Everybody's Talkin' at Me: Levels of Majority Language Acquisition by Minority Language Speakers
Online Appendix

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Online Appendix: Omitted Proofs and Calculations

Proof of Lemma 1. Let $\mathcal{F}, \mathcal{P}, \mathcal{N}$ be the measures of measurable (possibly empty) sets of agents in each minority group that choose F, P and N respectively in σ^*.[38]

Convexity: Suppose $\sigma_i^*(\theta) = \sigma_i^*(\theta') = F$, i.e., for $\tilde{\theta} \in \{\theta, \theta'\}$, F is a best response

$$\begin{aligned}
1+\lambda+(n-1)\mathcal{F}+\alpha(n-1)\mathcal{P}-\ell_f\tilde{\theta} &\geq \left\{1, 1+\alpha\lambda+\alpha(n-1)\mathcal{F}+\alpha^2(n-1)\mathcal{P}-\ell_p\tilde{\theta}\right\} \text{ or} \\
\ell_f\tilde{\theta} &\leq \lambda+(n-1)\mathcal{F}+\alpha(n-1)\mathcal{P} \text{ and} \qquad (35) \\
(\ell_f-\ell_p)\tilde{\theta} &\leq (1-\alpha)\left[\lambda+(n-1)\mathcal{F}+\alpha(n-1)\mathcal{P}\right] \qquad (36)
\end{aligned}$$

Conditions (35) and (36) then imply that $\forall\delta \in (0,1)$,

$$\begin{aligned}
\ell_f(\delta\theta+(1-\delta)\theta') &\leq \lambda+(n-1)\mathcal{F}+\alpha(n-1)\mathcal{P} \text{ and} \\
(\ell_f-\ell_p)(\delta\theta+(1-\delta)\theta') &\leq (1-\alpha)\left[\lambda+(n-1)\mathcal{F}+\alpha(n-1)\mathcal{P}\right]
\end{aligned}$$

Hence, F is also a best response for type "$\delta\theta+(1-\delta)\theta'$." The analysis for strategies P and N is analogous.

Monotonicity: We only consider the case where $\sigma_i^*(\theta) = F$and $\sigma_i^*(\theta') = P$, and the remaining cases are analogous. Recall that $\mathcal{F}, \mathcal{P}, \mathcal{N}$ are connected intervals. Suppose instead $\theta > \theta'$, and $\sigma_i^*(\theta') = P$, $\sigma_i^*(\theta) = F$. Then

$$\begin{aligned}
\theta \text{ prefers } F \text{ to } P\text{, or } (\ell_f-\ell_p)\theta &\leq (1-\alpha)\left[\lambda+(n-1)\mathcal{F}+\alpha(n-1)\mathcal{P}\right] \\
\theta' \text{ prefers } P \text{ to } F\text{, or } (\ell_f-\ell_p)\theta' &\geq (1-\alpha)\left[\lambda+(n-1)\mathcal{F}+\alpha(n-1)\mathcal{P}\right]
\end{aligned}$$

which contradicts $\theta > \theta'$. We hence have $\theta' \geq \theta$.

Positivity: First, F is a dominant strategy for type 0, i.e., $\sigma_i^*(0) = F$. Now consider

[38] Here the measurable structure on each set of agents is its collection of Borel subsets.

type $\varepsilon \in \left[0, \min\left\{\frac{\lambda}{\ell_f}, \frac{(1-\alpha)\lambda}{\ell_f - \ell_p}\right\}\right]$. Given $\lambda > 1, \alpha > 0$ and $\ell_f > \ell_p$, we have

$$\begin{aligned} \ell_f \varepsilon &\leq \lambda + (n-1)\mathcal{F} + \alpha (n-1)\mathcal{P} \\ (\ell_f - \ell_p)\varepsilon &\leq (1-\alpha)\left[\lambda + (n-1)\mathcal{F} + \alpha(n-1)\mathcal{P}\right] \end{aligned}$$

regardless of $\mathcal{F}$ and $\mathcal{P}$. Hence, F is also a dominant strategy for type ε. Convexity then implies that all types in $[0, \varepsilon]$ choose F in any symmetric equilibrium. ■

Proof of Proposition 2. First, for equilibrium $\sigma^{\mathbb{FN}}$, the analysis is exactly identical to that for the binary language acquisition setting (Proposition 1) and we similarly have insufficient language acquisition.

Now consider equilibrium $\sigma^{\mathbb{FP}}$. We similarly write down the social welfare function when minority agents are either partially or fully learning the majority language with cutoff θ as:

$$W^{\mathbb{FP}}(\theta) = n \left[\begin{array}{c} 2\lambda H(\theta) + (n-1)(H(\theta))^2 + 2\alpha\lambda(1 - H(\theta)) + (n-1)\alpha^2 [1 - H(\theta)]^2 \\ +2\alpha(n-1)[1 - H(\theta)] H(\theta) - \ell_f \int_0^\theta t dH(t) - \ell_p \int_\theta^1 t dH(t) \end{array} \right]$$

Differentiate $W^{\mathbb{FP}}(\theta)$ and evaluate the derivative at $\theta = \theta_f$ in (9) to obtain

$$\begin{aligned} \left.\frac{dW^{\mathbb{FP}}(\theta)}{d\theta}\right|_{\theta=\theta_f} &\propto \left.\left[\begin{array}{c} 2(1-\alpha)\lambda + 2(n-1)(1-\alpha)H(\theta) \\ +2\alpha(1-\alpha)(n-1)(1-H(\theta)) - \ell_f\theta + \ell_p\theta \end{array} \right]\right|_{\theta=\theta_f} \\ &\propto \left[(1-\alpha)\lambda + (n-1)(1-\alpha)H(\theta_f)\alpha(1-\alpha)(n-1)(1-H(\theta_f))\right] \\ &> 0 \end{aligned}$$

which implies that increasing θ_f strictly increases social welfare. Hence, there is insufficient learning in equilibrium $\sigma^{\mathbb{FP}}$.

Finally, consider equilibrium $\sigma^{\mathbb{FPN}}$. Given an arbitrary pair of cutoffs (θ_F, θ_P) with $0 < \theta_F < \theta_P < 1$, the total social welfare in such a learning outcome can be written as

$$W^{\mathbb{FPN}}(\theta_F, \theta_P) = n \left[\begin{array}{c} 2\lambda H(\theta_F) + (n-1)(H(\theta_F))^2 + (n-1)\alpha^2 (H(\theta_P) - H(\theta_F))^2 \\ +2\alpha\lambda(H(\theta_P) - H(\theta_F)) + 2\alpha(n-1)(H(\theta_P) - H(\theta_F))H(\theta_F) \\ -\ell_f \int_0^{\theta_F} t dH(t) - \ell_p \int_{\theta_F}^{\theta_P} t dH(t) \end{array} \right]$$

We similarly obtain two first-order partial derivatives, evaluated as the equilibrium cutoff

(θ_f, θ_p) defined in (5) and (6), as

$$\left.\frac{\partial W^{\mathbb{FPN}}(\theta_F, \theta_P)}{\partial \theta_F}\right|_{(\theta_f, \theta_p)} \propto \left[\begin{array}{c}(1-\alpha)\lambda + (1-\alpha)(n-1) H(\theta_f) \\ +\alpha(1-\alpha)(n-1)(H(\theta_p) - H(\theta_f))\end{array}\right]$$
$$\left.\frac{\partial W^{\mathbb{FPN}}(\theta_F, \theta_P)}{\partial \theta_P}\right|_{(\theta_f, \theta_p)} \propto \left[\alpha\lambda + \alpha(n-1) H(\theta_f) + \alpha^2 (n-1)(H(\theta_p) - H(\theta_f))\right]$$

And we immediately obtain

$$\left.\frac{\partial W^{\mathbb{FPN}}(\theta_F, \theta_P)}{\partial \theta_F}\right|_{(\theta_f, \theta_p)} > 0 \text{ and } \left.\frac{\partial W^{\mathbb{FPN}}(\theta_F, \theta_P)}{\partial \theta_P}\right|_{(\theta_f, \theta_p)} > 0,$$

or there is insufficient acquisition at both the partial and the full learning levels. ■

Proof of Proposition 3. Consider Part [I], i.e., $L_p \geq 1$.

First, condition (14) in the uniform setting reduces to $L_f \leq \min\{1, 1-\alpha(1-L_p)\}$. Hence, given $L_p \geq 1$, equilibrium $\mathbb{F}$ exists if $L_f \leq 1$.

For equilibrium $\mathbb{FN}$, i.e., condition (13), $\theta_f = \frac{\lambda}{\ell_f - (n-1)} \in (0,1)$, is equivalent to $L_f > 1$, while $u_i\left(P; \sigma^{\mathbb{FN}}, \theta_f\right) \leq u_i\left(N; \sigma^{\mathbb{FN}}, \theta_f\right)$ reduces to

$$\frac{\lambda}{\ell_f - (n-1)} \geq \frac{\alpha\lambda}{\ell_p - \alpha(n-1)} \Leftrightarrow \frac{\ell_p}{\alpha} \geq \ell_f \Rightarrow L_p \geq L_f.$$

Hence, equilibrium $\mathbb{FN}$ arises if $1 < L_f \leq L_p$.

Equilibrium $\mathbb{FP}$ requires two conditions $\theta_f < 1$ and $u_i\left(P; \sigma^{\mathbb{FP}}, 1\right) \geq 1$. For the cutoff θ_f from (9), shown explicitly in (37), to be in $(0,1)$, we need $L_f > 1-\alpha(1-L_p)$. While the second condition $u_i\left(P; \sigma^{\mathbb{FP}}, 1\right) \geq 1$, i.e., type $\theta = 1$ prefers P to N, reduces to

$$\theta_f = \frac{(1-\alpha)[\lambda + \alpha(n-1)]}{(\ell_f - \ell_p) - (1-\alpha)^2(n-1)} \geq \frac{\ell_p - \alpha\lambda - \alpha^2(n-1)}{\alpha(1-\alpha)(n-1)} \equiv \frac{R(\alpha)}{\alpha(1-\alpha)(n-1)}. \tag{37}$$

It can be verified that $R(\alpha)/[\alpha(1-\alpha)(n-1)] < 1$ is equivalent to $L_p < 1$, contradicting $L_p \geq 1$. Hence, equilibrium $\mathbb{FP}$ cannot exist if $L_p \geq 1$.

Finally, consider equilibrium $\mathbb{FPN}$. Recall that the requirement for this equilibrium to exist is $0 < \theta_f < \theta_p < 1$, where the interior cutoffs are calculated in (15) and (16). Observe that $L_f > L_p$ (or $\ell_f > \frac{\ell_p}{\alpha}$) implies $\theta_f < \theta_p$. Moreover, since $\ell_f > \ell_p$, we have

$$\begin{aligned} L_p &\geq 1 \Leftrightarrow \alpha\lambda(\ell_f - \ell_p) \leq (\ell_f - \ell_p)[\ell_p - \alpha(n-1)], \\ \ell_f &> \frac{\ell_p}{\alpha} \Leftrightarrow (\ell_f - \ell_p)[\ell_p - \alpha(n-1)] \leq \ell_p(\ell_f - \ell_p) + (n-1)\left(2\alpha\ell_p - \alpha^2\ell_f - \ell_p\right), \end{aligned}$$

which jointly imply that $\theta_p < 1$, and the denominator of θ_f is positive, i.e., $\theta_f > 0$.

Now consider Part [II], i.e., $L_p < 1$. We will use the following key parameters:

$$\bar{\alpha} = \frac{\sqrt{\lambda^2 + 4(n-1)\ell_p} - \lambda}{2(n-1)}, \; \overline{L}_p = \frac{\ell_p}{\bar{\alpha}(\lambda + n - 1)}, \; G = \frac{\ell_p^2 - \alpha\lambda\ell_p - (n-1)(2\alpha\ell_p - \ell_p)}{[\ell_p - \alpha\lambda - (n-1)\alpha^2](\lambda + n - 1)}. \tag{38}$$

For equilibrium $\mathbb{F}$, condition (14) $L_f \le \min\{1, 1 - \alpha(1 - L_p)\}$ now reduces to

$$L_f \le 1 - \alpha(1 - L_p).$$

For equilibrium $\mathbb{FN}$, our discussion in Part [I] shows that this equilibrium exists if and only if $1 < L_f \le L_p$, which cannot hold when $L_p < 1$.

Now consider equilibrium $\mathbb{FP}$. As discussed in Part [I], "$\theta_f < 1$" is equivalent to $L_f > 1 - \alpha(1 - L_p)$, while "$u_i\left(P; \sigma^{\mathbb{FP}}, 1\right) \ge 1$" is shown in (37), where $R(\alpha)/[\alpha(1-\alpha)(n-1)] < 1$, given $L_p < 1$. In addition, we verify that $R(\bar{\alpha}) = 0$ (we ignore the other negative root of $R(\alpha) = 0$). Depending on the sign of $R(\alpha)$, we have two cases: If $\alpha \ge \bar{\alpha}$ or $L_p \le \overline{L}_p = \frac{\ell_p}{\bar{\alpha}(\lambda+n-1)}$, we have $R(\alpha) \le 0$ and hence (37) is automatic. Notice that $L_p \le \overline{L}_p$ is always true if $\overline{L}_p \ge 1$ since we have assumed $L_p < 1$. If $\overline{L}_p < 1$ and $L_p \in \left(\overline{L}_p, 1\right)$, we have $R(\alpha) > 0$ and (37) reduces to

$$\ell_f \le \frac{\alpha\lambda\ell_p - \ell_p^2 + (n-1)(2\alpha\ell_p - \ell_p)}{\alpha\lambda - \ell_p + (n-1)\alpha^2} \Longleftrightarrow L_f \le G. \tag{39}$$

One can verify that $G > 1 - \alpha(1 - L_p)$, so that $1 - \alpha(1 - L_p) < L_f \le G$ is well defined.

Finally, consider equilibrium $\mathbb{FPN}$. As before, $\theta_p > \theta_f$ is equivalent to $\ell_f > \frac{\ell_p}{\alpha}$, or $L_f > L_p$. We further need $\theta_f > 0$ and $\theta_p < 1$. Now rewrite $\theta_p < 1$ to be

$$\begin{aligned} \alpha\lambda(\ell_f - \ell_p) &< \ell_p(\ell_f - \ell_p) + (n-1)\left(2\alpha\ell_p - \alpha^2\ell_f - \ell_p\right) \text{ or} \\ \left(\ell_p - \alpha\lambda - \alpha^2(n-1)\right)\ell_f &= R(\alpha)\ell_f > \ell_p^2 - \alpha\lambda\ell_p + (1 - 2\alpha)(n-1)\ell_p \end{aligned} \tag{40}$$

Observe that (40) is similar to (39) for equilibrium $\mathbb{FP}$, since the two equilibria are similar. However, the ranges of α are different across the two equilibria. Recall that $R(\alpha) = \ell_p - \alpha\lambda - \alpha^2(n-1)$, with $R(\bar{\alpha}) = 0$, $R(\alpha) > 0$ for $\alpha < \bar{\alpha}$ and $R(\alpha) < 0$ for $\alpha > \bar{\alpha}$. We consider two familiar cases.

Case 1. $\overline{L}_p \ge 1$. Such an $\overline{L}_p$, together with $1 > L_p$, is equivalent to $\alpha > \frac{\ell_p}{\lambda+n-1} \ge \bar{\alpha}$, implying that $R(\alpha) < 0$. Hence condition (40) can be rewritten as

$$\ell_f < \frac{\ell_p^2 - \alpha\lambda\ell_p + (1-2\alpha)(n-1)\ell_p}{R(\alpha)} = \frac{\ell_p^2 - \alpha\lambda\ell_p + (1-2\alpha)(n-1)\ell_p}{\ell_p - \alpha\lambda - \alpha^2(n-1)}. \tag{41}$$

If the numerator of the RHS of (41) is non-negative, i.e., if $\ell_p^2 - \alpha\lambda\ell_p + (1-2\alpha)(n-1)\ell_p \geq 0$, then expression (41) can never hold since the RHS of (41) is non-positive and $\ell_f > 0$. While if the numerator is negative, which happens when $\alpha > \frac{\ell_p+n-1}{\lambda+2(n-1)}$, then

$$\frac{\ell_p}{\alpha} < \ell_f < \frac{\alpha\lambda\ell_p - \ell_p^2 - (1-2\alpha)(n-1)\ell_p}{\alpha\lambda - \ell_p + \alpha^2(n-1)}. \tag{42}$$

However, it can be verified that the above range for ℓ_f is empty given $\alpha > \frac{\ell_p}{\lambda+n-1}$. This discussion implies that equilibrium $\mathbb{FPN}$ does not exist if $\alpha > \frac{\ell_p}{\lambda+n-1} \geq \bar{\alpha}$ or equivalently if $\overline{L}_p \geq 1 > L_p$.

Case 2. $\overline{L}_p < 1$, or equivalently $\frac{\ell_p}{\lambda+n-1} < \bar{\alpha}$. We either have $L_p \in (\overline{L}_p, 1)$, i.e., $\alpha \in \left(\frac{\ell_p}{\lambda+n-1}, \bar{\alpha}\right)$, or $L_p \in \left(\frac{\ell_p}{\lambda+n-1}, \overline{L}_p\right]$, i.e., $\alpha \in [\bar{\alpha}, 1)$. For $\alpha \in \left(\frac{\ell_p}{\lambda+n-1}, \bar{\alpha}\right)$, $R(\alpha) > 0$, similar to our discussion for equilibrium $\mathbb{FP}$. Hence, condition (40) is equivalent to[39]

$$\ell_f > \frac{\ell_p^2 - \alpha\lambda\ell_p + (1-2\alpha)(n-1)\ell_p}{\ell_p - \alpha\lambda - \alpha^2(n-1)} \Longleftrightarrow L_f > G,$$

the opposite of that for equilibrium $\mathbb{FP}$. Next, for $\alpha \in [\bar{\alpha}, 1)$, an argument similar to the above (41) and (42) implies that equilibrium $\mathbb{FPN}$ cannot exist in this case.[40]

Summarizing, if $L_p < 1$, equilibrium $\mathbb{FPN}$ exists whenever we have $L_p \in (\overline{L}_p, 1)$ and

$$\ell_f > \frac{\ell_p}{\alpha} \text{ and } \ell_f > \frac{\ell_p^2 - \alpha\lambda\ell_p + (1-2\alpha)(n-1)\ell_p}{\ell_p - \alpha\lambda - \alpha^2(n-1)}.$$

Since

$$L_p < 1 \Longrightarrow \frac{\ell_p^2 - \alpha\lambda\ell_p + (1-2\alpha)(n-1)\ell_p}{\ell_p - \alpha\lambda - \alpha^2(n-1)} > \frac{\ell_p}{\alpha},$$

we conclude that equilibrium $\mathbb{FPN}$ arises when $L_p \in (\overline{L}_p, 1)$ and $L_f > G$.

Finally, uniqueness of equilibrium is immediate from the observation that the parameter constellations for the four equilibria for each case of $L_p \geq 1$ and $L_p < 1$ form a partition (exhaustive and mutually exclusive) of the entire parameter space. ■

[39] Notice that the numerator of G is positive, i.e., $\ell_p^2 - \alpha\lambda\ell_p + (1-2\alpha)(n-1)\ell_p > 0$, equivalently $\alpha < \frac{\ell_p+n-1}{\lambda+2(n-1)}$, given that $\alpha < \frac{\ell_p}{\lambda+n-1}$ $(L_p < 1)$.

[40] In our proof, we have not discussed the possibility where $\bar{\alpha} > 1$. This case is irrelevant and our arguments associated with such $\bar{\alpha}$ are then vacuously true.

Algebraic Representation for Figure 3(a)

Figure 3(a) is constructed with parameters: $\lambda = 2, n = 2, \ell_p = 1$. According to Proposition 3, the parameter constellations for the four equilibrium formats are algebraically as follows:

$$
\begin{aligned}
\text{Equilibrium } \mathbb{F} &: \quad \ell_f \le \min\{3, (4-3\alpha)\} \\
\text{Equilibrium } \mathbb{FP} &: \quad \left\{ \begin{array}{l} \text{If } \alpha > \sqrt{2}-1 > \frac{1}{3}, \text{ then } \ell_f > 4-3\alpha; \\ \text{If } \frac{1}{3} < \alpha \le \sqrt{2}-1, \text{ then } 4-3\alpha < \ell_f < \frac{2(1-2\alpha)}{1-2\alpha-\alpha^2}. \end{array} \right\} \\
\text{Equilibrium } \mathbb{FN} &: \quad \alpha \le \frac{1}{3}, \, 3 < \ell_f \le \frac{1}{\alpha}. \\
\text{Equilibrium } \mathbb{FPN} &: \quad \ell_f > \frac{2}{1+\alpha}, \ell_f > \frac{1}{\alpha}, \text{ and } \left\{ \begin{array}{l} \text{If } 0 < \alpha < \sqrt{2}-1, \text{ then } \ell_f > \frac{2(1-2\alpha)}{1-2\alpha-\alpha^2} \\ \text{If } \alpha \in \left[\sqrt{2}-1, \frac{1}{2}\right], \text{ then no solution} \\ \text{If } \alpha > \frac{1}{2}, \text{ then no solution} \end{array} \right\}
\end{aligned}
$$

Proof of Proposition 6. We establish the result by directly comparing the majority agents' total welfare before and after banning partial learning in the language economy.

First, observe that if currently the language equilibrium is either an $\mathbb{FP}$ or $\mathbb{FPN}$ equilibrium, banning partial learning will lead to an $\mathbb{FN}$ equilibrium with an *interior* cutoff

$$\theta_f^{FN} = \frac{\lambda}{\ell_f - (n-1)} \in (0,1).$$

And majority agents' total welfare after banning partial learning is:

$$W_M^B(\alpha) = \lambda n \theta_f^{FN} = \frac{n\lambda^2}{\ell_f - (n-1)}.$$

Now suppose the equilibrium before the ban is either an $\mathbb{FPN}$ equilibrium, with cutoffs θ_f^{FPN} and θ_p^{FPN} in (15) and (16), or an $\mathbb{FP}$ equilibrium, with cutoff $\theta_f^{\mathbb{FP}}$ in (9) or (37). The majority agents' welfare without the ban can be written as, respectively:

$$
\begin{aligned}
W_M^{\mathbb{FPN}}(\alpha) &= \lambda n[\theta_f^{FPN} + \alpha(\theta_p^{FPN} - \theta_f^{FPN})] = \frac{\lambda^2 n[\ell_p(1-\alpha)^2 + \alpha^2(\ell_f - \ell_p)]}{\ell_p(\ell_f - \ell_p) + (n-1)(2\alpha\ell_p - \alpha^2\ell_f - \ell_p)}, \\
W_M^{\mathbb{FP}}(\alpha) &= \lambda n[\theta_f^{FP} + \alpha(1-\theta_f^{FP})] = \lambda n \left\{ \frac{(1-\alpha)^2[\lambda + \alpha(n-1)]}{(\ell_f - \ell_p) - (1-\alpha)^2(n-1)} + \alpha \right\}.
\end{aligned}
$$

We first compare $W_M^B(\alpha)$ with $W_M^{\mathbb{FPN}}(\alpha)$. Note that $\theta_f^{FN} = \theta_p^{FPN} = \theta_f^{FPN}$ if $\alpha = \ell_p/\ell_f$, which implies that $W_M^{\mathbb{FPN}}(\alpha) = W_M^B(\alpha)$ if $\alpha = \ell_p/\ell_f$.[41] We can further calculate that

$$\frac{dW_M^{\mathbb{FPN}}(\alpha)}{d\alpha} = 0 \text{ if } \alpha = \ell_p/\ell_f \text{ and } \frac{d^2W_M^{\mathbb{FPN}}(\alpha)}{d\alpha^2} > 0.$$

Hence the welfare function $W_M^{\mathbb{FPN}}(\alpha)$ achieves its global minimum at $\alpha = \ell_p/\ell_f$, which implies that $W_M^{\mathbb{FPN}}(\alpha) > W_M^B(\alpha)$ for all $\alpha > \ell_p/\ell_f$.

Now we compare $W_M^B(\alpha)$ and $W_M^{\mathbb{FP}}(\alpha)$. First, given the $\mathbb{FP}$ equilibrium, it is sufficient to consider $\alpha > \ell_p/\ell_f$. Since $W_M^{\mathbb{FP}}(\alpha)$ is strictly increasing in ℓ_p and $\ell_p < \alpha\ell_f$, we have

$$W_M^{\mathbb{FP}}(\alpha) > W_M^{\mathbb{FP}}(\alpha)\big|_{\ell_p=\alpha\ell_f} = \hat{W}_M^{\mathbb{FP}}(\alpha) = \lambda n\left[\frac{(1-\alpha)(\lambda+\alpha(n-1))}{\ell_f-(1-\alpha)(n-1)}+\alpha\right].$$

We can also verify that[42]

$$\frac{d\hat{W}_M^{\mathbb{FP}}(\alpha)}{d\alpha} = \frac{\lambda n\ell_f(\ell_f-\lambda-(n-1))}{(\ell_f-\alpha-n+n\alpha+1)^2}+\lambda n > 0 \text{ and } \hat{W}_M^{\mathbb{FP}}(0) = W_M^B(\alpha).$$

Hence, we have $W_M^{\mathbb{FP}}(\alpha) > W_M^B(\alpha)$ for all $\alpha > \ell_p/\ell_f$ as well. ∎

Proof of Proposition 8. First consider the effect of changing λ on θ^*. Apply the Implicit Function Theorem to equation (20) to obtain:

$$\frac{\partial\theta^*}{\partial\lambda} = \frac{\partial r(\theta^*)/\partial\lambda}{1-\partial r(\theta^*)/\partial\theta} = \frac{1/\left(\ell_f e^{-\theta\phi}-n+1\right)}{1-\partial r(\theta^*)/\partial\theta}.$$

Since the steady state θ^* is stable, or $\frac{\partial r(\theta^*)}{\partial\theta} < 1$, the sign of $\frac{\partial\theta^*}{\partial\lambda}$ thus only depends on the derivative of $\frac{\partial r(\theta^*)}{\partial\lambda}$, which is strictly positive. We hence have $\frac{\partial\theta^*}{\partial\lambda} > 0$. We can similarly use the implicit function theorem to find that:

$$\begin{aligned}
\frac{\partial\theta^*}{\partial n} &= \frac{\partial r(\theta^*)/\partial n}{1-\partial r(\theta^*)/\partial\theta} > 0,\\
\frac{\partial\theta^*}{\partial\phi} &= \frac{\partial r(\theta^*)/\partial\phi}{1-\partial r(\theta^*)/\partial\theta} > 0,\\
\frac{\partial\theta^*}{\partial\ell_f} &= \frac{\partial r(\theta^*)/\partial\ell_f}{1-\partial r(\theta^*)/\partial\theta} < 0,
\end{aligned}$$

hence establishing the comparative statics result. ∎

[41] Recall that $\alpha = \ell_p/\ell_f$ marks the cutoff between $\mathbb{FPN}$ equilibrium and $\mathbb{FN}$ equilibrium in the parameter space (see Figure 3). And everything else fixed, as α increases from ℓ_p/ℓ_f to 1, we first enter the $\mathbb{FPN}$ equilibrium zone and then the $\mathbb{FP}$ equilibrium zone.

[42] Recall that since $0 < \theta_f^{FN} < 1$, we have $\ell_f > \lambda + n - 1$.