the physical dynamic parameters $(k_{0,z}^{\mathbb{P}}, K_{1,z}^{\mathbb{P}})$ simply from linear regressions,

$$z_t \sim k_{0,z}^{\mathbb{P}} + K_{1,z}^{\mathbb{P}} \cdot z_{t-1}$$

2. Then we feed these initial values to the whole estimation problem (E-3), and apply the optimization package in Matlab to optimize over the whole high-dimensional parameter space. We use the equivalent implementation of the CIP short rate (instead of the synthetic lending short rate), $r_t^{syn} - r_t^{ois}$, and the corresponding loading $\hat{\delta}_0 - \delta_0 + (\hat{\delta}_1 - \delta_1) z_t$.

After we finish estimating the key parameter set Θ, we proceed to obtain γ_0 and Γ_1 via a simple linear regressions,

$$x_t \sim \gamma_0 + \Gamma_1 z_t$$

We find that the residual standard errors for this linear regression are one order of magnitude smaller than Σ_z. In other words, we are able to obtain very accurate approximation of x_t through the state vector z_t, so adding the extra estimation error to the above approximation in the model will not cause much difference, but it requires augmenting the state space. For this reason, we make the assumption that x_t is spanned by z_t in the main model.

Finally with estimated Θ and (γ_0, Γ_1), we are able to obtain the Treasury net long and net short curves. We convert these curves into par curves to be comparable with the Treasury yield data.

E.7 Stationarity Restrictions

Treasury yields can be non-stationary, but the spread between an OIS rate and the matched-maturity Treasury yield cannot diverge due to arbitrage incentives in financial markets. Our main approach does not impose such a restriction for simplicity. In this subsection, we discuss how to impose stationarity on the process x_t and show that results are broadly similar.

First, our estimation reveals that z_t contains unit-root processes. In particular, the $\mathbb{Q}$-dynamics of z_t contains unit-root elements. Denote the eigenvalue decomposition of $K_{1,z}^{\mathbb{Q}}$ as

$$K_{1,z}^{\mathbb{Q}} = VDV^{-1}$$

where D is an diagonal matrix that contains all the eigenvalues of $K_{1,z}^{\mathbb{Q}}$, and V is the matrix of all the column eigenvectors for $K_{1,z}^{\mathbb{Q}}$. We find that two among the five eigenvalues have absolute values above 0.999, which is a strong sign of unit root.

To operationalize the stationarity restriction, we rotate the state vector z_t to $\tilde{z}_t = V^{-1}z_t$, and rewrite the $\mathbb{Q}$-dynamics in (17) as

$$\tilde{z}_{t+1} = V^{-1}k_{0,z}^{\mathbb{Q}} + D\tilde{z}_t + V^{-1}(\Sigma_z)^{1/2}\varepsilon_{z,t+1}^{\mathbb{Q}}, \varepsilon_{z,t+1}^{\mathbb{Q}} \sim N(0, I_N)$$

We denote the spread vector as

$$\hat{x}_t = x_t - \begin{pmatrix} r_t^{ois} + r_t^{cip} \\ r_t^{ois} - r_t^{cip} \\ r_t^{ois} \end{pmatrix}$$

Then we project $\hat{x}_t$ on the stationary components of $\tilde{z}_t$, i.e., three of five with (absolute values of) eigenvalues below 0.999. The loadings on the non-stationary components are set as zeros. Then we denote the whole projection as

$$\hat{x}_t = \tilde{\gamma}_0 + \tilde{\Gamma}_1\tilde{z}_t$$

Next, we rotate back to z_t,

$$\hat{x}_t = \tilde{\gamma}_0 + \tilde{\Gamma}_1 V^{-1} z_t$$

Thus, we obtain

$$x_t = \tilde{\gamma}_0 + \tilde{\Gamma}_1 V^{-1} z_t + \begin{pmatrix} r_t^{ois} + r_t^{cip} \\ r_t^{ois} - r_t^{cip} \\ r_t^{ois} \end{pmatrix}$$

In the implementation, we find that there are complex-number eigenvalues, so the resulting $\tilde{\gamma}_0$ and $\tilde{\Gamma}_1 V^{-1}$ are also complex numbers. Nevertheless, the imaginary parts are quite small so we only keep the real parts.

With the projection of x_t on z_t, we are able to derive the Treasury net long and net short curves. We illustrate the results in Figure A4. We find that results are very close to the baseline results in Figure 11. Furthermore, all other results, such as the relative yield index matching the movements in dealer position, are quite similar. For conciseness, we omit other results in this appendix.

F Proofs

F.1 Proof of Proposition 1 (Long Regime)

Define the function

$$f_1^{long}(y, r^{syn}; S^{bond}, \bar{q}, y_{\mathbb{Q}}, \varepsilon, \omega, \delta_U, \delta_H, \delta_{syn}) = e^{-(ny-(n-1)\varepsilon)} - \frac{\exp(-(n-1)y_{\mathbb{Q}})}{(1-h)(e^{r^{tri}} - e^{r^{ois}}) + e^{r^{syn}}}$$

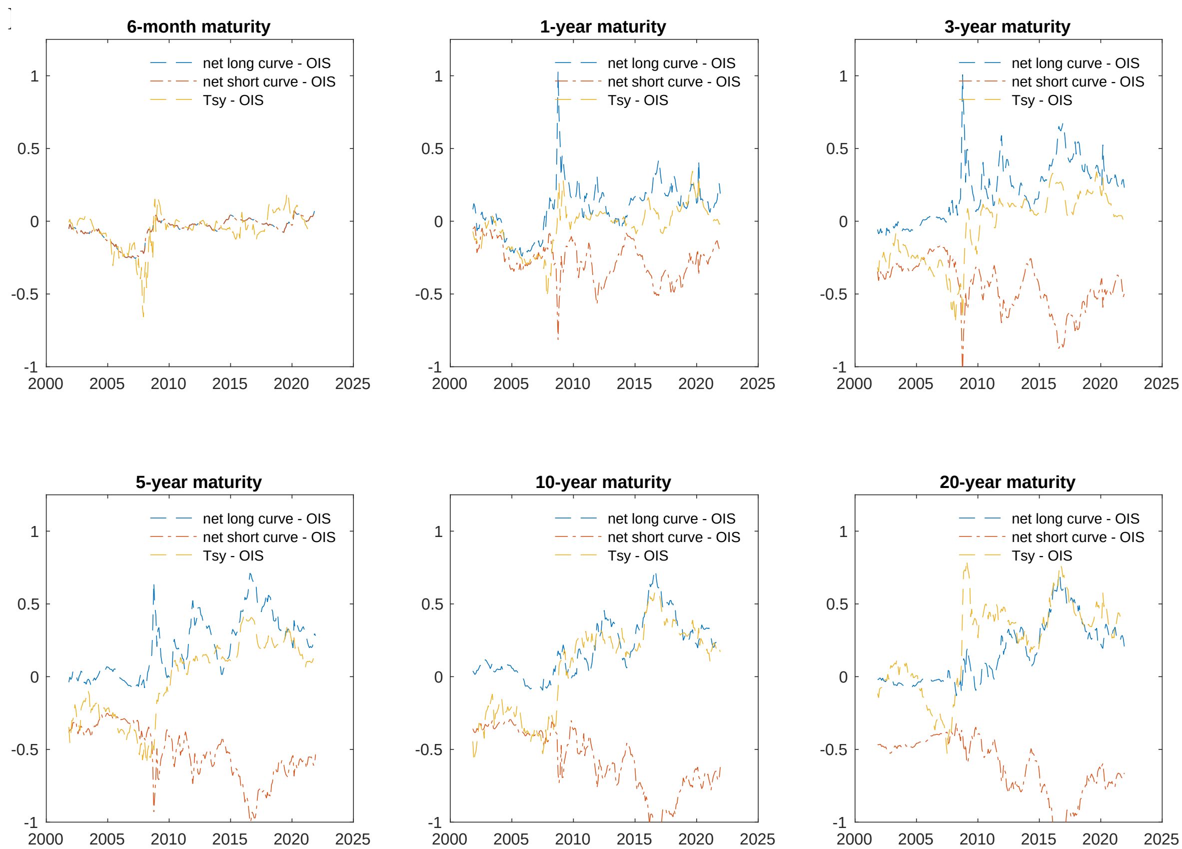
and the function

$$\begin{aligned} &f_2^{long}(y, r^{syn}; S^{bond}, \bar{q}, y_{\mathbb{Q}}, \varepsilon, \omega, \delta_U, \delta_H, \delta_{syn}) = \\ &\bar{q} - e^{-(ny-(n-1)(\varepsilon-\omega))} S^{bond} + D_U(ny - y^{bill} - (n-1)(y_{\mathbb{P}} + \varepsilon)) + \delta_U - \left(D^{syn}(r^{syn} - r^{ois}) + \delta_{syn}\right). \end{aligned}$$

By assumption, D_U and D^{syn} are continuously differentiable, and hence f_1 and f_2 are continuously differentiable.

Suppose there exists, given the exogenous values $y_{\mathbb{P}}, r^{ois}, r^{tri}, y^{bill}$ and some initial point ($S^{bond} > 0, \bar{q} > 0, y_{\mathbb{Q}}, \varepsilon = 0, \omega = 0, \delta_U = 0, \delta_H = 0, \delta_{syn} = 0$), a solution

$$\begin{bmatrix} f_1^{long}(y^*, r^{syn*}; S^{bond}, \bar{q}, y_{\mathbb{Q}}, 0,0,0,0,0) \\ f_2^{long}(y^*, r^{syn*}; S^{bond}, \bar{q}, y_{\mathbb{Q}}, 0,0,0,0,0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$



Notes: In this figure, we show the model-implied long and short Treasury curves minus the OIS rates for corresponding maturities, together with the actual Treasury–OIS spreads. We use the alternative projection method as in Appendix Section E.7. Data are from 2003 to 2021. All yields are par yields.

such that

$$D_H(ny^* - r^{syn*} - (n-1)y_{\mathbb{P}}) + D_U(ny - y^{bill} - (n-1)y_{\mathbb{P}}) < e^{-ny^*} S^{bond}.$$

Such a point constitutes an equilibrium.

Observe that

$$\frac{\partial f_1^{long}(\cdot)}{\partial y} < 0, \frac{\partial f_1^{long}(\cdot)}{\partial r^{syn}} > 0, \frac{\partial f_2^{long}(\cdot)}{\partial y} > 0, \frac{\partial f_2^{long}(\cdot)}{\partial r^{syn}} > 0,$$

and consequently

$$\begin{bmatrix} \frac{\partial f_1^{long}(\cdot)}{\partial y} & \frac{\partial f_1^{long}(\cdot)}{\partial r^{syn}} \\ \frac{\partial f_2^{long}(\cdot)}{\partial y} & \frac{\partial f_2^{long}(\cdot)}{\partial r^{syn}} \end{bmatrix}$$

is invertible (its determinant is strictly negative).

It follows that the equilibrium (y^*, r^{syn*}), if it exists, is unique. Suppose not and there exists another equilibrium (y, r^{syn}) in the long regime. If $r^{syn} > r^{syn*}$, then $\tilde{y} > y^*$ according to $f_1^{long}(\tilde{y}, r^{syn}) = 0$. By monotonicity of f_2^{long}, we have $f_2^{long}(y, r^{syn}) > f_2^{long}(y^*, r^{syn*}) = 0$, which contradicts to (y, r^{syn}) being an equilibrium. A symmetric argument rules out all $r^{syn} < r^{syn*}$. Thus, the equilibrium solution to y^* is unique. Strict monotonicity ensures the uniqueness of y^*.

By the implicit function theorem,

$$\begin{bmatrix} \frac{\partial y^*(\cdot)}{\partial x} \\ \frac{\partial r^{syn*}(\cdot)}{\partial x} \end{bmatrix} = - \begin{bmatrix} \frac{\partial f_1^{long}(\cdot)}{\partial y} & \frac{\partial f_1(\cdot)}{\partial r^{syn}} \\ \frac{\partial f_2(\cdot)}{\partial y} & \frac{\partial f_2(\cdot)}{\partial r^{syn}} \end{bmatrix}^{-1} \begin{bmatrix} \frac{\partial f_1^{long}(\cdot)}{\partial x} \\ \frac{\partial f_2^{long}(\cdot)}{\partial x} \end{bmatrix}$$

for any $x \in \{S^{bond}, \bar{q}, y_{\mathbb{Q}}, \varepsilon, \omega, \delta_U, \delta_H, \delta_{syn}\}$. Observe that the signs of the negative inverse matrix are

$$sgn\left(- \begin{bmatrix} \frac{\partial f_1^{long}(\cdot)}{\partial y} & \frac{\partial f_1^{long}(\cdot)}{\partial r^{syn}} \\ \frac{\partial f_2^{long}(\cdot)}{\partial y} & \frac{\partial f_2^{long}(\cdot)}{\partial r^{syn}} \end{bmatrix}^{-1} \right) = sign\left(\begin{bmatrix} \frac{\partial f_2^{long}(\cdot)}{\partial r^{syn}} & -\frac{\partial f_1^{long}(\cdot)}{\partial r^{syn}} \\ -\frac{\partial f_2^{long}(\cdot)}{\partial y} & \frac{\partial f_1^{long}(\cdot)}{\partial y} \end{bmatrix} \right) = \begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix}.$$

We solve for the comparative statics as follows:

1. An increase in S^{bond}: $\frac{\partial f_1^{long}(\cdot)}{\partial x} = 0, \frac{\partial f_2^{long}(\cdot)}{\partial x} < 0$, and therefore $\frac{\partial y^*(\cdot)}{\partial x} > 0$ and $\frac{\partial r^{syn*}(\cdot)}{\partial x} > 0$.

2. A decrease in $\bar{q}$ or a decrease in δ_U: $\frac{\partial f_1^{long}(\cdot)}{\partial x} = 0$ and

$$\frac{\partial f_2^{long}(\cdot)}{\partial x} = -\frac{\partial f_2^{long}(\cdot)}{e^{-ny} \cdot \partial (S^{bond})}.$$

   Thus, the decrease in $\bar{q}$ or δ_U is equivalent to the same same size expansion in the dollar supply of bonds.

3. An increase in δ_H has $\frac{\partial f_1^{long}(\cdot)}{\partial x} = 0, \frac{\partial f_2^{long}(\cdot)}{\partial x} = 0$, and therefore $\frac{\partial y^*(\cdot)}{\partial x} = 0$ and $\frac{\partial r^{syn*}(\cdot)}{\partial x} = 0$.

4. An increase in $y_{\mathbb{Q}}$ has $\frac{\partial f_1^{long}(\cdot)}{\partial x} > 0, \frac{\partial f_2^{long}(\cdot)}{\partial x} = 0$ and thus $\frac{\partial y^*(\cdot)}{\partial x} > 0$ and $\frac{\partial r^{syn*}(\cdot)}{\partial x} < 0$.

5. An increase of dy in both $y_{\mathbb{Q}}$ and $y_{\mathbb{P}}$ is equivalent to an increase ε by $\Delta\varepsilon = dy$ in both f_1 and f_2 and an increase in ω by $\Delta\omega = dy$. The increase in ε causes an $\frac{n-1}{n}\Delta\varepsilon$ increase in y and no change in r^{syn}. The increase in ω has $\frac{\partial f_1^{long}(\cdot)}{\partial x} = 0, \frac{\partial f_2^{long}(\cdot)}{\partial x} > 0$, and thus thus $\frac{\partial y^*(\cdot)}{\partial x} < 0$ and $\frac{\partial r^{syn*}(\cdot)}{\partial x} < 0$. Taking the two effects together, clearly r^{syn*} will decrease. To determine the sign on y^*, we can evaluate the change of dy in both $y_{\mathbb{Q}}$ and $y_{\mathbb{P}}$ directly and obtain $\frac{\partial f_1^{long}(\cdot)}{\partial x} > 0, \frac{\partial f_2^{long}(\cdot)}{\partial x} < 0$, which implies y^* will increase. In summary, we find that the increase of dy in both $y_{\mathbb{Q}}$ and $y_{\mathbb{P}}$ increases y^* by less than $\frac{n-1}{n}\Delta\varepsilon$ and decreases r^{syn*}. Furthermore, the absolute value of the effect of ω is smaller than that of ε, indicating that the total effect is still to increase bond yield.

   Taking the two effects together, we find that the increase of dy in both $y_{\mathbb{Q}}$ and $y_{\mathbb{P}}$ increase y^* by less than $\frac{n-1}{n}\Delta\varepsilon$ and decreases r^{syn*}.

6. An increase in δ_{syn} has $\frac{\partial f_1^{long}(\cdot)}{\partial x} = 0, \frac{\partial f_2^{long}(\cdot)}{\partial x} < 0$, and thus $\frac{\partial y^*(\cdot)}{\partial x} > 0$ and $\frac{\partial r^{syn*}(\cdot)}{\partial x} > 0$.

F.2 Proof of Proposition 2 (Short Regime)

Define the function

$$f_1^{short}(y, r^{syn}; S^{bond}, \bar{q}, y_{\mathbb{Q}}, \varepsilon, \omega, \delta_U, \delta_H, \delta_{syn}) = e^{-(ny-(n-1)\varepsilon)} - \frac{\exp(-(n-1)y_{\mathbb{Q}})}{e^{r^{sec}} + e^{r^{ois}} - e^{r^{syn}}}$$

and the function

$$\begin{aligned}f_2^{short}(y,r^{syn};S^{bond},\bar{q},y_{\mathbb{Q}},\varepsilon,\omega,\delta_U,\delta_H,\delta_{syn}) =&\bar{q}+e^{-(ny-(n-1)(\varepsilon-\omega))}S^{bond}\\ &-D_U(ny-y^{bill}-(n-1)(y_{\mathbb{P}}+\varepsilon))-\delta_U\\ &-\left(D^{syn}(r^{syn}-r^{ois})+\delta_{syn}\right)\\ &-2(D_H(ny-y^{syn}-(n-1)(y_{\mathbb{P}}+\varepsilon))+\delta_H)\end{aligned}$$

By assumption, D_U, D_H, and D^{syn} are continuously differentiable, and hence f_1^{short} and f_2^{short} are continuously differentiable.

Suppose there exists, given the exogenous values $y_{\mathbb{P}}, r^{ois}, r^{tri}, y^{bill}$ and some initial point ($S^{bond} > 0, \bar{q} > 0, y_{\mathbb{Q}}, \varepsilon = 0, \omega = 0, \delta_U = 0, \delta_H = 0, \delta_{syn} = 0$), a solution

$$\begin{bmatrix} f_1^{short}(y^*,r^{syn*};S^{bond},\bar{q},y_{\mathbb{Q}},0,0,0,0,0)\\ f_2^{short}(y^*,r^{syn*};S^{bond},\bar{q},y_{\mathbb{Q}},0,0,0,0,0)\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix},$$

such that

$$D_H(ny^*-r^{syn*}-(n-1)y_{\mathbb{P}})+D_U(ny-y^{bill}-(n-1)y_{\mathbb{P}}) > e^{-ny^*}S^{bond}.$$

Such a point constitutes an equilibrium.

Observe that

$$\frac{\partial f_1^{short}(\cdot)}{\partial y} < 0, \frac{\partial f_1^{short}(\cdot)}{\partial r^{syn}} < 0, \frac{\partial f_2^{short}(\cdot)}{\partial y} < 0, \frac{\partial f_2^{short}(\cdot)}{\partial r^{syn}} > 0,$$

and consequently

$$\begin{bmatrix} \frac{\partial f_1^{short}(\cdot)}{\partial y} & \frac{\partial f_1^{short}(\cdot)}{\partial r^{syn}}\\ \frac{\partial f_2^{short}(\cdot)}{\partial y} & \frac{\partial f_2^{short}(\cdot)}{\partial r^{syn}}\end{bmatrix}$$

is invertible (its determinant is strictly negative).

It follows that the equilibrium (y^*, r^{syn*}), if it exists, is unique. Suppose not and there exists another pair (r^{syn}, y) that satisfies the eqilibrium in the short regime. If $r^{syn} > r^{syn*}$, we must have $y < y^*$ due to $f_1^{short}(y, r^{syn}) = f_1^{short}(y^*, r^{syn*})$. (if no such $\tilde{y}$ exists, r^{syn} cannot be part of an equilibrium). It follows that $f_2^{short}(y, r^{syn}) > f_2^{short}(y^*, r^{syn*}) = 0$, and hence r^{syn} cannot be part of an equilibrium. A symmetric argument rules out all $r^{syn} < r^{syn*}$, and strict monotonicity ensures the uniqueness of y^*.

By the implicit function theorem,

$$\begin{bmatrix} \frac{\partial y^*(\cdot)}{\partial x} \\ \frac{\partial r^{syn*}(\cdot)}{\partial x} \end{bmatrix} = -\begin{bmatrix} \frac{\partial f_1^{short}(\cdot)}{\partial y} & \frac{\partial f_1^{short}(\cdot)}{\partial r^{syn}} \\ \frac{\partial f_2^{short}(\cdot)}{\partial y} & \frac{\partial f_2^{short}(\cdot)}{\partial r^{syn}} \end{bmatrix}^{-1} \begin{bmatrix} \frac{\partial f_1^{short}(\cdot)}{\partial x} \\ \frac{\partial f_2^{short}(\cdot)}{\partial x} \end{bmatrix}$$

for any $x \in \{S^{bond}, \bar{q}, y_{\mathbb{Q}}, \varepsilon, \omega, \delta_U, \delta_H\}$. Observe that the signs of the negative inverse matrix are

$$sgn\left(-\begin{bmatrix} \frac{\partial f_1^{short}(\cdot)}{\partial y} & \frac{\partial f_1^{short}(\cdot)}{\partial r^{syn}} \\ \frac{\partial f_2^{short}(\cdot)}{\partial y} & \frac{\partial f_2^{short}(\cdot)}{\partial r^{syn}} \end{bmatrix}^{-1}\right) = sign\left(\begin{bmatrix} \frac{\partial f_2^{short}(\cdot)}{\partial r^{syn}} & -\frac{\partial f_1^{short}(\cdot)}{\partial r^{syn}} \\ -\frac{\partial f_2^{short}(\cdot)}{\partial y} & \frac{\partial f_1^{short}(\cdot)}{\partial y} \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

We solve for the comparative statics as follows:

1. An increase in S^{bond}: $\frac{\partial f_1^{short}(\cdot)}{\partial x} = 0, \frac{\partial f_2^{short}(\cdot)}{\partial x} > 0$, and therefore $\frac{\partial y^*(\cdot)}{\partial x} > 0$ and $\frac{\partial r^{syn*}(\cdot)}{\partial x} < 0$. Thus, bond yield y^* increases, but the synthetic rate r^{syn*} decreases.

2. An increase in $\bar{q}$ or a decrease in δ_U (i.e., a parallel decrease in D_U): $\frac{\partial f_1^{short}(\cdot)}{\partial x} = 0$ and

$$\frac{\partial f_2^{short}(\cdot)}{\partial \bar{q}} = -\frac{\partial f_2^{short}(\cdot)}{\partial \delta_U} = \frac{\partial f_2^{short}(\cdot)}{e^{-ny} \cdot \partial(S^{bond})}.$$

   Thus, the increase in $\bar{q}$ or the same decrease in δ_U are equivalent to the same same size expansion in the dollar supply of bonds.

3. An increase in δ_H has $\frac{\partial f_1^{short}(\cdot)}{\partial x} = 0, \frac{\partial f_2^{short}(\cdot)}{\partial x} < 0$, and therefore $\frac{\partial y^*(\cdot)}{\partial x} < 0$ and $\frac{\partial r^{syn*}(\cdot)}{\partial x} > 0$.

4. An increase in $y_{\mathbb{Q}}$ has $\frac{\partial f_1^{short}(\cdot)}{\partial x} > 0, \frac{\partial f_2^{short}(\cdot)}{\partial x} = 0$ and thus $\frac{\partial y^*(\cdot)}{\partial x} > 0$ and $\frac{\partial r^{syn*}(\cdot)}{\partial x} > 0$.

5. An increase of dy in both $y_{\mathbb{Q}}$ and $y_{\mathbb{P}}$: this change is equivalent to an increase ε by dy in both f_1^{short} and f_2^{short} and an increase in ω by dy. The increase in ε causes an $\frac{n-1}{n}\Delta\varepsilon$ decrease in y and no change in r^{syn*}. The increase in ω has $\frac{\partial f_1^{short}(\cdot)}{\partial x} = 0, \frac{\partial f_2^{short}(\cdot)}{\partial x} < 0$, and thus thus $\frac{\partial y^*(\cdot)}{\partial x} < 0$ and $\frac{\partial r^{syn*}(\cdot)}{\partial x} > 0$. Taking the two effects together, clearly r^{syn*} will increase. To determine the sign on y^*, we can evaluate the change of dy in both $y_{\mathbb{Q}}$ and $y_{\mathbb{P}}$ directly and

obtain $\frac{\partial f_1^{short}(\cdot)}{\partial x} > 0, \frac{\partial f_2^{short}(\cdot)}{\partial x} > 0$, which implies y^* will increase. In summary, we find that the increase of dy in both $y_{\mathbb{Q}}$ and $y_{\mathbb{P}}$ increases y^* by less than $\frac{n-1}{n}\Delta\varepsilon$ and increases r^{syn*}.

6. An increase in δ_{syn} has $\frac{\partial f_1^{long}(\cdot)}{\partial x} = 0, \frac{\partial f_2^{long}(\cdot)}{\partial x} < 0$, and thus $\frac{\partial y^*(\cdot)}{\partial x} < 0$ and $\frac{\partial r^{syn*}(\cdot)}{\partial x} > 0$.

F.3 Proof of Proposition 3 (Intermediate Regime)

Define the functions

$$f_1^{int}(y, r^{syn}; S^{bond}, \bar{q}, y_{\mathbb{Q}}, \varepsilon, \omega, \delta_U, \delta_H, \delta_{syn}) = \bar{q} - e^{-(ny-(n-1)(\varepsilon-\omega))}S^{bond} - \left(D^{syn}(r^{syn} - r^{ois}) + \delta_{syn}\right) \\ + D_U(ny - y^{bill} - (n-1)(y_{\mathbb{P}} + \varepsilon)) + \delta_U$$

and

$$f_2^{int}(y, r^{syn}; S^{bond}, \bar{q}, y_{\mathbb{Q}}, \varepsilon, \omega, \delta_U, \delta_H, \delta_{syn}) = \bar{q} - (D_H(ny - r^{syn} - (n-1)(y_{\mathbb{P}} + \varepsilon)) + \delta_H) \\ - \left(D^{syn}(r^{syn} - r^{ois}) + \delta_{syn}\right).$$

By assumption, D_U, D_H, and D^{syn} are continuously differentiable, and hence f_1^{int} and f_2^{int} are continuously differentiable.

Suppose there exists, given the exogenous values $y_{\mathbb{P}}, r^{ois}, r^{tri}, y^{bill}$ and some initial point ($S^{bond} > 0, \bar{q} > 0, y_{\mathbb{Q}}, \varepsilon = 0, \omega = 0, \delta_U = 0, \delta_H = 0, \delta_{syn} = 0$), a solution

$$\begin{bmatrix} f_1^{int}(y^*, r^{syn*}; S^{bond}, \bar{q}, y_{\mathbb{Q}}, 0, 0, 0, 0, 0) \\ f_2^{int}(y^*, r^{syn*}; S^{bond}, \bar{q}, y_{\mathbb{Q}}, 0, 0, 0, 0, 0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

such that

$$y^s < y^* < y^l$$

Such a point constitutes an interior equilibrium.

Observe that

$$\frac{\partial f_1^{int}(\cdot)}{\partial y} = ne^{-ny}S^{bond} + nD_U' > 0$$

$$\frac{\partial f_1^{int}(\cdot)}{\partial r^{syn}} = -(D^{syn})' > 0$$

$$\frac{\partial f_2^{int}(\cdot)}{\partial y} = -nD'_H < 0$$

$$\frac{\partial f_2^{int}(\cdot)}{\partial r^{syn}} = D'_H - (D^{syn})' > 0$$

Then the determinant of the derivative matrix

$$\begin{bmatrix} \frac{\partial f_1^{int}(\cdot)}{\partial y} & \frac{\partial f_1^{int}(\cdot)}{\partial r^{syn}} \\ \frac{\partial f_2^{int}(\cdot)}{\partial y} & \frac{\partial f_2^{int}(\cdot)}{\partial r^{syn}} \end{bmatrix}$$

is positive, which implies that the derivative matrix is invertible.

It follows that the equilibrium (y^*, r^{syn*}), if it exists, is unique. Suppose not and there exists another pair (r^{syn}, y) that satisfies the equilibrium in the intermediate regime. If $r^{syn} > r^{syn*}$, we must have $y > y^*$ due to $f_2^{int}(y, r^{syn}) = f_2^{int}(y^*, r^{syn*})$. It follows that $f_1^{int}(y, r^{syn}) > f_1^{int}(y^*, r^{syn*}) = 0$, and hence r^{syn} cannot be part of an equilibrium. A symmetric argument rules out all $r^{syn} < r^{syn*}$. Strict monotonicity also guarantees the uniqueness of y^*.

By the implicit function theorem,

$$\begin{bmatrix} \frac{\partial y^*(\cdot)}{\partial x} \\ \frac{\partial r^{syn*}(\cdot)}{\partial x} \end{bmatrix} = - \begin{bmatrix} \frac{\partial f_1^{int}(\cdot)}{\partial y} & \frac{\partial f_1^{int}(\cdot)}{\partial r^{syn}} \\ \frac{\partial f_2^{int}(\cdot)}{\partial y} & \frac{\partial f_2^{int}(\cdot)}{\partial r^{syn}} \end{bmatrix}^{-1} \begin{bmatrix} \frac{\partial f_1^{int}(\cdot)}{\partial x} \\ \frac{\partial f_2^{int}(\cdot)}{\partial x} \end{bmatrix}$$

for any $x \in \{S^{bond}, \bar{q}, y_{\mathbb{Q}}, \varepsilon, \omega, \delta_U, \delta_H\}$. Observe that the signs of the negative inverse matrix are

$$sgn\left(- \begin{bmatrix} \frac{\partial f_1^{int}(\cdot)}{\partial y} & \frac{\partial f_1^{int}(\cdot)}{\partial r^{syn}} \\ \frac{\partial f_2^{int}(\cdot)}{\partial y} & \frac{\partial f_2^{int}(\cdot)}{\partial r^{syn}} \end{bmatrix}^{-1} \right) = sign\left(- \begin{bmatrix} \frac{\partial f_2^{int}(\cdot)}{\partial r^{syn}} & -\frac{\partial f_1^{int}(\cdot)}{\partial r^{syn}} \\ -\frac{\partial f_2^{int}(\cdot)}{\partial y} & \frac{\partial f_1^{int}(\cdot)}{\partial y} \end{bmatrix} \right) = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}.$$

We solve for the comparative statics as follows:

1. An increase in S^{bond}: $\frac{\partial f_1^{int}(\cdot)}{\partial x} < 0, \frac{\partial f_2^{int}(\cdot)}{\partial x} = 0$, and therefore $\frac{\partial y^*(\cdot)}{\partial x} > 0$ and $\frac{\partial r^{syn*}(\cdot)}{\partial x} > 0$. Thus, both the bond yield y^* and the synthetic rate r^{syn*} increase.
2. An increase in $\bar{q}$: $\frac{\partial f_1^{int}(\cdot)}{\partial x} > 0$ and $\frac{\partial f_2^{int}(\cdot)}{\partial \bar{q}} > 0$. Thus, we have $\frac{\partial r^{syn*}(\cdot)}{\partial x} < 0$. To determine the

sign of $\frac{\partial y^*(\cdot)}{\partial x}$, we note that

$$\frac{\partial f_2^{int}(\cdot)}{\partial r^{syn}} = D_H' - (D^{syn})' > \frac{\partial f_1^{int}(\cdot)}{\partial r^{syn}} = -(D^{syn})' > 0$$

$$\frac{\partial f_1^{int}(\cdot)}{\partial \bar{q}} = \frac{\partial f_2^{int}(\cdot)}{\partial \bar{q}} = 1$$

Thus,

$$\frac{\partial y^*(\cdot)}{\partial x} \propto \frac{\partial f_1^{int}(\cdot)}{\partial r^{syn}} - \frac{\partial f_2^{int}(\cdot)}{\partial r^{syn}} < 0$$

3. An increase in δ_U: $\frac{\partial f_1^{int}(\cdot)}{\partial x} > 0$ and $\frac{\partial f_2^{int}(\cdot)}{\partial \bar{q}} = 0$. Thus, we have $\frac{\partial y^*(\cdot)}{\partial x} < 0$ and $\frac{\partial r^{syn*}(\cdot)}{\partial x} < 0$.

4. An increase in δ_H has $\frac{\partial f_1^{int}(\cdot)}{\partial x} = 0$, $\frac{\partial f_2^{int}(\cdot)}{\partial x} < 0$, and therefore $\frac{\partial y^*(\cdot)}{\partial x} < 0$ and $\frac{\partial r^{syn*}(\cdot)}{\partial x} > 0$.

5. An increase in $y_{\mathbb{Q}}$ has $\frac{\partial f_1^{int}(\cdot)}{\partial x} = 0$, $\frac{\partial f_2^{int}(\cdot)}{\partial x} = 0$ and thus $\frac{\partial y^*(\cdot)}{\partial x} = 0$ and $\frac{\partial r^{syn*}(\cdot)}{\partial x} = 0$.

6. An increase of dy in both $y_{\mathbb{Q}}$ and $y_{\mathbb{P}}$: this change is equivalent to an increase ε by dy in both f_1^{int} and f_2^{int} and an increase in ω by dy. The increase in ε causes an $\frac{n-1}{n}dy$ increase in y and no change in r^{syn*}. The increase in ω has $\frac{\partial f_1^{short}(\cdot)}{\partial x} > 0$, $\frac{\partial f_2^{short}(\cdot)}{\partial x} = 0$, and thus $\frac{\partial y^*(\cdot)}{\partial x} < 0$ and $\frac{\partial r^{syn*}(\cdot)}{\partial x} > 0$. To determine the total effect on y^*, we can evaluate the change of dy in both $y_{\mathbb{Q}}$ and $y_{\mathbb{P}}$ directly and obtain $\frac{\partial f_1^{int}(\cdot)}{\partial x} < 0$, $\frac{\partial f_2^{int}(\cdot)}{\partial x} > 0$, which implies that the total effect on y^* is positive. In summary, we find that the increase of dy in both $y_{\mathbb{Q}}$ and $y_{\mathbb{P}}$ increases y^* (by less than $\frac{n-1}{n}dy$) and increases r^{syn*}.

7. An increase in δ_{syn} has $\frac{\partial f_1^{long}(\cdot)}{\partial x} = \frac{\partial f_2^{long}(\cdot)}{\partial x} = -1$, and thus $\frac{\partial y^*(\cdot)}{\partial x} \propto \frac{\partial f_2^{int}(\cdot)}{\partial r^{syn}} - \frac{\partial f_1^{int}(\cdot)}{\partial r^{syn}} > 0$ and $\frac{\partial r^{syn*}(\cdot)}{\partial x} > 0$.

F.4 Proof of Proposition 4

Propositions 1, 2, and 3 establish that there is at most one equilibrium in each regime. To proceed, we first prove that across all possible regimes, the equilibrium is unique. Then we show the existence of an equilibrium. Finally, we will show how bond supply S^{bond} and the risk premium $y_{\mathbb{Q}}$ affects the equilibrium regime.

Define

$$f_1(y, r^{syn}; S^{bond}, y_{\mathbb{Q}}) = e^{-ny} - \frac{\exp(-(n-1)y_{\mathbb{Q}})}{(1-h)(e^{r^{tri}} - e^{r^{ois}}) + e^{r^{syn}}}$$

$$f_2(y, r^{syn}; S^{bond}, y_{\mathbb{Q}}) = \bar{q} - e^{-ny} S^{bond} - D^{syn}(r^{syn} - r^{ois}) + D_U(ny - y^{bill} - (n-1)y_{\mathbb{P}}).$$

$$f_3(y, r^{syn}; S^{bond}, y_{\mathbb{Q}}) = e^{-ny} S^{bond} - D_H(ny - r^{syn} - (n-1)y_{\mathbb{P}}) - D_U(ny - y^{bill} - (n-1)y_{\mathbb{P}}).$$

$$f_4(y, r^{syn}; S^{bond}, y_{\mathbb{Q}}) = e^{-ny} - \frac{\exp(-(n-1)y_{\mathbb{Q}})}{e^{r^{sec}} + e^{r^{ois}} - e^{r^{syn}}}$$

$$\begin{aligned} f_5(y, r^{syn}; S^{bond}, y_{\mathbb{Q}}) = \bar{q} + e^{-ny} S^{bond} - D^{syn}(r^{syn} - r^{ois}) - D_U(ny - y^{bill} - (n-1)y_{\mathbb{P}}) \\ - 2D_H(ny - r^{syn} - (n-1)y_{\mathbb{P}}). \end{aligned}$$

where f_1 is the residual of long-regime dealer indifference equation (31), f_2 is the residual of the long-regime market indifference curve (32), f_3 is the residual of the bond-market clearing condition in (27), f_4 is the residual of short-regime dealer indifference equation (34), and f_5 is the residual of the short-regime market indifference curve (35).

In equilibrium, bond market clearing (27) and synthetic lending market clearing (30) implies

$$f_3 = q^{bond}$$

$$D_H + D^{syn} = q^{syn}$$

By assumption, $r^{ois} > r^{tri} > r^{sec}$, and in any equilibrium, $r^{syn} \geq r^{ois}$. It follows that

$$2e^{r^{syn}} \geq 2e^{r^{ois}} > e^{r^{sec}} + 2e^{r^{ois}} - e^{r^{tri}} > e^{r^{sec}} + e^{r^{ois}} + (1-h)(e^{r^{ois}} - e^{r^{tri}}),$$

and hence that

$$e^{r^{syn}} + (1-h)(e^{r^{tri}} - e^{r^{ois}}) > e^{r^{sec}} + e^{r^{ois}} - e^{r^{syn}}.$$

It follows that

$$f_4(y, r^{syn}; S^{bond}, y_{\mathbb{Q}}) < f_1(y, r^{syn}; S^{bond}, y_{\mathbb{Q}}).$$

In a long-regime equilibrium, $q^{bond} > 0$, so

$$\bar{q} = q^{bond} + q^{syn}$$

Therefore,

$$\begin{aligned} f_5(y, r^{syn}; S^{bond}, y_{\mathbb{Q}}) &= \bar{q} + f_3(y, r^{syn}; S^{bond}, y_{\mathbb{Q}}) - D^{syn}(r^{syn} - r^{ois}) - D_H(ny - r^{syn} - (n-1)y_{\mathbb{P}}) \\ &= \bar{q} + q^{bond} - q^{syn} \\ &= 2q^{bond} \\ &> 0 \end{aligned}$$

Furthermore, the equilibrium conditions in the long equilibrium indicates

$$f_1 = f_2 = 0, \quad f_3 = q^{bond} > 0$$

In a short-regime equilibrium, $q^{bond} < 0$, so

$$\bar{q} = -q^{bond} + q^{syn}$$

Therefore,

$$\begin{aligned} f_2(y, r^{syn}; S^{bond}, y_{\mathbb{Q}}) &= \bar{q} - f_3(y, r^{syn}; S^{bond}, y_{\mathbb{Q}}) - D^{syn}(r^{syn} - r^{ois}) - D_H(ny - r^{syn} - (n-1)y_{\mathbb{P}}) \\ &= \bar{q} - q^{bond} - D^{syn}(r^{syn} - r^{ois}) - D_H(ny - r^{syn} - (n-1)y_{\mathbb{P}}) \\ &= \bar{q} - q^{bond} - q^{syn} \\ &= -2q^{bond} \\ &> 0 \end{aligned}$$

Furthermore, the equilibrium conditions in the short equilibrium indicates

$$f_4 = f_5 = 0, \quad f_3 = q^{bond} < 0$$

In an intermediate-regime equilibrium, $f_3 = q^{bond} = 0$, so

$$\bar{q} = q^{syn}$$

and

$$f_2(y, r^{syn}; S^{bond}, y_{\mathbb{Q}}) = \bar{q} - q^{bond} - q^{syn} = 0$$

$$f_5(y, r^{syn}; S^{bond}, y_{\mathbb{Q}}) = \bar{q} + q^{bond} - q^{syn} = 0$$

Furthermore, the intermediate-regime equilibrium requires that the yield is between the long and short thresholds, so

$$f_1 \geq 0 \geq f_4$$

Note that f_1, f_3, and f_5 are decreasing in y and increasing in r^{syn}, whereas f_2 is increasing in both y and r^{syn} and f_4 is decreasing in both in both y and r^{syn}.

We next show that the existence of either a long or a short equilibrium rules out the existence of another kind of equilibrium. Since all equilibria involve $q^{bond} > 0, q^{bond} < 0$, or $q^{bond} = 0$, it follows that an intermediate-regime equilibrium cannot coexist with other equilibria as well (i.e., the uniqueness of the intermediate-regime equilibrium holds once we prove the other two). Thus, the equilibrium if exists must be unique.

F.4.1 Uniqueness of a Long Regime Equilibrium

Suppose there is a $(y_{long}, r^{syn}_{long})$ that is a long equilibrium. Equilibrium conditions imply

$$f_4(y_{long}, r^{syn}_{long}; \cdot) < 0 = f_1(y_{long}, r^{syn}_{long}; \cdot) = f_2(y_{long}, r^{syn}_{long}; \cdot)$$

$$f_3(y_{long}, r^{syn}_{long}; \cdot) > 0, \quad f_5(y_{long}, r^{syn}_{long}; \cdot) > 0$$

The goal is to show that there cannot be another equilibrium in the short or the intermediate regime.

1. Now suppose there is another equilibrium (y, r^{syn}) that is in the intermediate regime, which implies

$$f_2(y, r^{syn}; \cdot) = f_3(y, r^{syn}; \cdot) = f_5(y, r^{syn}; \cdot) = 0$$

$$f_1(y, r^{syn}; \cdot) \geq 0 \geq f_4(y, r^{syn}; \cdot)$$

If $r^{syn} > r^{syn}_{long}$, we must have $y > y_{long}$ by $f_3(y, r^{syn}; \cdot) < f_3(y_{long}, r^{syn}_{long}; \cdot)$, but in this case,

$$f_2(y, r^{syn}; \cdot) > f_2(y_{long}, r^{syn}_{long}; \cdot) = 0,$$

which results in a contradiction.

If $r^{syn} < r^{syn}_{long}$, we have have $y < y_{long}$ by $f_1(y, r^{syn}; \cdot) > f_1(y_{long}, r^{syn}_{long}; \cdot)$, but in this case

$$f_2(y, r^{syn}; \cdot) < f_2(y_{long}, r^{syn}_{long}; \cdot) = 0,$$

whcih reuslts in a contradiction.

If $r^{syn} = r^{syn}_{long}$, it is not possible to simultaneously increase f_1 and decrease f_3 by changing y, and therefore no intermediate equilibrium exists.

Consequently, there is no alternative equilibrium in the intermediate regime.

2. Now suppose there is another equilibrium (y, r^{syn}) that is in the short regime,which implies

$$f_1(y, r^{syn}; \cdot) > 0 = f_4(y, r^{syn}; \cdot) = f_5(y, r^{syn}; \cdot)$$

$$f_2(y, r^{syn}; \cdot) > 0, \quad f_3(y, r^{syn}; \cdot) < 0$$

If $r^{syn} > r^{syn}_{long}$, we must have $y > y_{long}$ by $f_3(y, r^{syn}) < f_3(y_{long}, r^{syn}_{long}; \cdot)$, but in this case $f_4(y, r^{syn}; \cdot) < f_4(y_{long}, r^{syn}_{long}; \cdot) < 0$, which leads to a contradiction.

If $r^{syn} < r^{syn}_{long}$, we have have $y < y_{long}$ by $f_1(y, r^{syn}; \cdot) > f_1(y_{long}, r^{syn}_{long}; \cdot)$, but in this case $f_2(y, r^{syn}; \cdot) < f_2(y_{long}, r^{syn}_{long}; \cdot) = 0$, which again leads to a contradiction.

If $r^{syn} = r^{syn}_{long}$, it is not possible to simultaneously increase f_1 and decrease f_3 by changing y, and therefore no short equilibrium exists.

Consequently, there is no alternative equilibrium in the long regime.

F.4.2 Uniqueness of a Short Regime Equilibrium

Suppose there is a $(y_{short}, r^{syn}_{short})$ that is a short equilibrium. Equilibrium conditions imply

$$f_1(y_{short}, r^{syn}_{short}; \cdot) > 0 = f_4(y_{short}, r^{syn}_{short}; \cdot) = f_5(y_{short}, r^{syn}_{short}; \cdot)$$

$$f_2(y_{short}, r^{syn}_{short}; \cdot) > 0, \quad f_3(y_{short}, r^{syn}_{short}; \cdot) < 0$$

1. Now suppose there is another equilibrium (y, r^{syn}) in the intermediate regime, which implies

$$f_2(y, r^{syn}; \cdot) = f_3(y, r^{syn}; \cdot) = f_5(y, r^{syn}; \cdot) = 0$$

$$f_1(y, r^{syn}; \cdot) \geq 0 \geq f_4(y, r^{syn}; \cdot)$$

If $r^{syn} > r^{syn}_{short}$, we must have $y > y_{short}$ by $f_5(y, r^{syn}; \cdot) = f_5(y_{short}, r^{syn}_{short}; \cdot)$, but in this case $f_2(y, r^{syn}; \cdot) > f_2(y_{short}, r^{syn}_{short}; \cdot) > 0$, which leads to a contradiction.

If $r^{syn} < r^{syn}_{short}$, we have have $y < y_{short}$ by $f_5(y, r^{syn}; \cdot) = f_5(y_{short}, r^{syn}_{short}; \cdot)$, but in this case $f_4(y, r^{syn}; \cdot) > f_4(y_{short}, r^{syn}_{short}; \cdot) = 0$, which leads to a contradiction.

If $r^{syn} = r^{syn}_{short}$, then by $f_5(y, r^{syn}; \cdot) = 0$ we must have $y = y_{short}$. However, then this leads to $f_3(y, r^{syn}; \cdot) = f_3(y_{short}, r^{syn}_{short}; \cdot) < 0$, which is a contradiction.

Consequently, there is no alternative equilibrium in the intermediate regime.

2. Now suppose there is another equilibrium (y, r^{syn}) in the long regime, which implies

$$f_4(y, r^{syn}; \cdot) < 0 = f_1(y, r^{syn}; \cdot) = f_2(y, r^{syn}; \cdot)$$

$$f_3(y, r^{syn}; \cdot) > 0, \quad f_5(y, r^{syn}; \cdot) > 0$$

If $r^{syn} > r^{syn}_{short}$, we have have $y > y_{short}$ by $f_1(y, r^{syn}; \cdot) < f_1(y_{short}, r^{syn}_{short}; \cdot)$, but in this case $f_2(y, r^{syn}; \cdot) > f_2(y_{short}, r^{syn}_{short}; \cdot) > 0$, which is a contradiction.

If $r^{syn} < r^{syn}_{short}$, we must have $y < y_{short}$ by $f_3(y, r^{syn}) > f_3(y_{short}, r^{syn}_{short}; \cdot)$, but in this case $f_4(y, r^{syn}; \cdot) > f_4(y_{short}, r^{syn}_{short}; \cdot) = 0$, which is a contradiction.

If $r^{syn} = r^{syn}_{short}$, it is not possible to simultaneously increase f_3 and decrease f_1 by changing y, and therefore no long equilibrium exists.

Consequently, there is no alternative equilibrium in the short regime.

F.4.3 Equilibrium Existence

Next, we prove the existence of the equilibrium. The high-level idea is to construct the equilibrium as a convex mapping from a compact and convex set to itself, and then apply the Kakutani fixed-point theorem.

First, we show the compactness of the state space (y, r^{syn}).

Compactness of the y dimension

In any equilibrium, we must have

$$f_3(y, r^{ois}; S^{bond}, y_{\mathbb{Q}}) \leq f_3(y, r^{syn}; S^{bond}, y_{\mathbb{Q}}) \leq \bar{q}.$$

Because f_3 is decreasing in y, there is a y_{min} such that

$$f_3(y_{min}, r^{ois}; S^{bond}, y_{\mathbb{Q}}) > \bar{q},$$

and any equilibrium must have $y \geq y_{min}$. We must also have, in any equilibrium,

$$f_3(y, r^{syn}; S^{bond}, y_{\mathbb{Q}}) \geq -\bar{q},$$

which yields, by $D_H \geq 0$,

$$e^{-ny} S^{bond} - D_U(ny - y^{bill} - (n-1)y_{\mathbb{P}}) \geq -\bar{q}.$$

Defining y_{max} by

$$e^{-ny_{max}} S^{bond} - D_U(ny_{max} - y^{bill} - (n-1)y_{\mathbb{P}}) = -\bar{q},$$

it follows that $y \leq y_{max}$.

Compactness of the r^{syn} dimension

Define r^{min} as

$$D^{syn}(r^{min} - r^{ois}) = \bar{q},$$

By assumption $D^{syn}(0) > \bar{q}$ and D^{syn} is a strictly decreasing function, we have $r^{min} - r^{ois} > 0$. For any $r^{syn} < r^{min}$,

$$D^{syn}(r^{syn} - r^{ois}) > \bar{q},$$

which violates the synthetic market clearing condition in (30). Consequently, in any equilibrium, $r^{syn} \geq r^{min}$.

Next, we will find an upper bound r^{max} such that for any $r^{syn} > r^{max}$, one of the market clearing conditions are violated. First, we note that there exists a r_1^{max} such that for all $r^{syn} > r_1^{max}$, for any feasible y we consider, i.e. $y \in [y_{min}, y_{max}]$,

$$e^{-ny}(1-h)(e^{r^{tri}} - e^{r^{ois}}) + e^{r^{syn}} > \exp(-(n-1)y_{\mathbb{Q}}) > e^{-ny}(e^{r^{sec}} + e^{r^{ois}} - e^{r^{syn}}).$$

which says that $y \in (y^s, y^l)$ and thus the equilibrium is in the intermediate regime and dealer chooses $q^{bond} = 0$, and supply $\bar{q}$ to the synthetic lending market. We will show that if r^{syn} is too large, the synthetic lending market demand will fall below this supply.

Define synthetic lending demand as

$$m(y, r^{syn}) = D^{syn}(r^{syn} - r^{ois}) + D_H(ny - r^{syn} - (n-1)y_{\mathbb{P}}).$$

which decreases in r^{syn}. There exists a $r^{max} \geq r_1^{max}$ such that, for all $r^{syn} > r^{max}$ and $y \in [y_{min}, y_{max}]$,

$$m(y, r^{syn}) < \bar{q},$$

which breaks the synthetic lending market clearing condition.

Consequently, if $r^{syn} \geq r^{max}$, no equilibrium can exist.

Convex and Closed Correspondence

So far we have found a compact and convex space $\mathscr{C} = [y_{min}, y_{max}] \times [r^{min}, r^{max}]$ where the equilibrium (y, r^{syn}) must belong. Next, we define the correspondence for the equilibrium and prove that it is convex and closed.

The mapping we construct will constitute four dimensions, including (y, r^{syn}), the dealer bond position q^{bond}, and dealer synthetic lending q^{syn}.

From the dealer optimization problem, the demand correspondence only depends on (y, r^{syn}) and is defined as follows

$$Q(y, r^{syn}) = \begin{cases} \{(\bar{q}, 0)\} & \text{if } f_1(y, r^{syn}; S^{bond}, y_{\mathbb{Q}}) < 0 \\ \{(q^{bond}, q^{syn}) \in \mathbb{R}_+^2 : q^{bond} + q^{syn} = \bar{q}\} & \text{if } f_1(y, r^{syn}; S^{bond}, y_{\mathbb{Q}}) = 0, \\ \{(q^{bond}, q^{syn}) \in \mathbb{R}_- \times \mathbb{R}_+ : -q^{bond} + q^{syn} = \bar{q}\} & \text{if } f_4(y, r^{syn}; S^{bond}, y_{\mathbb{Q}}) = 0, \\ \{(-\bar{q}, 0)\} & \text{if } f_4(y, r^{syn}; S^{bond}, y_{\mathbb{Q}}) > 0, \\ \{(0, \bar{q})\} & otherwise. \end{cases}$$

The first case $f_1 < 0$ is the only-long region where $y > y^l$. The second case $f_1 = 0$ is the long region where $y = y^l$. The third case $f_4 = 0$ is the sell region where $y = y^s$. The fourth case $f_4 > 0$ is the sell-only region where $y < y^s$. The fifth case is the intermediate region where $y^s < y < y^l$.

Define the aggregate excess demand correspondence as

$$Z(y, r^{syn}) = \{(z_1, z_2\} \in \mathbb{R}^2 : (z_1 + f_3(y, r^{syn}; \cdot), m(y, r^{syn}) - z_2) \in Q(y, r^{syn})\}.$$

Here, z_1 represents the excess demand for bonds, and z_2 is the excess demand for synthetic loans. By definition, $f_3(\cdot)$ is the bond supply less non-intermediary demand, and hence $f_3(\cdot) + z_1$ must equal the intermediary demand q^{bond}. Likewise, $m(\cdot)$ is synthetic loan demand, and $m(\cdot) - z_2$ must equal the synthetic loan supply q^{syn}.

Note that this correspondence is non-empty, u.h.c. (by the u.h.c. property of q, which ultimately arises from the continuity of f_1, f_4, and the continuity of f_3 and m). Note that it is also convex-valued, a property it inherits from Q. Define the maximum and minimum possible excess demands by

$$\begin{aligned}
z^{bond}_{max} &= \max_{(y, r^{syn}) \in [y_{min}, y_{max}] \times [r^{min}, r^{max}]} \bar{q} - f_3(y, r^{syn}; \cdot), \\
z^{bond}_{min} &= \min_{(y, r^{syn}) \in [y_{min}, y_{max}] \times [r^{min}, r^{max}]} -\bar{q} - f_3(y, r^{syn}; \cdot), \\
z^{syn}_{max} &= \max_{(y, r^{syn}) \in [y_{min}, y_{max}] \times [r^{min}, r^{max}]} m(y, r^{syn}), \\
z^{syn}_{\min} &= \min_{(y, r^{syn}) \in [y_{min}, y_{max}] \times [r^{min}, r^{max}]} m(y, r^{syn}) - \bar{q},
\end{aligned}$$

Now define a price player, who solves, given any vector $(z_1, z_2) \in [z^{bond}_{min}, z^{bond}_{max}]$,

$$\max_{(y, r^{syn}) \in [y_{min}, y_{max}] \times [r^{min}, r^{max}]} (y, r^{syn}) \cdot \begin{bmatrix} -z_1 \\ z_2 \end{bmatrix}.$$

Let $p^*(z)$ be the optimal policy correspondence, and note that it is non-empty, u.h.c., and convex-valued (which follows from the concavity of the objective).

Now define the correspondence

$$g(y, r^{syn}, z) = \begin{bmatrix} p^*(z) \\ Z(y, r^{syn}) \end{bmatrix}$$

which maps $[y_{min}, y_{max}] \times [r^{min}, r^{max}] \times [z^{bond}_{min}, z^{bond}_{max}] \times [z^{syn}_{min}, z^{syn}_{max}]$ to itself. Note that this set is compact, and by the u.h.c. properties of p^* and Z and the compactness of this set, g has a closed graph. Consequently, by Kakutani's fixed point theorem, a fixed point (y^*, r^{syn*}, z^*) exists.

By construction, at y_{min},

$$f_3(y_{min}, r^{ois}; S^{bond}, y_{\mathbb{Q}}) > \bar{q},$$

and consequently all values $Z_1(y, r^{syn})$ are negative. The best response of the price player at this point would be y_{max}, and hence there cannot be a fixed point with $y^* = y_{min}$. Essentially the same logic rules out $y^* = y_{max}$. Similarly, if $r^{syn*} = r^{min}$, then all values of $Z_2(y^*, r^{syn*})$ are positive, and the price player's best response is r_{max}, and hence this cannot be a fixed point. Likewise, if $r^{syn*} = r^{max}$, then all values of $Z_2(y^*, r^{syn*})$ are negative, and the price player's best response is $r^{syn} = r^{min}$. It follows that the fixed point is interior, and hence that $z^* = Z(y^*, r^{syn*}) = (0,0)$. Note that a fixed point with $z^* = (0,0)$ cannot exist in which there is no supply of synthetic lending; consequently, the equilibrium is either a long regime equilibrium,

$$f_4(y, r^{syn}; S^{bond}, y_{\mathbb{Q}}) < f_1(y, r^{syn}; S^{bond}, y_{\mathbb{Q}}) = 0,$$

a short regime equilibrium,

$$0 = f_4(y, r^{syn}; S^{bond}, y_{\mathbb{Q}}) < f_1(y, r^{syn}; S^{bond}, y_{\mathbb{Q}}),$$

or an intermediate equilibrium,

$$f_4(y, r^{syn}; S^{bond}, y_{\mathbb{Q}}) < 0 < f_1(y, r^{syn}; S^{bond}, y_{\mathbb{Q}}).$$

F.4.4 Bond Supply and Equilibrium Regime

To prove that the existence of cutoffs S_S and S_B with $0 \leq S_S \leq S_B \leq \infty$, such that the short-regime, the intermediate regime, and the long-regime fall into the three regions, we simply prove that there is a ranking of the equilibrium along the supply of bonds S.

Consider $S^{bond} = S$. According to the previous proofs, an equilibrium (y, r^{syn}) exists and must be unique.

Long Equilibrium

First, we show that if S corresponds to a long-regime equilibrium, then for any $\bar{S} > S$, the equilibrium $(\bar{y}, \bar{r}^{syn})$ must also be a long equilibrium.

Suppose instead the equilibrium for $S^{bond} = \bar{S}$ is a short-regime equilibrium with $(\bar{y}, \bar{r}^{syn})$. Then we must have $y = y^l > y^s = \bar{y}$. Furthermore, $f_2(y, r^{syn}; S, y_{\mathbb{Q}}) = 0$ and $f_2(\bar{y}, \bar{r}^{syn}; S, y_{\mathbb{Q}}) > f_2(\bar{y}, \bar{r}^{syn}; \bar{S}, y_{\mathbb{Q}}) > 0$. By monotonicity of f_2, we must have $\bar{r}^{syn} > r^{syn}$. Therefore, by monotonicty of f_5, we have

$$f_5(\bar{y}, \bar{r}^{syn}; \bar{S}, y_{\mathbb{Q}}) > f_5(y, r^{syn}; \bar{S}, y_{\mathbb{Q}})$$

However, in the long regime, $f_5(\bar{y}, \bar{r}^{syn}; \bar{S}, y_{\mathbb{Q}}) = 0$, and in the short regime, $f_5(y, r^{syn}; \bar{S}, y_{\mathbb{Q}}) > f_5(y, r^{syn}; S, y_{\mathbb{Q}}) > 0$, which leads to a contradiction. Thus, $\bar{S}$ cannot correspond to a short equilibrium.

Next, suppose that $\bar{S}$ corresponds to an intermediate-regime equilibrium. Then we have $y = y^l \geq \bar{y}$, $f_3(\bar{y}, \bar{r}^{syn}; \bar{S}, y_{\mathbb{Q}}) = 0$. For the equilibrium of S, we have $f_3(y, r^{syn}; \bar{S}, y_{\mathbb{Q}}) > f_3(y, r^{syn}; S, y_{\mathbb{Q}}) > 0$. By the monotonicity of f_3, $r^{syn} > \bar{r}^{syn}$. Thus, $f_2(\bar{y}, \bar{r}^{syn}; S, y_{\mathbb{Q}}) < f_2(y, r^{syn}; S, y_{\mathbb{Q}})$.

However, by the properties of long and intermediate regimes, we also have $f_2(y, r^{syn}; S, y_{\mathbb{Q}}) > 0$ and $f_2(\bar{y}, \bar{r}^{syn}; S, y_{\mathbb{Q}}) > f_2(\bar{y}, \bar{r}^{syn}; \bar{S}, y_{\mathbb{Q}}) = 0$, which leads to a contradiction.

In summary, if S is a long-regime equilibrium, for any $\bar{S} > S$, the equilibrium $(\bar{y}, \bar{r}^{syn})$ must also be a long-regime equilibrium.

Short Equilibrium

Second, we show that if S corresponds to a short-regime equilibrium, then for any $\underline{S} < S$, the equilibrium solution $(\underline{y}, \underline{r}^{syn})$ must also be a short-regime equilibrium.

Suppose that instead the equilibrium for $S^{bond} = \underline{S}$ is a long-regime equilibrium. Then we must have $y = y^s < y^l = \underline{y}$. Furthermore, $f_5(\underline{y}, \underline{r}^{syn}; S, y_{\mathbb{Q}}) > f_5(\underline{y}, \underline{r}^{syn}; \underline{S}, y_{\mathbb{Q}}) > 0$, and $f_5(y, r^{syn}; S, y_{\mathbb{Q}}) = 0$. By monotonicity of f_5, we get $\underline{r}^{syn} > r^{syn}$. Thus, by monotonicity of f_2, we obtain

$$f_2(y, r^{syn}; S, y_{\mathbb{Q}}) < f_2(\underline{y}, \underline{r}^{syn}; S, y_{\mathbb{Q}})$$

However, by the properties of long and short regimes, we must have $f_2(\underline{y}, \underline{r}^{syn}; S, y_{\mathbb{Q}}) < f_2(\underline{y}, \underline{r}^{syn}; \underline{S}, y_{\mathbb{Q}}) = 0$, and $f_2(y, r^{syn}; S, y_{\mathbb{Q}}) > 0$, which leads to a contradiction.

Suppose that the equilibrium for $S^{bond} = \underline{S}$ is an intermediate-regime equilibrium. Then we must have $y = y^s \leq \underline{y}$. Furthermore, $f_5(\underline{y}, \underline{r}^{syn}; S, y_{\mathbb{Q}}) > f_5(\underline{y}, \underline{r}^{syn}; \underline{S}, y_{\mathbb{Q}}) = 0$, and $f_5(y, r^{syn}; S, y_{\mathbb{Q}}) = 0$. By monotonicity of f_5, we get $\underline{r}^{syn} > r^{syn}$. Thus, by monotonicity of f_2, we obtain

$$f_2(y, r^{syn}; S, y_{\mathbb{Q}}) < f_2(\underline{y}, \underline{r}^{syn}; S, y_{\mathbb{Q}})$$

However, by the properties of short and intermediate regimes, we must have $f_2(y, r^{syn}; S, y_{\mathbb{Q}}) > 0$ and $f_2(\underline{y}, \underline{r}^{syn}; S, y_{\mathbb{Q}}) < f_2(\underline{y}, \underline{r}^{syn}; \underline{S}, y_{\mathbb{Q}}) = 0$, which leads to a contradiction.

In summary, if S is a short-regime equilibrium, for any $\underline{S} < S$, the equilibrium $(\underline{y}, \underline{r}^{syn})$ must also be a short-regime equilibrium.

Regime Ranking

From the above discussions, we know that there must be cutoffs S_S and S_B with $0 \leq S_S \leq S_B \leq \infty$, such that a short-regime equilibrium exists in the left region, an intermediate-regime equilibrium exists in the middle region, and a long-regime equilibrium exists in the right region. However, we still have to prove whether the intervals are open or closed.

Intervals for Regimes

We now show that the interval of long-regime equilibrium should be (S_B, ∞) instead of $[S_B, \infty)$. Suppose that $S^{bond} = S$ is a long-regime equilibrium, with solutions (y, r^{syn}), and q^{bond}. By definition, $q^{bond} > 0$.

In the long regime, $f_3(y, r^{syn}; S, y_{\mathbb{Q}}) = q^{bond}$. We know that $q^{bond} > 0$ increases in the total supply of bond and the mapping is continuous. Therefore, there exists a smaller bond supply $S^{bond} = S - \varepsilon$ for $\varepsilon > 0$, such that the new equilibrium still has $q^{bond} > 0$. Consequently, the interval of S^{bond} for the long-regime equilibrium must be an open set.

Similarly, the interval for the short-regime equilibrium must also be an open set, $(-\infty, S_B)$.

F.4.5 Term Premium and Equilibrium Regime

Next, we study how the term premium $y_{\mathbb{Q}}$ affects the equilibrium. Consider $y_{\mathbb{Q}}$. According to previous proofs, an equilibrium solution (y, r^{syn}) exists and is unique.

Long Equilibrium

First, we show that if $y_{\mathbb{Q}}$ corresponds to a long-regime equilibrium, then for any $\bar{y}_{\mathbb{Q}} > y_{\mathbb{Q}}$, the equilibrium $(\bar{y}, \bar{r}^{syn})$ must also be a long equilibrium.

Suppose instead the equilibrium for $\bar{y}_{\mathbb{Q}}$ is a short-regime equilibrium with $(\bar{y}, \bar{r}^{syn})$. Then we must have $y = y^l > y^s = \bar{y}$. Furthermore, $f_2(y, r^{syn}; S^{bond}, y_{\mathbb{Q}}) = 0$ and $f_2(\bar{y}, \bar{r}^{syn}; S^{bond}, y_{\mathbb{Q}}) = f_2(\bar{y}, \bar{r}^{syn}; S^{bond}, \bar{y}_{\mathbb{Q}}) > 0$. By monotonicity of f_2, we must have $\bar{r}^{syn} > r^{syn}$. Therefore, by monotonicty of f_5, we have

$$f_5(\bar{y}, \bar{r}^{syn}; S^{bond}, y_{\mathbb{Q}}) > f_5(y, r^{syn}; S^{bond}, y_{\mathbb{Q}})$$

However, in the short regime, $f_5(\bar{y}, \bar{r}^{syn}; S^{bond}, \bar{y}_{\mathbb{Q}}) = f_5(\bar{y}, \bar{r}^{syn}; S^{bond}, y_{\mathbb{Q}}) = 0$, and in the long regime, $f_5(y, r^{syn}; S^{bond}, y_{\mathbb{Q}}) > 0$, which leads to a contradiction. Thus, $\bar{y}_{\mathbb{Q}}$ cannot correspond to a short equilibrium.

Next, suppose that $\bar{y}_{\mathbb{Q}}$ corresponds to an intermediate-regime equilibrium. Then we have $y = y^l \geq \bar{y}$, $f_3(\bar{y}, \bar{r}^{syn}; S^{bond}, y_{\mathbb{Q}}) = f_3(\bar{y}, \bar{r}^{syn}; S^{bond}, \bar{y}_{\mathbb{Q}}) = 0$. For the long-regime equilibrium of $y_{\mathbb{Q}}$, we have $f_3(y, r^{syn}; S^{bond}, y_{\mathbb{Q}}) > 0$. By the monotonicity of f_3, $r^{syn} > \bar{r}^{syn}$. Thus, $f_2(\bar{y}, \bar{r}^{syn}; S^{bond}, y_{\mathbb{Q}}) < f_2(y, r^{syn}; S^{bond}, y_{\mathbb{Q}})$.

However, by the properties of long and intermediate regimes, we also have $f_2(y, r^{syn}; S^{bond}, y_{\mathbb{Q}}) = 0$ and $f_2(\bar{y}, \bar{r}^{syn}; S^{bond}, y_{\mathbb{Q}}) = f_2(\bar{y}, \bar{r}^{syn}; S^{bond}, \bar{y}_{\mathbb{Q}}) = 0$, which leads to a contradiction.

In summary, if $y_{\mathbb{Q}}$ is a long-regime equilibrium, for any $\bar{y}_{\mathbb{Q}} > y_{\mathbb{Q}}$, the equilibrium $(\bar{y}, \bar{r}^{syn})$ must also be a long equilibrium.

Short Equilibrium

Second, we show that if $y_{\mathbb{Q}}$ corresponds to a short-regime equilibrium, then for any $\underline{y}_{\mathbb{Q}} < y_{\mathbb{Q}}$, the equilibrium $(\underline{y}, \underline{r}^{syn})$ must also be a short-regime equilibrium.

Suppose that instead the equilibrium for $\underline{y}_{\mathbb{Q}}$ is a long-regime equilibrium. Then we must have $y = y^s < y^l = \underline{y}$. Furthermore, $f_5(\underline{y}, \underline{r}^{syn}; S^{bond}, y_{\mathbb{Q}}) = f_5(\underline{y}, \underline{r}^{syn}; S^{bond}, \underline{y}_{\mathbb{Q}}) > 0$, and $f_5(y, r^{syn}; S^{bond}, y_{\mathbb{Q}}) = 0$. By monotonicity of f_5, we get $\underline{r}^{syn} > r^{syn}$. Thus, by monotonicity of f_2, we obtain

$$f_2(y, r^{syn}; S^{bond}, y_{\mathbb{Q}}) < f_2(\underline{y}, \underline{r}^{syn}; S^{bond}, y_{\mathbb{Q}})$$

However, by the properties of long and short regimes, we must have

$$f_2(\underline{y}, \underline{r}^{syn}; S^{bond}, y_{\mathbb{Q}}) = f_2(\underline{y}, \underline{r}^{syn}; S^{bond}, \underline{y}_{\mathbb{Q}}) = 0,$$

and $f_2(y, r^{syn}; S^{bond}, y_{\mathbb{Q}}) > 0$, which leads to a contradiction.

Suppose that the equilibrium for $\underline{y}_{\mathbb{Q}}$ is an intermediate-regime equilibrium. Then we must have $y = y^s \leq \underline{y}$. Furthermore, $f_3(\underline{y}, \underline{r}^{syn}; S^{bond}, y_{\mathbb{Q}}) = f_3(\underline{y}, \underline{r}^{syn}; S^{bond}, \underline{y}_{\mathbb{Q}}) = 0$, and $f_3(y, r^{syn}; S^{bond}, y_{\mathbb{Q}}) < 0$. By monotonicity of f_3, we get $\underline{r}^{syn} > r^{syn}$. Thus, by monotonicity of f_2, we obtain

$$f_2(y, r^{syn}; S^{bond}, y_{\mathbb{Q}}) < f_2(\underline{y}, \underline{r}^{syn}; S^{bond}, y_{\mathbb{Q}})$$

However, by the properties of short and intermediate regimes, we must have $f_2(y, r^{syn}; S^{bond}, y_{\mathbb{Q}}) > 0$ and $f_2(\underline{y}, \underline{r}^{syn}; S^{bond}, y_{\mathbb{Q}}) = f_2(\underline{y}, \underline{r}^{syn}; S^{bond}, \underline{y}_{\mathbb{Q}}) = 0$, which leads to a contradiction.

In summary, if $y_{\mathbb{Q}}$ is a short-regime equilibrium, for any $\underline{y}_{\mathbb{Q}} < y_{\mathbb{Q}}$, the equilibrium $(\underline{y}, \underline{r}^{syn})$ must also be a short-regime equilibrium.

Regime Ranking

From the above discussions, we know that there must be cutoffs y_S and y_B with $0 \leq y_S \leq y_B \leq \infty$, such that a short-regime equilibrium, an intermediate-regime equilibrium, and a long-regime equilibrium exits in the left, middle and right regions. However, we still need to determine whether those intervals are open or closed sets.

Intervals for Regimes

We now show that the interval of long-regime equilibrium should be (y_B, ∞) instead of $[y_B, \infty)$. Suppose that $y_{\mathbb{Q}}$ is a long-regime equilibrium, with solutions (y, r^{syn}), and q^{bond}. By definition, $q^{bond} > 0$.

In the long regime, we know that $q^{bond} > 0$ is a continuous function of $y_{\mathbb{Q}}$. Therefore, there exists a smaller risk-neutral expectation $y_{\mathbb{Q}} - \varepsilon$, where the new equilibrium is still in the long regime with $q^{bond} > 0$. Consequently, the interval of $y_{\mathbb{Q}}$ for the long-regime equilibrium must be an open set.

Similarly, the interval of $y_{\mathbb{Q}}$ for the short-regime equilibrium must also be an open set.