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ON THE EXISTENCE OF MONOTONE PURE-STRATEGY EQUILIBRIA IN BAYESIAN GAMES

BY PHILIP J. RENY¹

We generalize Athey's (2001) and McAdams' (2003) results on the existence of monotone pure-strategy equilibria in Bayesian games. We allow action spaces to be compact locally complete metric semilattices and type spaces to be partially ordered probability spaces. Our proof is based on contractibility rather than convexity of best-reply sets. Several examples illustrate the scope of the result, including new applications to multi-unit auctions with risk-averse bidders.

KEYWORDS: Bayesian games, monotone pure strategies, equilibrium existence, multi-unit auctions, risk aversion.

1. INTRODUCTION

ATHEY (2001) ESTABLISHES the important result that a monotone pure-strategy equilibrium exists whenever a Bayesian game satisfies a Spence–Mirlees single crossing property. Athey's result is now a central tool for establishing the existence of monotone pure-strategy equilibria in auction theory (see, e.g., Athey (2001), Reny and Zamir (2004)). Recently, McAdams (2003) shows that Athey's results, which exploit the assumed total ordering of the players' one-dimensional type and action spaces, can be extended to settings in which type and action spaces are multidimensional and only partially ordered. This permits new existence results in auctions with multidimensional types and multi-unit demands (see McAdams (2003, 2006)). The techniques employed by Athey and McAdams, while ingenious, have their limitations and do not appear to easily extend beyond the environments they consider. We therefore introduce a new approach.

The approach taken here exploits an important unrecognized property of a large class of Bayesian games. In these games, the players' pure-strategy best-reply sets, while possibly nonconvex, are always contractible.² This observation permits us to generalize the results of Athey and McAdams in several directions. First, we permit infinite-dimensional type spaces and infinite-dimensional action spaces. Both can occur, for example, in share auctions, where a bidder's type is a function that expresses his marginal valuation at any quantity of the good and where a bidder's action is a downward-sloping

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²A set is contractible if it can be continuously deformed, within itself, to a single point. Convex sets are contractible, but contractible sets need not be convex (e.g., the symbol “+” viewed as a subset of \mathbb{R}^2).

demand schedule. Second, even when type and action spaces are subsets of Euclidean space, we permit more general joint distributions over types, allowing one player to have private information about the support of another's private information, as well as permitting positive probability on lower dimensional subsets, which can be useful when modeling random demand in auctions. Third, our approach allows general partial orders on both type spaces and action spaces. This can be especially helpful because, while single crossing may fail for one partial order, it might nonetheless hold for another, in which case our existence result can still be applied (see Section 5 for several such applications). Finally, while single crossing is helpful in establishing the hypotheses of our main theorem, it is not necessary; our hypotheses are satisfied even in instances where single crossing fails (e.g., as in Reny and Zamir (2004)).

The key to our approach is to employ a more powerful fixed-point theorem than those employed in Athey (2001) and McAdams (2003). Both papers consider the game's best reply correspondence: Kakutani's theorem is used in Athey (2001); Glicksberg's theorem is used in McAdams (2003). In both cases, essentially all of the effort is geared toward proving that sets of monotone pure-strategy best replies are convex. Our central observation is that this impressive effort is unnecessary and, more importantly, that the additional structure imposed to achieve the desired convexity (i.e., *Euclidean* type spaces with the *coordinatewise* partial order, *Euclidean sublattice* action spaces, *absolutely continuous* type distributions) is unnecessary as well.

The fixed-point theorem on which our approach is based is due to Eilenberg and Montgomery (1946) and does not require the correspondence in question to be convex-valued. Rather, the correspondence need only be contractible-valued. Consequently, we need only demonstrate that monotone pure-strategy best-reply sets are contractible. While this task need not be straightforward in general, it turns out to be essentially trivial in the class of Bayesian games of interest here. To gain a sense of this, note first that a pure strategy—a function from types to actions—is a best reply for a player if and only if it is a pointwise interim best reply for almost every type of that player. Consequently, any piecewise combination of two best replies—i.e., a strategy equal to one of the best replies on some subset of types and equal to the other best reply on the remainder of types—is also a best reply. Thus, by reducing the set of types on which the first best reply is employed and increasing the set of types on which the second is employed, it is possible to move from the first best reply to the second, all the while remaining within the set of best replies. With this simple observation, the set of best replies can be shown to be contractible.³

Because contractibility of best-reply sets follows almost immediately from the pointwise almost everywhere optimality of interim best replies, we are able

³Because we are concerned with *monotone* pure-strategy best replies, some care must be taken to ensure that one maintains monotonicity throughout the contraction. Further, continuity of the contraction requires appropriate assumptions on the distribution over players' types. In particular, there can be no atoms.

to expand the domain of analysis well beyond Euclidean type and action spaces, and most of our additional effort is directed here. In particular, we require and prove two new results about the space of monotone functions from partially ordered probability spaces into compact metric semilattices. The first of these results (Lemma A.10) is a generalization of Helly's selection theorem, stating that, under suitable conditions, any sequence of monotone functions possesses a pointwise almost everywhere convergent subsequence. The second result (Lemma A.16) provides conditions under which the space of monotone functions is an absolute retract, a property that, like convexity, renders a space amenable to fixed-point analysis.

Our main result, Theorem 4.1, is as follows. Suppose that action spaces are compact convex locally convex semilattices or compact locally complete metric semilattices, that type spaces are partially ordered probability spaces, that payoffs are continuous in actions for each type vector, and that the joint distribution over types induces atomless marginals for each player assigning positive probability only to sets that can be order-separated by a fixed countable set of his types.⁴ If, whenever the others employ monotone pure strategies, each player's set of monotone pure-strategy best replies is nonempty and join-closed,⁵ then a monotone pure-strategy equilibrium exists.

We provide several applications that yield new existence results. First, we consider both uniform-price and discriminatory multi-unit auctions with independent private information. We depart from standard assumptions by permitting bidders to be risk averse. Under risk aversion, McAdams (2007) contains a uniform-price auction example having no monotone pure-strategy equilibrium, suggesting that a general existence result is simply unavailable. However, we show that this negative result stems from the use of the coordinate-wise partial order over types. By employing a distinct (and more economically relevant) partial order over types—a technique novel to our methods—we are able to demonstrate the existence of a monotone pure-strategy equilibrium with respect to this alternative partial order in both uniform-price and discriminatory auctions. Another application considers a price-competition game between firms selling differentiated products. Firms have private information about their constant marginal cost as well as private information about market demand. While it is natural to assume that costs may be affiliated, in the context we consider, it is less natural to assume that information about market demand is affiliated because information that improves demand for some firms may worsen it for others. Nonetheless, and again through a judicious choice of a partial order over types, we are able to establish the existence of a pure-strategy equilibrium that is monotone in players' costs, but not necessarily

⁴One set is order-separated by another if the one set contains two points between which lies a point in the other.

⁵A subset of strategies is join-closed if the pointwise supremum of any pair of strategies in the set is also in the set.

monotone in their private information about demand. Our final application establishes the existence of monotone *mixed* strategy equilibria when type spaces have atoms.⁶

If the actions of distinct players are strategic complements—an assumption we do not impose—even stronger results can be obtained. Indeed, in Van Zandt and Vives (2007), it is shown that monotone pure-strategy equilibria exist under somewhat more general distributional and action-space assumptions than we employ here, and that such an equilibrium can be obtained through iterative application of the best reply-map.⁷ The existence result in Van Zandt and Vives (2007) is perhaps the strongest possible for Bayesian games with strategic complementarities. Of course, while many interesting economic games exhibit strategic complements, many do not. Indeed, many auction games satisfy the hypotheses required to apply our result here, but fail to satisfy the strategic complements condition.⁸ The two approaches are therefore complementary.

The remainder of the paper is organized as follows. Section 2 presents the essential ideas as well as the corollary of Eilenberg and Montgomery's (1946) fixed-point theorem that is central to our approach. Section 3 describes the formal environment, including semilattices and related issues. Section 4 contains our main result, Section 6 contains its proof, and Section 5 provides several applications. Some readers interested in specific applications may find it sufficient to skip ahead to Corollary 4.2—a special case of our main result—which requires little in the way of preparation.

2. THE MAIN IDEA

As already mentioned, the proof of our main result is based on a fixed-point theorem that permits the correspondence for which a fixed point is sought—here, the product of the players' monotone pure best-reply correspondences—to have contractible rather than convex values.

In this section, we introduce this fixed-point theorem and illustrate the ease with which contractibility can be established, focusing on the most basic case in which type spaces are $[0, 1]$, action spaces are subsets of $[0, 1]$, and the marginal distribution over each player's type space is atomless.

⁶A player's mixed strategy is monotone if every action in the totally ordered support of one of his types is greater than or equal to every action in the totally ordered support of any lower type.

⁷Related results can be found in Milgrom and Roberts (1990) and Vives (1990).

⁸In a first-price independent private-value auction, for example, a bidder might increase his bid if his opponent increases her bid slightly when her private value is high. However, for sufficiently high increases in her bid at high private values, the bidder might be better off reducing his bid (and chance of winning) to obtain a higher surplus when he does win. Such strictly optimal nonmonotonic responses to increases in the opponent's strategy are not possible under strategic complements.

A subset X of a metric space is *contractible* if for some $x_0 \in X$ there is a continuous function $h: [0, 1] \times X \rightarrow X$ such that for all $x \in X$, $h(0, x) = x$ and $h(1, x) = x_0$. We then say that h is a *contraction* for X .

Note that every convex set is contractible since, choosing any point x_0 in the set, the function $h(\tau, x) = (1 - \tau)x + \tau x_0$ is a contraction. On the other hand, there are contractible sets that are not convex (e.g., the symbol “+”). Hence, contractibility is a strictly more permissive condition than convexity.

A subset X of a metric space Y is said to be a *retract* of Y if there is a continuous function mapping Y onto X leaving every point of X fixed. A metric space (X, d) is an *absolute retract* if for every metric space (Y, δ) containing X as a closed subset and preserving its topology, X is a retract of Y .⁹ Examples of absolute retracts include closed convex subsets of Euclidean space or of any metric space, and many nonconvex sets as well (e.g., any contractible polyhedron).¹⁰ The fixed-point theorem we make use of is the following corollary of an even more general result due to Eilenberg and Montgomery (1946).¹¹

THEOREM 2.1: *Suppose that a compact metric space (X, d) is an absolute retract and that $F: X \rightarrow X$ is an upper-hemicontinuous, nonempty-valued, contractible-valued correspondence.¹² Then F has a fixed point.*

For our purposes, the correspondence F is the product of the players' monotone pure-strategy best-reply correspondences and X is the product of their sets of monotone pure strategies. While we must eventually establish all of the properties necessary to apply Theorem 2.1, our modest objective for the remainder of this section is to show, with remarkably little effort, that in the simple environment considered here, F is contractible-valued, i.e., that monotone pure best-reply sets are contractible.

Suppose that player 1's type is drawn uniformly from the unit interval $[0, 1]$ and that $A \subseteq [0, 1]$ is player 1's compact action set. Fix monotone pure strategies for the other players and suppose that $\bar{s}: [0, 1] \rightarrow A$ is the largest monotone best reply for player 1 in the sense that if s is any other monotone

⁹It is not necessary to understand the concept of an absolute retract to apply any of our results: none of our hypotheses requires checking that a space is an absolute retract. However, to prove our main result using Theorem 2.1, we must (and do) demonstrate that under our hypotheses, each player's space of monotone pure strategies is an absolute retract (see Lemma A.16).

¹⁰Indeed, a compact subset, X , of Euclidean space is an absolute retract if and only if it is contractible and locally contractible. The latter means that for every $x_0 \in X$ and every neighborhood U of x_0 , there is a neighborhood V of x_0 and a continuous $h: [0, 1] \times V \rightarrow U$ such that $h(0, x) = x$ and $h(1, x) = x_0$ for all $x \in V$.

¹¹Theorem 2.1 follows directly from Eilenberg and Montgomery (1946, Theorem 1), because every absolute retract is a contractible absolute neighborhood retract (Borsuk (1966, V, (2.3))) and every nonempty contractible set is acyclic (Borsuk (1966, II, (4.11))).

¹²By upper hemicontinuous, we always mean that the correspondence in question has a closed graph.

best reply, then $\bar{s}(t) \geq s(t)$ for every type t of player 1.¹³ We now provide a contraction that continuously shrinks player 1's entire set of monotone best replies, within itself, to the largest monotone best reply \bar{s} . The simple, but key, observation is that a pure strategy is a best reply for player 1 if and only if it is a pointwise best reply for almost every type $t \in [0, 1]$ of player 1.

Consider the following candidate contraction. For $\tau \in [0, 1]$ and any monotone best reply, s , for player 1, define $h(\tau, s) : [0, 1] \rightarrow \mathcal{A}$ as

$$h(\tau, s)(t) = \begin{cases} s(t), & \text{if } t \leq 1 - \tau \text{ and } \tau < 1, \\ \bar{s}(t), & \text{otherwise.} \end{cases}$$

Note that $h(0, s) = s$, $h(1, s) = \bar{s}$, and $h(\tau, s)(t)$ is always either $\bar{s}(t)$ or $s(t)$ and so is a best reply for almost every t . Hence, by the key observation in the previous paragraph, $h(\tau, s)(\cdot)$ is a best reply. The pure strategy $h(\tau, s)(\cdot)$ is monotone because it is the smaller of two monotone functions for low values of t and the larger of them for high values of t . Moreover, because the marginal distribution over player 1's type is atomless, the monotone pure strategy $h(\tau, s)(\cdot)$ varies continuously in the arguments τ and s when the distance between two strategies of player 1 is defined to be the integral with respect to his type distribution of their absolute pointwise difference (see Section 6).¹⁴ Consequently, h is a contraction under this metric, and so player 1's set of monotone best replies is contractible. It is that simple.

Figure 2.1 shows how the contraction works when player 1's set of actions \mathcal{A} happens to be finite, so that his set of monotone best replies cannot be convex in the usual sense unless it is a singleton. Three monotone functions are shown in each panel, where 1's actions are on the vertical axis and 1's types are on the horizontal axis. The dotted line step function is s , the solid line step function is \bar{s} , and the thick solid line step function is the step function determined by the contraction h .

In panel (a), $\tau = 0$ and h coincides with s . The position of the vertical line appearing in each panel represents the value of τ . The vertical line in each panel intersects the horizontal axis at the point $1 - \tau$. When $\tau = 0$, the vertical line is at the far right-hand side, as shown in panel (a). As indicated by the arrow, the vertical line moves continuously toward the origin as τ moves from 0 to 1. The thick step function determined by the contraction h is $s(t)$ for values of t to the left of the vertical line and is $\bar{s}(t)$ for values of t to the right; see panels (b) and (c). The step function h therefore changes continuously with τ because the areas between strategies change continuously. In panel (d),

¹³Such a largest monotone best reply exists under the hypotheses of our main result.

¹⁴This particular metric is important because it renders a player's payoff continuous in his strategy choice.

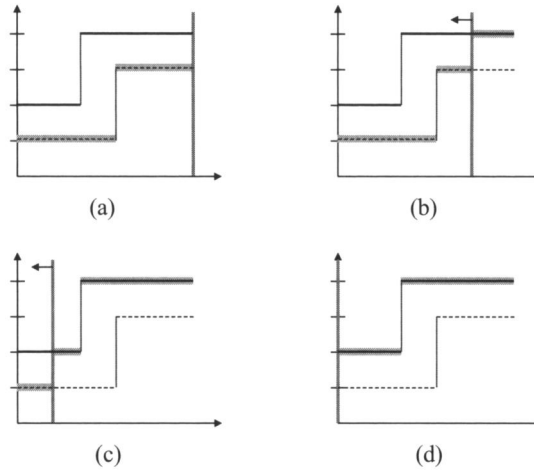


FIGURE 2.1.—The contraction.

$\tau = 1$ and h coincides with \bar{s} . So altogether, as τ moves continuously from 0 to 1, the image of the contraction moves continuously from s to \bar{s} .

Two points are worth mentioning before moving on. First, single crossing plays no role in establishing the contractibility of sets of monotone best replies. As we shall see, ensuring the existence of monotone pure-strategy best replies is where single crossing can be helpful. Thus, the present approach clarifies the role of single crossing insofar as the existence of monotone pure-strategy equilibrium is concerned.¹⁵ Second, the action spaces employed in the above example are totally ordered, as in Athey (2001). Consequently, if two actions are optimal for some type of player 1, then the maximum of the two actions, being one or the other of them, is also optimal. The optimality of the maximum of two optimal actions is important for ensuring that a largest monotone best reply exists.

When action spaces are only partially ordered (e.g., when actions are multi-dimensional with, say, the coordinatewise partial order), the maximum of two optimal actions need not even be well defined, let alone optimal. Therefore, to also cover partially ordered action spaces, we assume in the sequel (see Section 3.2) that action spaces are semilattices—i.e., that for every pair of actions there is a least upper bound—and that the least upper bound of two optimal actions is optimal. Stronger versions of both assumptions are employed in McAdams (2003).

¹⁵In both Athey (2001) and McAdams (2003) single crossing is employed to help establish the existence of monotone best replies *and* to establish the convexity of the set of monotone best replies. The single crossing conditions in Athey (2001) and McAdams (2003) are therefore more restrictive than necessary. See Section 4.1.

3. THE ENVIRONMENT

In order as to speak about monotone pure strategies, the players' type and action spaces must come equipped with partial orders. Moreover, as mentioned just above, action spaces require the additional structure of a semilattice. The following section provides the order-related concepts we need for both type spaces and action spaces.

3.1. *Partial Orders, Lattices, and Semilattices*

Let X be a nonempty set partially ordered by \geq .¹⁶ If x, y , and z are members of X , we say that y lies between x and z if $x \geq y \geq z$. If X is endowed with a sigma algebra of subsets \mathcal{A} , then the partial order \geq on X is called *measurable* if $\{(x, y) \in X \times X : x \geq y\}$ is a member of $\mathcal{A} \times \mathcal{A}$.¹⁷ If X is endowed with a topology, then the partial order \geq on X is called *closed* if $\{(x, y) \in X \times X : x \geq y\}$ is closed in the product topology. The partial order \geq on X is called *convex* if X is a subset of a real vector space and $\{(x, y) \in X \times X : x \geq y\}$ is convex. Note that if the partial order on X is convex, then X is convex because $x \geq x$ for every $x \in X$. Say that X is *upper-bound-convex* if it contains the convex combination of any two members whenever one of them, \bar{x} say, is an upper bound for X , i.e., $\bar{x} \geq x$ for every $x \in X$.¹⁸ Every convex set is upper-bound-convex.

For $x, y \in X$, if the set $\{x, y\}$ has a least upper bound in X , then it is unique and will be denoted by $x \vee y$, the *join* of x and y . In general, such a bound need not exist. However, if every pair of points in X has a least upper bound in X , then we shall say that X is a *semilattice*. It is straightforward to show that, in a semilattice, every finite set, $\{x, y, \dots, z\}$, has a least upper bound, which we denote by $\vee\{x, y, \dots, z\}$ or $x \vee y \vee \dots \vee z$.

If the set $\{x, y\}$ has a greatest lower bound in X , then it too is unique and it will be denoted by $x \wedge y$, the *meet* of x and y . Once again, in general, such a bound need not exist. If every pair of points in X has both a least upper bound and a greatest lower bound in X , then we say that X is a *lattice*.¹⁹ A semilattice (lattice) in \mathbb{R}^m endowed with the coordinatewise partial order will be called a *Euclidean* semilattice (lattice).

Clearly, every lattice is a semilattice. However, the converse is not true. For example, under the coordinatewise partial order, the set of vectors in \mathbb{R}^2 whose sum is at least 1 is a semilattice, but not a lattice.

¹⁶Hence, \geq is transitive ($x \geq y$ and $y \geq z$ imply $x \geq z$), reflexive ($x \geq x$), and antisymmetric ($x \geq y$ and $y \geq x$ imply $x = y$).

¹⁷Recall that $\mathcal{A} \times \mathcal{A}$ is the smallest sigma algebra containing all sets of the form $A \times B$ with A, B in \mathcal{A} .

¹⁸Sets without upper bounds are trivially upper-bound-convex.

¹⁹Defining a semilattice in terms of the join operator, \vee , rather than the meet operator, \wedge , is entirely a matter of convention.

A *metric semilattice* is a semilattice, X , endowed with a metric under which the join operator, \vee , is continuous as a function from $X \times X$ into X . A metric semilattice in \mathbb{R}^m endowed with the coordinatewise partial order and the Euclidean metric will be called a *Euclidean metric semilattice*. Because in a semilattice $x \geq y$ if and only if $x \vee y = x$, a partial order in a metric semilattice is necessarily closed.²⁰

A semilattice X is *complete* if every nonempty subset S of X has a least upper bound, $\vee S$, in X . A metric semilattice X is *locally complete* if for every $x \in X$ and every neighborhood U of x , there is a neighborhood W of x contained in U such that every nonempty subset S of W has a least upper bound, $\vee S$, contained in U . Lemma A.18 establishes that a compact metric semilattice X is locally complete if and only if for every $x \in X$ and every sequence $x_n \rightarrow x$, $\lim_m(\bigvee_{n \geq m} x_n) = x$.²¹ A distinct sufficient condition for local completeness is given in Lemma A.20.

Some examples of compact locally complete metric semilattices are as follows.

- Finite semilattices.
- Compact sublattices of the Euclidean lattice \mathbb{R}^m —because in a Euclidean sublattice, the join of any two points is their coordinatewise maximum.
- Compact Euclidean metric semilattices (Lemma A.19).
- Compact upper-bound-convex Euclidean semilattices (Lemmas A.17 and A.19).
- The space of continuous functions $f: [0, 1] \rightarrow [0, 1]$ satisfying for some $\lambda > 0$ the Lipschitz condition $|f(x) - f(y)| \leq \lambda|x - y|$, endowed with the maximum norm $\|f\| = \max_x |f(x)|$ and partially ordered by $f \geq g$ if $f(x) \geq g(x)$ for all $x \in [0, 1]$.

The last example in the above list is an infinite-dimensional, compact, locally complete metric semilattice. In general, and unlike compact Euclidean metric semilattices, infinite-dimensional metric semilattices need not be locally complete even if they are compact and convex.²²

3.2. A Class of Bayesian Games

There are N players, $i = 1, 2, \dots, N$. Player i 's type space is T_i and his action space is A_i , and both are nonempty and partially ordered. In addition, A_i is

²⁰The converse does not hold. For example, the set $X = \{(x_1, x_2) \in \mathbb{R}_+^2 : x_1 + x_2 = 1\} \cup \{(1, 1)\}$ is a semilattice with the coordinatewise partial order, and this order is closed under the Euclidean metric. But X is not a metric semilattice because whenever $x_n \neq y_n$ and $x_n, y_n \rightarrow x$, we have $(1, 1) = \lim(x_n \vee y_n) \neq (\lim x_n) \vee (\lim y_n) = x$.

²¹Hence, compactness and metrizability of a lattice under the order topology (see Birkhoff (1967, p. 244)) are sufficient, but not necessary, for local completeness of the corresponding semilattice.

²²No \mathcal{L}_p space is locally complete when $p < +\infty$ and endowed with the pointwise partial order. See Hart and Weiss (2005) for a compact metric semilattice that is not locally complete. Their example can be modified so that the space is, in addition, convex and locally convex.

endowed with a metric. Unless a notational distinction is helpful, all partial orders, although possibly distinct, will be denoted by \geq . Player i 's payoff function is $u_i: A \times T \rightarrow \mathbb{R}$, where $A = \times_{i=1}^N A_i$ and $T = \times_{i=1}^N T_i$. For each player i , \mathcal{T}_i is a sigma algebra of subsets of T_i , and members of \mathcal{T}_i will often be referred to simply as *measurable sets*. The common prior over the players' types is a countably additive probability measure μ defined on $T_1 \times \dots \times T_N$. Let G denote this Bayesian game.

For each player i , we let μ_i denote the marginal of μ on T_i , and hence the domain of μ_i is T_i .

As functions from types into actions, best replies for any player i are determined only up to μ_i measure zero sets. This leads us to the following definitions. A *pure strategy* for player i is a function, $s_i: T_i \rightarrow A_i$, that is μ_i -a.e. (almost everywhere) equal to a measurable function and is *monotone* if $t'_i \geq t_i$ implies $s_i(t'_i) \geq s_i(t_i)$ for all $t_i, t'_i \in T_i$.^{23,24} Let S_i denote player i 's set of pure strategies and let $S = \times_{i=1}^N S_i$.

A vector of pure strategies, $(\hat{s}_1, \dots, \hat{s}_N) \in S$ is a *Bayesian–Nash equilibrium* or simply an *equilibrium* if for every player i and every pure strategy s'_i for player i ,

$$\int_T u_i(\hat{s}(t), t) d\mu(t) \geq \int_T u_i(s'_i(t_i), \hat{s}_{-i}(t_{-i}), t) d\mu(t),$$

where the left-hand side, henceforth denoted by $U_i(\hat{s})$, is player i 's payoff given the joint strategy \hat{s} , and the right-hand side is his payoff when he employs s'_i and the others employ \hat{s}_{-i} .²⁵

It will sometimes be helpful to speak of the payoff to player i 's type t_i from the action a_i given the strategies of the others, s_{-i} . This payoff, which we refer to as i 's *interim* payoff, is

$$V_i(a_i, t_i, s_{-i}) \equiv \int_{T_{-i}} u_i(a_i, s_{-i}(t_{-i}), t) d\mu_i(t_{-i}|t_i),$$

where $\mu_i(\cdot|t_i)$ is a version of the conditional probability on T_{-i} given t_i .²⁶ A single such version is fixed for each player i once and for all. Consequently, $(\hat{s}_1, \dots, \hat{s}_N) \in S$ is an equilibrium according to our definition above if and only if for each player i and μ_i -a.e. $t_i \in T_i$,

$$V_i(\hat{s}_i(t_i), t_i, \hat{s}_{-i}) \geq V_i(a_i, t_i, \hat{s}_{-i}) \quad \text{for every } a_i \in A_i,$$

²³Recall that a property $P(t_i)$ holds μ_i -a.e. if the set of t_i for which $P(t_i)$ holds contains a measurable subset having μ_i measure 1.

²⁴The measurable subsets of the metric space A_i are the Borel sets.

²⁵This definition of pure-strategy Bayesian–Nash equilibrium coincides, for example, with that implicit in Milgrom and Weber (1985).

²⁶The conditional, $\mu_i(\cdot|t_i)$, will not otherwise appear in the sequel and should not be confused with the marginal, μ_i , which will appear throughout.

that is, if and only if for each player i , $\hat{s}_i(t_i)$ is an interim best reply against \hat{s}_{-i} for μ_i -a.e. $t_i \in T_i$.

We make use of the following additional assumptions on the Bayesian game G . For every player i :

G.1. The partial order on T_i is measurable.

G.2. The probability measure μ_i on T_i is atomless.²⁷

G.3. There is a countable subset T_i^0 of T_i such that every set in \mathcal{T}_i assigned positive probability by μ_i contains two points between which lies a point in T_i^0 .

G.4. A_i is a compact metric space and a semilattice with a closed partial order.²⁸

G.5. Either (i) A_i is a convex subset of a locally convex topological vector space and the partial order on A_i is convex or (ii) A_i is a locally complete metric semilattice.²⁹

G.6. $u_i(a, t)$ is bounded, jointly measurable, and continuous in $a \in A$ for every $t \in T$.

Conditions G.1–G.5 are very general, covering a wide variety of situations. To reassure more applied readers, we illustrate that G.1–G.5 hold in settings that are not uncommon. The proof of the following proposition can be found in Appendix A.6.

PROPOSITION 3.1:

(i) *Conditions G.1–G.3 are satisfied, in particular, when both of the following conditions (a) and (b) hold: (a) each player i 's type space, $T_i = [\underline{\tau}_i, \bar{\tau}_i]^{n_i}$, is the union $T_{i1} \cup T_{i2} \cup \dots$ of a finite or countably infinite number of nondegenerate Euclidean cubes, $T_{ik} = [\underline{\tau}_{ik}, \bar{\tau}_{ik}]^{n_{ik}}$, of possibly different dimensions and where the partial order on T_i is the coordinatewise partial order; and (b) according to player i 's marginal, μ_i , each one of the cubes T_{ik} is chosen with probability p_{ik} and then $t_i \in T_{ik}$ is chosen according to the probability density f_{ik} on T_{ik} , which need not be everywhere positive.*

(ii) *Conditions G.4 and G.5 are satisfied, in particular, when each player's set of actions is a compact subset of Euclidean space endowed with the coordinatewise partial order, and the coordinatewise maximum of any two actions is itself a feasible action.*

In Athey (2001) and McAdams (2003) it is assumed that each A_i is a compact sublattice of Euclidean space, that each T_i is a Euclidean cube $[\underline{\tau}_i, \bar{\tau}_i]^{n_i}$ endowed with the coordinatewise partial order, and that μ is absolutely con-

²⁷For every $t_i \in T_i$, the singleton set $\{t_i\}$ is in \mathcal{T}_i by G.1. See Appendix A.1.

²⁸Note that G.4 does not require A_i to be a metric semilattice—its join operator need not be continuous.

²⁹It is permissible for (i) to hold for some players and (ii) to hold for others. A topological space is locally convex if for every open set U , every point in U has a convex open neighborhood contained in U .

tinuous with respect to Lebesgue measure, a situation strictly covered by conditions (i) and (ii) of Proposition 3.1.³⁰ Hence, their hypotheses, which also include action continuity of utility functions, are strictly more restrictive than G.1–G.6. The additional structure they impose is, in fact, necessary for their Kakutani–Glicksberg-based approach.³¹

In addition to permitting infinite-dimensional type spaces, assumption G.1 permits the partial order on player i 's type space to be distinct from the usual coordinatewise partial order when T_i is Euclidean. As we shall see, this flexibility is very helpful in providing new equilibrium existence results for multi-unit auctions with risk-averse bidders.

Assumption G.2 is used to establish the contractibility of the players' sets of monotone best replies and, in particular, to construct an associated contraction that is continuous in a topology in which payoffs are continuous as well.

Assumption G.3 connects the partial order on a player's type space with his marginal distribution, and it implies, in particular, that no atomless subset of a player's type space having positive probability can be totally unordered. For example, if $T_i = [0, 1]^2$ is endowed with the Borel sigma algebra and the coordinatewise partial order, G.3 requires μ_i to assign probability 0 to any atomless negatively sloped line in T_i . In fact, whenever T_i happens to be a separable metric space and \mathcal{T}_i contains the open sets, G.3 holds if every atomless set having positive μ_i measure contains two "strictly ordered" points (Lemma A.21).³²

Together with G.1 and G.4, G.3 ensures the compactness of the players' sets of monotone pure strategies (Lemma A.10) in a topology in which payoffs are continuous.³³ Thus, although G.3 is logically unrelated to Milgrom and Weber's (1985) absolute-continuity assumption on the joint distribution over

³⁰In McAdams (2003) it is assumed further that the joint density over types is everywhere strictly positive, and in Athey (2001) it is assumed that each $n_i = 1$.

³¹Indeed, suppose a player's action set is the semilattice $\mathcal{A} = \{(1, 0), (1/2, 1/2), (0, 1), (1, 1)\}$ in \mathbb{R}^2 with the coordinatewise partial order and note that \mathcal{A} is not a sublattice of \mathbb{R}^2 . It is not difficult to see that this player's set of monotone pure strategies from $[0, 1]$ into \mathcal{A} , endowed with the metric $d(f, g) = \int_0^1 |f(x) - g(x)| dx$, is homeomorphic to three line segments joined at a common endpoint. Consequently, this strategy set is not homeomorphic to a convex set and so neither Kakutani's nor Glicksberg's theorems can be directly applied. On the other hand, this strategy set is an absolute retract (see Lemma A.16), which is sufficient for our approach.

³²Two points in a partially ordered metric space are strictly ordered if they are contained in disjoint open sets such that every point in one set is greater than or equal to every point in the other.

³³Indeed, without G.3, a player's type space could be the negative diagonal in $[0, 1]^2$ endowed with the coordinatewise partial order. But then every measurable function from types to actions would be monotone, because no two distinct types are ordered. Compactness in a useful topology is then effectively precluded.

types, it plays the same compactness role for monotone pure strategies as the Milgrom–Weber assumption plays for distributional strategies.^{34,35}

Assumption G.5 is used to ensure that the set of monotone pure strategies is an absolute retract and therefore amenable to fixed-point analysis.

Assumption G.6 is used to ensure that best replies are well defined and that best-reply correspondences are upper hemicontinuous. Assumption G.6 is trivially satisfied when action spaces are finite. Thus, for example, it is possible to consider auctions here by supposing that players' bid spaces are discrete. We do so in Section 5, where we also consider auctions with continuum bid spaces by considering limits of ever finer discretizations.

4. THE MAIN RESULT

Call a subset of player i 's pure strategies *join-closed* if for any pair of strategies, s_i, s'_i , in the subset, the strategy taking the action $s_i(t_i) \vee s'_i(t_i)$ for each $t_i \in T_i$ is also in the subset.³⁶ We can now state our main result, whose proof is provided in Section 6.

THEOREM 4.1: *If G.1–G.6 hold, and each player's set of monotone pure best replies is nonempty and join-closed whenever the others employ monotone pure strategies, then G possesses a monotone pure-strategy equilibrium.*

Once again, for readers interested in certain applications, it may be sufficient to have access to the following more basic—although substantially less powerful—corollary of Theorem 4.1.³⁷ See Remark 3 for the proof.

COROLLARY 4.2: *Suppose that conditions (i) and (ii) of Proposition 3.1 hold, and that each player's payoff function is continuous in the joint vector of actions*

³⁴To see that even G.2 and G.3 together do not imply the Milgrom and Weber (1985) restriction that μ is absolutely continuous with respect to the product of its marginals $\mu_1 \times \cdots \times \mu_n$, note that G.2 and G.3 hold when there are two players, each with unit interval type space with the usual order, and where the players' types are drawn according to Lebesgue measure on the diagonal of the unit square.

³⁵One might wonder whether G.3 can be weakened by requiring, instead, merely that every atomless set in \mathcal{T}_i assigned positive probability by μ_i contains two distinct ordered points. The answer is “no,” in the sense that this weakening permits examples in which every measurable function from $[0, 1]$ into $[0, 1]$ is monotone, precluding compactness of the set of monotone pure strategies in a useful topology.

³⁶Note that when the join operator is continuous, as it is in a metric semilattice, the resulting function is a.e.-measurable, being the composition of a.e.-measurable and continuous functions. But even when the join operator is not continuous, because the join of two monotone pure strategies is monotone, it is a.e.-measurable under the hypotheses of Lemma A.11 and hence under the hypotheses of Theorem 4.1.

³⁷Indeed, insofar as applications are concerned, Theorem 4.1, in particular, permits one to tailor the partial orders to the structure of the problem, a technique that can be very useful (see, e.g., Examples 5.1, 5.2, and 5.3). In contrast, the corollary insists on the coordinatewise partial order.

for any joint vector of types. Suppose, in addition, that the coordinatewise minimum of any two feasible actions is itself a feasible action, that for each player i and for every monotone joint pure strategy, s_{-i} , of the others, player i 's interim payoff $V_i(\cdot, s_{-i})$ is defined and twice continuously differentiable on an open ball, U_i , containing $A_i \times T_i$, and that for every $(a_i, t_i) \in U_i$,³⁸

(a) $\partial^2 V_i(a_i, t_i, s_{-i}) / \partial a_{ij} \partial a_{il} \geq 0$ for all $j \neq l$,

(b) $\partial^2 V_i(a_i, t_i, s_{-i}) / \partial a_{ij} \partial t_{il} \geq 0$ for all j, l .

Then G possesses a monotone pure-strategy equilibrium, \hat{s} . In particular, for every player i and every pair of types t'_i, t_i in T_i , if every coordinate of t'_i is weakly greater than the corresponding coordinate of t_i , then every coordinate of i 's equilibrium action $\hat{s}_i(t_i)$ when his type is t_i is weakly greater than the corresponding coordinate of his equilibrium action $\hat{s}_i(t'_i)$ when his type is t'_i .

A strengthening of Theorem 4.1 can be helpful when one wishes to demonstrate not merely the existence of a monotone pure-strategy equilibrium, but the existence of a monotone pure-strategy equilibrium within a particular subset of strategies. For example, in a uniform-price auction for m units, a strategy mapping a player's nonincreasing m vector of marginal values into a vector of m bids is undominated only if his bid for a k th unit is no greater than his marginal value for a k th unit. As formulated, Theorem 4.1 does not directly permit one to demonstrate the existence of an undominated equilibrium.³⁹ The next result takes care of this. Its proof is a straightforward extension of the proof of Theorem 4.1 and is provided in Remark 7.

A subset of player i 's pure strategies is called *pointwise-limit-closed* if whenever s_i^1, s_i^2, \dots are each in the set and $s_i^n(t_i) \rightarrow_n s_i(t_i)$ for μ_i almost every $t_i \in T_i$, then s_i is also in the set. A subset of player i 's pure strategies is called *piecewise-closed* if whenever s_i and s'_i are in the set, then so is any strategy s''_i such that for every $t_i \in T_i$ either $s''_i(t_i) = s_i(t_i)$ or $s''_i(t_i) = s'_i(t_i)$.

THEOREM 4.3: *Under the hypotheses of Theorem 4.1, if for each player i , C_i is a join-closed, piecewise-closed, and pointwise-limit-closed subset of pure strategies containing at least one monotone pure strategy, and the intersection of C_i with i 's set of monotone pure best replies is nonempty whenever every other player j employs a monotone pure strategy in C_j , then G possesses a monotone pure-strategy equilibrium in which each player i 's pure strategy is in C_i .*

REMARK 1: When player i 's action space is a semilattice with a closed partial order (as implied by G.4) and C_i is defined by any collection of weak inequalities, i.e., if \mathcal{F}_i and \mathcal{G}_i are arbitrary collections of measurable functions from T_i

³⁸This formulation permits A_i to be finite or, more generally, disconnected.

³⁹Note that it is not possible to restrict the action space alone to ensure that the player chooses an undominated strategy, since the bids that he must be permitted to choose will depend on his private type, i.e., his vector of marginal values.

into A_i and $C_i = \bigcap_{f \in \mathcal{F}_i, g \in \mathcal{G}_i} \{s_i \in \mathcal{S}_i : g(t_i) \leq s_i(t_i) \leq f(t_i) \text{ for } \mu_i \text{ a.e. } t_i \in T_i\}$, then C_i is join-closed, piecewise-closed, and pointwise-limit-closed.

The next section provides conditions that are sufficient for the hypotheses of Theorem 4.1.

4.1. *Sufficient Conditions for Nonempty and Join-Closed Sets of Monotone Best Replies*

In both Athey (2001) and McAdams (2003), quasisupermodularity and single-crossing conditions are put to good use within the confines of a lattice. We now provide weaker versions of both of these conditions, as well as a single condition that is weaker than their combination.

Suppose that player i 's action space, A_i , is a lattice. We say that player i 's interim payoff function V_i is *weakly quasisupermodular* if for all monotone pure strategies s_{-i} of the others, all $a_i, a'_i \in A_i$, and every $t_i \in T_i$,

$$V_i(a_i, t_i, s_{-i}) \geq V_i(a_i \wedge a'_i, t_i, s_{-i}) \quad \text{implies} \\ V_i(a_i \vee a'_i, t_i, s_{-i}) \geq V_i(a'_i, t_i, s_{-i}).$$

In McAdams (2003), the stronger assumption of quasisupermodularity—introduced in Milgrom and Shannon (1994)—is imposed, requiring, in addition, that the second inequality must be strict if the first happens to be strict.⁴⁰ It is well known that V_i is supermodular in actions—hence weakly quasisupermodular—when the coordinates of a player's *own* action vector are complementary, that is, when $A_i = [0, 1]^K$ is endowed with the coordinatewise partial order and the second cross-partial derivatives of $V_i(a_{i1}, \dots, a_{iK}, t_i, s_{-i})$ with respect to distinct action coordinates are nonnegative.⁴¹

We say that i 's interim payoff function V_i satisfies *weak single crossing* if for all monotone pure strategies s_{-i} of the others, for all player i action pairs $a'_i \geq a_i$, and for all player i type pairs $t'_i \geq t_i$,

$$V_i(a'_i, t_i, s_{-i}) \geq V_i(a_i, t_i, s_{-i}) \quad \text{implies} \\ V_i(a'_i, t'_i, s_{-i}) \geq V_i(a_i, t'_i, s_{-i}).$$

In Athey (2001) and McAdams (2003) it is assumed that V_i satisfies the slightly more stringent single crossing condition in which, in addition to the

⁴⁰When actions are totally ordered, as in Athey (2001), interim payoffs are automatically supermodular, and hence both quasisupermodular and weakly quasisupermodular.

⁴¹Complementarities between the actions of distinct *players* is not implied. This is useful because, for example, many auction games satisfy only own-action complementarity.

above, the second inequality is strict whenever the first one is.⁴² We next present a condition that will be shown to be weaker than the combination of weak quasisupermodularity and weak single crossing.

Return now to the case in which A_i is merely a semilattice. For any joint pure strategy of the others, player i 's interim best-reply correspondence is a mapping from his type into the set of optimal actions—or interim best replies—for that type. Say that player i 's interim best-reply correspondence is *monotone* if for every monotone joint pure strategy of the others, whenever action a_i is optimal for player i when his type is t_i , and a'_i is optimal when his type is $t'_i \geq t_i$, then $a_i \vee a'_i$ is optimal when his type is t'_i .⁴³

The following result relates the above conditions to the hypotheses of Theorem 4.1.

PROPOSITION 4.4: *The hypotheses of Theorem 4.1 are satisfied if G.1–G.6 hold, and if for each player i and for each monotone joint pure strategy of the other players, at least one of the following three conditions is satisfied.*⁴⁴

(i) *Player i 's action space is a lattice and i 's interim payoff function is weakly quasisupermodular and satisfies weak single crossing.*

(ii) *Player i 's interim best-reply correspondence is nonempty-valued and monotone.*

(iii) *Player i 's set of monotone pure-strategy best replies is nonempty and join-closed.*

Furthermore, the three conditions are listed in increasing order of generality, that is, (i) \implies (ii) \implies (iii).

PROOF: Because, under G.1–G.6, the hypotheses of Theorem 4.1 hold if condition (iii) holds for each player i , it suffices to show that (i) \implies (ii) \implies (iii). So, fix some player i and some monotone pure strategy for every player but i for the remainder of the proof.

(i) \implies (ii). Suppose i 's action space is a lattice. By G.4 and G.6, for each of i 's types, his interim payoff function is continuous on his compact action space. Player i therefore possesses an optimal action for each of his types and so his interim best-reply correspondence is nonempty-valued. Suppose that action a_i is optimal for i when his type is t_i and a'_i is optimal when his type is $t'_i \geq t_i$. Then because $a_i \wedge a'_i$ is no better than a_i when i 's type is t_i , weak quasisupermodularity implies that $a_i \vee a'_i$ is at least as good as a'_i when i 's type is t_i . Weak single

⁴²For conditions on the joint distribution of types, μ , and the players' payoff functions, $u_i(a, t)$, that imply the more stringent condition, see Athey (2001, pp. 879–881), McAdams (2003, p. 1197), and Van Zandt and Vives (2007).

⁴³This is strictly weaker than requiring the interim best-reply correspondence to be increasing in the strong set order, which in any case requires the additional structure of a lattice (see Milgrom and Shannon (1994)).

⁴⁴Which of the three conditions is satisfied is permitted to depend both on the player, i , and on the joint pure strategy employed by the others.

crossing then implies that $a_i \vee a'_i$ is at least as good as a'_i when i 's type is t'_i . Since a'_i is optimal when i 's type is t'_i , so too must be $a_i \vee a'_i$. Hence, i 's interim best-reply correspondence is monotone.

(i) \implies (iii). Let $B_i: T_i \rightarrow A_i$ denote i 's interim best-reply correspondence. If a_i and a'_i are in $B_i(t_i)$, then $a_i \vee a'_i$ is also in $B_i(t_i)$ by the monotonicity of $B_i(\cdot)$ (set $t_i = t'_i$ in the definition of a monotone correspondence). Consequently, $B_i(t_i)$ is a subsemilattice of i 's action space for each t_i and, therefore, i 's set of monotone pure-strategy best replies is join-closed (measurability of the pointwise join of two strategies follows as in footnote 32). It remains to show that i 's set of monotone pure best replies is nonempty.

Let $\bar{a}_i(t_i) = \vee B_i(t_i)$, which is well defined because G.4 and Lemma A.6 imply that A_i is a complete semilattice. Because i 's interim payoff function is continuous in his action, $B_i(t_i)$ is compact. Hence $B_i(t_i)$ is a compact subsemilattice of A_i and so $B_i(t_i)$ is itself complete by Lemma A.6. Therefore, $\bar{a}_i(t_i)$ is a member of $B_i(t_i)$, implying that $\bar{a}_i(t_i)$ is optimal for every t_i . It remains only to show that $\bar{a}_i(\cdot)$ is monotone (measurability in t_i can be ensured by Lemma A.11).

So, suppose that $t'_i \geq t_i$. Because $\bar{a}_i(t_i) \in B_i(t_i)$ and $\bar{a}_i(t'_i) \in B_i(t'_i)$, the monotonicity of $B_i(\cdot)$ implies that $\bar{a}_i(t_i) \vee \bar{a}_i(t'_i) \in B_i(t'_i)$. Therefore, because $\bar{a}_i(t'_i)$ is the largest member of $B_i(t'_i)$, we have $\bar{a}_i(t'_i) = \bar{a}_i(t_i) \vee \bar{a}_i(t'_i) \geq \bar{a}_i(t_i)$, as desired. *Q.E.D.*

REMARK 2: The environments considered in Athey (2001) and McAdams (2003) are strictly more restrictive than G.1–G.6 permit. Moreover, their conditions on interim payoffs are strictly more restrictive than condition (i) of Proposition 4.4. Theorem 4.1 is, therefore, a strict generalization of their main results.

REMARK 3: We can now prove Corollary 4.2. Conditions G.1–G.5 hold by Proposition 3.1, G.6 holds by assumption, and the coordinatewise minimum condition and (ii) imply that i 's action space is a lattice. Furthermore, when others use monotone pure strategies, (a) implies that i 's interim payoff function is weakly quasisupermodular and (b) implies that it satisfies weak single crossing. Hence, by Proposition 4.4, the hypotheses of Theorem 4.1 are satisfied and the result follows.

When G.1–G.6 hold, it is often possible to apply Theorem 4.1 by verifying condition (i) of Proposition 4.4. But there are important exceptions. For example, Reny and Zamir (2004) have shown in the context of asymmetric first-price auctions that when bidders have distinct and finite bid sets, monotone best replies exist even though weak single crossing fails. Furthermore, since action sets (i.e., real-valued bids) there are totally ordered, best-reply sets are necessarily join-closed and so the hypotheses of Theorem 4.1 are satisfied even though condition (i) of Proposition 4.4 is not. A similar situation arises in the context of multi-unit discriminatory auctions with risk-averse bidders (see Section 5.2 below). There, under constant absolute risk aversion (CARA), weak

quasisupermodularity fails but sets of monotone best replies are nonetheless nonempty and join-closed because condition (ii) of Proposition 4.4 is satisfied.

4.2. Symmetric Games

We very briefly provide a companion result for symmetric Bayesian games. If $x = (x_1, \dots, x_N)$ is an N vector and π is a permutation of $1, \dots, N$, let x_π denote the N vector whose i th coordinate is $x_{\pi(i)}$. Also, let $u(a, t)$ denote the N vector of the players' payoffs when the vector of actions and types is (a, t) .

The Bayesian game G defined above is *symmetric* if for every permutation, π , of $1, 2, \dots, N$, the following conditions hold:

(i) $\mathcal{T}_1 = \dots = \mathcal{T}_N$ (hence, $T_1 = \dots = T_N$) and the partial orders on all the T_i are the same.

(ii) $A_1 = \dots = A_N$ and the partial orders on all the A_i are the same.

(iii) $\mu(D) = \mu(t_\pi \in T : t \in D)$ for every $D \in \mathcal{T}$.⁴⁵

(iv) $u(a_\pi, t_\pi) = u_\pi(a, t)$ for every $(a, t) \in A \times T$.

A pure-strategy equilibrium is *symmetric* if each player employs the same pure strategy.

THEOREM 4.5: *If G is symmetric, then it possesses a symmetric monotone pure-strategy equilibrium if G.1–G.6 hold, and each player's set of monotone pure strategies is nonempty and join-closed whenever the others employ the same monotone pure strategy.*⁴⁶

We now turn to several applications of our results.

5. APPLICATIONS

The first two of our four applications are to uniform-price and discriminatory auctions with risk-averse bidders who possess independent private information. The novelty is in permitting risk aversion. We consider separately the case in which bids are restricted to a finite grid and the case in which they are not. In the uniform-price auction, values are permitted to be interdependent when bid grids are finite, but are restricted to be private when bids can be any nonnegative number. In each of these cases it is currently not known whether a pure-strategy equilibrium exists.

⁴⁵Because $\mathcal{T}_1 = \dots = \mathcal{T}_N$, $D \in \mathcal{T}$ implies $\{t_\pi \in T : t \in D\} \in \mathcal{T}$.

⁴⁶To prove Theorem 4.5, let M_1 denote player 1's (and hence each player's) set of monotone pure strategies and consider the correspondence $B: M_1 \rightarrow M_1$, where $B(s_1)$ is the set of monotone pure-strategy best replies of player 1 when all other players employ the monotone pure strategy $s_1 \in M_1$. By following steps analogous to those in the proof of Theorem 4.1, one shows that the hypotheses of Theorem 2.1 are satisfied, so that B has a fixed point $\hat{s}_1 \in M_1$. The conditions defining a symmetric game ensure that $(\hat{s}_1, \dots, \hat{s}_1)$ is then a symmetric monotone pure-strategy equilibrium.

In the discriminatory auction, we restrict values to be private both when bid grids are finite and when they are not. In the finite-grid case, Theorem 4 of Milgrom and Weber (1985) implies the existence of a pure-strategy equilibrium. However, the existence of a *monotone* pure-strategy equilibrium remains an important open question. In particular, monotonicity in the finite-grid case is crucial for establishing the existence of a (monotone) pure-strategy equilibrium in the unrestricted bid case, where the existence of a pure-strategy equilibrium (monotone or otherwise) is an open question. Indeed, our technique for establishing existence with unrestricted bid sets is to consider limits of finite-grid equilibria as the grid becomes ever finer. Without monotonicity, one cannot ensure the existence of a convergent subsequence of pure strategies and the technique would fail.

For the uniform-price auction, McAdams (2007) contains a counterexample to the existence of a *monotone* pure-strategy equilibrium when bidders are risk averse. This, it turns out, is due to the use of the coordinatewise partial order over the bidders' types. However, the economics of the auction setting (both uniform-price and discriminatory) calls for a partial order over types that ensures, for each k , that when a bidder's type "increases," so too does his marginal utility of winning a k th unit of the good. Only then can one reasonably expect that a bidder will bid more for each unit when his type rises. The coordinatewise partial order enjoys this property only under risk neutrality, while the partial order we introduce—which reduces to the coordinatewise partial order under risk neutrality—always has this property. Using our methods, which permit flexibility in the partial orders employed, we establish the existence of pure-strategy equilibria that are monotone in a new, but economically meaningful, partial order over types in both the uniform-price and discriminatory multi-unit auctions whether bids are restricted to a finite grid or not.

Our third application illustrates how the existence of a pure-strategy equilibrium can be established in a multidimensional type setting when the players' interim payoff functions exhibit strict single crossing in even a single coordinate of their type. The example is economically interesting because it yields a pure-strategy equilibrium in an oligopoly setting without substitute goods. It is technically interesting because one cannot easily obtain the existence of a pure-strategy equilibrium through alternative means. For example, one might first apply Theorem 1 of Milgrom and Weber (1985) to correctly conclude that the game possesses an equilibrium in distributional strategies. One might then hope to conclude that strict single crossing, even in just one coordinate, implies that all such equilibria must be pure. But this second step can fail because, in the example, strict single crossing is sure to hold only when the other players employ monotone pure strategies, and need not hold when, for example, they employ arbitrary distributional strategies.

The final application is to Bayesian games with type spaces containing atoms, where it is shown that our main result establishes the existence of what we call monotone mixed-strategy equilibria.

5.1. Uniform-Price Multi-Unit Auctions With Risk-Averse Bidders

Consider a uniform-price auction with n bidders and m homogeneous units of a single good for sale. Each bidder i simultaneously submits a bid, $b_i = (b_{i1}, \dots, b_{im})$, where $b_{i1} \geq \dots \geq b_{im}$ and each b_{ik} is taken from the set $B \subseteq [0, 1]$, where B contains both 0 and 1. Call b_{ik} bidder i 's k th unit bid. The uniform price, p , is the $m + 1$ st highest of all nm unit bids. Each unit bid above p wins a unit at price p , and any remaining units are awarded to unit bids equal to p according to a random-bidder-order tie-breaking rule.⁴⁷ We begin by considering the case in which the bid set B is finite.

Bidder i 's private type is a nonincreasing vector $t_i = (t_{i1}, \dots, t_{im}) \in [0, 1]^m$, so that his type space is $T_i = \{t_i \in [0, 1]^m : t_{i1} \geq \dots \geq t_{im}\}$. Bidder i is risk averse with utility function for money $u_i : [-m, m] \rightarrow \mathbb{R}$, where $u'_i > 0$, $u''_i \leq 0$. When the vector of bidder types is $t = (t_1, \dots, t_n)$, bidder i 's marginal value for a k th unit is $v_i(t_{ik}, t_{-i})$, where $v_i : [0, 1]^{m(n-1)+1} \rightarrow [0, 1]$ is continuous, and $\partial v_i(t_{ik}, t_{-i}) / \partial t_{ik}$ is continuous and strictly positive. Consequently, bidder i 's expected utility of winning k units at price p may depend on the types of others and is given by $u_i(\sum_{j=1}^k v_i(t_{ij}, t_{-i}) - kp)$. For notational simplicity, we specialize our arguments, but not our results, to the case in which values are private, i.e., where $v_i(t_{ik}, t_{-i}) = t_{ik}$.⁴⁸ Types are chosen independently across bidders, and bidder i 's type vector is chosen according to the density f_i , which need not be positive on all of T_i .

Multi-unit uniform-price auctions always have trivial equilibria in weakly dominated strategies in which some player always bids very high on all units and all others always bid zero. We wish to establish the existence of monotone pure-strategy equilibria that are not trivial in this sense. But observe that, because the set of feasible bids is finite, bidding above one's marginal value on some unit need not be weakly dominated. Indeed, it might be a strict best reply for bidder i of type t_i to bid $b_{ik} > t_{ik}$ for a k th unit as long as there is no feasible bid in $[t_{ik}, b_{ik})$. Such a k th unit bid might permit bidder i to win a k th unit and earn a surplus with high probability rather than risk losing the unit by bidding below t_{ik} . On the other hand, in this instance there is never any gain, and there might be a loss, from bidding above b_{ik} on a k th unit.

Call a monotone pure-strategy equilibrium *nontrivial* if for each bidder i , for f_i almost every t_i , and for every k , bidder i 's k th unit bid does not exceed the smallest feasible unit bid greater than or equal to t_{ik} .⁴⁹ As shown by McAdams

⁴⁷As in McAdams (2003), the tie-breaking rule is as follows. Bidders are ordered randomly and uniformly. Then one bidder at a time according to this order—each bidder's *total* remaining demand (i.e., his number of bids equal to p) or as much as possible—is filled at price p per unit until supply is exhausted.

⁴⁸Interdependent values introduce no substantive complications.

⁴⁹Alternatively, in the case of interdependent values, the smallest feasible unit bid greater than or equal to $\sup_{t_{-i}} v_i(t_{ik}, t_{-i})$.

(2007), under the coordinatewise partial order on type and action spaces, non-trivial monotone pure-strategy equilibria need not exist when bidders are risk averse, as we permit here. Nonetheless, we will demonstrate that a nontrivial monotone pure-strategy equilibrium does exist under an economically meaningful partial order on type spaces that differs from the coordinatewise partial order; we maintain the coordinatewise partial order on the action space B^m of m vectors of unit bids.

Before introducing the new partial order, it is instructive to see what goes wrong with the coordinatewise partial order on types. The heart of the matter is that single crossing fails. To see why, it is enough to consider the case of two units. Fix monotone pure strategies for the other bidders and consider two bids for bidder i , $\bar{b}_i = (\bar{b}_{i1}, \bar{b}_{i2})$ and $\underline{b}_i = (\underline{b}_{i1}, \underline{b}_{i2})$, where $\bar{b}_{ik} > \underline{b}_{ik}$ for $k = 1, 2$. Suppose that when bidder i employs the high bid, \bar{b}_i , he is certain to win both units and pay \bar{p} for each, while he is certain to win only one unit when he employs the low bid, \underline{b}_i . Further suppose that the low bid yields a price for the one unit he wins that is either \underline{p} or $\underline{p}' > \underline{p}$, each being equally likely. Thus, the expected difference in his payoff from employing the high bid versus the low one can be written as

$$\begin{aligned} & \frac{1}{2}[u_i(t_{i1} + t_{i2} - 2\bar{p}) - u_i(t_{i1} - \underline{p}')] \\ & + \frac{1}{2}[u_i(t_{i1} + t_{i2} - 2\bar{p}) - u_i(t_{i1} - \underline{p})], \end{aligned}$$

where we suppose that the first square-bracketed term is positive and the second is negative. Single crossing requires the above average of the bracketed terms, when nonnegative, to remain nonnegative when bidder i 's type increases according to the coordinatewise partial order, i.e., when t_{i1} and t_{i2} increase. But this can fail when risk aversion is strict because the first bracketed term, being positive, strictly falls when t_{i1} increases. Consequently, the average of the bracketed terms can become negative since, even though the negative second bracketed term increases with t_{i1} , it may not increase by much.

The economic intuition for the failure of single crossing is straightforward. Under risk aversion, the marginal utility of winning a second unit falls when the dollar value of a first unit rises, giving the bidder an incentive to reduce his second unit bid so as to reduce the price paid on the first unit. We now turn to the new partial order, which ensures that a higher type implies a higher marginal utility of winning each additional unit. Thus, this new partial order has economic content and is not merely a technical device used to establish the existence of a pure-strategy equilibrium.

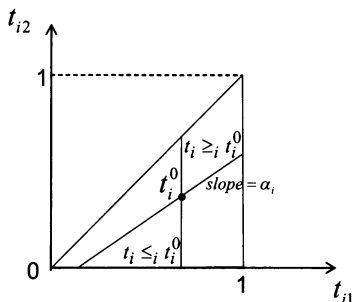


FIGURE 5.1.—Types that are ordered with t_i^0 are bounded between two lines through t_i^0 , one line being vertical and the other having slope α_i .

For each bidder i , let $\alpha_i = \frac{u_i^{(-m)}}{u_i^{(m)}} - 1 \geq 0$ and consider the partial order, \geq_i , on T_i defined as follows: $t'_i \geq_i t_i$ if

$$(5.1) \quad \begin{aligned} t'_{i1} &\geq t_{i1} \quad \text{and} \\ t'_{ik} - \alpha_i(t'_{i1} + \cdots + t'_{ik-1}) \\ &\geq t_{ik} - \alpha_i(t_{i1} + \cdots + t_{ik-1}) \quad \text{for all } k \in \{2, \dots, m\}.^{50} \end{aligned}$$

Figure 5.1 shows the types that are greater than and less than a typical type, t_i^0 , when types are two-dimensional, i.e., when $m = 2$. In that case, one type is considered greater than another if the one type is coordinatewise greater and if, in addition, the increase in the second coordinate of the type vector is sufficiently high relative to the increase in the first coordinate. Only then will the bidder's marginal utilities of winning both a first and second unit increase, and only then will he have an incentive to increase his first and second unit bids.

Under the Euclidean metric and Borel sigma algebra on the type space, the partial order \geq_i defined by (5.1) is clearly closed so that G.1 is satisfied. Because the marginal distribution of each player's type has a density, G.2 is satisfied as well. To see that G.3 is satisfied, let T_i^0 be the set of points in T_i with rational coordinates and suppose that $\int_B f_i(t_i) dt_i > 0$ for some Borel subset B of T_i . Then B must have positive Lebesgue measure in \mathbb{R}^m . Consequently, by Fubini's theorem, there exists $z \in \mathbb{R}^m$ (indeed there is a positive Lebesgue measure of such z 's) such that the line defined by $z + \mathbb{R}((1 + \alpha_i), (1 + \alpha_i)^2, \dots, (1 +$

⁵⁰Under interdependent values, this second condition becomes

$$\begin{aligned} v_i(t'_{ik}, t_{-i}) - \alpha_i(v_i(t'_{i1}, t_{-i}) + \cdots + v_i(t'_{ik-1}, t_{-i})) \\ \geq v_i(t_{ik}, t_{-i}) - \alpha_i(v_i(t_{i1}, t_{-i}) + \cdots + v_i(t_{ik-1}, t_{-i})) \end{aligned}$$

for all t_{-i} and all $k \in \{2, \dots, m\}$.

$\alpha_i)^m$) intersects B in a set of positive one-dimensional Lebesgue measure on the line. Therefore, we may choose two distinct points, t_i and t'_i in B that are on this line. Hence, $t'_i - t_i = \beta((1 + \alpha_i), (1 + \alpha_i)^2, \dots, (1 + \alpha_i)^m)$ for some $\beta > 0$. But then $t'_{i1} - t_{i1} = \beta(1 + \alpha_i) > 0$ and for $k \in \{2, \dots, m\}$,

$$\begin{aligned} t'_{ik} - t_{ik} &= \beta(1 + \alpha_i)^k \\ &= \beta\{1 + \alpha_i[1 + (1 + \alpha_i) + (1 + \alpha_i)^2 + \dots + (1 + \alpha_i)^{k-1}]\} \\ &= \beta(1 + \alpha_i) + \alpha_i[\beta(1 + \alpha_i) + \beta(1 + \alpha_i)^2 + \dots + \beta(1 + \alpha_i)^{k-1}] \\ &= \beta(1 + \alpha_i) + \alpha_i[(t'_{i1} - t_{i1}) + (t'_{i2} - t_{i2}) + \dots + (t'_{ik-1} - t_{ik-1})] \\ &> \alpha_i[(t'_{i1} - t_{i1}) + (t'_{i2} - t_{i2}) + \dots + (t'_{ik-1} - t_{ik-1})]. \end{aligned}$$

Consequently, for any $t_i^0 \in T_i^0$ close enough to $(t'_i + t_i)/2$,

$$t'_i \geq_i t_i^0 \geq_i t_i,$$

according to the partial order \geq_i defined by (5.1). Hence, G.3 is satisfied.

As noted in Section 4.1, action spaces, being finite sublattices, are compact locally complete metric semilattices. Hence, G.4 and G.5(ii) hold. Also, G.6 holds because action spaces are finite. Thus, we have so far verified G.1–G.6.

In McAdams (2003) it is shown that for any fixed order of players for tie-breaking purposes, the pair of auction outcomes associated with any pair of joint bid vectors b and b' is identical to the pair of outcomes associated with $b \vee b'$ and $b \wedge b'$. This implies that each bidder's ex post (and hence interim) payoff function is modular and hence quasisupermodular, even under risk aversion.⁵¹

By condition (i) of Proposition 4.4, the hypotheses of Theorem 4.1 will, therefore, be satisfied if interim payoffs satisfy weak single crossing, which we now demonstrate. It is here where the new partial order \geq_i in (5.1) is fruitfully employed.

To verify weak single crossing, it suffices to show that ex post payoffs satisfy increasing differences. So fix the strategies of the other bidders, a realization of their types, and an ordering of the players for the purposes of tie-breaking. With these fixed, suppose that the bid, \bar{b} , chosen by bidder i of type t_i wins k units at the price \bar{p} per unit, while the coordinatewise-lower bid, \underline{b} , wins $j \leq k$ units at the price $p \leq \bar{p}$ per unit. The difference in i 's ex post utility from bidding \bar{b} versus \underline{b} is then

$$(5.2) \quad u_i(t_{i1} + \dots + t_{ik} - k\bar{p}) - u_i(t_{i1} + \dots + t_{ij} - jp).$$

⁵¹The particular tie-break rule used both here and in McAdams (2003) is important for this result.

Assuming that $t'_i \geq_i t_i$ in the sense of (5.1), it suffices to show that (5.2) is weakly greater at t'_i than at t_i . Noting that (5.1) implies that $t'_{il} \geq t_{il}$ for every l , it can be seen that if $j = k$, then (5.2), being negative, is weakly greater at t'_i than at t_i by the concavity of u_i . It, therefore, remains only to consider the case in which $j < k$, where we have

$$\begin{aligned} & u_i(t'_{i1} + \cdots + t'_{ik} - k\bar{p}) - u_i(t_{i1} + \cdots + t_{ik} - k\bar{p}) \\ & \geq u'_i(m)[(t'_{i1} - t_{i1}) + \cdots + (t'_{ik} - t_{ik})] \\ & \geq u'_i(m)[(t'_{i1} - t_{i1}) + \cdots + (t'_{ij+1} - t_{ij+1})] \\ & \geq u'_i(-m)[(t'_{i1} - t_{i1}) + \cdots + (t'_{ij} - t_{ij})] \\ & \geq u_i(t'_{i1} + \cdots + t'_{ij} - j\underline{p}) - u_i(t_{i1} + \cdots + t_{ij} - j\underline{p}), \end{aligned}$$

where the first and fourth inequalities follow from the concavity of u_i and because a bidder's surplus lies between m and $-m$, and the third inequality follows because $t'_i \geq_i t_i$ in the sense of (5.1). We conclude that weak single crossing holds and so the hypotheses of Theorem 4.1 are satisfied.

Finally, for each player i , let C_i denote the subset of his pure strategies such that for f_i almost every t_i and for every k , bidder i 's k th unit bid does not exceed $\phi(t_{ik})$, the smallest feasible unit bid greater than or equal to t_{ik} .⁵² By Remark 1, each C_i is join-closed, piecewise-closed, and pointwise-limit-closed. Further, because the hypotheses of Theorem 4.1 are satisfied, whenever the others employ monotone pure strategies, player i has a monotone best reply, $b'_i(\cdot)$, say. Defining $b_i(t_i)$ to be the coordinatewise minimum of $b'_i(t_i)$ and $(\phi(t_{i1}), \dots, \phi(t_{im}))$ for all $t_i \in T_i$ implies that $b_i(\cdot)$ is a monotone best reply contained in C_i . This is because, ex post, any units won by employing $b'_i(\cdot)$ that are also won by employing $b_i(\cdot)$ are won at a weakly lower price with $b_i(\cdot)$, and any units won by employing $b'_i(\cdot)$ that are not won by employing $b_i(\cdot)$ cannot be won at a positive surplus. Hence, the hypotheses of Theorem 4.3 are satisfied and we conclude that a nontrivial monotone pure-strategy equilibrium exists. We may therefore state the following proposition.

PROPOSITION 5.1: *Consider an independent private information, interdependent-value, uniform-price multi-unit auction as above with the random-bidder-order tie-breaking rule. Suppose that bids are restricted to a finite grid, that each bidder i 's nonincreasing type vector is chosen according to the density f_i , and that each bidder is weakly risk averse. Then there is a pure-strategy equilibrium of the auction with the following properties for each bidder i :*

(i) *The equilibrium is monotone under the type-space partial order \geq_i defined by (5.1) and under the usual coordinatewise partial order on bids.*

⁵²In the case of interdependent values, $\phi(t_{ik})$ is the smallest feasible unit bid greater than or equal to $\sup_{t_{-i}} v_i(t_{ik}, t_{-i})$.

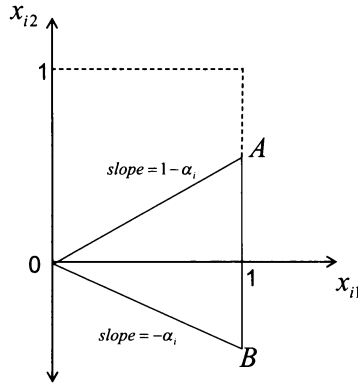


FIGURE 5.2.—After performing the change of variable from t_i to x_i as described in Remark 5, bidder i 's new type space is triangle OAB and it is endowed with the coordinatewise partial order. The figure is drawn for the case in which $\alpha_i \in (0, 1)$.

(ii) *The equilibrium is nontrivial in the sense that for f_i almost all of his types t_i and for every k , bidder i 's k th unit bid does not exceed the smallest feasible unit bid greater than or equal to $\sup_{t_{-i}} v_i(t_{ik}, t_{-i})$.*

REMARK 4: The partial order defined by (5.1) reduces to the usual coordinatewise partial order under risk neutrality (i.e., when $\alpha_i = 0$), but is distinct from the coordinatewise partial order under strict risk aversion (i.e., when $\alpha_i > 0$), in which case McAdams (2003) does not apply since the coordinatewise partial order is employed there.

REMARK 5: In the private values case, the partial order defined by (5.1) can instead be thought of as a change of variable from t_i to say x_i , where $x_{i1} = t_{i1}$ and $x_{ik} = t_{ik} - \alpha_i(t_{i1} + \dots + t_{ik-1})$ for $k > 1$, and where the coordinatewise partial order is applied to the new type space. Our results apply equally well using this change-of-variable technique. In contrast, McAdams (2003) still does not apply because the resulting type space is not the product of intervals, an assumption maintained in McAdams (2003) together with a strictly positive joint density.⁵³ See Figure 5.2 for the case in which $m = 2$. In the more general interdependent values case, there is no obvious change of variable that would render the coordinatewise partial order equivalent to the partial order we use here.

⁵³Indeed, starting with the partial order defined by (5.1), there is no change of variable that, when combined with the coordinatewise partial order, is order-preserving and maps to a product of intervals. This is because, in contrast to a product of intervals with the coordinatewise partial order, under the new partial order, there is never a smallest element of the type space and there is no largest element when $\alpha_i > 1$.

In the private-values case, by considering finer and finer finite grids of bids, one can permit unit bids to be any nonnegative real number. The proof of the following corollary of Proposition 5.1 is given in the Appendix.

COROLLARY 5.2: *If all the conditions of Proposition 5.1 hold except that bidders' unit bids are permitted to be any nonnegative real number and if, in addition, values are private (i.e., $v_i(t_{ik}, t_{-i}) = t_{ik}$), then for any tie-break rule,⁵⁴ a weakly undominated pure-strategy equilibrium exists that is monotone in the sense described in Proposition 5.1. Moreover, ties occur with probability 0 in every such equilibrium.*

5.2. Discriminatory Multi-Unit Auctions With CARA Bidders

Consider the same finite bid set and private-values setup as in Section 5.1 with three exceptions. First, change the payment rule so that each bidder pays his k th unit bid for a k th unit won. Second, assume that each bidder's utility function, u_i , exhibits constant absolute risk aversion. Third, assume that values are private, i.e., that $v_i(t_{ik}, t_{-i}) = t_{ik}$.

Despite these changes, single crossing still fails under the coordinatewise partial order on types for the same underlying reason as in a uniform-price auction with risk-averse bidders. Nonetheless, the same methods in the previous section demonstrate that assumptions G.1–G.6 hold here and that each bidder i 's interim payoff function satisfies weak single crossing under the partial order, \geq_i , defined in (5.1).⁵⁵

For the remainder of this section, we therefore employ the type-space partial order \geq_i defined in (5.1) and the coordinatewise partial order on the space of feasible bid vectors. Monotonicity of pure strategies is then defined in terms of these partial orders.

If it can be shown that interim payoffs are quasipermodular, condition (i) of Proposition 4.4 would permit us to apply Theorem 4.1. However, quasipermodularity does not hold in discriminatory auctions with strictly risk-averse bidders—even CARA bidders.

The intuition for the failure of quasipermodularity is as follows. Suppose there are two units and let b_k denote a k th unit bid. Fixing b_2 , suppose that b_1 is chosen to maximize a bidder's interim payoff when his type is (t_1, t_2) , namely,

$$P_1(b_1)[u(t_1 - b_1) - u(0)] \\ + P_2(b_2)[u((t_1 - b_1) + (t_2 - b_2)) - u(t_1 - b_1)],$$

⁵⁴A tie-break rule specifies, possibly randomly, how any units that remain after awarding a unit to each unit bid above the $m + 1$ st highest are distributed among the unit bids equal to the $m + 1$ st highest.

⁵⁵This statement remains true with any risk-averse utility function. The CARA utility assumption is required for a different purpose, which will be revealed shortly.

where $P_k(b_k)$ is the probability of winning at least k units.⁵⁶

There are two benefits from increasing b_1 . First, the probability, $P_1(b_1)$, of winning at least one unit increases. Second, when risk aversion is strict, the marginal utility, $u((t_1 - b_1) + (t_2 - b_2)) - u(t_1 - b_1)$, of winning a second unit increases. The cost of increasing b_1 is that the marginal utility, $u(t_1 - b_1) - u(0)$, of winning a first unit decreases. Optimizing over the choice of b_1 balances this cost with the two benefits. For simplicity, suppose that the optimal choice of b_1 satisfies $b_1 > t_2$.

Now suppose that b_2 increases. Indeed, suppose that b_2 increases to t_2 . Then the marginal utility of winning a second unit vanishes. Consequently, the second benefit from increasing b_1 is no longer present and the optimal choice of b_1 may fall—even with CARA utility.

This illustrates that the change in utility from increasing one's first unit bid may be positive when one's second unit bid is low, but negative when one's second unit bid is high. Thus, the different coordinates of a bidder's bid are not necessarily complementary, and weak quasisupermodularity can fail. We therefore cannot appeal to condition (i) of Proposition 4.4.

Fortunately, we can instead appeal to condition (ii) of Proposition 4.4, owing to the following lemma, whose proof is given in the Appendix. It is here where we employ the assumption of CARA utility.

LEMMA 5.3: *Fix any monotone pure strategies for other bidders and suppose that the vector of bids b_i is optimal for bidder i when his type vector is t_i , and that b'_i is optimal when his type is $t'_i \geq_i t_i$, where \geq_i is the partial order defined in (5.1). Then the vector of bids $b_i \vee b'_i$ is optimal when his type is t'_i .*

Because Lemma 5.3 establishes condition (ii) of Proposition 4.4, we may apply Theorem 4.1 to conclude that a monotone pure-strategy equilibrium exists. Thus, despite the failure—even with CARA utilities—of both single crossing with the coordinatewise partial order on types and of weak quasisupermodularity with the coordinatewise partial order on bids, we have established the following proposition.

PROPOSITION 5.4: *Consider an independent private-value discriminatory multi-unit auction as above with the random-bidder-order tie-breaking rule and in which bids are restricted to a finite grid. Suppose that each bidder i 's vector of marginal values is nonincreasing and chosen according to the density f_i , and that each bidder is weakly risk averse and exhibits constant absolute risk aversion. Then there is a pure-strategy equilibrium that is monotone under the type-space partial order \geq_i defined by (5.1) and under the usual coordinatewise partial order on bids.*

⁵⁶Our tie-breaking rule ensures that, given the others' strategies, the probability of winning at least k units depends only on one's k th unit bid.

The proof of the following corollary is provided in the Appendix.

COROLLARY 5.5: *When the bidders' unit bids are permitted to be any nonnegative real number, the conclusions of Proposition 5.4 remain valid for any tie-break rule.⁵⁷ Moreover, ties occur with probability 0 in every equilibrium.*

The two applications provided so far demonstrate that it is useful to have flexibility in defining the partial order on the type space, since the mathematically natural partial order (in this case the coordinatewise partial order on the original type space) may not be the partial order that corresponds best to the economics of the problem. The next application shows that even when single crossing cannot be established for all coordinates of the type space jointly, it is enough for the existence of a pure-strategy equilibrium if single crossing holds strictly even for a single coordinate of the type space.

5.3. Price Competition With Nonsubstitutes

Consider an n -firm differentiated-product price-competition setting. Firm i chooses price $p_i \in [0, 1]$ and receives two pieces of private information—his constant marginal cost $c_i \in [0, 1]$ and information $x_i \in [0, 1]$ about the state of demand in each of the n markets. The demand for firm i 's product is $D_i(p, x)$ when the vector of prices chosen by all firms is $p \in [0, 1]^n$ and when their joint vector of private information about market demand is $x \in [0, 1]^n$. Demand functions are assumed to be twice continuously differentiable, strictly positive when own-price is less than 1, and strictly downward-sloping, by which we mean $\partial D_i(p, x)/\partial p_i < 0$.

Some products may be substitutes, but others need not be. More precisely, the n firms are partitioned into two subsets N_1 and N_2 .⁵⁸ Products produced by firms within each subset are substitutes, and so we assume that $D_i(p, x)$ and $\partial D_i(p, x)/\partial p_i$ are nondecreasing in p_j whenever i and j are in the same N_k . In addition, marginal costs are affiliated among firms within each N_k and are independent across the two subsets of firms. The joint density of costs is given by the continuously differentiable density $f(c)$ on $[0, 1]^n$. Information about market demand may be correlated across firms, but is independent of all marginal costs and has continuously differentiable joint density $g(x)$ on $[0, 1]^n$. We do not assume that market demands are nondecreasing in x because we wish to permit the possibility that information that increases demand for some products might decrease it for others.

⁵⁷See footnote 54.

⁵⁸The extension to any finite number of subsets is straightforward.

Given pure strategies $p_j(c_j, x_j)$ for the others, firm i 's interim profits are

$$(5.3) \quad v_i(p_i, c_i, x_i) = \int (p_i - c_i) D_i(p_i, p_{-i}(c_{-i}, x_{-i}), x) g_i(x_{-i}|x_i) f_i(c_{-i}|c_i) dx_{-i} dc_{-i},$$

so that

$$(5.4) \quad \frac{\partial^2 v_i(p_i, c_i, x_i)}{\partial c_i \partial p_i} = -E\left(\frac{\partial D_i}{\partial p_i} \middle| c_i, x_i\right) + \frac{\partial}{\partial c_i} E(D_i | c_i, x_i) + (p_i - c_i) \frac{\partial}{\partial c_i} E\left(\frac{\partial D_i}{\partial p_i} \middle| c_i, x_i\right).$$

Note that both partial derivatives with respect to c_i on the right-hand side of (5.4) are nonnegative. For example, consider the expectation in the first partial derivative (the second is similar) and suppose that $i \in N_1$. Then

$$E(D_i | c_i, x_i) = E[E(D_i(p_i, p_{-i}(c_{-i}, x_{-i}), x) | c_i, x_i, (c_j, x_j)_{j \in N_2}) | c_i, x_i].$$

The inner expectation is nondecreasing in c_i because the vector of marginal costs for firms in N_1 are affiliated, their prices are nondecreasing in their costs, and their goods are substitutes. That the entire expectation is nondecreasing in c_i follows from the independence of (c_i, x_i) and $(c_j, x_j)_{j \in N_2}$.

Therefore, if $p_j(c_j, x_j)$ is nondecreasing in c_j for each firm $j \neq i$ and every x_j , then

$$(5.5) \quad \frac{\partial^2 v_i(p_i, c_i, x_i)}{\partial c_i \partial p_i} \geq -E\left(\frac{\partial D_i}{\partial p_i} \middle| c_i, x_i\right) > 0$$

for all $p_i, c_i, x_i \in [0, 1]$ such that $p_i \geq c_i$.

Thus, according to (5.5), when $p_i \geq c_i$, single crossing holds strictly for the marginal cost coordinate of the type space. On the other hand, single crossing need not hold for the market-demand coordinate, x_i , since we have made no assumptions about how x_i affects demand.⁵⁹ Nonetheless, we shall now define a partial order on firm i 's type space $T_i = [0, 1]^2$ under which a monotone pure-strategy equilibrium exists.

Note that because $-\partial D_i / \partial p_i$ is positive and continuous on its compact domain, it is bounded strictly above zero with a bound that is independent of the pure strategies, $p_j(c_j, x_j)$, employed by other firms. Hence, because our continuity assumptions imply that $\partial^2 v_i(p_i, c_i, x_i) / \partial x_i \partial p_i$ is bounded, there exists

⁵⁹We cannot simply restrict attention to strategies $p_i(c_i, x_i)$ that are monotone in c_i and jointly measurable in (c_i, x_i) , because this set of pure strategies is not compact in a topology that renders ex ante payoffs continuous.

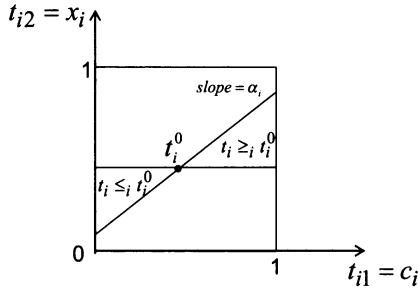


FIGURE 5.3.—Types that are greater than and less than t_i^0 are bounded between two lines through t_i^0 , one line being horizontal, the other having slope α_i .

$\alpha_i > 0$ such that for all $\beta \in [0, \alpha_i]$ and all pure strategies $p_j(c_j, x_j)$ nondecreasing in c_j ,

$$(5.6) \quad \frac{\partial^2 v_i(p_i, c_i, x_i)}{\partial c_i \partial p_i} + \beta \frac{\partial^2 v_i(p_i, c_i, x_i)}{\partial x_i \partial p_i} > 0$$

for all $p_i, c_i, x_i \in [0, 1]$ such that $p_i \geq c_i$.

Inequality (5.6) implies that when $p_i \geq c_i$, the marginal gain from increasing one’s price, namely,

$$\frac{\partial v_i(p_i, c_i, x_i)}{\partial p_i},$$

is strictly increasing along lines in (c_i, x_i) space with slope $\beta \in [0, \alpha_i]$. This provides a basis for defining a partial order under which players possess monotone best replies.

For each player i , define the partial order \geq_i on $T_i = [0, 1]^2$ as follows: $(c'_i, x'_i) \geq_i (c_i, x_i)$ if $\alpha_i c'_i - x'_i \geq \alpha_i c_i - x_i$ and $x'_i \geq x_i$. Figure 5.3 shows those types greater than and less than a typical type $t_i^0 = (c_i^0, x_i^0)$.

Under the partial order \geq_i , assumptions G.1–G.3 hold as in Example 5.1. The action-space assumption G.4 clearly holds while G.5(ii) holds by Lemma A.19 given the usual partial order over the reals. Assumption G.6 holds by our continuity assumption on demand. Also, because the action space $[0, 1]$ is totally ordered, the set of monotone best replies is join-closed because the join of two best replies is, at every t_i , equal to one of them or to the other. Finally, as is shown in the Appendix (see Lemma A.22), under the type-space partial order, \geq_i , firm i possesses a monotone best reply when the others employ monotone pure strategies.

Therefore, by Theorem 4.1, there exists a pure-strategy equilibrium in which each firm’s price is monotone in (c_i, x_i) according to \geq_i . In particular, there is a pure-strategy equilibrium in which each firm’s price is nondecreasing in his marginal cost, the coordinate in which strict single crossing holds.

5.4. Type Spaces With Atoms

When type spaces contain atoms, assumption G.2 fails and there may not exist a pure-strategy equilibrium, let alone a monotone pure-strategy equilibrium. Thus, one must permit mixing and we show here how our results can be used to ensure the existence of a monotone mixed-strategy equilibrium.

We follow Aumann (1964) and define a mixed-strategy for player i to be a measurable function $m_i: T_i \times [0, 1] \rightarrow A_i$, where $[0, 1]$ is endowed with the Borel sigma algebra \mathcal{B} and $T_i \times [0, 1]$ is endowed with the product sigma algebra $\mathcal{T}_i \times \mathcal{B}$.

Mixed strategies m_1, \dots, m_N for the N players are implemented as follows. The players' types t_1, \dots, t_N are drawn jointly according to μ and then, independently, each player i privately draws ω_i from $[0, 1]$ according to a uniform distribution. Player i knowing t_i and ω_i takes the action $m_i(t_i, \omega_i)$. Player i 's payoff given the mixed strategies m_1, \dots, m_N is therefore, $\int_T \int_{[0,1]^N} u_i(m(t, \omega), t) d\omega d\mu$, where $m(t, \omega) = (m_1(t_1, \omega_1), \dots, m_N(t_N, \omega_N))$.

Call a mixed strategy $m_i: T_i \times [0, 1] \rightarrow A_i$ *monotone* if the image of $m_i(t_i, \cdot)$, i.e., the set $m_i(t_i, [0, 1])$, is a totally ordered subset of A_i for every $t_i \in T_i$ and if every member of the image of $m_i(t_i, \cdot)$ is greater than or equal to every member of the image of $m_i(t'_i, \cdot)$ whenever $t_i \geq t'_i$.⁶⁰ Loosely, a mixed strategy is monotone if whenever a player's type randomizes over actions, any two actions in the support of his mixture are ordered. Moreover, every action in the support of one type's mixture is greater than every action in the support of any lower type's mixture.

The following result permits a player's marginal type distribution to contain atoms, even countably many.

THEOREM 5.6: *If G.1 and G.3–G.6 hold, and each player's set of monotone pure best replies is nonempty and join-closed whenever the others employ monotone mixed strategies, then G possesses a monotone mixed-strategy equilibrium.*

PROOF: For each player i , let T_i^* denote the set of atoms of μ_i . Consider the following surrogate Bayesian game. Player i 's type space is $Q_i = [(T_i \setminus T_i^*) \times \{0\}] \cup (T_i^* \times [0, 1])$ and the sigma algebra on Q_i is generated by all sets of the form $(B \setminus T_i^*) \times \{0\}$ and $(B \cap T_i^*) \times C$, where $B \in \mathcal{T}_i$ and C is a Borel subset of $[0, 1]$. The joint distribution on types, ν , is determined as follows. Nature first chooses $t \in T$ according to the original type distribution μ . Then, for each i , Nature independently and uniformly chooses $x_i \in [0, 1]$ if $t_i \in T_i^*$ and chooses $x_i = 0$ if $t_i \in T_i \setminus T_i^*$.⁶¹ Hence, ν_i , the marginal distribution on Q_i , is atomless.

⁶⁰A subset of a partially ordered space is totally ordered if any two members of the subset are ordered. Such a subset is sometimes also called a chain.

⁶¹In particular, if for each player i , $B_i \in \mathcal{T}_i$ and C_i is a Borel subset of $[0, 1]$, and $D = \times_{i \in I} [(B_i \setminus T_i^*) \times \{0\}] \times \times_{i \in I'} [(B_i \cap T_i^*) \times C_i]$, then $\nu(D) = \mu(\{ \times_{i \in I} (B_i \setminus T_i^*) \} \times \{ \times_{i \in I'} (B_i \cap T_i^*) \}) \prod_{i \in I'} \lambda(C_i)$, where λ is Lebesgue measure on $[0, 1]$.

Player i is informed of $q_i = (t_i, x_i)$. Action spaces are unchanged. The x_i are payoff irrelevant and so payoff functions are as before. This completes the description of the surrogate game.

The partial order on Q_i is the lexicographic partial order. That is, $q'_i = (t'_i, x'_i) \geq (t_i, x_i) = q_i$ if either $t'_i \geq t_i$ and $t'_i \neq t_i$ or $t'_i = t_i$ and $x'_i \geq x_i$. The metrics and partial orders on the players' action spaces are unchanged.

It is straightforward to show that under the hypotheses above, all the hypotheses of Theorem 4.1 but perhaps G.3 hold in the surrogate game.⁶² We now show that G.3 too holds in the surrogate game.

For each player i , let T_i^0 denote the countable subset of T_i that can be used to verify G.3 in the original game and define the countable set $Q_i^0 = [T_i^0 \times \{0\}] \cup [T_i^* \times R]$, where R denotes the set of rationals in $[0, 1]$. Suppose that for some player i , $\nu_i(B) > 0$ for some measurable subset B of Q_i . Then either $\nu_i(B \cap [(T_i \setminus T_i^*) \times \{0\}]) > 0$ or $\nu_i(B \cap (\{t_i^*\} \times [0, 1])) > 0$ for some $t_i^* \in T_i^*$. In the former case, $\mu_i(\{t_i \in T_i \setminus T_i^* : (t_i, 0) \in B\}) > 0$ and G.3 in the original game implies the existence of t'_i and t''_i in $\{t_i \in T_i \setminus T_i^* : (t_i, 0) \in B\}$ and t_i^0 in T_i^0 such that $t''_i \geq t_i^0 \geq t'_i$ according to the partial order on T_i . But then $(t''_i, 0) \geq (t_i^0, 0) \geq (t'_i, 0)$ according to the lexicographic partial order on Q_i , and where $(t''_i, 0)$ and $(t'_i, 0)$ are in B and $(t_i^0, 0)$ is in Q_i^0 . In the latter case, there exist x'_i, x_i in $[0, 1]$ with $x'_i > x_i > 0$, such that (t_i^*, x_i) and (t_i^*, x'_i) are in B . But for any rational r between x'_i and x_i , we have $(t_i^*, x'_i) \geq (t_i^*, r) \geq (t_i^*, x_i)$ according to the lexicographic order on Q_i and where (t_i^*, r) is in Q_i^0 . Thus, the surrogate game satisfies G.3 and we may conclude, by Theorem 4.1, that it possesses a monotone pure-strategy equilibrium. But any such equilibrium induces a monotone mixed-strategy equilibrium of the original game. *Q.E.D.*

REMARK 6: The proof of Theorem 5.6, in fact, demonstrates that players need only randomize when their type is an atom.

6. PROOF OF THEOREM 4.1

Let M_i denote the nonempty set of monotone functions from T_i into A_i , and let $M = \prod_{i=1}^N M_i$. By Lemma A.11, every element of M_i is equal μ_i almost everywhere to a measurable monotone function, and so M_i coincides with player i 's set of monotone pure strategies. Let $\mathbf{B}_i : M_{-i} \rightarrow M_i$ denote player i 's best-reply correspondence when all players must employ monotone pure strategies. Because, by hypothesis, each player possesses a monotone best reply (among all strategies) when the others employ monotone pure strategies, any fixed point of $\prod_{i=1}^n \mathbf{B}_i : M \rightarrow M$ is a monotone pure-strategy equilibrium. The following steps demonstrate that such a fixed point exists.

⁶²Observe that a monotone pure strategy in the surrogate game induces a monotone mixed strategy in the original game, and that a monotone pure strategy in the original game defines a monotone pure strategy in the surrogate game by viewing it to be constant in x_i .

Step I— M Is a Nonempty, Compact, Metric, Absolute Retract: Without loss, we may assume for each player i that the metric d_i on A_i is bounded.⁶³ Given d_i , define the metric δ_i on M_i by⁶⁴

$$\delta_i(s_i, s'_i) = \int_{T_i} d_i(s_i(t_i), s'_i(t_i)) d\mu_i(t_i).$$

By Lemmas A.13 and A.16, each (M_i, δ_i) is a compact absolute retract.⁶⁵ Consequently, under the product topology—metrized by the sum of the δ_i — M is a nonempty compact metric space and, by Borsuk (1966, IV, (7.1)), an absolute retract.

Step II— $\times_{i=1}^n \mathbf{B}_i$ Is Nonempty-Valued and Upper Hemicontinuous: We first demonstrate that, given the metric spaces (M_j, δ_j) , each player i 's payoff function, $U_i: M \rightarrow \mathbb{R}$, is continuous under the product topology. To see this, suppose that s^n is a sequence of joint strategies in M and that $s^n \rightarrow s \in M$. By Lemma A.12, for each player i , $s_i^n(t_i) \rightarrow s_i(t_i)$ for μ_i almost every $t_i \in T_i$. Consequently, $s^n(t) \rightarrow s(t)$ for μ almost every $t \in T$.⁶⁶ Hence, since u_i is bounded, Lebesgue's dominated convergence theorem yields

$$U_i(s^n) = \int_T u_i(s^n(t), t) d\mu(t) \rightarrow \int_T u_i(s(t), t) d\mu(t) = U_i(s),$$

establishing the continuity of U_i .

Because each M_i is compact, Berge's theorem of the maximum implies that $\mathbf{B}_i: M_{-i} \rightarrow M_i$ is nonempty-valued and upper hemicontinuous. Hence, $\times_{i=1}^n \mathbf{B}_i$ is nonempty-valued and upper hemicontinuous as well.

Step III— $\times_{i=1}^n \mathbf{B}_i$ Is Contractible-Valued: According to Lemma A.3, for each player i , assumptions G.1–G.3 imply the existence of a monotone and measurable function $\Phi_i: T_i \rightarrow [0, 1]$ such that $\mu_i\{t_i \in T_i: \Phi_i(t_i) = c\} = 0$ for every $c \in [0, 1]$.⁶⁷ Fixing such a function Φ_i permits the construction of a contraction map as follows.

⁶³For any metric, $d(\cdot, \cdot)$, a topologically equivalent bounded metric is $\min(1, d(\cdot, \cdot))$.

⁶⁴Formally, the resulting metric space (M_i, δ_i) is the space of equivalence classes of functions in M_i that are equal μ_i almost everywhere, i.e., two functions are in the same equivalence class if the set on which they coincide contains a measurable subset having μ_i measure 1. Nevertheless, analogous to the standard treatment of \mathcal{L}_p spaces, in the interest of notational simplicity, we focus on the elements of the original space M_i rather than on the equivalence classes themselves.

⁶⁵One cannot improve on Lemma A.16 by proving, for example, that M_i , metrized by δ_i , is homeomorphic to a convex set. It need not be (e.g., see footnote 31).

⁶⁶This is because if Q_1, \dots, Q_n are such that $\mu(Q_i \times T_{-i}) = \mu_i(Q_i) = 1$ for all i , then $\mu(\times_i Q_i) = \mu(\bigcap_i (Q_i \times T_{-i})) = 1$.

⁶⁷For example, if $T_i = [0, 1]^2$ and μ_i is absolutely continuous with respect to Lebesgue measure, we may take $\Phi_i(t_i) = (t_{i1} + t_{i2})/2$.

To construct the contraction, we require each player i to have pointwise *everywhere* largest best replies, not merely best replies that are pointwise μ_i *almost everywhere* largest. The existence of such best replies is established next.

Fix some monotone pure strategy, s_{-i} , for players other than i , and consider player i 's set of monotone pure best replies, $\mathbf{B}_i(s_{-i})$. We wish to show that there exists $\bar{s}_i \in \mathbf{B}_i(s_{-i})$ such that $\bar{s}_i(t_i) \geq s_i(t_i)$ for every $t_i \in T_i$ and every $s_i \in \mathbf{B}_i(s_{-i})$. A natural idea is to define $\bar{s}_i(t_i) = \vee s_i(t_i)$ for each $t_i \in T_i$, where the join is taken over all $s_i \in \mathbf{B}_i(s_{-i})$. However, because each $s_i \in \mathbf{B}_i(s_{-i})$ is an interim best reply against s_{-i} only for μ_i a.e. t_i , it is not at all clear that \bar{s}_i , so defined, is a member of $\mathbf{B}_i(s_{-i})$. Thus, we must proceed more carefully.

Because $\mathbf{B}_i(\cdot)$ is upper hemicontinuous, it is closed-valued and, therefore, $\mathbf{B}_i(s_{-i})$ is compact, being a closed subset of the compact metric space M_i . By hypothesis, $\mathbf{B}_i(s_{-i})$ is nonempty and join-closed, and so $\mathbf{B}_i(s_{-i})$ is a compact semilattice under the partial order defined by $s_i \geq s'_i$ if $s_i(t_i) \geq s'_i(t_i)$ for μ_i a.e. $t_i \in T_i$. By Lemma A.12, this partial order is closed. Therefore, Lemma A.6 implies that $\mathbf{B}_i(s_{-i})$ is a complete semilattice so that $\bar{s}_i = \vee \mathbf{B}_i(s_{-i})$ is a well defined member of $\mathbf{B}_i(s_{-i})$. Consequently for every $s_i \in \mathbf{B}_i(s_{-i})$, $\bar{s}_i(t_i) \geq s_i(t_i)$ for μ_i a.e. $t_i \in T_i$. By Lemma A.14, there exists $\tilde{s}_i \in M_i$ such that $\bar{s}_i(t_i) = \tilde{s}_i(t_i)$ for μ_i a.e. t_i (and hence $\tilde{s}_i \in \mathbf{B}_i(s_{-i})$) and such that $\tilde{s}_i(t_i) \geq s_i(t_i)$ for every $t_i \in T_i$ and every s_i that is μ_i a.e. less than or equal to \tilde{s}_i , and, therefore, in particular for every $s_i \in \mathbf{B}_i(s_{-i})$. This yields the desired pointwise everywhere upper bound, \bar{s}_i , for $\mathbf{B}_i(s_{-i})$.

Define $h : [0, 1] \times \mathbf{B}_i(s_{-i}) \rightarrow \mathbf{B}_i(s_{-i})$ as follows: For every $t_i \in T_i$,

$$(6.1) \quad h(\tau, s_i)(t_i) = \begin{cases} s_i(t_i), & \text{if } \Phi_i(t_i) \leq 1 - \tau \text{ and } \tau < 1, \\ \bar{s}_i(t_i), & \text{otherwise.} \end{cases}$$

Note that $h(0, s_i) = s_i$, $h(1, s_i) = \bar{s}_i$, and $h(\tau, s_i)(t_i)$ is always either $\bar{s}_i(t_i)$ or $s_i(t_i)$, and so is an interim best reply for μ_i almost every t_i . Moreover, $h(\tau, s_i)$ is monotone because Φ_i is monotone and $\bar{s}_i(t_i) \geq s_i(t_i)$ for every $t_i \in T_i$. Hence, $h(\tau, s_i) \in \mathbf{B}_i(s_{-i})$. Therefore, h will be a contraction for $\mathbf{B}_i(s_{-i})$ and $\mathbf{B}_i(s_{-i})$ will be contractible if $h(\tau, s_i)$ is continuous, which we establish next.⁶⁸

Suppose $\tau_n \in [0, 1]$ converges to τ and $s_i^n \in \mathbf{B}_i(s_{-i})$ converges to s_i , both as $n \rightarrow \infty$. By Lemma A.12, there is a measurable subset, D , of i 's types such that $\mu_i(D) = 1$ and for all $t_i \in D$, $s_i^n(t_i) \rightarrow s_i(t_i)$. Consider any $t_i \in D$. There are three cases: (a) $\Phi_i(t_i) < 1 - \tau$, (b) $\Phi_i(t_i) > 1 - \tau$, and (c) $\Phi_i(t_i) = 1 - \tau$. In case (a), $\tau < 1$ and $\Phi_i(t_i) < 1 - \tau_n$ for n large enough and so $h(\tau_n, s_i^n)(t_i) = s_i^n(t_i) \rightarrow s_i(t_i) = h(\tau, s_i)$. In case (b), $\Phi_i(t_i) > 1 - \tau_n$ for n large enough and so for such

⁶⁸With Φ_i defined as in footnote 67, Figure 6.1 provides snapshots of the resulting $h(\tau, s_i)$ as τ moves from 0 to 1. The axes are the two dimensions of the type vector (t_{i1}, t_{i2}) , and the arrow within the figures depicts the direction in which the negatively sloped line, $(t_{i1} + t_{i2})/2 = 1 - \tau$, moves as τ increases. For example, panel (a) shows that when $\tau = 0$, $h(\tau, s_i)(t_i)$ is equal to $s_i(t_i)$ for all t_i in the unit square. On the other hand, panel (c) shows that when $\tau = 3/4$, $h(\tau, s_i)(t_i)$ is equal to $s_i(t_i)$ for t_i below the negatively sloped line and equal to $\bar{s}_i(t_i)$ for t_i above it.

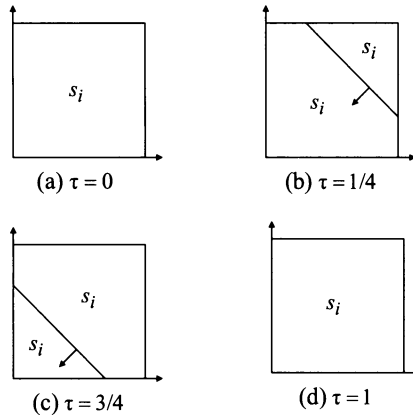


FIGURE 6.1.— $h(\tau, s_i)$ as τ varies from 0 (panel (a)) to 1 (panel (d)) and the domain is the unit square.

large enough n , $h(\tau_n, s_i^n)(t_i) = \bar{s}_i(t_i) = h(\tau, s_i)(t_i)$. Because the remaining case (c) occurs only if t_i is in a set of types having μ_i measure 0, we have shown that $h(\tau_n, s_i^n)(t_i) \rightarrow h(\tau, s_i)(t_i)$ for μ_i a.e. t_i , which, by Lemma A.12 implies that $h(\tau_n, s_i^n) \rightarrow h(\tau, s_i)$, establishing the continuity of h .

Thus, for each player i , the correspondence $\mathbf{B}_i: M_{-i} \rightarrow M_i$ is contractible-valued. Under the product topology, $\times_{i=1}^n \mathbf{B}_i$ is therefore contractible-valued as well.

Steps I–III establish that $\times_{i=1}^n \mathbf{B}_i$ satisfies the hypotheses of Theorem 2.1 and, therefore, possesses a fixed point. *Q.E.D.*

REMARK 7: The proof of Theorem 4.3 mimics that of Theorem 4.1, but where each M_i is replaced with $M_i \cap C_i$ and where each correspondence $\mathbf{B}_i: M_{-i} \rightarrow M_i$ is replaced with the correspondence $\mathbf{B}_i^*: M_{-i} \cap C_{-i} \rightarrow M_i \cap C_i$ defined by $\mathbf{B}_i^*(s_{-i}) = \mathbf{B}_i(s_{-i}) \cap C_i$. The proof goes through because the hypotheses of Theorem 4.3 imply that each $M_i \cap C_i$ is compact, nonempty, join-closed, piecewise-closed, and pointwise-limit-closed (and hence the proof that each $M_i \cap C_i$ is an absolute retract mimics the proof of Lemma A.16), and that each correspondence \mathbf{B}_i^* is upper hemicontinuous, nonempty-valued, and contractible-valued (the contraction is once again defined by (6.1)). The result then follows from Theorem 2.1.

APPENDIX

To simplify the notation, we drop the subscript i from T_i , μ_i , and A_i throughout the Appendix. Thus, in this appendix, T , μ , and A should be thought of as the type space, marginal distribution, and action space, respectively, of any one of the players, not as the joint type spaces, joint distribution, and joint action

spaces of all the players. For convenience, we rewrite here without subscripts the assumptions from Section 3.2 that will be used in this appendix.

G.1. T is endowed with a sigma algebra of subsets, \mathcal{T} , a measurable partial order, and a countably additive probability measure μ .

G.2. The probability measure μ is atomless.

G.3. There is a countable subset T^0 of T such that every set in \mathcal{T} assigned positive probability by μ contains two points between which lies a point in T^0 .

G.4. A is a compact metric space and a semilattice with a closed partial order.

G.5. Either (i) A is a convex subset of a locally convex linear topological space and the partial order on A is convex or (ii) A is a locally complete metric semilattice.

A.1. Partially Ordered Probability Spaces

Preliminaries. We say that $\Psi = (T, \mathcal{T}, \mu, \geq)$ is a *partially ordered probability space* if G.1 holds, i.e., if \mathcal{T} is a sigma algebra of subsets of T , \geq is a measurable partial order on T , and μ is a countably additive probability measure with domain \mathcal{T} . If, in addition, G.2 holds, we say that Ψ is a *partially ordered atomless probability space*.

If $\Psi = (T, \mathcal{T}, \mu, \geq)$ is a partially ordered probability space, Lemma 5.1.1 of Cohn (1980) implies that the sets $\geq(t) = \{t' \in T : t' \geq t\}$ and $\leq(t) = \{t' \in T : t \geq t'\}$ are in \mathcal{T} for each $t \in T$. Hence, for all $t, t' \in T$, the interval $[t, t'] = \{t'' \in T : t' \geq t'' \geq t\}$ is a member of \mathcal{T} , being the intersection of $\geq(t)$ and $\leq(t')$. In particular, the singleton set $\{t\}$, being a degenerate interval, is a member of \mathcal{T} for every $t \in T$.

LEMMA A.1: *Suppose that $(T, \mathcal{T}, \mu, \geq)$ is a partially ordered probability space satisfying G.3 and that $D \in \mathcal{T}$ has positive measure under μ . Then there are sequences $\{t_n\}_{n=1}^\infty$ in T^0 and $\{t'_n\}_{n=1}^\infty$ in D , such that μ assigns positive measure to the intervals $[t_n, t'_n]$ and $[t'_n, t_{n+1}]$ for every n .*

PROOF: For each of the countably many t^0 in T^0 , remove from D all members of $\geq(t^0)$ if $D \cap \geq(t^0)$ has μ measure 0 and remove from D all members of $\leq(t^0)$ if $D \cap \leq(t^0)$ has μ measure 0. Having removed from D countably many subsets each with μ measure 0, we are left with a set D' with the same positive measure as D . Applying G.3 to D' , there exist t, t' in D' and \tilde{t}_1 in T^0 such that $t' \geq \tilde{t}_1 \geq t$. Hence, t' is a member of both D' and $\geq(\tilde{t}_1)$, implying that $\mu(D \cap \geq(\tilde{t}_1)) > 0$, and t is a member of both D' and $\leq(\tilde{t}_1)$, implying that $\mu(D \cap \leq(\tilde{t}_1)) > 0$.

Setting $D_0 = D$, we may inductively apply the same argument, for each $k \geq 1$, to the positive μ measure set $D_k = D_{k-1} \cap \geq(\tilde{t}_k)$, yielding $\tilde{t}_{k+1} \in T^0$ such that $\mu(D_k \cap \geq(\tilde{t}_{k+1})) > 0$ and $\mu(D_k \cap \leq(\tilde{t}_{k+1})) > 0$.

Define the sequence $\{t_n\}_{n=1}^\infty$ in T^0 by setting $t_n = \tilde{t}_{3n-2}$ and define the sequence $\{t'_n\}_{n=1}^\infty$ in D by letting t'_n be any member of $D \cap [\tilde{t}_{3n-1}, \tilde{t}_{3n}]$. The latter set is always nonempty because for every $k \geq 1$,

$$\begin{aligned} \text{(A.1)} \quad \mu(D \cap [\tilde{t}_k, \tilde{t}_{k+1}]) &\geq \mu([D_{k-1} \cap \geq(\tilde{t}_k)] \cap \leq(\tilde{t}_{k+1})) \\ &= \mu(D_k \cap \leq(\tilde{t}_{k+1})) \\ &> 0, \end{aligned}$$

where the first line follows because D contains D_{k-1} and the second line follows from the definition of D_k . Hence the two sequences, $\{t_n\}$ in T^0 and $\{t'_n\}$ in D , are well defined.

Finally, for every $n \geq 1$, (A.1) implies $\mu([t_n, t'_n]) \geq \mu([\tilde{t}_{3n-2}, \tilde{t}_{3n-1}]) \geq \mu(D \cap [\tilde{t}_{3n-2}, \tilde{t}_{3n-1}]) > 0$ and $\mu([t'_n, t_{n+1}]) \geq \mu([\tilde{t}_{3n}, \tilde{t}_{3n+1}]) \geq \mu(D \cap [\tilde{t}_{3n}, \tilde{t}_{3n+1}]) > 0$, as desired. Q.E.D.

COROLLARY A.2: *Under the hypotheses of Lemma A.1, if $\mu([a, b]) > 0$, then $\mu([a, t^*]) > 0$ and $\mu([t^*, b]) > 0$ for some $t^* \in T^0$.*

PROOF: Let $D = [a, b]$, and obtain sequences $\{t_n\}$ in T^0 and $\{t'_n\}$ in $[a, b]$ satisfying the conclusion of Lemma A.1. Then letting $t^* = t_2 \in T^0$, for example, yields $\mu([a, t^*]) \geq \mu([t'_1, t_2]) > 0$, where the first inequality follows because $t'_1 \in [a, b]$ implies $[a, t^*]$ contains $[t'_1, t^*] = [t'_1, t_2]$, and yields $\mu([t^*, b]) \geq \mu([t_2, t'_2]) > 0$, where the first inequality follows because $t'_2 \in [a, b]$ implies $[t^*, b]$ contains $[t^*, t'_2] = [t_2, t'_2]$. Q.E.D.

LEMMA A.3: *If $(T, \mathcal{T}, \mu, \geq)$ is a partially ordered atomless probability space satisfying G.3, then there is a monotone and measurable function $\Phi: T \rightarrow [0, 1]$ such that $\mu(\Phi^{-1}(\alpha)) = 0$ for every $\alpha \in [0, 1]$.*

PROOF: Let $T^0 = \{t_1, t_2, \dots\}$ be the countable subset of T in G.3. Define $\Phi: T \rightarrow [0, 1]$ by

$$\text{(A.2)} \quad \Phi(t) = \sum_{k=1}^\infty 2^{-k} \mathbf{1}_{\geq(t_k)}(t).$$

Clearly, Φ is monotone and measurable, being the pointwise convergent sum of monotone and measurable functions. It remains to show that $\mu(\Phi^{-1}(\alpha)) = 0$ for every $\alpha \in [0, 1]$.

Suppose, by way of contradiction, that $\mu(\Phi^{-1}(\alpha)) > 0$. Because μ is atomless, $\mu(\Phi^{-1}(\alpha) \setminus T^0) = \mu(\Phi^{-1}(\alpha)) > 0$, and so applying G.3 to $\Phi^{-1}(\alpha) \setminus T^0$ yields t', t'' in $\Phi^{-1}(\alpha) \setminus T^0$ and $t_k \in T^0$ such that $t'' \geq t_k \geq t'$. But then $\alpha = \Phi(t'') \geq \Phi(t') + 2^{-k} > \Phi(t') = \alpha$, a contradiction. Q.E.D.

A.2. Semilattices

The standard proofs of the next two lemmas are omitted.

LEMMA A.4: *If G.4 holds, and a_n, b_n, c_n are sequences in A such that $a_n \leq b_n \leq c_n$ for every n and both a_n and c_n converge to a , then b_n converges to a .*

LEMMA A.5: *If G.4 holds, then every nondecreasing sequence and every non-increasing sequence in A converges.*

LEMMA A.6: *If G.4 holds, then A is a complete semilattice.*

PROOF: Let S be a nonempty subset of A . Because A is a compact metric space, S has a countable dense subset, $\{a_1, a_2, \dots\}$. Let $a^* = \lim_n a_1 \vee \dots \vee a_n$, where the limit exists by Lemma A.5. Suppose that $b \in A$ is an upper bound for S and let a be an arbitrary element of S . Then some sequence, a_{n_k} , converges to a . Moreover, $a_{n_k} \leq a_1 \vee a_2 \vee \dots \vee a_{n_k} \leq b$ for every k . Taking the limit as $k \rightarrow \infty$ yields $a \leq a^* \leq b$. Hence, $a^* = \vee S$. *Q.E.D.*

A.3. The Space of Monotone Functions From T Into A

In this section, we introduce a metric, δ , under which the space \mathcal{M} of monotone functions from T into A will be shown to be a compact metric space. Furthermore, it will be shown that under suitable conditions, the metric space (\mathcal{M}, δ) is an absolute retract. Some preliminary results are required.

Recall that a property $P(t)$ is said to hold for μ a.e. $t \in T$ if the set of $t \in T$ on which $P(t)$ holds contains a measurable subset having μ measure 1. We next introduce an important definition.

DEFINITION A.7: Given a partially ordered probability space $\Psi = (T, \mathcal{T}, \mu, \geq)$ and a partially ordered metric space A , say that a monotone function $f: T \rightarrow A$ is Ψ approachable at $t \in T$ if there are sequences $\{t_n\}$ and $\{t'_n\}$ in T such that $\lim_n f(t_n) = \lim_n f(t'_n) = f(t)$ and the intervals $[t_n, t]$ and $[t, t'_n]$ have positive μ measure for every n .

REMARK 8: (i) The positive measure condition implies that the intervals are nonempty, i.e., that $t'_n \geq t \geq t_n$ for every n . (ii) Because we have not endowed T with a topology, neither $\{t_n\}$ nor $\{t'_n\}$ is required to converge. (iii) f is Ψ approachable at every atom t of μ because we can set $t_n = t'_n = t$ for all n .

LEMMA A.8: *Suppose that $\Psi = (T, \mathcal{T}, \mu, \geq)$ is a partially ordered probability space satisfying G.3, that A satisfies G.4, and that $f: T \rightarrow A$ is measurable and monotone. Then the set of points at which f is Ψ approachable is measurable.*

PROOF: Suppose that f is Ψ approachable at $t \in T$, and that the sequences $\{t_n\}$ and $\{t'_n\}$ satisfy the conditions in Definition A.7. Then, by Corollary A.2, for each n there exist $\tilde{t}_n, \tilde{t}'_n$ in T^0 such that the intervals $[t_n, \tilde{t}_n]$, $[\tilde{t}_n, t]$, $[t, \tilde{t}'_n]$, and $[\tilde{t}'_n, t'_n]$ each have positive μ measure. In particular, $t_n \leq \tilde{t}_n \leq t$ implies $f(t_n) \leq f(\tilde{t}_n) \leq f(t)$ and $t \leq \tilde{t}'_n \leq t'_n$ implies $f(t) \leq f(\tilde{t}'_n) \leq f(t'_n)$. Consequently, by Lemma A.4, $\lim_n f(\tilde{t}_n) = \lim_n f(\tilde{t}'_n) = f(t)$. We conclude that the definition of Ψ -approachability at any $t \in T$ would be unchanged if the sequences $\{t_n\}$ and $\{t'_n\}$ were required to be in T^0 .

Let d be the metric on A , and for every $t_1, t_2 \in T$ and every $n = 1, 2, \dots$, define

$$T^n_{t_1, t_2} = \left\{ t \in T : \mu([t_1, t]) > 0, \mu([t, t_2]) > 0, \right. \\ \left. d(f(t_1), f(t)) < \frac{1}{n}, d(f(t_2), f(t)) < \frac{1}{n} \right\}.$$

Then according to the conclusion drawn in the preceding paragraph, the set of points at which f is Ψ approachable is

$$\bigcap_{n \geq 1} \bigcup_{t_1, t_2 \in T^0} T^n_{t_1, t_2}.$$

Consequently, it suffices to show that each $T^n_{t_1, t_2}$ is measurable, and for this it suffices to show that, as functions of t , the functions $\mu([t_1, t])$, $\mu([t, t_2])$, $d(f(t_1), f(t))$, and $d(f(t_2), f(t))$ are measurable.

The functions $d(f(t_1), f(t))$ and $d(f(t_2), f(t))$ are measurable in t because the metric d is continuous in its arguments and f is measurable. For the measurability of $\mu([t_1, t])$, let $E = \{(t', t'') \in T \times T : t' \geq t''\} \cap (T \times \geq(t_1))$. Then E is in $\mathcal{T} \times \mathcal{T}$ by the measurability of \geq , and $[t_1, t] = E_t$ is the slice of E in which the first coordinate is t . Proposition 5.1.2 of Cohn (1980) states that $\mu(E_t)$ is measurable in t . A similar argument shows that $\mu([t, t_2])$ is measurable in t . *Q.E.D.*

LEMMA A.9: *Suppose that G.1, G.3, and G.4 hold, i.e., that $\Psi = (T, \mathcal{T}, \mu, \geq)$ is a partially ordered probability space satisfying G.3 and that A satisfies G.4. If $f : T \rightarrow A$ is measurable and monotone, then f is Ψ approachable at μ a.e. $t \in T$.*

PROOF: Let D denote the set of points at which f is not Ψ approachable. By Lemma A.8, D is a member of \mathcal{T} . It suffices to show that $\mu(D) = 0$.

Define $T^n_{t_1, t_2}$ as in the proof of Lemma A.8 so that

$$D = \bigcup_{n=1} \bigcap_{t_1, t_2 \in T^0} (T^n_{t_1, t_2})^c,$$

and suppose, by way of contradiction, that $\mu(D) > 0$. Then, for some $N \geq 1$, $\mu(D_N) > 0$, where $D_N = \bigcap_{t_1, t_2 \in T^0} (T^N_{t_1, t_2})^c$.

Let d denote the metric on A . Then for every $t \in D_N$ and every $t_1, t_2 \in T^0$ such that the intervals $[t_1, t]$ and $[t, t_2]$ have positive μ measure, either

$$(A.3) \quad d(f(t_1), f(t)) \geq \frac{1}{N} \quad \text{or} \quad d(f(t_2), f(t)) \geq \frac{1}{N}.$$

By Lemma A.1, there are sequences $\{t_n\}_{n=1}^\infty$ in T^0 and $\{t'_n\}_{n=1}^\infty$ in D_N such that μ assigns positive measure to the intervals $[t_n, t'_n]$ and $[t'_n, t_{n+1}]$ for every n . Consequently, for every n , (A.3) implies that either

$$(A.4) \quad d(f(t_n), f(t'_n)) \geq \frac{1}{N} \quad \text{or} \quad d(f(t_{n+1}), f(t'_n)) \geq \frac{1}{N}.$$

On the other hand, because for every n , the intervals $[t_n, t'_n]$ and $[t'_n, t_{n+1}]$ —having positive μ measure—are nonempty, we have $t_1 \leq t'_1 \leq t_2 \leq t'_2 \leq \dots$. Hence, the monotonicity of f implies that

$$f(t_1) \leq f(t'_1) \leq f(t_2) \leq f(t'_2) \leq \dots$$

is a monotone sequence of points in A and must, therefore, converge by Lemma A.5. But then both $d(f(t_n), f(t'_n))$ and $d(f(t_{n+1}), f(t'_n))$ converge to zero, contradicting (A.4), and so we conclude that $\mu(D) = 0$. *Q.E.D.*

LEMMA A.10—A Generalized Helly Selection Theorem: *Suppose that G.1, G.3, and G.4 hold, i.e., that $\Psi = (T, \mathcal{T}, \mu, \geq)$ is a partially ordered probability space satisfying G.3 and that A satisfies G.4. If $f_n : T \rightarrow A$ is a sequence of monotone functions—not necessarily measurable—then there is a subsequence, f_{n_k} , and a measurable monotone function, $f : T \rightarrow A$, such that $f_{n_k}(t) \rightarrow_k f(t)$ for μ a.e. $t \in T$.*

PROOF: Let $T^0 = \{t_1, t_2, \dots\}$ be the countable subset of T satisfying G.3. Choose a subsequence, f_{n_k} , of f_n such that, for every i , $\lim_k f_{n_k}(t_i)$ exists. Define $f(t_i) = \lim_k f_{n_k}(t_i)$ for every i , and extend f to all of T by defining $f(t) = \vee \{a \in A : a \leq f(t_i) \text{ for all } t_i \geq t\}$.⁶⁹ By Lemma A.6, this is well defined because $\{a \in A : a \leq f(t_i) \text{ for all } t_i \geq t\}$ is nonempty for each t since it contains any limit point of $f_{n_k}(t)$. Indeed, if $f_{n_{k_j}}(t) \rightarrow_j a$, then $a = \lim_j f_{n_{k_j}}(t) \leq \lim_j f_{n_{k_j}}(t_i) = f(t_i)$ for every $t_i \geq t$. Furthermore, as required, the extension to T is monotone and leaves the values of f on $\{t_1, t_2, \dots\}$ unchanged, where the latter follows because the monotonicity of f on $\{t_1, t_2, \dots\}$ implies that $\{a \in A : a \leq f(t_i) \text{ for all } t_i \geq t_k\} = \{a \in A : a \leq f(t_k)\}$. To see that f is measurable, note first that $f(t) = \lim_m g_m(t)$, where $g_m(t) = \vee \{a \in A : a \leq f(t_i) \text{ for all } i = 1, \dots, m \text{ such that } t_i \geq t\}$ and where the limit exists by Lemma A.5. Because

⁶⁹Hence, $f(t) = \vee A$ if no $t_i \geq t$.

the partial order on T is measurable, each g_m is a measurable simple function. Hence, f is measurable, being the pointwise limit of measurable functions.

Let f be Ψ approachable at $t \in T$. By Lemma A.9, it suffices to show that $f_{n_k}(t) \rightarrow f(t)$. So suppose that $f_{n_{k_j}}(t) \rightarrow a \in A$ for some subsequence n_{k_j} of n_k . By the compactness of A , it suffices to show that $a = f(t)$.

Because f is Ψ approachable at $t \in T$, the argument in the first paragraph of the proof of Lemma A.8 implies that there exist sequences $\{t_{i_n}\}$ and $\{t'_{i_n}\}$ in T^0 such that $\lim_n f(t_{i_n}) = \lim_n f(t'_{i_n}) = f(t)$, and such that the intervals $[t_{i_n}, t]$ and $[t, t'_{i_n}]$ have positive μ measure for every n . In particular, the intervals $[t_{i_n}, t]$ and $[t, t'_{i_n}]$ are always nonempty, and so $t_{i_n} \leq t \leq t'_{i_n}$, implying by the monotonicity of each f_{n_k} that

$$f_{n_k}(t_{i_n}) \leq f_{n_k}(t) \leq f_{n_k}(t'_{i_n})$$

for every k and n . Because the partial order on A is closed, taking the limit first in k yields

$$f(t_{i_n}) \leq a \leq f(t'_{i_n}),$$

and taking the limit next in n yields

$$f(t) \leq a \leq f(t),$$

from which we conclude that $a = f(t)$, as desired.

Q.E.D.

By setting $\{f_n\}$ in Lemma A.10 equal to a constant sequence, we obtain the following lemma.

LEMMA A.11: *Under G.1, G.3, and G.4, every monotone function from T into A is μ almost everywhere equal to a measurable monotone function.*

We now introduce a metric on \mathcal{M} , the space of monotone functions from T into A . Denote the metric on A by d and assume without loss that $d(a, b) \leq 1$ for all $a, b \in A$. Define the metric, δ , on \mathcal{M} by

$$\delta(f, g) = \int_T d(f(t), g(t)) d\mu(t),$$

which is well defined by Lemma A.11.

Formally, the resulting metric space (\mathcal{M}, δ) is the space of equivalence classes of monotone functions that are equal μ almost everywhere, i.e., two functions are in the same equivalence class if there is a measurable subset of T having μ measure 1 on which they coincide. Nevertheless, and analogous to the standard treatment of \mathcal{L}_p spaces, we focus on the elements of the original space \mathcal{M} rather than on the equivalence classes themselves.

LEMMA A.12: *Under G.1, G.3, and G.4, $\delta(f_k, f) \rightarrow 0$ if and only if $d(f_k(t), f(t)) \rightarrow 0$ for μ a.e. $t \in T$.*

PROOF: Only If. Suppose that $\delta(f_k, f) \rightarrow 0$. By Lemma A.9, it suffices to show that $f_k(t) \rightarrow f(t)$ for all Ψ -approachability points, t , of f .

Let t_0 be a Ψ -approachability point of f . Because A is compact, it suffices to show that an arbitrary convergent subsequence, $f_{k_j}(t_0)$, of $f_k(t_0)$ converges to $f(t_0)$. So suppose that $f_{k_j}(t_0)$ converges to $a \in A$. By Lemma A.10, there is a further subsequence, $f_{k'_j}$ of f_{k_j} , and a monotone measurable function, $g: T \rightarrow A$, such that $f_{k'_j}(t) \rightarrow g(t)$ for μ a.e. t in T . Because d is bounded, the dominated convergence theorem implies that $\delta(f_{k'_j}, g) \rightarrow 0$. But $\delta(f_{k'_j}, f) \rightarrow 0$ then implies that $\delta(f, g) = 0$ and so $f_{k'_j}(t) \rightarrow f(t)$ for μ a.e. t in T .

Because t_0 is a Ψ -approachability point of f , there are sequences $\{t_n\}_{n=1}^\infty$ and $\{t'_n\}_{n=1}^\infty$ in T such that $\lim_n f(t_n) = \lim_n f(t'_n) = f(t_0)$, and the intervals $[t_n, t_0]$ and $[t_0, t'_n]$ have positive μ measure for every $n \geq 1$.

Consequently, because $f_{k'_j}(t) \rightarrow f(t)$ for μ a.e. t in T , and because the intervals $[t_n, t_0]$ and $[t_0, t'_n]$ have positive μ measure, for every n there exist \tilde{t}_n and \tilde{t}'_n such that $t_n \leq \tilde{t}_n \leq t_0 \leq \tilde{t}'_n \leq t'_n$, $f_{k'_j}(\tilde{t}_n) \rightarrow_j f(\tilde{t}_n)$ and $f_{k'_j}(\tilde{t}'_n) \rightarrow_j f(\tilde{t}'_n)$. Consequently, $f_{k'_j}(\tilde{t}_n) \leq f_{k'_j}(t_0) \leq f_{k'_j}(\tilde{t}'_n)$, and taking the limit as $j \rightarrow \infty$ yields $f(\tilde{t}_n) \leq a \leq f(\tilde{t}'_n)$, so that $f(t_n) \leq f(\tilde{t}_n) \leq a \leq f(\tilde{t}'_n) \leq f(t'_n)$ and, therefore, $f(t_n) \leq a \leq f(t'_n)$. Taking the limit of the latter inequality as $n \rightarrow \infty$ yields $f(t_0) \leq a \leq f(t_0)$, so that $a = f(t_0)$, as desired.

If. To complete the proof, suppose that $f_k(t)$ converges to $f(t)$ for μ a.e. $t \in T$. Then because d is bounded, the dominated convergence theorem implies that $\delta(f_k, f) \rightarrow 0$. Q.E.D.

Combining Lemmas A.10 and A.12, we obtain the following lemma.

LEMMA A.13: *Under G.1, G.3, and G.4, the metric space (\mathcal{M}, δ) is compact.*

LEMMA A.14: *Suppose that G.1, G.3, and G.4 hold, and that $f: T \rightarrow A$ is monotone. If for every $t \in T$, $\bar{f}(t) = \vee g(t)$, where the join is taken over all monotone $g: T \rightarrow A$ such that $g(t) \leq f(t)$ for μ a.e. $t \in T$, then $\bar{f}: T \rightarrow A$ is monotone and $\bar{f}(t) = f(t)$ for μ a.e. $t \in T$.⁷⁰*

PROOF: Note that $\bar{f}(t)$ is well defined for each $t \in T$ by Lemma A.6, and \bar{f} is monotone, being the pointwise join of monotone functions. It remains only to show that $\bar{f}(t) = f(t)$ for μ a.e. $t \in T$.

⁷⁰It can be further shown that for all $t \in T$, $\bar{f}(t) = \vee \{a \in A: a \leq f(t') \text{ for all } t' \geq t \text{ such that } t' \in T \text{ is a } \Psi\text{-approachability point of } f\}$. But we will not need this result.

Suppose first that f is measurable. Let C denote the measurable (by Lemma A.8) set of Ψ -approachability points of f and let L_f denote the set of monotone $g : T \rightarrow A$ such that $g(t) \leq f(t)$ for μ a.e. $t \in T$. By Lemma A.9, $\mu(C) = 1$.

We claim that $f(t) \geq g(t)$ for every $t \in C$ and every $g \in L_f$. To see this, fix $g \in L_f$ and let D be a measurable set with μ measure 1 such that $g(t) \leq f(t)$ for every $t \in D$. Consider $t \in C$. Because t is a Ψ -approachability point of f , there are sequences $\{t_n\}$ and $\{t'_n\}$ in T such that $\lim_n f(t_n) = \lim_n f(t'_n) = f(t)$, and such that the intervals $[t_n, t]$ and $[t, t'_n]$ have positive μ measure for every n . Therefore, in particular, the set $D \cap [t, t'_n]$ has positive μ measure for every n . Consequently, for every n , we may choose $\tilde{t}_n \in D \cap [t, t'_n]$ and, therefore, $f(t'_n) \geq f(\tilde{t}_n) \geq g(\tilde{t}_n) \geq g(t)$ for all n . In particular, $f(t'_n) \geq g(t)$ for all n , so that $f(t) = \lim_n f(t'_n) \geq g(t)$, proving the claim.

Consequently, $f(t) \geq \bigvee_{g \in L_f} g(t)$ for every $t \in C$. Hence, because f itself is a member of L_f , $f(t) = \bigvee_{g \in L_f} g(t) = \tilde{f}(t)$ for every $t \in C$ and, therefore, for μ a.e. $t \in T$.

If f is not measurable, then by Lemma A.11, we can repeat the argument, replacing f with a measurable and monotone $\tilde{f} : T \rightarrow A$ that is μ almost everywhere equal to f , concluding that $\tilde{f}(t) = \bigvee_{g \in L_f} g(t)$ for μ a.e. $t \in T$. But $L_f = L_{\tilde{f}}$ then implies that for μ a.e. $t \in T$, $f(t) = \tilde{f}(t) = \bigvee_{g \in L_f} g(t) = \bigvee_{g \in L_f} g(t) = \tilde{f}(t)$. Q.E.D.

LEMMA A.15: Assume G.1, G.3, and G.4. Suppose that the join operator on A is continuous and that $\Phi : T \rightarrow [0, 1]$ is a monotone and measurable function such that $\mu(\Phi^{-1}(c)) = 0$ for every $c \in [0, 1]$. Define $h : [0, 1] \times \mathcal{M} \times \mathcal{M} \rightarrow \mathcal{M}$ by defining for every $t \in T$,

$$(A.5) \quad h(\tau, f, g)(t) = \begin{cases} f(t), & \text{if } \Phi(t) \leq |1 - 2\tau| \text{ and } \tau < 1/2, \\ g(t), & \text{if } \Phi(t) \leq |1 - 2\tau| \text{ and } \tau \geq 1/2, \\ f(t) \vee g(t), & \text{if } \Phi(t) > |1 - 2\tau|. \end{cases}$$

Then $h : [0, 1] \times \mathcal{M} \times \mathcal{M} \rightarrow \mathcal{M}$ is continuous.

PROOF: Suppose that $(\tau_k, f_k, g_k) \rightarrow (\tau, f, g) \in [0, 1] \times \mathcal{M} \times \mathcal{M}$. By Lemma A.12, there is a μ measure 1 subset, D , of T such that $f_k(t) \rightarrow f(t)$ and $g_k(t) \rightarrow g(t)$ for every $t \in D$. There are three cases: $\tau = 1/2$, $\tau > 1/2$, and $\tau < 1/2$.

Suppose that $\tau < 1/2$. For each $t \in D$ such that $\Phi(t) < |1 - 2\tau|$, we have $\Phi(t) < |1 - 2\tau_k|$ for all k large enough. Hence, $h(\tau_k, f_k, g_k)(t) = f_k(t)$ for all k large enough, and so $h(\tau_k, f_k, g_k)(t) = f_k(t) \rightarrow f(t) = h(\tau, f, g)(t)$. Similarly, for each $t \in D$ such that $\Phi(t) > |1 - 2\tau|$, $h(\tau_k, f_k, g_k)(t) = f_k(t) \vee g_k(t) \rightarrow f(t) \vee g(t) = h(\tau, f, g)(t)$, where the limit follows because \vee is continuous. Because $\mu(\{t \in T : \Phi(t) = |1 - 2\tau|\}) = 0$, we have, therefore, shown that if

$\tau < 1/2$, then $h(\tau_k, f_k, g_k)(t) \rightarrow h(\tau, f, g)(t)$ for μ a.e. $t \in T$ and so, by Lemma A.12, $h(\tau_k, f_k, g_k) \rightarrow h(\tau, f, g)$.

Because the case $\tau > 1/2$ is similar to $\tau < 1/2$, we consider only the remaining case in which $\tau = 1/2$. In this case, $|1 - 2\tau_k| \rightarrow 0$. Consequently, for any $t \in T$ such that $\Phi(t) > 0$, we have $h(\tau_k, f_k, g_k)(t) = f_k(t) \vee g_k(t)$ for k large enough and so $h(\tau_k, f_k, g_k)(t) = f_k(t) \vee g_k(t) \rightarrow f(t) \vee g(t) = h(1/2, f, g)(t)$. Hence, because $\mu(\{t \in T : \Phi(t) = 0\}) = 0$, we have shown that $h(\tau_k, f_k, g_k)(t) \rightarrow h(1/2, f, g)(t)$ for μ a.e. $t \in T$, and so again by Lemma A.12, $h(\tau_k, f_k, g_k) \rightarrow h(\tau, f, g)$. Q.E.D.

LEMMA A.16: *Under G.1–G.5, the metric space (\mathcal{M}, δ) is an absolute retract.*

PROOF: Define $h : [0, 1] \times \mathcal{M} \times \mathcal{M} \rightarrow \mathcal{M}$ by $h(\tau, s, s')(t) = \tau s(t) + (1 - \tau)s'(t)$ for all $t \in T$ if G.5(i) holds, and by (A.5) if G.5(ii) holds, where the monotone function $\Phi(\cdot)$ appearing in (A.5) is defined by (A.2). Note that h maps into \mathcal{M} in case G.5(i) holds because A is convex (which itself follows because the partial order on A is convex). We claim that, in each case, h is continuous. Indeed, if G.5(ii) holds, the continuity of h follows from Lemmas A.3 and A.15. If G.5(i) holds and the sequence $(\tau_n, s_n, s'_n) \in [0, 1] \times \mathcal{M} \times \mathcal{M}$ converges to (τ, s, s') , then by Lemma A.12, $s_n(t) \rightarrow s(t)$ and $s'_n(t) \rightarrow s'(t)$ for μ a.e. $t \in T$. Hence, because A is a convex subset of a linear topological space, $\tau_n s_n(t) + (1 - \tau_n)s'_n(t) \rightarrow \tau s(t) + (1 - \tau)s'(t)$ for μ a.e. $t \in T$. But then Lemma A.12 implies $\tau_n s_n + (1 - \tau_n)s'_n \rightarrow \tau s + (1 - \tau)s'$, as desired.

One consequence of the continuity of h is that for any $g \in \mathcal{M}$, $h(\cdot, \cdot, g)$ is a contraction for \mathcal{M} so that (\mathcal{M}, δ) is contractible. Hence, by Borsuk (1966, IV, (9.1)) and Dugundji (1965), it suffices to show that for each $f' \in \mathcal{M}$, every neighborhood U of f' contains a neighborhood V of f' such that the sets V^n , $n \geq 1$, defined inductively by $V^1 = h([0, 1], V, V)$, $V^{n+1} = h([0, 1], V, V^n)$, are all contained in U .

We shall establish this by way of contradiction. Specifically, let us suppose to the contrary that for some neighborhood U of $f' \in \mathcal{M}$, there is no open set V containing f' and contained in U such that all the V^n as defined above are contained in U . In particular, for each $k = 1, 2, \dots$, taking V to be $B_{1/k}(f')$, the $1/k$ ball around f' , there exists n_k such that some $g_k \in V^{n_k}$ is not in U . We derive a contradiction separately for each of the two cases, G.5(i) and G.5(ii).

Case I. Suppose G.5(i) holds. For each n , $V^{n+1} \subset \text{co } V$, so that for every $k = 1, 2, \dots$, $g_k \in V^{n_k} \subset \text{co } B_{1/k}(f')$. Hence, for each k there exist $f_1^k, \dots, f_{n_k}^k$ in $B_{1/k}(f')$ and nonnegative weights $\lambda_1^k, \dots, \lambda_{n_k}^k$ summing to one such that $g_k = \sum_{j=1}^{n_k} \lambda_j^k f_j^k \notin U$. Hence, $g_k(t) = \sum_{j=1}^{n_k} \lambda_j^k f_j^k(t)$ for μ a.e. $t \in T$ and so for all t in some measurable set E having μ measure 1. Moreover, the sequence $f_1^1, \dots, f_{n_1}^1, f_1^2, \dots, f_{n_2}^2, \dots$ converges to f' . Consequently, by Lemma A.12 the sequence $f_1^1(t), \dots, f_{n_1}^1(t), f_1^2(t), \dots, f_{n_2}^2(t), \dots$ converges to $f'(t)$ for μ a.e. $t \in T$ and so for all t in some measurable set D having μ measure 1. But then for each $t \in D \cap E$ and every convex neighborhood W_i of $f'(t)$, each

of $f_1^k(t), \dots, f_{n_k}^k(t)$ is in W_t for all k large enough and, therefore, $g_k(t) = \sum_{j=1}^{n_k} \lambda_j^k f_j^k(t)$ is in W_t for k large enough as well. But this implies, by the local convexity of A , that $g_k(t) \rightarrow f'(t)$ for every $t \in D \cap E$ and hence for μ a.e. $t \in T$. Lemma A.12 then implies that $g_k \rightarrow f'$, contradicting that no g_k is in U .

Case II. Suppose G.5(ii) holds. As a matter of notation, for $f, g \in \mathcal{M}$, write $f \leq g$ if $f(t) \leq g(t)$ for μ a.e. $t \in T$. Also, for any sequence of monotone functions f_1, f_2, \dots , in \mathcal{M} , denote by $f_1 \vee f_2 \vee \dots$ the monotone function taking the value $\lim_n [f_1(t) \vee f_2(t) \vee \dots \vee f_n(t)]$ for each t in T . This is well defined by Lemma A.5.

If $g \in V^1$, then $g = h(\tau, f_0, f_1)$ for some $\tau \in [0, 1]$ and some $f_0, f_1 \in V$. Hence, by the definition of h , we have $g \leq f_0 \vee f_1$ and either $f_0 \leq g$ or $f_1 \leq g$. We may choose the indices so that $f_0 \leq g \leq f_0 \vee f_1$. Inductively, it can similarly be seen that if $g \in V^n$, then there exist $f_0, f_1, \dots, f_n \in V$ such that

$$(A.6) \quad f_0 \leq g \leq f_0 \vee \dots \vee f_n.$$

Hence, for each $k = 1, 2, \dots$, $g_k \in V^{n_k}$ and (A.6) imply that there exist $f_0^k, \dots, f_{n_k}^k \in V = B_{1/k}(f')$ such that

$$(A.7) \quad f_0^k \leq g_k \leq f_0^k \vee \dots \vee f_{n_k}^k.$$

Consider the sequence $f_0^1, \dots, f_{n_1}^1, f_0^2, \dots, f_{n_2}^2, \dots$. Because f_j^k is in $B_{1/k}(f')$, this sequence converges to f' . Let us reindex this sequence as f_1, f_2, \dots . Hence, $f_j \rightarrow f'$.

Because for every n , the set $\{f_n, f_{n+1}, \dots\}$ contains the set $\{f_0^k, \dots, f_{n_k}^k\}$ whenever k is large enough, we have

$$f_0^k \vee \dots \vee f_{n_k}^k \leq \bigvee_{j \geq n} f_j$$

for every n and all large enough k . Combined with (A.7), this implies that

$$(A.8) \quad f_0^k \leq g_k \leq \bigvee_{j \geq n} f_j$$

for every n and all large enough k .

Now $f_0^k \rightarrow f'$ as $k \rightarrow \infty$. Hence, by Lemma A.12, $f_0^k(t) \rightarrow f'(t)$ for μ a.e. $t \in T$. Consequently, if for μ a.e. $t \in T$, $\bigvee_{j \geq n} f_j(t) \rightarrow f'(t)$ as $n \rightarrow \infty$, then (A.8) and Lemma A.4 imply that $g_k(t) \rightarrow f'(t)$ for μ a.e. $t \in T$. But then Lemma A.12 implies that $g_k \rightarrow f'$, once again contradicting that no g_k is in U .

It, therefore, remains only to establish that for μ a.e. $t \in T$, $\bigvee_{j \geq n} f_j(t) \rightarrow f'(t)$ as $n \rightarrow \infty$. But by Lemma A.18, because A is locally complete, this will follow if $f_j(t) \rightarrow_j f'(t)$ for μ a.e. t , which follows from Lemma A.12 because $f_j \rightarrow f'$. Q.E.D.

A.4. Locally Complete Metric Semilattices

We denote the partially ordered set by A in this section because the results to follow, while applicable to any partially ordered set, are applied in the main text to the players' action sets.

LEMMA A.17: *If A is an upper-bound-convex Euclidean semilattice and compact in the Euclidean metric, then A is a Euclidean metric semilattice, i.e., \vee is continuous.*

PROOF: Suppose that $a_n \rightarrow a$, $b_n \rightarrow b$, $a \vee b = c$, and $a_n \vee b_n \rightarrow d$, where all of these points are in A . We must show that $c = d$. Because $a_n \leq a_n \vee b_n$, taking limits implies $a \leq d$. Similarly, $b \leq d$, so that $c = a \vee b \leq d$. Thus, it remains only to show that $c \geq d$.

Let $\bar{a} = \vee A$ denote the largest element of A , which is well defined by Lemma A.6. By the upper-bound-convexity of A , $\varepsilon \bar{a} + (1 - \varepsilon)c \in A$ for every $\varepsilon \in [0, 1]$. Because the coordinatewise partial order is closed, it suffices to show that $\varepsilon \bar{a} + (1 - \varepsilon)c \geq d$ for every $\varepsilon > 0$ sufficiently small. So fix $\varepsilon \in (0, 1)$ and consider the k th coordinate, c_k , of c . If for some n , $a_{kn} > c_k$, then because $\bar{a}_k \geq a_{kn}$, we have $\bar{a}_k > c_k$ and, therefore, $\varepsilon \bar{a}_k + (1 - \varepsilon)c_k > c_k$. Consequently, because $a_{kn} \rightarrow_n a_k \leq c_k$, we have $\varepsilon \bar{a}_k + (1 - \varepsilon)c_k > a_{kn}$ for all n sufficiently large. On the other hand, suppose that $a_{kn} \leq c_k$ for all n . Then because $\bar{a}_k \geq c_k$, we have $\varepsilon \bar{a}_k + (1 - \varepsilon)c_k \geq a_{kn}$ for all n . So in either case, $\varepsilon \bar{a}_k + (1 - \varepsilon)c_k \geq a_{kn}$ for all n sufficiently large. Therefore, because k is arbitrary, $\varepsilon \bar{a} + (1 - \varepsilon)c \geq a_n$ for all n sufficiently large. Similarly, $\varepsilon \bar{a} + (1 - \varepsilon)c \geq b_n$ for all n sufficiently large. Therefore, because $\varepsilon \bar{a} + (1 - \varepsilon)c \in A$, $\varepsilon \bar{a} + (1 - \varepsilon)c \geq a_n \vee b_n$ for all n sufficiently large. Taking limits in n gives $\varepsilon \bar{a} + (1 - \varepsilon)c \geq d$. Q.E.D.

LEMMA A.18: *If G.4 holds, then A is locally complete if and only if for every $a \in A$ and every sequence a_n converging to a , $\lim_n(\bigvee_{k \geq n} a_k) = a$.*

PROOF: We first demonstrate the "only if" direction. Suppose that A is locally complete, that U is a neighborhood of $a \in A$, and that $a_n \rightarrow a$. By local completeness, there is a neighborhood W of a contained in U such that every subset of W has a least upper bound in U . In particular, because for n large enough, $\{a_n, a_{n+1}, \dots\}$ is a subset of W , the least upper bound of $\{a_n, a_{n+1}, \dots\}$, namely $\bigvee_{k \geq n} a_k$, is in U for n large enough. Since U was arbitrary, this implies $\lim_n(\bigvee_{k \geq n} a_k) = a$.

We now turn to the "if" direction. Fix any $a \in A$ and let $B_{1/n}(a)$ denote the open ball around a with radius $1/n$. For each n , $\bigvee B_{1/n}(a)$ is well defined by Lemma A.6. Moreover, because $\bigvee B_{1/n}(a)$ is nonincreasing in n , $\lim_n \bigvee B_{1/n}(a)$ exists by Lemma A.5. We first argue that $\lim_n \bigvee B_{1/n}(a) = a$. For each n , construct as in the proof of Lemma A.6 a sequence $\{a_{n,m}\}$ of points in $B_{1/n}(a)$ such that $\lim_m(a_{n,1} \vee \dots \vee a_{n,m}) = \bigvee B_{1/n}(a)$. We can, therefore, choose m_n sufficiently large so that the distance between $a_{n,1} \vee \dots \vee a_{n,m_n}$ and $\bigvee B_{1/n}(a)$ is less

than $1/n$. Consider now the sequence $\{a_{1,1}, \dots, a_{1,m_1}, a_{2,1}, \dots, a_{2,m_2}, a_{3,1}, \dots, a_{3,m_3}, \dots\}$. Because $a_{n,m}$ is in $B_{1/n}(a)$, this sequence converges to a . Consequently, by hypothesis,

$$\lim_n (a_{n,1} \vee \dots \vee a_{n,m_n} \vee a_{(n+1),1} \vee \dots \vee a_{(n+1),m_{(n+1)}} \vee \dots) = a.$$

But because every $a_{k,j}$ in the join in parentheses on the left-hand side above (denote this join by b_n) is in $B_{1/n}(a)$, we have

$$a_{n,1} \vee \dots \vee a_{n,m_n} \leq b_n \leq \vee B_{1/n}(a).$$

Therefore, because for every n the distance between $a_{n,1} \vee \dots \vee a_{n,m_n}$ and $\vee B_{1/n}(a)$ is less than $1/n$, Lemma A.4 implies that $\lim_n \vee B_{1/n}(a) = \lim_n b_n$. But since $\lim_n b_n = a$, we have $\lim_n \vee B_{1/n}(a) = a$. Next, for each n , let S_n be an arbitrary nonempty subset of $B_{1/n}(a)$ and choose any $s_n \in S_n$. Then $s_n \leq \vee S_n \leq \vee B_{1/n}(a)$. Because $s_n \in B_{1/n}(a)$, Lemma A.4 implies that $\lim_n \vee S_n = a$. Consequently, for every neighborhood U of a , there exists n large enough such that $\vee S$ (well defined by Lemma A.6) is in U for every subset S of $B_{1/n}(a)$. Since A was arbitrary, A is locally complete. Q.E.D.

LEMMA A.19: *Every compact Euclidean metric semilattice is locally complete.*

PROOF: Suppose that $a_n \rightarrow a$ with every a_n and a in the semilattice, which we assume to be a subset of \mathbb{R}^K . By Lemma A.18, it suffices to show that $\lim_n (\bigvee_{k \geq n} a_k) = a$. By Lemma A.5, $\lim_n (\bigvee_{k \geq n} a_k)$ exists and is equal to $\lim_n \lim_m (a_n \vee \dots \vee a_m)$ since $a_n \vee \dots \vee a_m$ is nondecreasing in m and $\lim_m (a_n \vee \dots \vee a_m)$ is nonincreasing in n . For each dimension $k = 1, \dots, K$, let $a_{n,m}^k$ denote the first among a_n, a_{n+1}, \dots, a_m with the largest k th coordinate. Hence, $a_n \vee \dots \vee a_m = a_{n,m}^1 \vee \dots \vee a_{n,m}^K$, where the right-hand side consists of K terms. Because $a_n \rightarrow a$, $\lim_m a_{n,m}^k$ exists for each k and n , and $\lim_n \lim_m a_{n,m}^k = a$ for each k . Consequently, $\lim_n \lim_m (a_n \vee \dots \vee a_m) = \lim_n \lim_m (a_{n,m}^1 \vee \dots \vee a_{n,m}^K) = (\lim_n \lim_m a_{n,m}^1) \vee \dots \vee (\lim_n \lim_m a_{n,m}^K) = a \vee \dots \vee a = a$, as desired. Q.E.D.

LEMMA A.20: *If G.4 holds and for all $a \in A$, every neighborhood of a contains a' such that $b' \leq a'$ for all b' close enough to a , then A is locally complete.*

PROOF: Suppose that $a_n \rightarrow a$. By Lemma A.18, it suffices to show that $\lim_n (\bigvee_{k \geq n} a_k) = a$. For every n and m , $a_m \leq a_m \vee a_{m+1} \vee \dots \vee a_{m+n}$, and so taking the limit first as $n \rightarrow \infty$ and then as $m \rightarrow \infty$ gives $a \leq \lim_m \bigvee_{k \geq m} a_k$, where the limit in n exists by Lemma A.5 because the sequence is monotone. Hence, it suffices to show that $\lim_m \bigvee_{k \geq m} a_k \leq a$.

Let U be a neighborhood of a and let a' be chosen as in the statement of the lemma. Then because $a_m \rightarrow a$, $a_m \leq a'$ for all m large enough. Consequently, for m large enough and for all n , $a_m \vee a_{m+1} \vee \dots \vee a_{m+n} \leq a'$. Tak-

ing the limit first in n and then in m yields $\lim_m \bigvee_{k \geq m} a_k \leq a'$. Because for every neighborhood U of a this holds for some a' in \bar{U} , $\lim_m \bigvee_{k \geq m} a_k \leq a$, as desired. Q.E.D.

A.5. Assumption G.3

Say that two points in a partially ordered metric space are *strictly ordered* if they are contained in disjoint open sets and every member of one set is greater or equal to every member of the other. The following lemma provides a sufficient condition for G.3 to hold when T happens to be a separable metric space.

LEMMA A.21: *Suppose that $(T, \mathcal{T}, \mu, \geq)$ is a partially ordered probability space, that T is a separable metric space, and that \mathcal{T} contains the open sets. Then G.3 holds if every atomless set having positive μ measure contains two strictly ordered points.*

PROOF: Let T^0 be the union of a countable dense subset of T and the countable set of atoms of μ , and suppose that $D \in \mathcal{T}$ has positive μ measure. We must show that $t_1 \geq t_0 \geq t_2$ for some $t_1, t_2 \in D$ and some $t_0 \in T^0$.

If D contains an atom, t_0 , of μ , then we may set $t_1 = t_2 = t_0$ and we are done. Hence, we may assume that D is atomless.

Without loss, we may assume that $\mu(D \cap U) > 0$ for every open set U whose intersection with D is nonempty.⁷¹ Because $\mu(D) > 0$, there exist $t'_1, t_2 \in D$ and open sets U'_1 containing t'_1 and U_2 containing t_2 such that every member of U'_1 is greater than or equal to every member of U_2 , which we write as $U'_1 \geq U_2$.

Because $D \cap U'_1$ is nonempty (e.g., it contains t'_1), $\mu(D \cap U'_1) > 0$. Consequently, there exist $t_1, t''_1 \in D \cap U'_1$ and open sets U_1 containing t_1 and U''_1 containing t''_1 such that $U_1 \geq U''_1$. Hence, $U_1 \cap U'_1 \geq U''_1 \cap U'_1 \geq U_2$. Therefore, because the open set $U''_1 \cap U'_1$ is nonempty (e.g., it contains t''_1), it contains some t_0 in the dense set T^0 . Hence, $t_1 \geq t_0 \geq t_2$, because $t_1 \in U_1 \cap U'_1$ and $t_2 \in U_2$. Noting that t_1 and t_2 are members of D completes the proof. Q.E.D.

A.6. Sufficient Conditions for G.1–G.5

PROOF OF PROPOSITION 3.1: Suppose that each player i 's type space and marginal distribution satisfy the hypotheses of the lemma. Then G.1 and G.2 are immediate. To see that G.3 holds, for each i and k , let T_{ik}^0 be a countable dense subset of T_{ik} . Consequently, if $\mu_i(B) > 0$, then by Fubini's theorem, there exist k and $t_i \in (\underline{\tau}_{ik}, \bar{\tau}_{ik})^{n_{ik}}$ such that $B \cap L_{ik}(t_i)$ contains a continuum of members, any two of which define an interval of types containing a member of

⁷¹Otherwise replace D with $D \cap V^c$, where V is the largest open set whose intersection with D has μ -measure 0. To see that V is well defined, let $\{U_i\}$ be a countable base of open sets. Then V is the union of all the U_i satisfying $\mu(U_i \cap D) = 0$.

T_{ik}^0 , where $L_{ik}(t_i)$ is the line joining the lowest point in T_{ik} , i.e., $(\tau_{ik}, \dots, \tau_{ik})$, with t_i . Hence, G.3 holds by setting $T_i^0 = T_{i1}^0 \cup T_{i2}^0 \cup \dots$.

Suppose next that each player's action space satisfies the hypotheses of the lemma. If the coordinatewise maximum of any two actions is a feasible action, the join of any two points is their coordinatewise maximum and, hence, the join operator is continuous in the Euclidean metric. Each player's action space is then a compact Euclidean metric semilattice and, by Lemma A.19, locally complete. Conditions G.4 and G.5 are therefore satisfied. *Q.E.D.*

A.7. Proofs From Section 5

PROOF OF COROLLARY 5.2: Consider the uniform-price auction but where unit bids can be any nonnegative real number. Because marginal values are between 0 and 1, without loss we may restrict attention to unit bids in $[0, 1]$. The resulting game is discontinuous. Remark 3.1 in Reny (1999) establishes that if this game is better-reply secure, then the limit of a convergent sequence of pure-strategy ε equilibria, as ε tends to zero, is a pure-strategy equilibrium. Hence, in view of Lemma A.13, it suffices to show that the auction game is better-reply secure (when players employ monotone pure strategies) and that it possesses, for every $\varepsilon > 0$, an ε equilibrium in monotone pure strategies.

An argument analogous to that given in the first paragraph on page 1046 in Reny (1999) shows that, regardless of the tie-break rule, the uniform-price auction game with unit bid space $[0, 1]$ is better-reply secure when bidders employ weakly undominated monotone pure strategies and that ties occur with probability 0 in every such equilibrium. Fix $\varepsilon > 0$. By Proposition 5.1, for each $k = 1, 2, \dots$, there is a nontrivial monotone pure-strategy equilibrium, b^k , of the uniform-price auction when unit bids are restricted to the finite set $\{0, 1/k, 2/k, \dots, k/k\}$. It suffices to show that for all k sufficiently large, b^k is an ε equilibrium of the game in which unit bids can be chosen from $[0, 1]$.

Fix player i . Let D denote the set of nonincreasing bid vectors in $[0, 1]^m$. It suffices to show that for all k sufficiently large and all monotone pure strategies $b: T_i \rightarrow D$ for player i , there is a monotone pure strategy $b': T_i \rightarrow D \cap \{0, 1/k, 2/k, \dots, k/k\}^m$ that yields player i utility within ε of $b(\cdot)$ uniformly in the others' strategies. By weak dominance, it suffices to consider monotone pure strategies $b: T_i \rightarrow D$ for player i such that each unit bid, $b_j(t_i)$, is in $[0, t_{ij}]$ for every $t_i = (t_{i1}, \dots, t_{im}) \in T_i$. So let $b(\cdot)$ be such a monotone pure strategy and let $b': T_i \rightarrow D \cap \{0, 1/k, 2/k, \dots, k/k\}^m$ be such that for every $t_i \in T_i$, $b'_j(t_i)$ is the smallest member of $\{0, 1/k, \dots, k/k\}$ greater than or equal to $b_j(t_i)$. Hence, $b'(\cdot)$ is monotone and $b'_1(t_i) \geq \dots \geq b'_m(t_i)$ for every $t_i \in T_i$, so that $b'(\cdot)$ is a feasible monotone pure strategy. If bidder i employs $b'(\cdot)$ instead of $b(\cdot)$, then regardless of his type, and for any strategies the others might employ and for each $j = 1, \dots, m$, bidder i will win a j th unit whenever $b(\cdot)$ would have won a j th unit although the price might be higher because his bid vector is higher, and he may win a j th unit when $b(\cdot)$ would not have. The increase in

the price caused by the at most $1/k$ increase in each of his unit bids can be no greater than $1/k$, and because $b_j(t_i) \leq t_{ij}$ for every $t_i \in T_i$, the ex post surplus lost on each additional unit won from employing $b'(\cdot)$ instead of $b(\cdot)$ can be no greater than $1/k$. Hence, the total ex post loss in surplus as a result of the strategy change can be no greater than $2m/k$, which can be made arbitrarily small for k sufficiently large, regardless of the others' strategies. Hence, i 's expected utility loss from employing $b'(\cdot)$ instead of $b(\cdot)$ is, for k large enough, less than ε , and this holds uniformly in the others' strategies. *Q.E.D.*

REMARK 9: An alternative proof method is to consider the limit of a sequence of finite-grid monotone pure-strategy equilibria (which exist by Proposition 5.1) as the grid becomes increasingly fine. Then techniques as in Jackson, Simon, Swinkels, and Zame (2002) can be used to show that any limit strategies (which, by Lemma A.13, exist along a subsequence, and are monotone and pure) form an equilibrium with an endogenous tie-break rule. Theorem 6 of Jackson and Swinkels (2005) then implies that ties occur with probability 0 and that the same strategies constitute an equilibrium for any tie-break rule.

The proof of Corollary 5.5 is analogous to the proof of Corollary 5.2 above.

PROOF OF LEMMA 5.3: Fix monotone pure strategies for all players but i . For the remainder of this proof, we omit most subscripts i to keep the notation manageable. Let $v(b, t)$ denote bidder i 's expected payoff from employing the bid vector $b = (b_1, \dots, b_m)$ when his type vector is $t = (t_1, \dots, t_m)$. Then letting $P_k(b_k)$ denote the probability that bidder i wins at least k units—which, owing to our tie-breaking rule, depends only on his k th unit bid b_k —we have, where 1_k is an m vector of k ones followed by $m - k$ zeros,

$$\begin{aligned} v(b, t) &= u(0) + \sum_{k=1}^m P_k(b_k) (u((t - b) \cdot 1_k) - u((t - b) \cdot 1_{k-1})) \\ &= \frac{1}{r} \sum_{k=1}^m e^{r(b_1 + \dots + b_{k-1})} P_k(b_k) (1 - e^{-r(t_k - b_k)}) e^{-r(t_1 + \dots + t_{k-1})}, \end{aligned}$$

where $u(x) = \frac{1 - e^{-rx}}{r}$ is bidder i 's utility function with constant absolute risk aversion parameter $r \geq 0$, where it is understood that $u(x) = x$ when $r = 0$. Note that the dependence of r on i has been suppressed.

From now on, we proceed as if $r > 0$, because all of the formulae employed here have well defined limits as r tends to 0 that correspond to the risk neutral case $u(x) = x$.

Letting $w_k(b_k, t) = \frac{1}{r} P_k(b_k) (1 - e^{-r(t_k - b_k)}) e^{-r(t_1 + \dots + t_{k-1})}$, we can write

$$v(b, t) = \sum_{k=1}^m e^{r(b_1 + \dots + b_{k-1})} w_k(b_k, t).$$

As shown in (5.2) (and setting $\bar{p} = \underline{p} = 0$ there), for each $k = 2, \dots, m$,

$$(A.9) \quad u(t_1 + \dots + t_k) - u(t_1 + \dots + t_{k-1}) = \frac{1}{r}(1 - e^{-rt_k})e^{-r(t_1 + \dots + t_{k-1})},$$

is nondecreasing in t according to the partial order \geq_i defined in (5.1). Henceforth, we employ the partial order \geq_i on i 's type space. We next demonstrate the following facts.

- (i) $w_k(\underline{b}_k, t)$ is nondecreasing in t .
 - (ii) $w_k(\bar{b}_k, t) - w_k(\underline{b}_k, t)$ is nondecreasing in t for all $\bar{b}_k \geq \underline{b}_k$.
- To see (i), write

$$\begin{aligned} w_k(\underline{b}_k, t) &= \frac{1}{r}P_k(\underline{b}_k)(1 - e^{-r(t_k - \underline{b}_k)})e^{-r(t_1 + \dots + t_{k-1})} \\ &= \frac{1}{r}P_k(\underline{b}_k)(1 - e^{-rt_k})e^{-r(t_1 + \dots + t_{k-1})} \\ &\quad + \frac{1}{r}P_k(\underline{b}_k)(e^{r\underline{b}_k} - 1)(-e^{-r(t_1 + \dots + t_k)}). \end{aligned}$$

The first term in the sum is nondecreasing in t according to \geq_i by (A.9) and the second term, being nondecreasing in the coordinatewise partial order, is a fortiori nondecreasing in t according to \geq_i .

Turning to (ii), if $P_k(\underline{b}_k) = 0$, then $w_k(\underline{b}_k, t) = 0$ and (ii) follows from (i). So assume $P_k(\underline{b}_k) > 0$. Then

$$\begin{aligned} w_k(\bar{b}_k, t) - w_k(\underline{b}_k, t) &= \frac{1}{r}P_k(\bar{b}_k)(1 - e^{-r(t_k - \bar{b}_k)})e^{-r(t_1 + \dots + t_{k-1})} \\ &\quad - \frac{1}{r}P_k(\underline{b}_k)(1 - e^{-r(t_k - \underline{b}_k)})e^{-r(t_1 + \dots + t_{k-1})} \\ &= \left(\frac{P_k(\bar{b}_k)}{P_k(\underline{b}_k)} - 1\right)w_k(\underline{b}_k, t) \\ &\quad + \frac{1}{r}P_k(\bar{b}_k)(e^{r\bar{b}_k} - e^{r\underline{b}_k})(-e^{-r(t_1 + \dots + t_k)}). \end{aligned}$$

The first term in the sum is nondecreasing in t according to \geq_i by (i) and the second term, being nondecreasing in the coordinatewise partial order, is a fortiori nondecreasing in t according to \geq_i . This proves (ii).

Suppose now that the vector of bids b is optimal for bidder i when his type vector is t and that b' is optimal when his type is $t' \geq_i t$. We must argue that $b \vee b'$ is optimal when his type is t' . If $b_k \leq b'_k$ for all k , then $b \vee b' = b'$ and we are done. Hence, we may assume that there is a maximal set of consecutive

coordinates of b that are strictly greater than those of b' . That is, there exist coordinates j and l with $j \leq l$ such that $b_k > b'_k$ for $k = j, \dots, l$, $b_{j-1} \leq b'_{j-1}$, and $b_{l+1} \leq b'_{l+1}$, where the first of the last two inequalities is ignored if $j = 1$ and the second is ignored if $l = m$.

Let \hat{b} be the bid vector obtained from b by replacing its coordinates j through l with the coordinates j through l of b' . Because b is optimal at t , and \hat{b} is nonincreasing and therefore feasible, $v(b, t) - v(\hat{b}, t)$ is nonnegative. Dividing $v(b, t) - v(\hat{b}, t)$ by $e^{r(b_1 + \dots + b_j)}$ implies

$$\begin{aligned} 0 \leq & w_j(b_j, t) - w_j(b'_j, t) + \sum_{k=j+1}^l e^{r(b_j + \dots + b_{k-1})} (w_k(b_k, t) - w_k(b'_k, t)) \\ & + (e^{r(b_j + \dots + b_l)} - e^{r(b'_j + \dots + b'_l)}) \\ & \times [w_{l+1}(b_{l+1}, t) + e^{rb_{l+1}} w_{l+2}(b_{l+2}, t) + \dots \\ & + e^{r(b_{l+1} + \dots + b_{m-1})} w_m(b_m, t)]. \end{aligned}$$

Consequently, for $t' \geq t$, (i) and (ii) imply

$$\begin{aligned} \text{(A.10)} \quad 0 \leq & w_j(b_j, t') - w_j(b'_j, t') + \sum_{k=j+1}^l e^{r(b_j + \dots + b_{k-1})} (w_k(b_k, t') - w_k(b'_k, t')) \\ & + (e^{r(b_j + \dots + b_l)} - e^{r(b'_j + \dots + b'_l)}) \\ & \times [w_{l+1}(b_{l+1}, t') + e^{rb_{l+1}} w_{l+2}(b_{l+2}, t') + \dots \\ & + e^{r(b_{l+1} + \dots + b_{m-1})} w_m(b_m, t')]. \end{aligned}$$

Focusing on the second term in square brackets in (A.10), we claim that

$$\begin{aligned} \text{(A.11)} \quad & w_{l+1}(b_{l+1}, t') + e^{rb_{l+1}} w_{l+2}(b_{l+2}, t') + \dots + e^{r(b_{l+1} + \dots + b_{m-1})} w_m(b_m, t') \\ & \leq w_{l+1}(b'_{l+1}, t') + e^{rb'_{l+1}} w_{l+2}(b'_{l+2}, t') + \dots + e^{r(b'_{l+1} + \dots + b'_{m-1})} w_m(b'_m, t'). \end{aligned}$$

To see this, note that because $b_{l+1} \leq b'_{l+1}$, the bid vector b'' obtained from b' by replacing its coordinates $l+1$ through m with the coordinates $l+1$ through m of b is a feasible (i.e., nonincreasing) bid vector. Consequently, because b' is optimal at t' , we must have $0 \leq v(b', t') - v(b'', t')$. But this difference in utilities is precisely the difference between the right-hand and left-hand sides of (A.11) multiplied by $e^{r(b_1 + \dots + b_l)}$, thereby establishing (A.11).

Thus, we may conclude, after making use of (A.11) in (A.10), that

$$0 \leq w_j(b_j, t') - w_j(b'_j, t') + \sum_{k=j+1}^l e^{r(b_j + \dots + b_{k-1})} (w_k(b_k, t') - w_k(b'_k, t'))$$

$$\begin{aligned}
 &+ (e^{r(b_j+\dots+b_l)} - e^{r(b'_j+\dots+b'_l)}) \\
 &\times [w_{l+1}(b'_{l+1}, t') + e^{rb'_{l+1}}w_{l+2}(b'_{l+2}, t') + \dots \\
 &+ e^{r(b'_{l+1}+\dots+b'_{m-1})}w_m(b'_m, t')] \\
 &= \frac{v(\tilde{b}, t') - v(b', t')}{e^{r(b'_1+\dots+b'_{j-1})}},
 \end{aligned}$$

where \tilde{b} is the nonincreasing and therefore feasible bid vector obtained from b' by replacing its coordinates j through l with the coordinates j through l of b . Hence, \tilde{b} is optimal at t' because $v(\tilde{b}, t') \geq v(b', t')$ and b' is optimal at t' .

Thus, we have shown that whenever j, \dots, l is a maximal set of consecutive coordinates such that $b_k > b'_k$ for all $k = j, \dots, l$, replacing in b' the unit bids b'_j, \dots, b'_l with the coordinate-by-coordinate larger unit bids b_j, \dots, b_l results in a bid vector that is optimal at t' . Applying this result finitely often leads to the conclusion that $b \vee b'$ is optimal at t' , as desired. *Q.E.D.*

LEMMA A.22: *Consider the price competition game from Section 5.3. Under the partial orders on types \geq_i defined there for each firm i , each firm possesses a monotone pure-strategy best reply when the other firms employ monotone pure strategies.*

PROOF: Suppose that all firms $j \neq i$ employ monotone pure strategies according to \geq_j defined in Section 5.3. Therefore, in particular, $p_j(c_j, x_j)$ is non-decreasing in c_j for each x_j , and (5.6) applies. For the remainder of this proof, we omit most subscripts i to keep the notation manageable.

Because firm i 's interim payoff function is continuous in his price for each of his types, and because his action space, $[0, 1]$, is totally ordered and compact, firm i possesses a largest best reply, $\hat{p}(c, x)$, for each of his types $(c, x) \in [0, 1]^2$. We will show that $\hat{p}(\cdot)$ is monotone according to \geq_i .

Let $\bar{t} = (\bar{c}, \bar{x})$ and $\underline{t} = (\underline{c}, \underline{x})$ in $[0, 1]^2$ be two types of firm i , and suppose that $\bar{t} \geq_i \underline{t}$. Hence, $\bar{c} \geq \underline{c}$ and $\bar{x} - \underline{x} = \beta(\bar{c} - \underline{c})$ for some $\beta \in [0, \alpha_i]$. Let $\bar{p} = \hat{p}(\bar{c}, \bar{x})$, $\underline{p} = \hat{p}(\underline{c}, \underline{x})$, and $t^\lambda = (1 - \lambda)\underline{t} + \lambda\bar{t}$ for $\lambda \in [0, 1]$. We wish to show that $\bar{p} \geq \underline{p}$.

By the fundamental theorem of calculus,

$$v_i(\underline{p}, t^\lambda) - v_i(\bar{p}, t^\lambda) = \int_{\bar{p}}^{\underline{p}} \frac{\partial v_i(p, t^\lambda)}{\partial p} dp,$$

so that

$$\frac{\partial [v_i(\underline{p}, t^\lambda) - v_i(\bar{p}, t^\lambda)]}{\partial \lambda}$$

$$\begin{aligned}
&= \int_{p'}^{\underline{p}} \frac{\partial^2 v_i(p, t^\lambda)}{\partial \lambda \partial p} dp \\
&= \int_{p'}^{\underline{p}} \left[\frac{\partial^2 v_i(p, t^\lambda)}{\partial c \partial p} (\bar{c} - \underline{c}) + \frac{\partial^2 v_i(p, t^\lambda)}{\partial x \partial p} (\bar{x} - \underline{x}) \right] dp \\
&= (\bar{c} - \underline{c}) \int_{p'}^{\underline{p}} \left[\frac{\partial^2 v_i(p, t^\lambda)}{\partial c \partial p} + \beta \frac{\partial^2 v_i(p, t^\lambda)}{\partial x \partial p} \right] dp \\
&\geq 0,
\end{aligned}$$

where the inequality follows by (5.6) if $\underline{p} \geq p' \geq \bar{c}$. Therefore, $v_i(\underline{p}, \bar{t}) - v_i(p', \bar{t}) \geq v_i(\underline{p}, \underline{t}) - v_i(p', \underline{t}) \geq 0$, where the first inequality follows because $t^0 = \underline{t}$, $t^1 = \bar{t}$, and the second follows because \underline{p} is a best reply at \underline{t} . Therefore, we have shown the following: If $\underline{p} \geq \bar{c}$, then

$$v_i(\underline{p}, \bar{t}) - v_i(p', \bar{t}) \geq 0 \quad \text{for all } p' \in [\bar{c}, \underline{p}].$$

Hence, if $\underline{p} \geq \bar{c}$, then $\hat{p}(\bar{t}) = \bar{p} \geq \underline{p} = \hat{p}(\underline{t})$ because $\hat{p}(\bar{t})$ is the largest best reply at \bar{t} and because no best reply at $\bar{t} = (\bar{c}, \bar{x})$ is below \bar{c} . On the other hand, if $\underline{p} < \bar{c}$, then $\bar{p} = \hat{p}(\bar{t}) \geq \bar{c} > \underline{p} = \hat{p}(\underline{t})$, where the first inequality again follows because no best reply at \bar{t} is below \bar{c} . We conclude that $\bar{p} \geq \underline{p}$, as desired. *Q.E.D.*

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