Self-Fulfilling Debt Crises: A Quantitative Analysis*

Luigi Bocola

Northwestern University and NBER

Alessandro Dovis

University of Pennsylvania and NBER

Abstract

This paper uses the information contained in the joint dynamics of government's debt maturity choices and interest rate spreads to quantify the importance of self-fulfilling expectations in sovereign bond markets. We consider a model of sovereign borrowing featuring endogenous debt maturity, risk averse lenders and self-fulfilling rollover crises á la Cole and Kehoe (2000). In this environment, interest rate spreads are driven by economic fundamentals and by expectations of future self-fulfilling defaults. These two sources of default risk have contrasting implications for the debt maturity choices of the government. Therefore, they can be indirectly inferred by tracking the evolution of the maturity structure of debt during a crisis. We fit the model to the Italian debt crisis of 2008-2012, finding that 12% of the spreads over this episode were due to rollover risk. Our results have implications for the effects of the liquidity provisions established by the European Central Bank during the summer of 2012.

Keywords: Sovereign Debt Crises, Rollover Risk, Maturity Choices, Risk Premia.

JEL codes: F34, E44, G12, G15

*First draft: February 12, 2015. This draft: September 27, 2016. We thank Mark Aguiar, Pooyan Ahmadi, Manuel Amador, Cristina Arellano, Juliane Begenau, David Berger, Anmol Bhandari, Javier Bianchi, Hal Cole, Russell Cooper, Satyajit Chatterjee, V.V. Chari, Cosmin Ilut, Patrick Kehoe, Thibaut Lamadon, Ellen McGrattan, Gaston Navarro, Monika Piazzesi, Jesse Schreger, Cédric Tille, Mark Wright, and seminar participants at Berkeley, Board of Governors, Columbia University, CREI, Duke University, Federal Reserve Bank of Chicago, Federal Reserve Bank of Minneapolis, McGill University, North Carolina State, Penn State University, University of Cambridge, University of Maryland, University of Notre Dame, University of Pennsylvania, University of Western Ontario, University of Wisconsin Madison, Chicago Booth junior International Macro conference, SCIEA 2015, University of Rochester conference on the European Sovereign Debt Crisis, Konstanz Seminar for Monetary Theory and Policy, Rome Junior conference on Macroeconomics, University of Zurich conference on the Economics of Sovereign Debt, SED 2015, NBER Summer Institute 2015, ITAM-Pier 2015, NBER within and across border meeting (fall 2015), ASSA 2016, Sciences Po Summer Workshop in International Finance and Macro Finance. Gaston Chaumont, Parisa Kamali, and Tommy Khouang provided excellent research assistance. All errors are our own.

1 Introduction

The idea that lenders' pessimistic beliefs about the solvency of a government can be self-fulfilling has been often used by economists to explain fluctuations in sovereign bond yields. For example, it has been a common explanation for the sharp increase in interest rate spreads of southern European economies in 2011, and for their subsequent decline upon the introduction of the Outright Monetary Transactions (OMT) bond-purchasing program by the European Central Bank (ECB).[1] According to this view, the policy intervention of the ECB was desirable because it eliminated non fundamental fluctuations in bond yields, protecting members of the euro-area from inefficient self-fulfilling crises.

However, assessing whether movements in interest rate spreads are self-fulfilling is challenging in practice, and this makes the interpretation of these "lender of last resort" types of policies difficult. The observed widening in interest rates spreads may have been due purely to the bad economic fundamentals of these economies, and their decline following the establishment of the OMT program may have reflected heightened expectations of future bailouts by the European authorities. Clearly, this alternative interpretation of the events may lead to a less favorable assessment of the program, as bailout guarantees can induce governments to overborrow and they can introduce balance sheet risk for the ECB.

The contribution of this paper is to provide the first quantitative analysis of a benchmark model of self-fulfilling debt crises, and to use it to measure fundamental and non fundamental fluctuations in interest rate spreads during the Eurozone crisis. In the model, the maturity structure of debt chosen by the government responds differently to these two sources of default risk. Our measurement strategy consists in using this restriction, along with observed maturity choices, to infer the likelihood of a self-fulfilling crisis. After fitting the model to Italian data, we find that 12% of the interest rate spreads during the 2008-2012 period were due, on average, to rollover risk. We then use this decomposition to assess the implications of the OMT program.

We consider the canonical model of sovereign borrowing in the tradition of Eaton and Gersovitz (1981), Aguiar and Gopinath (2006) and Arellano (2008). In our environment, a government issues debt of multiple maturities in order to smooth the effect of fluctuations in tax revenues on government expenditures. The government lacks commitment over future policies and, as in Cole and Kehoe (2000), it raises new debt before deciding whether to default or not. This last assumption leads to the possibility of self-fulfilling debt crises. Lenders, in fact, have no incentives to buy new bonds when they expect the government to default. As the debt market shuts down, the government may find it too costly to service

[1] The program, introduced in September 2012, allowed the ECB to purchases of sovereign bonds in secondary markets without explicit quantity limits. See Section 6.

the maturing debt exclusively out of its tax revenues, and it may thus decide to default, validating the lenders' pessimistic expectations. These rollover crises can arise in the model when the stock of debt coming due is sufficiently large and economic fundamentals weak.

In this set up, interest rate spreads vary over time because of “non-fundamental" and “fundamental" risk. Specifically, they may reflect the self-fulfilling expectations that lenders will not roll-over government debt in the near future, or they may be high because investors fear that the government will default purely because of its poor economic conditions. Our approach consists in indirectly inferring these sources of risk from observed changes in the maturity structure of government debt. The reason why the debt maturity choices made by the government provide information on these sources of default risk builds on basic properties of the model.

Consider first a scenario where high interest rates reflect mostly the possibility that lenders will not roll-over the debt in the near future. In this situation, the government lengthens debt maturity: by back-loading payments, the government lowers the stock of debt that needs to be rolled over, reducing in this fashion the possibility of a “run". This incentive to lengthen debt maturity in presence of rollover risk was originally emphasized in Cole and Kehoe (2000).

Consider now a scenario where high interest rates are not due to the fear of a rollover crisis, but rather reflect bad economic fundamentals. In the model, the government finds optimal to shorten debt maturity in this situation. By doing so, the government improves the terms at which it borrows from the lenders, and this is valuable in bad times because it allows the government to better smooth its consumption. As emphasized in Arellano and Ramanarayanan (2012) and Aguiar and Amador (2014b), short term debt is a better instrument than long term debt for disciplining the borrowing behavior of the government in the future. Therefore, by shortening debt maturity, the government can marginally reduce the risk of default going forward, and the associated default premia charged by the lenders. As shown in Dovis (2014), these gains are not necessarily offset by losses due to a decrease in the insurance provided by long term debt.

Because of these properties, changes in the maturity structure of government debt can be used to gauge the importance of rollover risk. Observing a government that lengthens maturity during a crisis is interpreted by the model as evidence of a quantitatively sizable role for rollover risk, while a shortening would suggest that the underlying sources are fundamental. In practice, however, debt maturity also depends on the compensation that lenders require to hold risky long term debt. These risk premia may vary over time and they may increase during a debt crisis,[2] thus confounding our measurement: rollover risk could

[2] Broner, Lorenzoni, and Schmukler (2013) document that risk premia on long term bonds increase systematically during sovereign crises in emerging markets.

be driving interest rate spreads and yet we could observe a shortening of debt maturity simply because lenders are not willing to hold long term assets. To address this issue, we allow for time-varying risk premia on long term bonds in the model by introducing shocks to the lenders' stochastic discount factor as in Ang and Piazzesi (2003).

After calibrating the model using Italian data and assessing its fit, we turn to our main quantitative experiment, which consists in measuring the rollover risk component of observed interest rate spreads during the 2008-2012 debt crisis. For this purpose, we apply a nonlinear filter to the model and extract the sequence of structural shocks that can explain the behavior of several data series over the sample. Equipped with this path, we construct the counterfactual interest rate spreads that would have emerged if the likelihood of a rollover crisis was zero throughout the episode. The rollover risk component is then the difference between the observed interest rate spreads and the counterfactual ones. We find that this component represents, on average, 12% of the interest rate spreads observed during the episode. We reach this conclusion because the Italian government predominantly issued short term securities at the height of the debt crisis, a behavior that led to a reduction of 0.5 years in the average life of outstanding debt.

We next proceed to an evaluation of the OMT program. We model this policy as a bond price floor schedule implemented by a deep pocketed central bank. When appropriately designed, this instrument eliminates the possibility of future rollover crises without the necessity to carry out bond purchases on the equilibrium path. This design, which results in a Pareto improvement, is our normative benchmark. We use our framework to test whether the OMT program is implementing this benchmark. Specifically, we construct the interest rate spread that would emerge in a world without rollover crises, and we compare it with the actual Italian spread observed after the OMT announcements. We find that this counterfactual spread is roughly 100 basis points above the observed one, this being consistent with the hypothesis that the policy may have fostered expectations of future bailouts on the equilibrium path.

There is a long literature on multiplicity of equilibria in models of sovereign debt. While the Eaton and Gersovitz (1981) model tends to generate a unique equilibrium,[3] the seminal papers of Alesina, Prati, and Tabellini (1989) and Cole and Kehoe (2000) show that the government's inability to commit to current repayments can lead to self-fulfilling rollover crises. Starting with Conesa and Kehoe (2012), Chatterjee and Eyigungor (2012) and Roch and Uhlig (2014), recent papers have introduced this feature in models with income shocks. In contemporaneous work, Aguiar, Chatterjee, Cole, and Stangebye (2016) show that the

[3]See Auclert and Rognlie (2014) for a proof of a unique equilibrium in the Eaton and Gersovitz (1981) model when the government can issue only short term debt. Multiple equilibria in that model may arise when the government issues long term debt, see Stangebye (2014) and Aguiar and Amador (2016).

introduction of time varying rollover risk allows models of sovereign debt to better capture the behavior of spreads and debt for emerging economies. Our paper is complementary to their analysis. Rather than studying the effect of rollover risk on average, we ask the question of how one can quantify its importance during a particular historical event, such as the European debt crisis. For this purpose, we enrich the workhorse model with maturity choices, and propose a measurement strategy based on the joint dynamics of interest rate spreads and debt maturity.

The idea of using agents' choices to learn about the types of risk they are facing has a long tradition in the literature. A classic example is the use of consumption data along with the logic of the permanent income hypothesis to separate between permanent and transitory income shocks. See Cochrane (1994) for an application on U.S. aggregate data, Aguiar and Gopinath (2007) for emerging markets, and Guvenen and Smith (2014) for a recent study using micro data. As it is the case for these approaches, our analysis relies heavily on the assumptions underlying the structural model, and it is not robust to misspecifications of the trade-offs that governs the maturity choices. Unfortunately, the literature is scant on systematic studies documenting the motives driving the management of public debt in practice. However, documents produced by Treasury departments around the world and historical episodes support the idea that governments actively manage debt maturity to prevent rollover crises, this being consistent with our key identifying restriction.[4]

More generally, our research is related to papers that study the quantitative properties of sovereign debt models. Closely related works include Arellano and Ramanarayanan (2012), Bianchi, Hatchondo, and Martinez (2014), Hatchondo, Martinez, and Sosa Padilla (2015), and Borri and Verdhelan (2013). Relative to the existing literature, we are the first to analyze a sovereign debt model with rollover risk, endogenous debt maturity, and risk aversion on the side of the lenders.[5] A second departure from the literature lies in the calibration. When applied to emerging markets, researchers have emphasized the role of impatience (a low discount factor) as a major rationale for government's borrowing. These calibrations tend to generate high average interest rate spreads because the government gravitates most of the time around the borrowing limits defined by default risk. Moreover, they lead to procyclical debt issuances because the implicit borrowing limits are laxer in high income states. Both

[4]For instance, the OECD discusses practical issues related to public debt management in its "Sovereign Borrowing Outlook". This is published yearly since 2009, and it can be downloaded at `http://www.oecd.org/finance/financial-markets/oecdsovereignborrowingoutlook.htm`. In Appendix E we detail an historical example of a Treasury department that extended the life of its debt in the face of a mounting rollover problem.

[5]Our modeling of the maturity choices differs from the formulation of Arellano and Ramanarayanan (2012) and builds on recent work by Sanchez, Sapriza, and Yurdagul (2015) and Bai, Kim, and Mihalache (2014). Specifically, we allow the government to issues portfolios of zero coupon bonds restricted to have an exponentially decaying repayment profile. The maturity choice is discrete, and it consists on the choice of the decaying factor. This feature simplifies the numerical analysis of the model.

of these predictions are inconsistent with the behavior of public finances in Europe over the period of analysis. We document that a simple modification of the government's utility function, namely the introduction of a consumption commitment, allows the model to fit the behavior of interest rate spreads and debt issuances in our sample.

From an econometric viewpoint, the environment we consider is an example of an *incomplete* model (Tamer, 2003), in which multiple equilibria leads to a region of the state space where outcomes are not unique. There are several approaches developed in the applied literature to deal with this complication. One could conduct inference by characterizing the predictions for outcome variables that are consistent with the full set of equilibria.[6] Alternatively, one could "complete" the model by introducing a rule that selects among the potential outcomes, and study the model predictions conditional on this selection device. We follow this second avenue. Our selection rule builds on Cole and Kehoe (2000), and it has been used extensively in subsequent studies: when outcomes are not unique, a sunspot determines (period by period) whether lenders desert the auction or not. This allows us to evaluate a likelihood function, and to filter the unobserved state variables using techniques routinely applied to models with a unique equilibrium (Fernández-Villaverde, Rubio-Ramírez, and Schorfheide, 2015). To best of our knowledge, we are the first in the macroeconomic literature to apply these tools to a model of this sort.[7]

Finally, our paper is an attempt to quantify the importance of a particular form of self-fulfilling expectations to the volatility of interest rate spreads. Multiple equilibria in sovereign debt models can arise through alternative mechanisms, such as the one emphasized in Calvo (1988) and recently studied by Lorenzoni and Werning (2013) and Navarro, Nicolini, and Teles (2015), or the one in Broner, Erce, Martin, and Ventura (2014). Our analysis is silent on whether these forces contributed to variation in bond yields during the European debt crisis.

Layout. The paper is organized as follows. We present the model in Section 2, and discuss our key identifying restriction in Section 3. We next turn in Section 4 to the calibration of the model, and a discussion of its properties. In Section 5 we use the calibrated model to measure the importance of rollover risk during the Italian sovereign debt crisis, and we assess the robustness of our results. We analyze the OMT program in Section 6. Section 7 concludes.

[6]See Lubik and Schorfheide (2004) for a general treatment of this approach in New Keynesian linear rational expectation models. Passadore and Xandri (2015) pursue this approach in the context of the Eaton and Gersovitz (1981) model.

[7]Aruoba, Cuba-Borda, and Schorfheide (2016) also consider a nonlinear macroeconomic model featuring multiple equilibria. In their New Keynesian model, however, indeterminacy is not confined to a particular region of the state space. Hence, the switch between "good" and "bad" outcomes is driven exclusively by the realization of the sunspot.

2 Model

2.1 Environment and recursive equilibrium

Preferences and endowments. Time is discrete and indexed by $t = 0, 1, 2, \ldots$. The exogenous state of the world is $s_t \in S$. We assume that s_t follows a Markov process with transition matrix $\mu(\cdot|s_{t-1})$. It is convenient to split the state into two components, $s_t = (s_{1,t}, s_{2,t})$ where $s_{1,t}$ is the *fundamental* component and $s_{2,t}$ is the *non-fundamental* component. The fundamental component affects endowments and preferences while the non-fundamental component collects coordination devices orthogonal to the fundamentals.

The economy is populated by a large number of lenders and a government. The government receives tax revenues every period and decides the path of spending G_t. Tax revenues are a constant share τ of the output produced in the economy, $Y_t = Y(s_{1,t})$. The government values a stochastic stream of spending $\{G_t\}_{t=0}^{\infty}$ according to

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U(G_t), \tag{1}$$

where the period utility function U is strictly increasing and concave.

The lenders value flows using the stochastic discount factor $M(s_{1,t}, s_{1,t+1})$. Hence the value of a stochastic stream of payments $\{d_t\}_{t=0}^{\infty}$ from time zero perspective is given by

$$\mathbb{E}_0 \sum_{t=0}^{\infty} M_{0,t} d_t, \tag{2}$$

where $M_{0,t} = \prod_{j=0}^{t} M_{j-1,j}$.

Market structure. The government can issue a portfolio of non-contingent defaultable bonds. For computational convenience, we restrict the government to issue portfolios of zero-coupon bonds (ZCB) indexed by (B_{t+1}, λ_{t+1}) for $\lambda_{t+1} \in [0, 1]$. A portfolio (B_{t+1}, λ_{t+1}) at the end of period t corresponds to a stock of $(1 - \lambda_{t+1})^{n-1} B_{t+1}$ ZCB that matures at $t + n$. The variable λ_{t+1} captures the maturity of the stock of debt: higher λ_{t+1} implies that the repayment profile is concentrated at shorter maturities. For instance, if $\lambda_{t+1} = 1$, then all the debt is due next period. The variable B_{t+1} controls the face value of debt, which is equal to B_{t+1}/λ_{t+1}. We let $q_{t,n}$ be the price of a ZCB of maturity n issued at time t.

The timing of events within the period follows Cole and Kehoe (2000): the government first issues new debt, lenders choose the price of newly issued debt, and finally the government decides to default or not, $\delta_t = 0$ or $\delta_t = 1$ respectively. We assume that if the government defaults, it is excluded from financial markets and it suffers losses in output.

We denote by $\underline{V}(s_{1,t})$ the value for the government conditional on a default. Lenders that hold inherited debt and newly issued debt do not receive any repayment.[8] Differently from the timing in Eaton and Gersovitz (1981), the government does not have the ability to commit not to default within the current period. As we will see, this assumption allows for the possibility of self-fulfilling debt crises.

The budget constraint for the government when it does not default is

$$G_t + B_t \leq \tau Y_t + \Delta_t, \tag{3}$$

where Δ_t is the net amount of resources that the government raises in the period,

$$\Delta_t = \sum_{n=1}^{\infty} q_{t,n} \left[(1-\lambda_{t+1})^{n-1} B_{t+1} - (1-\lambda_t)^n B_t \right]. \tag{4}$$

In the above expression, if a government enters the period with a portfolio (B_t, λ_t) and wants to exit it with a portfolio (B_{t+1}, λ_{t+1}), then it must issue additional $(1-\lambda_{t+1})^{n-1} B_{t+1} - (1-\lambda_t)^n B_t$ ZCB of maturity n. When $(1-\lambda_{t+1})^{n-1} B_{t+1} - (1-\lambda_t)^n B_t$ is negative, the government is buying back the ZCB of maturity n. Importantly, these are the only trades that the government executes, so our formulation does not require that the government buys back its entire portfolio and re-issues it every period.

Recursive equilibrium. We consider a recursive formulation of the equilibrium. Let $\mathbf{S} = (B, \lambda, s)$ be the state today and $\mathbf{S}'$ be the state tomorrow. The problem for the government that has not defaulted yet can be written as

$$V(\mathbf{S}) = \max_{B',\lambda',G,\delta \in \{0,1\}} \delta \left\{ U(G) + \beta \mathbb{E}[V(\mathbf{S}') \mid \mathbf{S}] \right\} + (1-\delta) \underline{V}(s_1) \tag{5}$$

subject to

$$\begin{aligned} G + B &\leq \tau Y(s_1) + \Delta(\mathbf{S}, B', \lambda'), \\ \Delta(\mathbf{S}, B', \lambda') &= \sum_{n=1}^{\infty} q_n(\mathbf{S}, B', \lambda') \left[(1-\lambda')^{n-1} B' - (1-\lambda)^n B \right], \end{aligned}$$

where $q_n(\mathbf{S}, B', \lambda')$ is the price of a defaultable ZCB of maturity n given the state $\mathbf{S}$ and the government's choices for the new portfolio (B', λ').

[8] The assumption of a zero recovery rate is made for tractability. One could obtain a non-zero recovery rate by modeling the debt restructuring process along the lines of Benjamin and Wright (2009) and Yue (2010). Note that, differently from Cole and Kehoe (2000), the government cannot use the funds raised in the issuance stage if it defaults. Our formulation simplifies the problem and it does not change its qualitative features. The same formulation has been adopted in other works, for instance Aguiar and Amador (2014b).

The lender's no-arbitrage conditions require that

$$q_n\left(\mathbf{S}, B', \lambda'\right) = \delta\left(\mathbf{S}\right) \mathbb{E}\left\{M\left(s_1, s_1'\right) \delta\left(\mathbf{S}'\right) q_{n-1}\left(\mathbf{S}', B'', \lambda''\right) | \mathbf{S}\right\} \text{ for } n \geq 1, \tag{6}$$

where B'' and λ'' are optimal debt and maturity choices given the state (B', λ', s'), and $q_0(\mathbf{S}, B', \lambda') = 1$. The presence of $\delta\left(\mathbf{S}\right)$ in equation (6) implies that new lenders receive a payout of zero in the event of a default today. Differently from the Eaton and Gersovitz (1981) timing convention, the pricing schedule does not only depend on the exogenous state s and on the end of the period debt portfolio (B', λ'), but it depends also on (B, λ). This is because the initial debt portfolio affects the current default decision $\delta\left(\mathbf{S}\right)$, which is a key determinant of the price of newly issued debt in our formulation.

A *recursive equilibrium* is a value function for the borrower V, associated decision rules $\{\delta, B', \lambda', G\}$ and a pricing function $q = \{q_n\}_{n \geq 1}$ such that $\{V, \delta, B', \lambda', G\}$ are a solution of the government problem (5) and q satisfies the no-arbitrage conditions (6).

2.2 Multiplicity of equilibria and Markov selection

This economy features multiple recursive equilibria, with outcomes not entirely determined by the fundamental state variables. As in Cole and Kehoe (2000), there are states of the world in which lenders' expectations of a default are self-fulfilling: if lenders expects the government to default today, and do not buy new bonds, the government finds it optimal to default while if lenders believe that the government will repay, and they roll-over the maturing debt, the government will indeed repay.

To understand how this situation can arise, it is convenient to define the price at which debt would be traded if in state (s, B, λ) the government would repay and issue (B', λ'). We refer to this price as the *fundamental price*,

$$q_n^{\text{fund}}\left(s, B', \lambda'\right) = \mathbb{E}\left\{M\left(s_1, s_1'\right) \delta\left(\mathbf{S}'\right) q_{n-1}\left(s', B'', \lambda''\right) | \mathbf{S}\right\}. \tag{7}$$

We also let

$$\Delta^{\text{fund}}\left(\mathbf{S}, B', \lambda'\right) = \sum_{n=1}^{\infty} q_n^{\text{fund}}\left(s, B', \lambda'\right) \left[(1-\lambda')^{n-1} B' - (1-\lambda)^n B\right]$$

be the amount of resources that the government can raise from lenders at such prices.

Consider now a state $\mathbf{S}$ where it is optimal for the government to repay if lenders expect that the government will not default in the current period. For this expectations to be valid, it must be that the government prefers repaying over defaulting,

$$\max_{B',\lambda'} \left\{ U\left(\tau Y\left(s_1\right) - B + \Delta^{\text{fund}}\left(\mathbf{S}, B', \lambda'\right)\right) + \beta \mathbb{E}\left[V\left(B', \lambda', s'\right)|\mathbf{S}\right] \right\} \geq \underline{V}\left(s_1\right). \tag{8}$$

Let's examine now this alternative scenario: at $\mathbf{S}$ the government tries to raise resources from the market, but lenders expect a default today, and they set the price of newly issued debt to zero. These expectations of the lenders are validated if the government prefers defaulting when it cannot issue new debt,[9]

$$\underline{V}\left(s_1\right) > U\left(\tau Y\left(s_1\right) - B\right) + \beta \mathbb{E}\left[V\left((1-\lambda)B, \lambda, s'\right)|\mathbf{S}\right]. \tag{9}$$

For these beliefs to trigger a default along the equilibrium path, it must also be the case that the government prefers to default rather than buying back part of its debt at the fundamental prices. That is,

$$\begin{aligned} \underline{V}\left(s_1\right) > \quad & \max_{B',\lambda'} \left\{ U\left(\tau Y\left(s_1\right) - B + \Delta^{\text{fund}}\left(\mathbf{S}, B', \lambda'\right)\right) + \beta \mathbb{E}\left[V\left(B', \lambda', s'\right)|\mathbf{S}\right] \right\} \\ & \text{subject to } \Delta^{\text{fund}}\left(\mathbf{S}, B', \lambda'\right) \leq 0. \end{aligned} \tag{10}$$

As we discuss in Appendix A, condition (10) implies condition (9).

For all λ and s there are intermediate values of B such that both (8) and (10) hold, see Proposition 1 in Aguiar and Amador (2014a) for a formal proof. When this happens, the default decisions of the government are indeterminate, and they depend on lenders' beliefs: lenders may extend credit to the government and there will be repayment, or they may not roll-over because they expect a default, in which case the government will not repay, validating lenders' expectations.

We follow most of the literature and use a parametric rule that selects among these possible outcomes. In order to explain the selection mechanism, it is useful to partition the state space in three regions. Following the terminology in Cole and Kehoe (2000), we say that the government is in the *safe zone*, $\mathcal{S}^{\text{safe}}$, if it does not default even if lenders are not willing to roll-over its debt. That is,

$$\mathcal{S}^{\text{safe}} = \left\{ \mathbf{S} : (10) \text{ does not hold} \right\}.$$

We say that the government is in the *crisis zone*, $\mathcal{S}^{\text{crisis}}$, if (B, λ, s) are such that it is not

[9]If condition (9) is not satisfied, instead, lenders' expectations cannot trigger a default. This is because it is optimal for the government to repay its debt even if it cannot raise additional resources in the market. Because of that, an individual lender has an incentive to buy government bonds at a positive price, this ruling out $q = 0$ as an equilibrium price.

optimal for the government to repay debt during a rollover crisis but it is optimal to repay if the lenders roll it over. That is,

$$\mathcal{S}^{\text{crisis}} = \Big\{ \mathbf{S} \, : \, (8) \text{ and } (10) \text{ hold } \Big\}.$$

Finally, the *default zone*, $\mathcal{S}^{\text{default}}$, is the region of the state space in which the government defaults on its debt irrespective of lenders expectations,

$$\mathcal{S}^{\text{default}} = \Big\{ \mathbf{S} \, : (8) \text{ does not hold } \Big\}.$$

Indeterminacy in outcomes arises only when the economy is in the crisis zone.[10]

We consider the following selection mechanism: let the non-fundamental state be $s_2 = (\pi, \xi)$. The variable π is the probability that there will be a rollover crisis in the *next* period conditional on the economy being in the crisis zone. We assume that π follows a first order Markov process. The variable ξ indicates whether a rollover crisis takes place in the current period. Whenever the economy is in the crisis zone, if $\xi = 0$ then lenders roll-over and there is no default. If $\xi = 1$, instead, the lenders do not roll-over and there is a default. Conditional on this selection rule, the outcome of the debt auctions are unique in the crisis zone. However, we cannot ensure that the equilibrium value function, decision rules and pricing functions are unique because the operator that implicitly defines a recursive equilibrium may have multiple fixed points, see Aguiar and Amador (2016). In the numerical analysis of the model, we iterate starting from the risk-free price schedule until we find a fixed point.

The equilibrium outcome is a stochastic process

$$\mathbf{y} = \{\lambda'(B_0, \lambda_0, s^t), B'(B_0, \lambda_0, s^t), \delta(B_0, \lambda_0, s^t), G(B_0, \lambda_0, s^t), q(B_0, \lambda_0, s^t)\}_{t=0}^{\infty}$$

naturally induced by the recursive equilibrium objects. The outcome path depends on properties of the selection, i.e. the process for $\{\pi_t\}$, and on the realization of the non-fundamental state $\{s_{2,t}\}$.

[10] It is in principle possible that outcomes are indeterminate in the safe zone too. This can happen when condition (9) holds while condition (10) does not. In such a case, two outcomes may arise in the safe zone. In the first, the government borrows at the fundamental prices. In the second, the government is prevented from borrowing at the fundamental prices, but it prefers to buy back some of its debt rather than defaulting. In our analysis, we abstract from this second scenario.

3 Measuring rollover risk: the role of maturity choices

In the environment presented in the previous section, interest rate spreads are driven by both fundamental and non-fundamental risk. The goal of our analysis is to measure the relative importance of these two forces. In this section, we discuss more formally this inference problem and explain our approach.

Rearranging equation (6), we can express the difference between the yield of an $n = 1$ bond issued by the government, $r_{1,t}$, and the risk-free rate, $r^*_t = 1/\mathbb{E}_t[M_{t,t+1}]$, as

$$\text{spr}_t \equiv \frac{r_{1,t} - r^*_t}{r_{1,t}} = \Pr_t\{\delta_{t+1} = 0\} + \text{Cov}_t\left(-\frac{M_{t,t+1}}{\mathbb{E}_t[M_{t,t+1}]}, \delta_{t+1}\right). \tag{11}$$

Interest rate spreads reflect both the probability of a future default by the government and the compensation that lenders demand for being exposed to such default risk. In the model, this risk arises because of two reasons. First, the government may be next period in the default zone, an event that occurs with probability $\Pr_t\{\mathbf{S}_{t+1} \in \mathcal{S}^{\text{default}}\}$. Second, there is a chance of a self-fulfilling rollover crisis at $t+1$, an event that occurs with probability π_t if the economy is in the crisis zone at $t+1$, $\Pr_t\{\mathbf{S}_{t+1} \in \mathcal{S}^{\text{crisis}}\} \times \pi_t$.

Ultimately, the goal of our analysis is to isolate the component of interest rate spreads that is due to the risk of a rollover crisis. We define this component as

$$\text{spr}^{\text{roll}}_t \equiv \text{spr}_t - \text{spr}|_{\pi_t=0}.$$

That is, the rollover risk component of interest rate spreads represents the difference between the actual spread spr_t, and the one that would arise if the likelihood of a rollover crisis next period was set to zero, $\text{spr}|_{\pi_t=0}$. The latter can be interpreted as the fundamental component of interest rate spreads.

The measurement problem arises because the counterfactual spreads $\text{spr}|_{\pi_t=0}$ and $\text{spr}^{\text{roll}}_t$ have no direct empirical counterpart. Our approach to overcome this issue consists in indirectly inferring these components by studying, through the lens of the model, the joint dynamics of interest rate spreads and debt maturity. As we will discuss next, the model suggests that the government has incentives to lengthen its debt maturity in the face of heightened rollover risk, while it should shorten it when default risk is mostly due to the fundamental component. Because of this property, changes in the maturity structure of government debt provide information on the relative importance of $\text{spr}|_{\pi_t=0}$ and $\text{spr}^{\text{roll}}_t$ in accounting for observed interest rate spreads: observing a government that extends the maturity of its debt while facing high interest rates is evidence of heightened rollover risk, while the opposite would point toward a more limited role for this component. In what follows, we review the trade-offs that the government faces when choosing debt maturity. Appendix B provides a more formal analysis within the context of a three-period version of the model.

Maturity choices and rollover risk. To understand how debt maturity responds to rollover risk in the model, it is important to note that the government can partly control the risk of facing a rollover crisis in the future, $\Pr_t\{\mathbf{S}_{t+1} \in \mathcal{S}^{\text{crisis}}\} \times \pi_t$. By managing its public debt, the government can alter the boundaries of the crisis zone defined by conditions (8) and (10), affecting in this fashion $\Pr_t\{\mathbf{S}_{t+1} \in \mathcal{S}^{\text{crisis}}\}$. Because rollover crises are costly, the government will respond to an increase in π_t by taking actions that reduce the risk of being in the crisis zone at $t+1$. As originally emphasized in Cole and Kehoe (2000), this can be achieved by lengthening the maturity structure of government debt.

To understand why lengthening debt maturity reduces the government exposure to future rollover crises, consider the condition defining the safe zone at $t+1$,[11]

$$U(\tau Y_{t+1} - B_{t+1}) + \beta \mathbb{E}_{t+1}[V((1-\lambda_{t+1})B_{t+1}, \lambda_{t+1}, s_{t+2})] \geq \underline{V}(s_{t+1}) \,. \tag{12}$$

Suppose that at time t the government extends the maturity of its debt while keeping constant the amount of resources that it raises. This is achieved by decreasing λ_{t+1} and reducing B_{t+1} by the appropriate amount. In this fashion, the government reduces the payments coming due in the next period at the cost of higher future payments. As a result, $U(\tau Y_{t+1} - B_{t+1})$ increases while the continuation value $\mathbb{E}_{t+1}[V((1-\lambda_{t+1})B_{t+1}, \lambda_{t+1}, s_{t+2})]$ reduces. However, the increase in current utility exceeds the reduction in the continuation value because the government is, by definition, credit constrained in the crisis zone and the marginal utility of current consumption is higher than the marginal reduction in expected future utility.[12] Because of that, the left hand side of (12) increases when the government lengthens debt maturity, enlarging the set of states for which the inequality is satisfied at $t+1$. From time t perspective, this implies an increase in $\Pr_t\{\mathbf{S}_{t+1} \in \mathcal{S}^{\text{safe}}\}$.

Extending debt maturity is thus a way for the government to reduce its exposure to rollover risk. In Appendix B, we formally isolate this incentive in a three-period version of the model. Proposition 2 shows that the government issues only long term debt in this economy when default risk arises exclusively because of the rollover problem ($\pi > 0$ and the set $\mathcal{S}^{\text{default}}$ is empty).

[11] For simplicity, we focus on condition (9) instead of (10), under the assumption that the government does not want to buy back its debt.

[12] To see this, note that condition (8) states that the government prefers to repay if it can freely choose a portfolio under fundamental prices. Conditions (9) and (10) state that the government prefers to default when facing fundamental prices under the restriction that $\Delta^{\text{fund}} \leq 0$. Hence it must be that the maximum in the left side of (8) is attained for a portfolio with $\Delta^{\text{fund}} > 0$, else the value of repaying would be the same in the two circumstances.

Maturity choices and fundamental risk. To understand how debt maturity responds to fundamental risk, we assume away rollover risk by considering a version of the model with $\pi_t = 0$ for all t and s^t. This is equivalent to adopting the timing convention in Eaton and Gersovitz (1981). The behavior of debt maturity in this environment has been previously studied theoretically by Aguiar and Amador (2014b), Dovis (2014) and Niepelt (2014), and quantitatively by Arellano and Ramanarayanan (2012), Sanchez, Sapriza, and Yurdagul (2015) and Hatchondo, Martinez, and Sosa Padilla (2015) among others. These papers have emphasized two channels as the main determinants of the maturity composition of debt: the *incentive* and the *insurance* channel.

The incentive channel makes short term debt desirable. In order to understand why, consider the price of a ZCB that matures in $n > 1$ periods in equation (6). The price depends not only on the possibility of a default tomorrow, but also on the issuance decisions of future governments: a higher B'' increases default risk going forward, and it depresses the reselling value of the long term bond today. This feature creates a time inconsistency problem. The future governments do not internalize the negative effects that new issuances have on the price of debt today, and they will tend to borrow more than what is optimal from the perspective of the current government. Importantly, short term debt is immune from this problem because, conditional on repayment, its value does not depend on future debt issuances, see equation (6) for $n = 1$. Therefore, by shortening debt maturity, the government is able to align the actions of future governments to its preferred spending path. In Appendix B we isolate this mechanism by considering a three period-version of the model without rollover risk and hedging motives. Proposition 3 shows that the government would issue only short term debt in such environment.

An alternative way of thinking about the incentive channel is to consider the effects that a change in the maturity structure has on the interest rates at which the government borrows today. Shortening debt maturity is, in fact, a way to discipline the borrowing behavior of future governments, and it allows the current government to reduce the interest rates at which it is borrowing. To understand why the maturity structure affects the borrowing incentives of future governments, consider a situation where one of such government inherits only short term debt. The government understands that any increase in interest rates will significantly reduce its consumption because the entire stock of debt will have to be refinanced at these higher interest rates. Hence, it will have less incentives to borrow and to be exposed to default risk. With a long term maturity structure, instead, these incentives are muted because the future government needs to refinance only a fraction of the stock of debt at the higher interest rates. Therefore, it will tend to borrow more, and this will be reflected into higher interest rates ex-ante.

While the incentive channel generates a motive to issue short term debt, the insurance channel makes long term debt desirable because it is a better instrument to provide insurance against shocks. To illustrate this point, consider a situation in which the government is hit by a negative shock to its tax revenues. Typically, the shock increases the likelihood of a default going forward and the interest rates on new issuances. If all inherited debt is short term, the government will have to refinance its stock of debt at the new high interest rates, and so either its current consumption or its continuation value must decline. If instead part of the inherited debt is long term, only a fraction of the stock of debt has to be refinanced at higher interest rates, and the government will be able to keep its current consumption relatively high without reducing its continuation value. The opposite happens in response to a positive shock to tax revenues. Therefore, a risk averse government would prefer issuing long term debt because this instrument reduces the volatility of its consumption. In Appendix B, proposition 4, we isolate this channel by showing that the government would issue only long term debt in this economy if the incentive channel was not operative (if the time 0 government could choose future debt issuances).

The relative strength of the incentive and of the insurance channel shapes the portfolio choices of the government. For our purposes, it is important to understand how the relative attractiveness of these instruments varies in response to adverse shocks that push the government closer to the default zone. While we are not aware of an analytical characterization of this comparative static exercise in the literature, typical calibrations of this model imply that the government shortens its debt maturity, see for example Arellano and Ramanarayanan (2012).[13] This result can be justified as follows.

First, when default risk increases, the incentive role of short term debt becomes more valuable from the government's perspective. High default risk states are, in fact, states in which the government would like to issue more debt for consumption smoothing motives. By shortening the maturity structure of its debt, the government can reduce at the margin future default probabilities and the interest rates that it faces when borrowing because lenders today price in the disciplining role that the new maturity structure exerts on future borrowing. This allows the government to raise more resources today and to better smooth consumption. Second, this shortening of debt maturity does not necessarily come at a cost of less insurance for the government. As discussed in Dovis (2014), the need to issue long term debt for insurance reasons falls when default risk increases.[14]

[13] In our model the government has to buy back some debt in order to shorten its debt maturity. This may seem at odds with the finding in Aguiar and Amador (2014b) that buy backs are never optimal. Our environment differs from theirs in two dimensions. First, our restriction of feasible portfolios requires that the government must buy back some debt to shorten the maturity. Second, insurance considerations play a role in our model while Aguiar and Amador (2014b) abstract from those.

[14] This is due to the fact that pricing functions in this class of models are more sensitive to shocks when the economy approaches the default region. The larger ex-post volatility of the price of long-term debt allows for more insurance because the market value of long term debt falls more in future bad states, making consumption

Summary and quantitative analysis. So far, we have argued that the dynamics of debt maturity provide information on the sources of default risk. In what follows, we will build on this insight and we will use the joint dynamics of interest rate spreads and debt maturity along with a calibrated version of the model to quantify the importance of rollover risk during the Italian debt crisis of 2008-2012.

Before proceeding further, though, it is important to discuss a potential pitfall in our strategy. While our approach emphasizes government incentives, the observed maturity choices also depend on lenders' preferences for the maturity of the bonds that they are purchasing. These preferences may vary over time, and they may be a confounding factor for our measurement strategy. For example, a government that is facing a rollover crisis may not be willing to lengthen debt maturity if at the same time lenders demand high compensation for holding long term bonds. Hence, rather than reflecting little rollover risk, a shortening of debt maturity may be the optimal response of a government that finds increasingly expensive to issue long term debt. This view finds support in the data, as previous research by Broner, Lorenzoni, and Schmukler (2013) has documented that risk premia on long term bonds systematically increase during debt crises. In our quantitative analysis we are going to control for these confounding factors by considering a stochastic discount factor for the lenders that can generate time variation in the risk premium on long term bonds.

4 Quantitative analysis

We now fit the model to Italian data during the pre-OMT period, 1999:Q1-2012:Q2. This section proceeds in four steps. Section 4.1 describes the parametrization and the calibration strategy. Section 4.2 reports the results of the calibration. Section 4.3 studies the fit of the model. Finally, Section 4.4 discusses the behavior of interest rate spreads and debt maturity in the calibrated model.

4.1 Parametrization and calibration strategy

We model the lenders' stochastic discount factor, $M_{t,t+1} = \exp\{m_{t,t+1}\}$, following Ang and Piazzesi (2003),

$$\begin{aligned} m_{t,t+1} &= -(\phi_0 + \phi_1 \chi_t) - \frac{1}{2}\kappa_t^2 \sigma_\chi^2 - \kappa_t \varepsilon_{\chi,t+1}, \\ \chi_{t+1} &= \mu_\chi (1-\rho_\chi) + \rho_\chi \chi_t + \varepsilon_{\chi,t+1} \qquad \varepsilon_{\chi,t+1} \sim \mathcal{N}(0, \sigma_\chi^2), \qquad (13) \\ \kappa_t &= \kappa_0 + \kappa_1 \chi_t. \end{aligned}$$

in the next period less sensitive to shocks.

In this formulation, expected excess returns on long term bonds are proportional to χ_t (see Appendix F), implying that shocks to this factor induce movements in risk premia on long term assets. For future reference, we index the parameters of the stochastic discount factor with $\theta_1 = [\phi_0, \phi_1, \kappa_0, \kappa_1, \mu_\chi, \rho_\chi, \sigma_\chi]$.

The government discounts future flow utility at the rate β. The utility function is

$$U(G_t) = \frac{(G_t - \underline{G})^{1-\sigma} - 1}{1-\sigma},$$

where $\underline{G}$ is the non-discretionary level of public spending. We interpret $\underline{G}$ as capturing the components of public spending that are hardly modifiable by the government in the short run, such us wages of public employees and pensions. As we will discuss in Section 4.3, this specification helps our model matching the cyclicality of public debt.

We introduce a utility cost for adjusting debt maturity,

$$\alpha \left(\frac{1}{4\lambda'} - \overline{d} \right)^2.$$

This adjustment cost serves two purposes. First, it leads to well defined maturity choices in regions of the state space where the government would have been otherwise indifferent over λ'. This ameliorates the convergence properties of the algorithm we use to numerically solve the model.[15] Second, it gives the model enough flexibility to match the level and volatility of debt maturity in the sample.

The output process, $Y_t = \exp\{y_t\}$, depends on the factor χ_t and on its innovations as follow,

$$y_{t+1} = \mu_y(1-\rho_y) + \rho_y y_t + \rho_{y\chi}(\chi_t - \mu_\chi) + \sigma_y \varepsilon_{y,t+1} + \sigma_{y\chi}\varepsilon_{\chi,t+1}, \qquad \varepsilon_{y,t+1} \sim \mathcal{N}(0,1). \quad (14)$$

We allow for correlation between χ_t and y_t in order to capture the cyclicality of risk premia.

If the government enters a default state, it is excluded from international capital markets for a random period, and it has a probability of re-entering capital markets equal to ψ. While in default, the government suffers a loss in tax revenues equal to d_t. This is motivated by evidence that sovereign defaults lead to severe financial and output disruptions (Hebert and Schreger, 2015; Bocola, 2016), and they should therefore imply a loss in fiscal revenues for the government. These costs are parametrized following Chatterjee and Eyigungor (2012),

$$d_t = \max\{0, d_0 \tau Y_t + d_1(\tau Y_t)^2\}. \quad (15)$$

[15] Maturity choices in the model are not determined absent default risk and with risk neutral lenders, see Arellano and Ramanarayanan (2012).

The convexity of d_t gives the model more flexibility to match the volatility of interest rate spreads, see Chatterjee and Eyigungor (2012). Mendoza and Yue (2012) offer a rationale for this assumption.

The probability of lenders not rolling over the debt in the crisis zone next period follows the stochastic process $\pi_t = \frac{\exp\{\tilde{\pi}_t\}}{1+\exp\{\tilde{\pi}_t\}}$, with $\tilde{\pi}_t$ given by

$$\tilde{\pi}_{t+1} = \pi^* + \sigma_\pi \varepsilon_{\pi,t+1}, \qquad \varepsilon_{\pi,t+1} \sim \mathcal{N}(0,1). \tag{16}$$

We let $\theta_2 = [\sigma, \tau, \underline{G}, \psi, \mu_y, \rho_y, \rho_{y\chi}, \sigma_y, \sigma_{y\chi}, \beta, d_0, d_1, \pi^*, \sigma_\pi, \overline{d}, \alpha]$ denote the parameters associated to the decision problem of the government.

Our strategy consists in calibrating $\theta = [\theta_1, \theta_2]$ in two steps. In the first step, we choose θ_1 to match the behavior of risk premia on *non-defaultable* long term bonds, measured using the term structure of German's treasuries. In the second step, and conditional on θ_1, we calibrate θ_2 by matching key facts about Italian public finances over the sample. Implicit in the first step is the assumption that the lenders in the model are the marginal investors for these assets as well: thus, we can measure their preferences for short versus long term bonds by studying the behavior of the term structure of German interest rates. We focus on bonds that are arguably not subject to default risk over the sample because of two reasons. First, the absence of a default during the event under analysis makes the measurement of risk premia on Italian bonds more challenging because of a "peso problem". Second, this approach allows us to calibrate θ_1 without solving the government decision problem, which is numerically complex. In Section 5.2 we assess the robustness of our results to this approach of modeling and measuring risk premia on long term bonds.

4.2 Calibration

We start by setting the parameters of the lenders' stochastic discount factor to fit the behavior of expected excess returns on long term German ZCB following the procedure developed by Cochrane and Piazzesi (2005). Let $q_t^{*,n}$ be the log price on a non-defaultable ZCB maturing in n quarters, $rx_{t+1}^n = q_{t+1}^{*,n-1} - q_t^{*,n} + q_t^{*,1}$ the associated *realized* excess log returns, $f_t^n = q_t^{*,n-1} - q_t^{*,n}$ the time t log forward rate for loans between $t+n-1$ and $t+n$, and $y_t^1 = -q_t^{*,1}$ the log yield on a ZCB maturing next quarter. We denote by $\mathbf{rx}_{t+1}$ and $\mathbf{f}_t$ vectors collecting, respectively, excess log returns and log forward rates for different maturities. Quarterly data (1973-2013) on the term structure of ZCB for German federal government securities is obtained from the Bundesbank online database, see Appendix D.

We proceed in two stages. In the first stage, we estimate by OLS a regression of the realized log excess returns averaged across maturities on all the forward rates in $\mathbf{f}_t$,

$$\overline{rx}_{t+1} = \gamma_0 + \gamma' \mathbf{f}_t + \overline{\eta}_t. \tag{17}$$

In the second stage, we estimate the regressions

$$rx^n_{t+1} = a_n + b_n(\hat{\gamma}_0 + \hat{\gamma}' \mathbf{f}_t) + \eta^n_t, \tag{18}$$

where $[\hat{\gamma}_0, \hat{\gamma}]$ is the OLS estimator derived in the first stage. Expected excess returns on a ZCB maturing in n period can then be measured using the fitted values of this second stage regression. Indeed, from equation (18) we have that $\mathbb{E}_t[rx^n_{t+1}] = \alpha_n + \beta_n(\hat{\gamma}_0 + \hat{\gamma}' \mathbf{f_t})$.

We choose θ_1 so that the pricing model defined by the equations in (2) and (13) fits key properties of short term real interest rates and expected excess returns on a bond with residual maturity of five years $(n = 20)$. Specifically, we select ϕ_0 and ϕ_1 to match the sample mean and standard deviation of the yields on the German bonds with a residual maturity of three months. The remaining parameters are chosen to match, in model simulated data, the coefficients of an AR(1) model estimated on $x_t = \hat{\gamma}_0 + \hat{\gamma}' \mathbf{f}_t$, and the OLS point estimates of the parameters in equation (18), $[\hat{a}_{20}, \hat{b}_{20}, \hat{\sigma}_{\eta^{20}}]$. Appendix F reports the results of the Cochrane and Piazzesi (2005) regressions and it describes in more details the calibration of θ_1. Panel A of Table 1 reports the numerical values of the calibrated parameters.

We can also use the model's restrictions to construct an empirical counterpart to χ_t. Expected excess returns on long term bonds are affine in χ_t, implying that

$$\chi_t = \frac{\mathbb{E}_t[rx^n_{t+1}] - \tilde{A}_n(\theta_1)}{\tilde{B}_n(\theta_1)}, \tag{19}$$

with $\tilde{A}_n(.)$ and $\tilde{B}_n(.)$ defined in Appendix F. We can therefore construct the time path of χ_t by substituting in the right hand side of equation (19) the fitted values of equation (18).

We next turn to the calibration of θ_2. We fix σ to 2, and we set $\psi = 0.05$, a value that implies an average exclusion from capital markets of 5.1 years following a default, in line with the evidence in Cruces and Trebesch (2013). The tax rate is set to 0.41, equal to the sample mean of tax revenues over GDP. We normalize μ_y to 0.89, so that tax revenues equal to 1 in a deterministic steady state. The spending requirement $\overline{G}$ is set to 0.68. This number replicates the sample mean of the ratio of wages of public employes and transfers to tax revenues during the 1999-2012 period, our measure of non-discretionary spending.

We map $\hat{y}_t = (y_t - \mu_y)$ to the deviations of Italian log real GDP from a linear trend. The real GDP series is obtained from OECD *Quarterly National Accounts*. We estimate the process in equation (14) for the 1999:Q1-2012:Q2 period using this series and the series for

Table 1: **Model calibration**

Parameter	Value	Targets
Panel A: Pricing model		
ϕ_0	0.002	Mean of risk-free rate
ϕ_1	1.473	Standard deviation of risk-free rate
$\kappa_0 \times \sigma_\chi$	-0.053	Method of Simulated Moments
$\kappa_1 \times \sigma_\chi$	-95.125	Method of Simulated Moments
μ_χ	0.002	Method of Simulated Moments
ρ_χ	0.449	Method of Simulated Moments
σ_χ	0.003	Method of Simulated Moments
Panel B: Government's decision problem		
σ	2.00	Conventional value
ψ	0.050	Cruces and Trebesch (2013)
τ	0.410	Tax revenues over GDP
$\underline{G}$	0.680	Non discretionary spending over tax revenues
μ_y	0.892	Normalization
ρ_y	0.970	Estimates of equation (14)
σ_y	0.008	Estimates of equation (14)
$\sigma_{y\chi}$	-0.002	Estimates of equation (14)
$\frac{\exp\{\pi^*\}}{1+\exp\{\pi^*\}} \times 400$	1.628	Method of Simulated Moments
σ_π	1.350	Method of Simulated Moments
β	0.970	Method of Simulated Moments
d_0	0.045	Method of Simulated Moments
d_1	0.082	Method of Simulated Moments
α	0.400	Method of Simulated Moments
$\overline{d}$	6.750	Method of Simulated Moments

Notes: We reparametrize the $d(.)$ function in equation (15) in order to make our calibration comparable with previous research. The parameter d_0 stands for the percentage loss in output after a default when output is three standard deviations below its average value. The parameter d_1 represents the percentage loss in output when the latter is at its average value.

χ_t obtained earlier. As $\rho_{y\chi}$ is not significantly different from zero, we impose the restriction $\rho_{y\chi} = 0$. The point estimates of this restricted model are $\rho_y = 0.970$, $\sigma_{y\chi} = -0.002$ and $\sigma_y = 0.008$.

The remaining parameters, $[\beta, d_0, d_1, \pi^*, \sigma_\pi, \overline{d}, \alpha]$, are chosen to match key features of the behavior of Italian public finances. We include in the set of empirical targets statistics that summarize the behavior of outstanding debt, interest rate spreads, and debt maturity. See

Appendix D for a detailed description of the data series.[16] Specifically, we consider the sample mean of the debt to output ratio, the correlation between the government deficit and detrended real GDP, and the mean, standard deviation and skewness of interest rate spreads on Italian ZCB with a residual maturity of five years. We also incorporate in the targets the sample mean and the standard deviation of an indicator of debt maturity, the weighted average life of Italian outstanding bonds.[17] These moments provide information on the parameters of the adjustment cost function.

There is, instead, little guidance in the literature on the choice of variables that provide information on π^* and σ_π. Our approach consists in targeting the adjusted R^2 of the following regression,

$$\begin{aligned} \text{spr}_t \;=\;& a_0 + a_1 \text{gdp}_t + a_2 \text{debt}_t + a_3 \hat{\chi}_t + a_4 \text{dur}_t + a_5(\text{gdp}_t \times \text{debt}_t) + a_6(\text{gdp}_t \times \hat{\chi}_t) \\ & + a_7(\text{gdp}_t \times \text{dur}_t) + a_8(\text{debt}_t \times \hat{\chi}_t) + a_9(\text{debt}_t \times \text{dur}_t) + a_{10}(\hat{\chi}_t \times \text{dur}_t) + e_t. \end{aligned} \tag{20}$$

The residual e_t measures variation in interest rate spreads that is orthogonal to the fundamental state variables in the model, and it should therefore provide information about the volatility of π_t. We estimate equation (20) by OLS, obtaining an adjusted R^2 of 82%.[18]

Because the numerical solution of the model is computationally costly, we first experiment with these seven parameters to obtain a range of values that is empirically relevant. We fix π^* to -5.5, a value that implies a 1.6% annualized probability of a rollover crisis conditional on the economy being in the crisis zone next period. We next solve the model on a grid of points for $[\beta, d_0, d_1, \sigma_\pi, \alpha, \overline{d}]$, and select the parametrization that minimizes a weighted distance between sample moments and their model implied counterparts. Model implied moments are computed on a long simulations ($T = 20000$), and we weight the distance between a sample moment and its model counterpart by the inverse of the sample moment absolute value. Appendix G presents the algorithm used for the numerical solution and for model simulations. Panel B of Table 1 reports the calibrated values for the model's parameters.

[16] Public debt is defined as the face value of outstanding bonds of the Italian central government. We use this indicator rather than the face value of outstanding debt because the latter includes components (such as direct loans by intermediaries) that are less subject to the rollover problem studied in this paper. We scale this indicator by annualized GDP at current prices.

[17] We use data from the Italian Treasury and construct the weighted average of the times of principal and coupon payments for outstanding bonds issued by the Italian central government. This indicator maps to $1/\lambda'$ in our model.

[18] The high explanatory power is mostly due to output, debt, and to their interaction. When including only these three terms in the regression, we obtain an adjusted R^2 of 68%.

4.3 Model fit

The first and second columns of Table 2 show that the model has good in sample fit. The face value of debt is 82% of annual GDP on average, close to the 88% in the sample. As in the data, the government deficit is negatively correlated with output. Interest rate spreads are typically close to zero in model simulations, with an annualized average value of 0.96% relative to the 0.59% observed in the sample. However, they can experience large spikes. Indeed, the distribution of interest rate spreads is right skewed, with a standard deviation that is in line with what observed in the Italian data. The model generates an empirically plausible relation between interest rate spreads and economic fundamentals, as captured by the R^2 of equation (20): 0.61 in the model relative to 0.82 in the data. Finally, debt maturity in model simulations is on average 6.79 years, with a standard deviation of 0.29. In the data, they are, respectively, 6.81 years and 0.16.

Table 2: **Calibration targets**

Statistic	Data	Baseline	$\underline{G} = 0, \beta = 0.90$
Average debt-to-output ratio	88.38	81.58	58.11
Correlation deficit and output	-0.25	-0.19	0.21
Average spread	0.59	0.96	10.23
Stdev of spread	1.16	1.68	8.29
Skewness of spread	2.53	8.52	1.96
Adj. R^2 of regression (20)	0.82	0.61	0.30
Average debt maturity	6.81	6.79	6.75
Stdev of debt maturity	0.16	0.29	0.05

Notes: See Appendix D for data definitions and sources. Sample moments are computed over the 1999:Q1-2012:Q2 period. Moments in the model are computed as described in Appendix G.1.

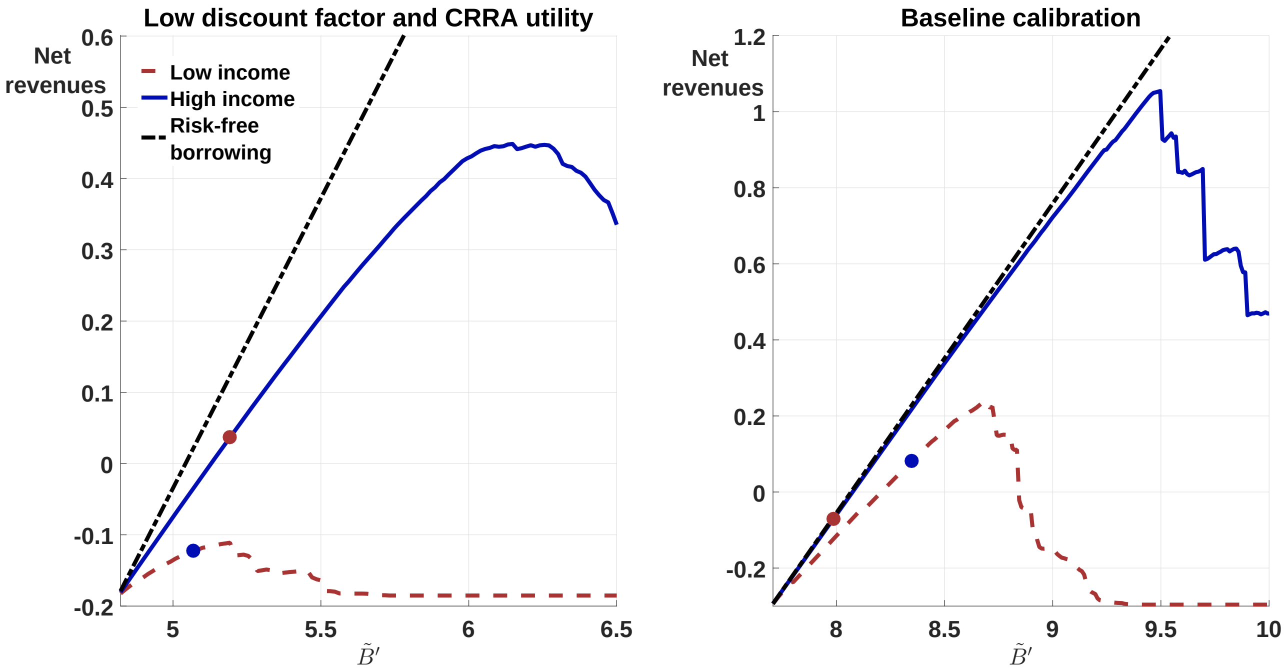
It is important to stress that our calibration differs substantially from the one typically pursued in the literature. Earlier studies that have fit this class of models to emerging markets produce pro-cyclical government deficits, with the government borrowing more when hit by positive y_t shocks. Moreover, in those calibrations the government is most of the times at risk of a default, with interest rate spreads being far away from zero even when output is above average. For example, in Chatterjee and Eyigungor (2012) the correlation between net exports (equal to the government surplus) and output is -0.44, while annualized interest rate spreads are on average 8%.[19] While consistent with the experience of emerging market economies, this pattern would be inconsistent with the Italian data, as interest rate

[19] The sovereign debt literature typically calibrates the model to match properties of total external debt, not public debt as in our case. The procyclicality of government's deficit is then essential to match, in a model without investment, the countercyclicality of net exports observed in the data.

spreads averaged few basis points prior to 2008, and debt increased by 20% during the 2008-2012 recession.

The main point of departure between our calibration and the one pursued in the literature lies in the parametrization of β and $\underline{G}$. Figure 1 illustrates this point by plotting the revenues from debt issuances net of coupon payments, $\Delta(\mathbf{S}, B', \lambda') - B$, as a function of the face value of debt, $\tilde{B}' = B'/\lambda'$. The figure plots this schedule for two output levels, "high" and "low", keeping the remaining state variables at their ergodic mean. For comparison, we also report the net revenues constructed under the assumption that the government can borrow at the risk-free rate.

Figure 1: **The cyclicality of debt issuances**



Notes: The figure reports $\Delta(\mathbf{S}, B', \lambda'^*) - B$ as a function of $\tilde{B}'/\lambda'^*$. The solid blue line plots this schedule for $y = 0.04$, while the dotted red line for $y = -0.04$. In both cases, we set the remaining state variables at their ergodic mean. The filled dots reports the net revenues associated to optimal debt and maturity choices for the government at $\mathbf{S}$. The dashed black line reports the net revenues under the assumption that the government can borrow at the risk-free rate. The left panel of the figure reports this information for a calibration that sets $\beta = 0.90$ and $\underline{G} = 0.00$. The right panel reports it for the baseline calibration of Table 1.

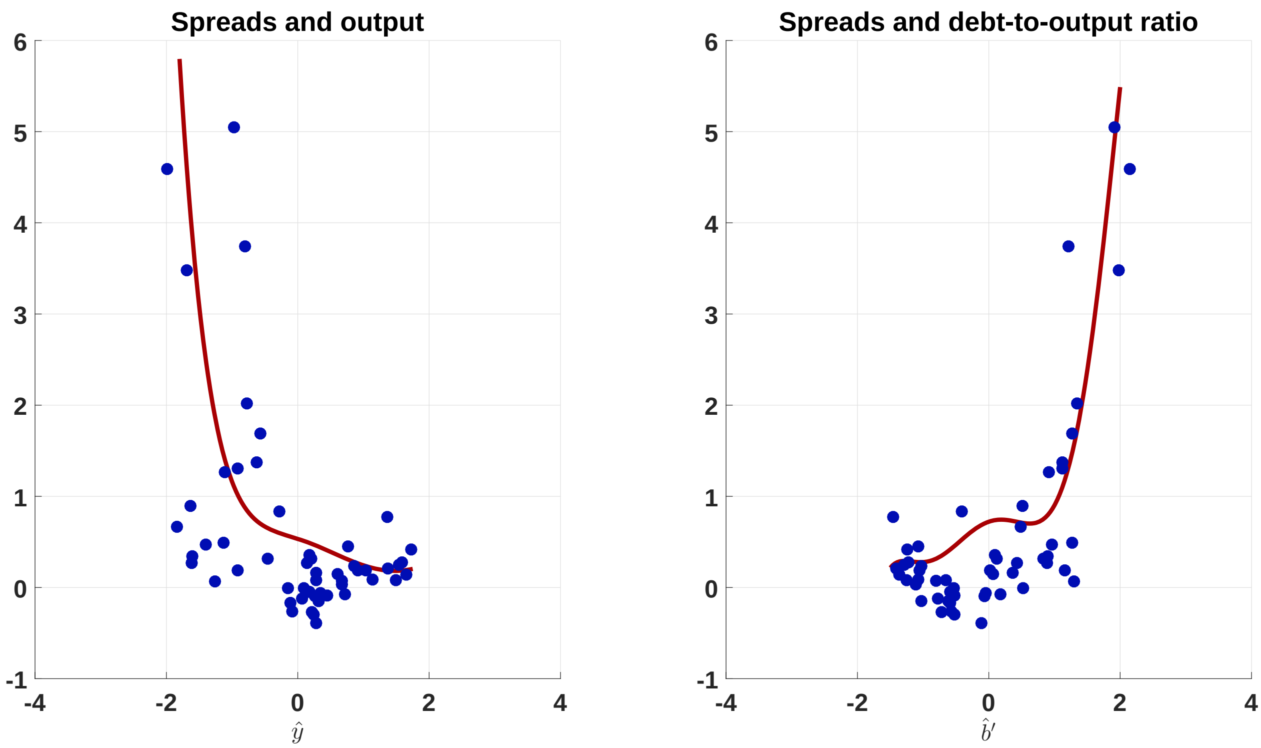
The left panel of the figure reports this information for a calibration that sets $\beta = 0.90$ and $\underline{G} = 0.00$. This mimics the typical calibration in the literature which sets a low discount factor for the government and a utility function that features constant relative risk aversion. We can observe that the net revenue schedule defines a "Laffer curve" for debt issuances. At low levels of $\tilde{B}'$, the government is able to increase the revenues he obtains from the lenders by issuing more debt. However, as $\tilde{B}'$ increases, the government becomes more at risk of a default in the future: lenders demand higher yields for holding government bonds, and the price of newly issued debt declines. This explains why net revenues increase in $\tilde{B}'$ at a decreasing rate. Eventually, the decline in bond prices dominates any increase in issued debt, and the net revenue schedule decreases with $\tilde{B}'$. Importantly, the figure also shows that the government can raise less revenues when income is low. This latter property is due to the persistence of the income process, and to the fact that the government defaults when income is low enough. Therefore, low income states are associated with a higher risk of a default in the future, lower prices for newly issued debt, and lower revenues for the government.

The filled circles in Figure 1 represent the face value of debt chosen by the government, $\tilde{B}'^{*}$, along with the associated net revenues, $\Delta(\mathbf{S}, B'^{*}, \lambda'^{*}) - B$. In the calibration typically pursued in the literature, the government is extremely impatient relative to the risk-free rate. Thus, it behaves myopically, and it uses the debt market to frontload consumption rather than to smooth it across states of the world. Because of this feature, the cyclicality of debt issuances mirrors that of the pricing schedule. In high income states, the risk of a default is small, and the government is able and willing to issue debt. Conversely, low income states are associated with tighter pricing and revenues schedules, and to less borrowing by the government. Hence, the impatience of the government coupled with the endogenous borrowing limits implied by default risk leads to procyclical debt issuances. Importantly, this behavior has implications for the interest rate spreads too. Because the government has strong incentives to borrow, it is always at risk of a default, and interest rate spreads are well above zero even in good times. One can verify that by noting that the slope of the "high income" Laffer curve evaluated at the optimal choices lies below the risk-free rate.

The right panel of Figure 1 reports the same information in our baseline calibration. The revenue schedules are qualitatively similar in the two parametrizations: they both define a Laffer curve for debt issuances, and they both shift out when the economy is hit by positive income shocks. The key difference is on the debt choices made by the government. In our parametrization, the government uses the debt markets mostly for consumption smoothing because it is more patient, and because the non-homotheticity in the utility function leads to stronger precautionary motives. Thus, the government pays back its debt in high income states, while it borrows in the face of bad income shocks. This behavior has two implications. First, and consistent with the Italian data, debt issuances are countercyclical. Second, the model generates interest rate spreads that are on average close to zero, and they jump to positive values only conditional on sufficiently low income realizations.

This latter implication can be better appreciated by looking at Figure 2. The solid line in the left panel reports the average relation between interest rate spreads and output in our baseline calibration while the dots report combinations of these two variables in the data. The model implied elasticity of interest rate spreads to output is highly nonlinear: in good times, interest rate spreads are close to zero and a decline in output in this region has essentially no effects on interest rate spreads. However, this elasticity achieves a value of -6 when output is one standard deviation below its mean. This elasticity is empirically plausible in terms of shape and magnitude. The right panel of the figure plots the same information for debt. Again, the implied elasticity of interest rate spreads to the debt-to-output ratio is highly nonlinear, and it well captures the relation between these two variables in the data. This state dependence in interest rate elasticities is what generates the right skewness of the interest rate spreads distribution documented in Table 2.

Figure 2: **Interest rate spreads sensitivity to output and debt**

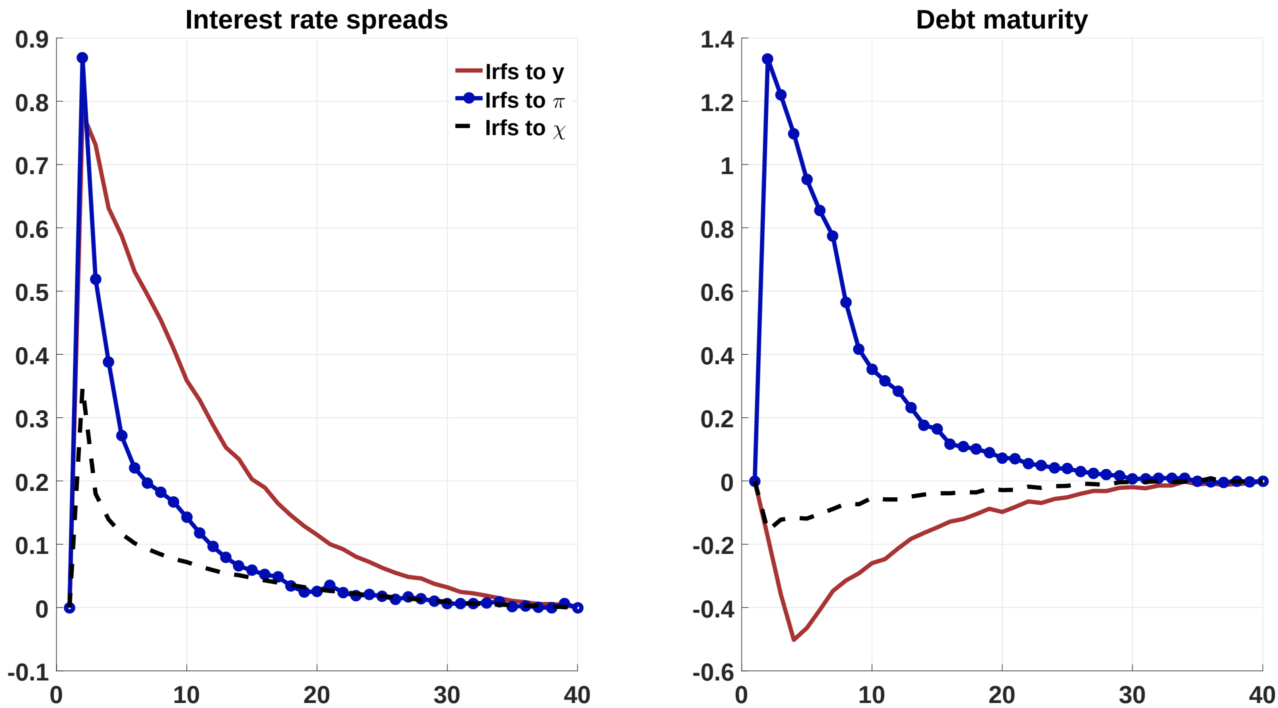


Notes: The left panel is constructed as follows. We simulate a $T = 20000$ *realization from the model. After standardizing the output series, we fit on the model simulated data a Chebyshev regression of polynomials up to the* 7^{th} *order. The red solid line represents the fitting curve of this regression. The dots are combinations of interest rate spreads and linearly detrended GDP (standardized) in the data. The right panel reports the same experiment, this time for the debt-to-output ratio.*

4.4 Sources of default risk and maturity choices

We now use impulse response analysis to discuss the behavior of interest rate spreads and debt maturity in the calibrated model. We start by studying how these two variables respond to shocks that increase the risk of a government default. We consider two scenarios. In the first scenario, given by the solid lines in Figure 3, we study the effects of a three standard deviations decline in output while setting $\pi_t = 0$ for all t. This first experiment captures the behavior of interest rate spreads and debt maturity conditional on an increase in fundamental default risk. In the second scenario, instead, we consider a persistent increase in π_t when the economy is currently in the crisis zone. This second experiment approximates the behavior of interest rate spreads and debt maturity conditional on an increase in rollover risk.

Figure 3: **The dynamics of interest rate spreads and debt maturity**



Notes: The solid line reports impulse response functions (IRFs) of interest rate spreads and debt maturity to a 3 standard deviations decline in y. The circled line reports the response to a 3 standard deviations increase in π, assuming a decaying rate of 0.95 for the shock. The dotted line reports the IRFs to a 3 standard deviations increase in χ. The IRFs to y and χ are initialized by setting the state variables at their ergodic mean, holding $\pi_t = 0$ for all t. The IRFs to π are initialized by setting the state variables at their mean conditional on the government being in the crisis zone. The IRFs are computed by simulations, see Appendix G. Interest rate spreads are expressed in annualized percentages while the weighted average life of outstanding bonds in years.

These two shocks increase the risk of a government default: in both cases, interest rate spreads on government bonds increase. However, the two impulses have different implications for the maturity structure of government debt. In the first experiment, where the increase in the risk of a government default is purely due to bad economic fundamentals, the government shortens the maturity of its debt. This is because the incentive benefits of short term debt becomes more valuable when the economy approaches the default zone: in our simulations, the average life of outstanding debt drops by 0.5 years on impact following the negative output shock.

In the second experiment, instead, the increase in the risk of a default occurs because of an increase in π_t. The government responds to this increase in the risk of a rollover crisis by lengthening its debt maturity: the average life of outstanding debt increases by 1.3 years in our simulations. As explained earlier, lengthening debt maturity is optimal in this circumstance because it reduces the exposure of the government to a future rollover crisis. These results confirms, in the calibrated model, the discussion of Section 3. Debt maturity responds differently depending on whether the increase in interest rate spreads is due to bad economic fundamentals or whether it is due to heightened rollover risk.

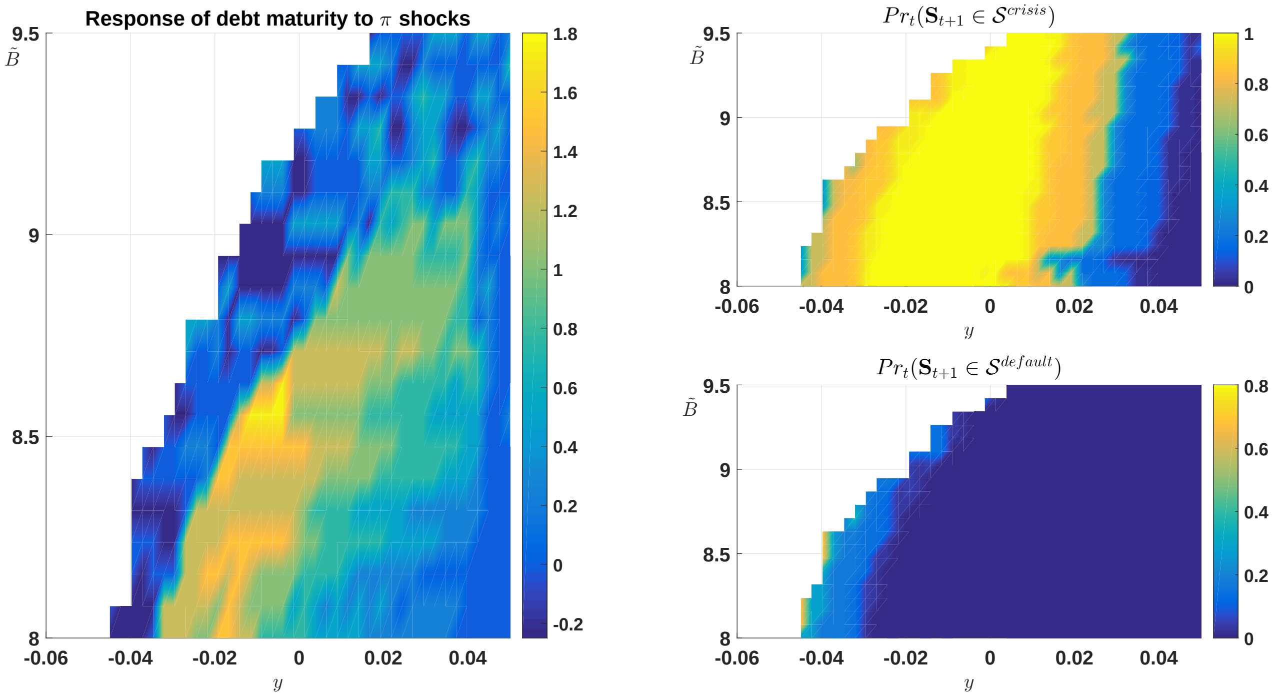
It is important to stress that these simulations reflect the *average* response of debt maturity to an increase in π_t. This response, however, is state dependent in the model. This could be problematic for our measurement strategy because there could be states under which debt maturity responds little to changes in π_t, making it difficult to detect rollover risk based on observed maturity choices.

Figure 4 explores this state dependency. In the left panel we report the (impact) response of debt maturity to a 3 standard deviations increase in π_t, as a function of the face value of inherited debt and output. Warmer colors in the heat map means that debt maturity responds more to an increase in π_t. We can see how this elasticity varies substantially with the level of inherited debt and output. In particular, the government does not respond to the increase in π_t when y is large enough and/or debt is sufficiently small. As we move toward a low output/high debt region, though, the government reacts more to the change in π_t, increasing its debt maturity up to 1.8 years. For sufficiently low output levels, however, the government changes little its debt maturity after the π_t shock.

To understand this non-monotonicity, the right panels of Figure 4 report $\Pr_t(\mathbf{S}_{t+1} \in \mathcal{S}^{\text{crisis}})$ and $\Pr_t(\mathbf{S}_{t+1} \in \mathcal{S}^{\text{default}})$ just before the π_t shock hits. In high output/low debt states, the government is unlikely to be in the crisis zone in the future. Hence, it has limited benefits from managing debt maturity after the increase in π_t. The same happens when the government has very low tax revenues or if it inherits a very large amount of debt. In those situations, the government ends up being at risk of a rollover crisis even when it issues the longest maturity in our grid. Moreover, in those states the government approaches the default zone, and lengthening debt maturity worsens the time inconsistency problem discussed in Section 3. Both of these forces curb the government's incentives to extend debt maturity after the increase in π_t.

For intermediate values of inherited debt and y, the response is the highest because the probability of being in the crisis zone is sensitive to changes in debt maturity, and the time inconsistency problem is not a concern as the government is far from the default boundary. Importantly, those are also the states where rollover risk accounts for a large share of interest rate spreads. Therefore, we should expect maturity choices to be more informative, and our

Figure 4: **The sensitivity of debt maturity to π_t across the state space**



Notes: The left panel reports the response of debt maturity to an increase in π_t, that is $1/\lambda(B, \lambda, y, \chi, \pi = .3) - 1/\lambda(B, \lambda, y, \chi, \pi = 0)$, as a function of B and y, fixing λ and χ at their ergodic mean. The white area corresponds to states in which the government defaults. The right panels report the probability of being in the crisis and default zone next period for the $\pi = 0$ case.

measurement more precise, when rollover risk matters the most.

We finally discuss how debt maturity choices are affected by a change in the compensation that lenders demand to hold long term bonds. The dashed lines in Figure 3 plots the response of interest rate spreads and debt maturity conditional on a 3 standard deviations increase in χ_t. As explained earlier, this shock increase the risk premium on long term assets. Accordingly, the government responds to this impulse by decreasing the average life of its outstanding debt. If not properly accounted for, these changes in χ_t may be problematic for our measurement strategy: debt maturity may change little, or even go down, after an increase in π_t if at the same time risk premia on long term assets increase. Therefore, an integral part of our measurement strategy consists in making sure that the model correctly reproduces the path of risk premia on long term bonds in the episode under analysis.

5 Decomposing Italian spreads

We now turn to the main quantitative experiment of the paper, and measure the importance of rollover risk during the debt crisis of 2008-2012. In Section 5.1 we combine the calibrated model with the data in order to retrieve the path for the exogenous shocks $\{y_t, \chi_t, \pi_t\}$. We use this path to measure the rollover risk component of interest rate spreads. Section 5.2 performs a robustness analysis.

5.1 Measuring rollover risk

The model defines the nonlinear state-space system

$$\begin{aligned} \mathbf{Y}_t &= \mathbf{g}(\mathbf{S}_t) + \eta_t \\ \mathbf{S}_t &= \mathbf{f}(\mathbf{S}_{t-1}, \varepsilon_t), \end{aligned} \tag{21}$$

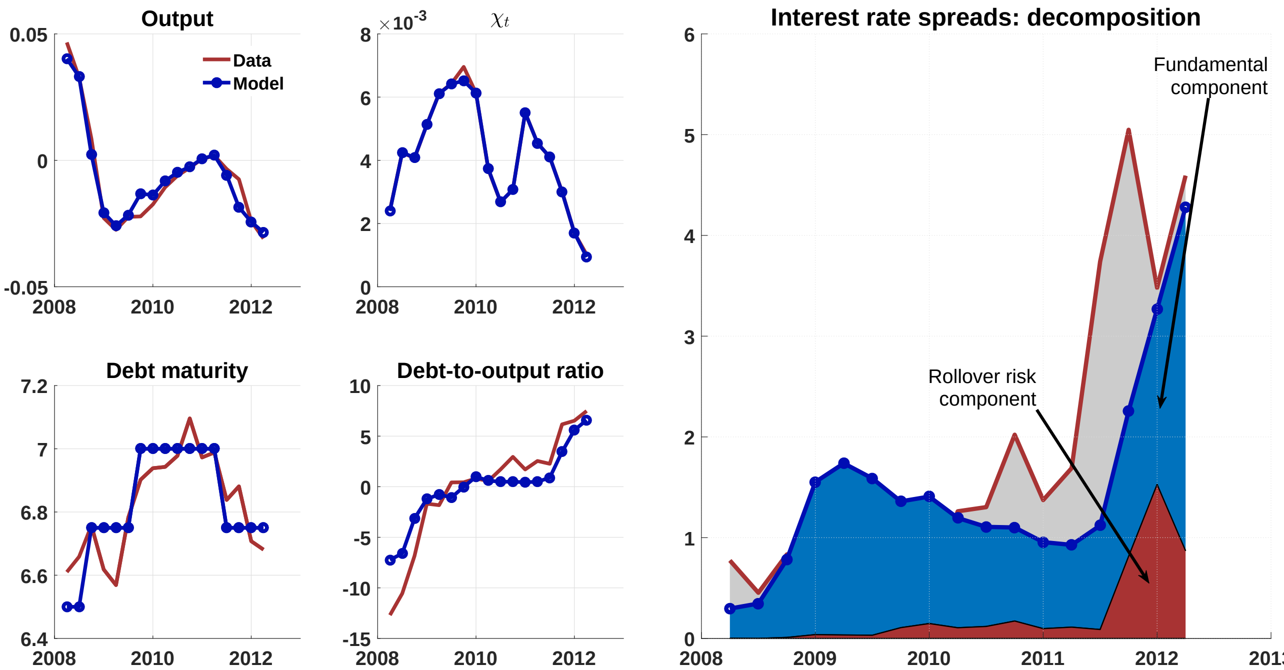
with $\mathbf{Y}_t$ being a vector of observable variables, $\mathbf{S}_t = [B_t, \lambda_t, y_t, \chi_t, \pi_t]$ the state vector, and ε_t the vector collecting the structural shocks. The vector η_t contains uncorrelated Gaussian errors. The functions $g(.)$ and $f(.)$ are obtained using the model's numerical solution.

The vector of observables includes detrended real GDP, the data counterpart to χ_t constructed using equation (19), the interest rate spread series, and the data counterpart to λ'. Given the time path of these variables over the 2008:Q1-2012:Q2 period, we estimate the realization of the state vector using the relation between states and observables implied by the system in (21). Technically, we carry out this step by applying the auxiliary particle filter to the above state-space model, see Appendix H for a description. It is important to stress that the inference on $[y_t, \chi_t]$ in our approach is disciplined by actual observations because the measurement equation incorporates the empirical counterparts of these shocks. The truly unobservable process is π_t.

Equipped with the path for the exogenous shocks, we next measure the contribution of rollover risk to interest rate spreads as defined in Section 3. We do so by feeding the model with a realization for the structural shocks equal to the one obtained earlier, with the exception that π_t is set to 0 throughout the sample. Therefore, rollover risk in this counterfactual is by construction zero, and the implied interest rate spreads reflect exclusively the impact of economic fundamentals. We label this the *fundamental component* of interest rate spreads. The difference between the filtered interest rate spread series and the counterfactual one nets out the impact of rollover risk. Accordingly, we label this difference the *rollover risk component* of interest rate spreads. Importantly, the model implied interest rate spreads are not necessarily equal to the one in the data because the system in (21) has more observables than structural shocks. Any difference between the observed interest rate spreads and the one produced by the model is captured by the errors $\eta_{\text{spread},t}$.

Figure 5 reports the results of this experiment. The solid lines in the left panels report the data series for output, χ_t, and debt maturity, while the circled lines the trajectories filtered with the structural model. The model replicates the time path of these observables accurately. We can also see that the model generates a trajectory for the debt-to-output ratio that tracks closely the one in the data, even though we did not include this variable in the set of observables. The right panel of Figure 5 reports observed interest rate spreads (solid line) along with their decomposition: the fundamental component (blue area), the rollover risk component (red area), and the residual component that we attribute to $\eta_{\text{spread},t}$ (light gray area). Overall, the model fits well Italian interest rate spreads during the event, although it cannot account for their sharp increase during the second half of 2011.

Figure 5: **Decomposition of interest rate spreads**



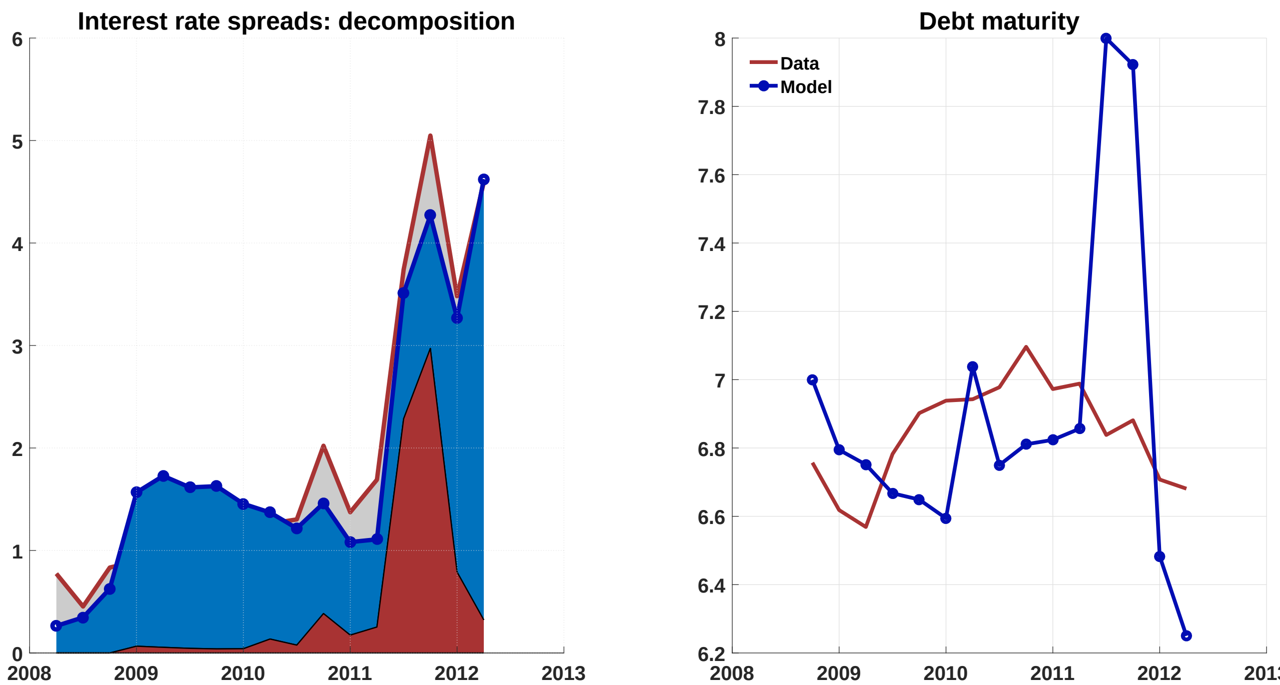
Notes: The left panels report Italian detrended real GDP, the χ_t series, the weighted average life of outstanding bonds, and the debt to output ratio in the data (solid lines) and associated filtered ones (circled lines). The debt to output ratio is de-meaned. The right panel plots yields differentials between an Italian and a German ZCB with a residual maturity of five year (circled line), along with its decomposition. See Appendix H.

Most of the increase in interest rate spreads during the episode is attributed to fundamental shocks. The fundamental component was essentially zero in 2008:Q2, while it was roughly 90% of observed interest rate spreads at the end of the episode. This pattern is the result of two main developments. First, the Italian economy experienced a major recession during this period: output went from being 5% above trend in 2008:Q1 to being 3% below trend at the end of the sample. Second, Italian debt increased by 20% of GDP over the 2008:Q1-2012:Q2 period, a fact that is captured by our model. Both of these developments push the government closer to the default zone, increasing in this fashion the fundamental component of interest rate spreads. Importantly, the increase in debt explains why output shocks have larger effects toward the end of the sample: the innovations to y in 2008:Q3 and 2012:Q2 are comparable in size, but their impact on interest rate spreads is higher in the latter period because the government was more levered at that point in time.

The model assigns a more limited role to rollover risk, on average 12% of the model implied interest rate spreads. This despite the fact that π_t is unobservable in our experiment, and it could be used in principle to fit variation in interest rate spreads that cannot be explained by the observable fundamental shocks. For example, the model has hard time fitting the increase in spreads observed in 2011 with fundamental shocks because output was recovering at the time. However, it attributes it to $\eta_{\text{spread},t}$ rather than to rollover risk.

In principle, there are two different explanations for this latter result. First, it might be that the Italian economy was far from the crisis zone in 2011, in which case changes in π_t would have limited effects on interest rate spreads. Second, it might be that the increase in π_t necessary to fit interest rate spreads would have counterfactual implications for other variables in $\mathbf{Y}_t$, specifically debt maturity. To further explore this issue, we repeat the experiment excluding debt maturity from the set of observable variables.

Figure 6: **Decomposition of interest rate spreads: no debt maturity**



Notes: The left panel plots yields differentials between an Italian and a German ZCB with a residual maturity of five year (solid line), along with its decomposition. The right panel plots the weighted average life of outstanding bonds in the data (solid line), and the one filtered by the model (circled line). See Appendix H.

Figure 6 reports the results. The left panel of the figure plots interest rate spreads along with their decomposition. Not surprisingly, the model now tracks more closely interest rate spreads, especially in 2011. Most of this improvement in fit is due to rollover risk. The right panel of the figure plots the model implied behavior for debt maturity in this experiment (circled line), along with the data counterpart (solid line). We can observe that heightened rollover risk in 2011 is associated to an increase in the average life of outstanding debt of 1 year, which is counterfactual because this indicator declined by 0.2 years in the data during the same period.

This experiment further clarifies the role of maturity choices in our measurement strategy. Absent data on debt maturity, the model has limited identifying restrictions to discipline the time path of π_t, and it attributes to this term variation in interest rate spreads that cannot be accounted by the fundamental shocks. By conditioning on observed maturity choices, instead, we impose discipline on the rollover risk component. Realizations of the state vector for which rollover risk accounts for a sizable fraction of spreads in 2011 imply an increase in the maturity of Italian debt. This variable, however, follows the opposite pattern in the data. Because of that, our measurement assigns a more limited role to this component.

5.2 Robustness

A major concern one could have at this stage is that of misspecification. The model we studied may miss potentially important determinants for debt maturity. Our approach could be understating the role of rollover risk if these omitted forces generated an incentive for the government to shorten debt maturity at the height of the debt crisis. In what follows, we perform a suggestive calculation and ask how sizable these omitted factors should be to overturn our findings.

To this end, we identify in the model a set of states $\mathbf{S}_t$ and debt choices B_t that replicate key features of the Italian economy in 2011:Q4 *and* generate a sizable role for rollover risk. We then quantify the welfare gains of extending debt maturity in such a scenario. Small gains would indicate that our result could be easily overturned by omitting factors, while large gains would imply that our results are more robust to misspecification. This experiment is carried out as follows. We set y_t, χ_t, λ_t, and λ_{t+1} to their data counterpart, and we consider pairs of (B_{t+1}, π_t) that replicate the observed Italian spreads in 2011:Q4. We then select the pair with lowest B_{t+1} and the highest π_t. This combination guarantees that rollover risk is the major driver of interest rate spreads in this experiment. Finally, we set B_t so that government deficit in the model equals to 3.5% of GDP, the Italian deficit in 2011:Q4. We then measure the net gains/losses that the government derives from a pure change in debt maturity - a variation in λ_{t+1} while adjusting B_{t+1} to keep current government spending constant.

In Panel A of Table 3 we report the certainty equivalent consumption associated with three portfolios of ZCB: an average maturity of 6.7 years (the actual choice in 2011:Q4), 5.5 years (the lower bound in our grid), and 8 years (the upper bound in our grid).[20] We can verify that welfare is increasing in the maturity of the debt portfolio. By lengthening the maturity of the stock of debt from 6.7 years to 8, the government would increase its certainty equivalent consumption by 0.80%. Hence, our baseline calibration suggests large welfare gains from increasing debt maturity when the government faces a substantial risk of a rollover crisis. This result indicates that omitted forces must be large for the observed decline in debt maturity to be consistent with a sizable role for rollover risk.

The reason why the model produces welfare gains of this magnitude can be explained as follows. First, in our experiment π equals 0.21. Thus, a government falling in the crisis zone next period faces a substantial risk of a rollover crisis. Second, in our calibration a default induces large output losses (roughly 8% of GDP), this making rollover crises costly from the perspective of the government. Third, lengthening debt maturity from 6.7 years to 8 years reduces the probability of falling in the crisis zone next quarter (from 71% to 0% in this experiment), thus having a first order effect on welfare.

A related concern is that our result could be sensitive to the way we modeled the pricing of risky long-term debt. In our approach, we posit a stochastic discount factor which is calibrated to match the behavior of the German (default-free) term structure. One may argue that the pricing of German long term bonds is not informative about risk premia on Italian securities, either because markets are segmented or because interest rate risk is fundamentally different than default risk. If this was the case, our approach could understate the costs of lengthening the debt maturity if issuing long term bonds was more costly than what implied by our stochastic discount factor, and it would bias upward the calculation for the net gains in Panel A of Table 3.

We can address this concern by performing a similar calculation to the one discussed above. Specifically, we can ask how sizable premia on long term bonds needed to be in 2011:Q4 to make the government *not* willing to lengthen debt maturity in presence of high rollover risk. To this end, we consider the following variant of the pricing equation (6):

$$q_n\left(\mathbf{S}, B', \lambda'\right) = \delta(\mathbf{S}) \frac{M}{1+\alpha_n M} \mathbb{E}\left[\delta(\mathbf{S}') q'_{n-1}\right] \tag{22}$$

[20]The certainty equivalent for a portfolio $\tilde{\lambda}$ is computed as follows. Given $\mathbf{S}_t$ and $(B'(\tilde{\lambda}), \tilde{\lambda})$, we first compute the value for the government, $V^{B'(\tilde{\lambda}),\tilde{\lambda}}$. The certainty equivalent is G^* solving

$$\frac{1}{1-\beta} \frac{(G^* - \underline{G})^{1-\sigma}}{1-\sigma} = V^{B'(\tilde{\lambda}),\tilde{\lambda}}.$$

Table 3: **Gains from lengthening debt maturity in presence of high rollover risk**

Panel A: Baseline Model			
	Debt Maturity		
	5.5 years	6.7 years	8 years
Baseline	0.9114	0.9144	0.9217
Panel B: Pricing Function (22)			
$\gamma_1 = 0.00\%$	0.9192	0.9178	0.9222
$\gamma_1 = 0.50\%$	0.9119	0.9111	0.9174
$\gamma_1 = 1.00\%$	0.9059	0.9054	0.9075
$\gamma_1 = 1.50\%$	0.9034	0.9025	0.9055
$\gamma_1 = 1.75\%$	0.9040	0.9006	0.8967

Notes: The table reports the certainty equivalent consumption for the government as a function of debt maturity in states of the world characterized by high rollover risk. See the text and footnote 20 for the details of the calculation. Panel A reports this information for the baseline model. Panel B reports this information for the model with the alternative pricing function defined in equation (22).

where M is the inverse of the risk-free rate, assumed constant, and the parameter α_n controls the excess returns on a ZCB that matures in n periods relative to the risk-free rate. In fact, conditional on not having a default today, the expected return for holding this bond equals

$$\frac{\mathbb{E}\left[\delta(\mathbf{S}')q'_{n-1}\right]}{q_n\left(\mathbf{S}, B', \lambda'\right)} = \frac{1}{M} + \alpha_n.$$

Rather than parametrizing $\{\alpha_n\}_{n\geq 1}$, we let $\gamma(\lambda)$ be the expected excess return on a portfolio of ZCB with weighted average life equals to $1/\lambda$ quarters,

$$\gamma(\lambda) = \lambda\left(\sum_{j=1}^{\infty}(1-\lambda)^{j-1}\alpha_j\right),$$

and express it as a linear function of λ,

$$\gamma(\lambda) = \gamma_0 + \frac{\gamma_1}{400}(4\times\lambda)^{-1}.$$

The parameter γ_1 represents the slope of the expected excess return curves. If the government lengthens the maturity of its outstanding debt by one year, it must pay an additional return of γ_1 percentage points in annualized terms. Our objective is to measure how large γ_1 should be in order to counteract the gains from lengthening debt maturity reported in panel A of Table 3.

Panel B of Table 3 replicates the previous exercise with this alternative lenders' discount factor. As a baseline, we set γ_0 so that the excess return for the portfolio with the lowest maturity on the grid equals zero. All the parameters not connected to the pricing functions are set in the same way as in the baseline, while we set $1/M$ to the average risk-free rate implied by our calibration. We solve the model for different values of γ_1, select states and portfolio choices as we described earlier for the $\gamma_1 = 0.00\%$ case, and we finally calculate the certainty equivalent consumption for portfolios with an average life of 5.5, 6.7, and 8 years.

Our results show that it is advantageous for the government to choose the highest maturity on the grid (8 years) as long as γ_1 is smaller than 1.75%. That is, in order to rationalize a shortening of debt maturity in presence of high rollover risk, the expected return on the 8 years portfolio must be at least 4% higher than the one for the 5.5 years portfolio.

To understand whether these values of γ_1 are empirically plausible, we follow Broner, Lorenzoni, and Schmukler (2013) and calculate realized holding period returns for Italian government bonds of maturities up to 10 years for the period 1999:M1-2011:M9.[21] We then calculate the empirical counterpart to $\gamma(\lambda)$ as

$$\gamma_{\lambda,\text{data}} = \lambda \left[\sum_{j=1}^{39} (1-\lambda)^{j-1} \overline{R}_j + \frac{(1-\lambda)^{40-1}}{\lambda} \overline{R}_{40} \right], \tag{23}$$

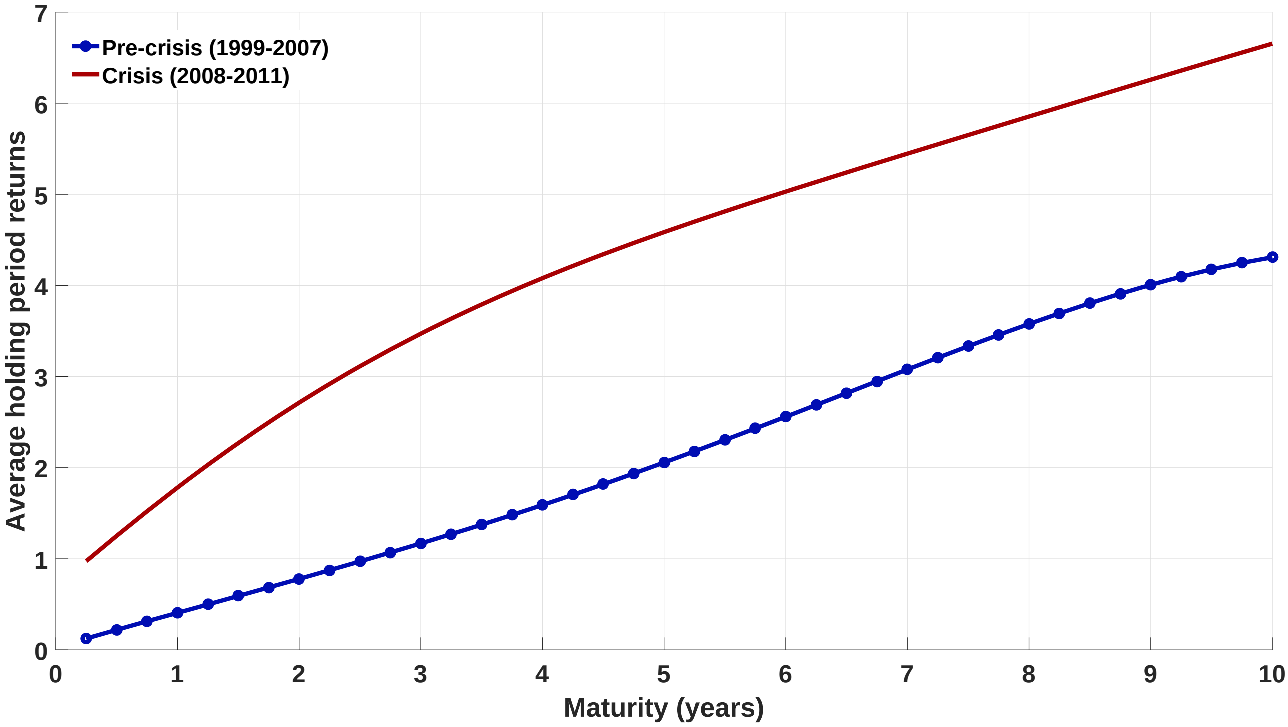
where $\overline{R}_j$ is the average holding period return for a bond with a residual maturity of j quarters relative to the one period risk-free return, approximated with the yields on a German government bond with a residual maturity of one quarter.

Figure 7 plots $\gamma_{\lambda,\text{data}}$ as a function of the maturity of the debt portfolio (expressed in years) for two different sub-periods: a pre-crisis period (1999-2007), and a crisis period (2008-2011). The figure shows that the level and the slope of average realized excess returns increase in the second sub-period.[22] Therefore, the concern that an increase in term premia may invalidate our result finds support in the data. However, it is evident that the minimal slope necessary to justify a shortening of debt maturity in presence of high rollover risk is substantially larger than the one observed in the data. In the crisis period, the average realized excess returns on the 8 years portfolio are only 1% higher than the one on the 5.5 years portfolio, four times smaller than the one implied by $\gamma_1 = 1.75\%$. We interpret this result as suggesting that risk premia on long term bonds must be implausibly large to curb the incentives to lengthen debt maturity in the presence of a sizable role for rollover risk.

[21] We end the crisis period in 2011:M9 in order to minimize the impact of the extremely low excess returns observed during the summer of 2011 on the sample averages.

[22] Interestingly, the measured average realized excess returns on Italian bonds during the crisis period are of the same order of magnitude than the ones documented by Broner, Lorenzoni, and Schmukler (2013) for emerging market economies.

Figure 7: **Term structure of holding period returns for Italian government debt**



Notes: We compute holding period returns for zero-coupon equivalent Italian government securities with residual maturities between one and forty quarters. We then take the difference between these holding period returns and the yields on a German government security with a residual maturity of one quarter. We average over time these excess holding period returns and use equation (23) to construct the empirical counterpart of $\gamma(\lambda)$. The figure plots the implied γ as a function of debt maturity, expressed in years. The circled line report this information for the 1999-2007 period, while the solid line reports the same information for the 2008-2011 period. See Appendix D for data definitions and sources.

6 Evaluating OMT announcements

We now turn to analyze the effects of the Outright Monetary Transaction (OMT) program through the lens of our model. As a response to soaring interest rate spreads in the euro-area periphery, the Governing Council of the European Central Bank (ECB) announced during the summer of 2012 that it would consider outright bond purchases in secondary sovereign bond markets. The technical framework of these operations was formulated on September 6 of the same year. The OMT program replaced the Security Market Program as a mean through which the ECB could intervene in sovereign bond markets.[23]

Even though the ECB never purchased government bonds within the OMT framework, the mere announcement of the program had significant effects on interest rate spreads of

[23]OMTs consist in direct purchases of sovereign bonds of members of the euro-area in secondary markets. These operations are considered by the ECB once a member state asks for financial assistance, and upon the fulfillment of a set of conditions. There are two main characteristics of these purchases. First, *no ex ante quantitative limits* are set on their size. Second, OMTs are *conditional* on the country being in a European Financial Stability Facility/European Stability Mechanism macroeconomic adjustment or precautionary program.

peripheral countries. Altavilla, Giannone, and Lenza (2014) estimate that OMT announcements decreased the Italian and Spanish 2 years government bonds by 200 basis points. This decline in interest rate spreads was widely interpreted by economists and policy makers as a reflection of the success of this program in reducing belief-driven inefficient fluctuations in sovereign bond markets of euro-area peripheral countries. Here we evaluate this interpretation in our calibrated model given the recovered path of exogenous shocks.

We model OMT as follows. At the beginning of each period, after all uncertainty is realized, the government can ask for assistance. In such case, the Central Bank (CB) commits to buy government bonds in secondary markets at a price $q_{n,CB}(\mathbf{S})$ that may depend on the state of the economy, $\mathbf{S}$. The CB assistance is conditional on the face value of government debt at the end of the period, B', being below a limit $\bar{B}_{CB}(\mathbf{S}, \lambda') < \infty$, also set by the CB. The limit can depend on the state of the economy and on the maturity of the stock of the debt portfolio, and it captures the conditionality of the assistance in the secondary markets. OMT is therefore fully characterized by a policy rule $(q_{n,CB}(\mathbf{S}), \bar{B}_{CB}(\mathbf{S},\lambda'))$. We assume that the CB finances bond purchases with a lump sum tax levied on the lenders, and that such transfers are small enough that they do not affect the stochastic discount factor $M_{t,t+1}$.

The problem for the government described in (5) changes as follows. Letting $a \in \{0,1\}$ be the decision to request CB assistance, with $a = 1$ for the case in which assistance is requested, the government problem is:

$$V(\mathbf{S}) = \max_{\delta \in \{0,1\}, B', \lambda', G, a \in \{0,1\}} \delta \left\{ U(G) + \beta \mathbb{E}[V(\mathbf{S}') | \mathbf{S}] \right\} + (1-\delta) \underline{V}(s_1) \tag{24}$$

subject to

$$\begin{aligned} G + B &\leq \tau Y(s_1) + \Delta\left(\mathbf{S}, a, B', \lambda'\right), \\ \Delta\left(\mathbf{S}, a, B', \lambda'\right) &= \sum_{n=1}^{\infty} q_n\left(\mathbf{S}, a, B', \lambda'\right)\left[(1-\lambda')^{n-1} B' - (1-\lambda)^n B\right] \\ B' &\leq \bar{B}_{CB}\left(\mathbf{S}, \lambda'\right) \quad \text{if } a = 1. \end{aligned}$$

The lenders no-arbitrage condition requires that for $n \geq 1$

$$q_n\left(\mathbf{S}, a, B', \lambda'\right) = \max\{a q_{n,CB}(\mathbf{S}) \mathbb{I}_{\{B' \leq \bar{B}_{CB}(\mathbf{S}, \lambda')\}}; \delta(\mathbf{S}) \mathbb{E}\left\{M\left(s_1, s_1'\right) \delta\left(\mathbf{S}'\right) q'_{n-1} | \mathbf{S}\right\} \tag{25}$$

where $q'_{n-1} = q_{n-1}(s', B'', \lambda'')$ with $B'' = B'(s', B', \lambda')$, $\lambda'' = \lambda'(s', B', \lambda')$, $a' = a(s', B', \lambda')$, and the convention $q'_0 = 1$. The max operator on the right hand side of equation (25) reflects the option that lenders now have to sell the bond to CB at the price $q_{n,CB}$ in case the government asks for assistance ($a = 1$). Because of that, pricing schedules now depend on current *and* future decisions of the government to activate OMTs. Given a policy rule

$(q_{CB}, \bar{B}_{CB})$, the recursive competitive equilibrium with OMT is defined as in Section 2.

We now turn to show that an appropriately designed policy rule can uniquely implement the equilibrium outcome that would arise in absence of rollover risk, that is, if $\pi_t = 0$ for all possible histories. We refer to such outcome as the *fundamental equilibrium outcome* and denote the objects of a recursive competitive equilibrium associated with it with a superscript "$*$". The fundamental equilibrium outcome is our normative benchmark.[24]

Proposition 1. *The OMT rule can be chosen such that the fundamental equilibrium outcome is uniquely implemented and assistance is never activated along the path. In such case, the equilibrium with OMT is a weak Pareto improvement relative to the equilibrium without it, strict if the equilibrium outcome without OMT does not coincide with the fundamental equilibrium.*

The proof for the proposition is provided in Appendix C. Intuitively, by setting a floor on bond prices, the CB allows the government to access financial markets even when lenders are not rolling over the debt. This access to credit market allows the government to repay the maturing debt, and it eliminates the self-fulfilling aspect of rollover crises. Quantity limits on debt issuances guarantee that the government does not choose a B' that is higher than the one arising in the fundamental equilibrium.[25] In this fashion, the CB can achieve a Pareto improvement without actually carrying out bond purchases on the equilibrium path.

The drop in interest rate spreads of Southern European economies observed after the introduction of OMT is consistent with this interpretation. However it does not provide by itself evidence that the policy operated only through this channel. In fact, a decline in interest rate spreads following the OMT announcements is also consistent with the interpretation that the policy raised bondholders' expectations of future bailouts for euro-area peripheral countries. To understand this point, suppose that in a given state in the future the fundamental price for the portfolio of debt is $q^{*\prime}$. Assume now that the CB sets an assistance price $q'_{CB} > q^{*\prime}$. From equation (25), the announcement of this policy leads to an increase in the price today (equivalently, a reduction in interest rate spreads) even if the CB is not currently purchasing bonds ($a = 0$). In this second interpretation, OMT announcements would have different welfare implications relative to the ones described in Proposition 1 because the policy would entail a redistribution of resources from the lenders to the government.

We now propose a procedure to evaluate whether the reduction in interest rate spreads observed after the OMT announcements reflects solely the elimination of rollover crises. Sup-

[24] We abstract from policy interventions that aim to ameliorate inefficiencies arising from incomplete markets and consider OMT rules targeted at eliminating "bad" equilibria. Such features will also survive in models with complete markets or in environment where some notion of constrained efficiency can be achieved as in Dovis (2014).

[25] Under OMT the government acts as a price taker and it has an incentive to borrow more relative to the fundamental equilibrium outcome.

pose that the CB credibly commits to a policy that implements the fundamental equilibrium. The announcement of this intervention eliminates rollover risk in every state of the world, and interest rate spreads jump to their value in the fundamental equilibrium. This fundamental spread represents a lower bound on the post-OMT spread under the hypothesis that the program was directed exclusively to prevent rollover crises. We can then compare the spreads observed in the data after the OMT announcements to their fundamental value: if the latter are higher than the observed ones, it would be evidence against the hypothesis that the policy operated exclusively through a reduction in rollover risk.

Table 4: **Actual and fundamental interest rate spreads in Italy**

	Actual spreads	**Fundamental spread**
2012:Q3	354.13	386.76
2012:Q4	285.03	386.97

Notes: The fundamental spreads are constructed as follows. We first obtain the decision rules from the fundamental equilibrium by solving the model with π_t identically equal to zero. We then obtain the counterfactual fundamental spread for 2012:Q3 and 2012:Q4 by feeding in these decision rules the empirical counterparts to χ_t and y_t. We initialize the endogenous state variables at their filtered value in 2012:Q2.

Table 4 reports the results for this test using the calibrated model. In the first column we have the Italian spreads observed after the OMT announcements, while the second column reports the fundamental spread constructed from the model.[26] We can see that the observed spreads are below their fundamental value. In particular, in 2012:Q4, the observed spread was 285 basis points, while our model suggests that the spread should have been 386 basis points if the program was exclusively eliminating rollover risk. Hence, our calculations suggest that the decline in the spreads observed after the OMT announcements partly reflects the anticipation of a future intervention of the ECB in secondary sovereign debt markets, over and beyond the potential reduction of rollover risk.

7 Conclusion

This paper has proposed a strategy to bring to the data the sovereign debt model of Eaton and Gersovitz (1981) modified to allow for self-fulfilling debt crises as in Cole and Kehoe (2000). In this class of models, the observed maturity choices of the government helps distinguishing between fundamental and non-fundamental sources of variation in interest rate spreads. We apply this identification strategy to Italian data during the debt crisis of 2008-2012. Our results indicate that fluctuations in rollover risk accounted for a modest fraction of the increase in sovereign borrowing costs. This finding suggests that the sharp reduction in

[26]See the note in Table 4 for a precise description of the exercise.

spreads observed upon the establishment of the OMT program partly reflects the expectation of future bailouts by European authorities.

Our approach is not limited to sovereign bond markets, and it could be applied in other environments where self-fulfilling expectations may be important drivers of default risk. For example, one could use changes in the liability and asset structure of financial intermediaries in periods like the Great Depression to assess whether bankruptcies of these institutions were driven by insolvency, or whether they were due to “bank runs” á la Diamond and Dybvig (1983) or Gertler and Kiyotaki (2015).

References

Aguiar, Mark and Manuel Amador. 2014a. *Sovereign Debt*. North-Holland, 647–87.

———. 2014b. "Take the Short Route: How to Repay and Restructure Sovereign Debt with Multiple Maturities." Manuscript, Federal Reserve Bank of Minneapolis.

———. 2016. "Maturity and Indeterminacy in Sovereign Debt Models." Manuscript, Federal Reserve Bank of Minneapolis.

Aguiar, Mark, Satyajit Chatterjee, Hal Cole, and Zachary Stangebye. 2016. "Quantitative Models of Sovereign Debt Crises." Forthcoming, Handbook of Macroeconomics, vol. 2.

Aguiar, Mark and Gita Gopinath. 2006. "Defaultable Debt, Interest Rates and the Current Account." *Journal of International Economics* 69:64–83.

———. 2007. "Emerging Market Business Cycles: The Cycle is the Trend." *Journal of Political Economy* 115:69–102.

Alesina, Alberto, Alessandro Prati, and Guido Tabellini. 1989. "Public Confidence and Debt Management: A Modela and a Case Study of Italy." NBER Working Paper, No. 3135.

Altavilla, Carlo, Domenico Giannone, and Michele Lenza. 2014. "The Financial and Macroeconomic Effects of OMT Announcements." ECB Working Paper Series, No 1707.

Ang, Adrew and Monika Piazzesi. 2003. "A No-Arbitrage Vector Autoregression of Term Structure Dynamics with Macroeconomic and Latent Variables." *Journal of Monetary Economics* 50:745–787.

Arellano, Cristina. 2008. "Default Risk and Income Fluctuations in Emerging Economies." *The American Economic Review* 98 (3):690–712.

Arellano, Cristina and A. Ramanarayanan. 2012. "Default and the Maturity Structure in Sovereign Debt." *Journal of Political Economy* 120:187–232.

Aruoba, Borağan S., Pablo Cuba-Borda, and Frank Schorfheide. 2016. "Macroeconomic Dynamics near the ZLB: A Tale of Two Countries." Manuscript, University of Pennsylvania.

Auclert, Adrien and Matthew Rognlie. 2014. "Unique Equilibrium in the Eaton-Gersovitz Model of Sovereign Debt." Manuscript, M.I.T.

Bai, Yan, Seon Tae Kim, and Gabriel Mihalache. 2014. "The Maturity and Payment Schedule of Sovereign Debt." Manuscript, University of Rochester.

Benjamin, David and Mark LJ Wright. 2009. "Recovery before redemption: A theory of delays in sovereign debt renegotiations."

Bianchi, Javier, Juan Carlos Hatchondo, and Leonardo Martinez. 2014. "International Reserves and Rollover Risk." Manuscript, Wisconsin University.

Bocola, Luigi. 2016. "The Pass-Through of Sovereign Risk." *Journal of Political Economy* 124 (4):879–926.

Borri, Nicola and Adrien Verdhelan. 2013. "Sovereign Risk Premia." Manuscript, MIT Sloan.

Broner, Fernando, Aitor Erce, Alberto Martin, and Jaume Ventura. 2014. "Sovereign debt markets in turbulent times: Creditor discrimination and crowding-out effects." *Journal of Monetary Economics* 61:114–142.

Broner, Fernando A., Guido Lorenzoni, and Sergio L. Schmukler. 2013. "Why Do Emerging Economies Borrow Short Term?" *Journal of the European Economic Association* 11:67–100.

Calvo, Guillermo. 1988. "Servicing the Public Debt: The Role of Expectations." *The American Economic Review* 78:647–661.

Campanaro, Alessandra and Dimitri Vittas. 2004. "Greco-Roman Lessons for Public Debt Management and Debt Market Development." World Bank Policy Research Working Paper 3414.

Chatterjee, Satyajit and Burcu Eyigungor. 2012. "Maturity, Indebtedness and Default Risk." *The American Economic Review* 6:2674–2699.

Cochrane, John and Monika Piazzesi. 2005. "Bond Risk Premia." *American Economic Review* 95:138–160.

Cochrane, John H. 1994. "Permanent and Transitory Components of GNP and Stock Prices." *Quarterly Journal of Economics* 109:241–265.

Cole, Harald L. and Timothy J. Kehoe. 2000. "Self-Fulfilling Debt Crisis." *Review of Economic Studies* 67:91–116.

Conesa, Juan Carlos and Timothy J. Kehoe. 2012. "Gambling for Redemption and Self-Fulfilling Crises." Research Department Staff Report 465, Federal Reserve Bank of Minneapolis.

Cruces, Juan J. and Christoph Trebesch. 2013. "Sovereign defauls: the price of haircuts." *American Economic Journal: Macroeconomics* 5:85–117.

Diamond, Douglas W and Philip H Dybvig. 1983. "Bank runs, deposit insurance, and liquidity." *The Journal of Political Economy* :401–419.

Dovis, Alessandro. 2014. "Efficient Sovereign Default." Manuscript, Penn State University.

Eaton, Jonathan and Mark Gersovitz. 1981. "Debt with potential repudiation: Theoretical and empirical analysis." *Review of Economic Studies* 48:289–309.

Fernández-Villaverde, Jesús, Juan Rubio-Ramírez, and Frank Schorfheide. 2015. "Solution and Estimation Methods for DSGE Models." Manuscript, University of Pennsylvania.

Gertler, Mark and Nobuhiro Kiyotaki. 2015. "Banking, Liquidity, and Bank Runs in an Infinite Horizon Economy." *American Economic Review* 105 (7):2011–2043.

Guvenen, Fatih and Anthony A. Smith. 2014. "Inferring Labor Income Risk and Partial Insurance From Economic Choices." *Econometrica* 82:2085–2129.

Hatchondo, Juan Carlos, Leonardo Martinez, and Cesar Sosa Padilla. 2015. "Debt Dilution and Sovereign Default Risk." Forthcoming, Journal of Political Economy.

Hebert, Benjamin and Jesse Schreger. 2015. "The Costs of Sovereign Default: Evidence from Argentina." Manuscript, Harvard Business School.

Kitagawa, Genshiro. 1996. "Monte Carlo Filter and Smoother for Non-Gaussian Nonlinear State Space Models." *Journal of Computational and Graphical Statistics* 5:1–25.

Lorenzoni, Guido and Ivan Werning. 2013. "Slow Moving Debt Crisis." Manuscript, Northwestern University.

Lubik, Thomas A. and Frank Schorfheide. 2004. "Testing for Indeterminacy: An Application to U.S. Monetary Policy." *American Economic Review* 94:190–217.

Mendoza, Enrique G and Vivian Z Yue. 2012. "A General Equilibrium Model of Sovereign Default and Business Cycles*." *Quarterly Journal of Economics* 127 (2).

Missale, Alessandro and Olivier Jean Blanchard. 1994. "The Debt Burden and Debt Maturity." *American Economic Review* 84:309–319.

Navarro, Gaston, Junapa Nicolini, and Pedro Teles. 2015. "Sovereign Defaults: The Role of Expectations." Manuscript, Federal Reserve Bank of Minneapolis.

Nelson, Charles R. and Andrew F. Siegel. 1987. "Parsimonious Modeling of Yield Curves." *Journal of Business* 60:473–489.

Niepelt, Dirk. 2014. "Debt maturity without commitment." *Journal of Monetary Economics* 68:S37–S54.

Pagano, Marco. 1988. "The Management of Public Debt and Financial Markets." In *High public debt: the Italian experience*, edited by Francesco Giavazzi and Luigi Spaventa. Cambridge University Press.

Passadore, Juan and Juan Xandri. 2015. "Robust Conditional Predictions in Dynamic Games: An Application to Sovereign Debt." Manuscript, M.I.T.

Pitt, Michael K. and Neil Shephard. 1999. "Filtering via Simulation: Auxiliary Particle Filters." *Journal of the American Statistical Association* 94:590–599.

Roch, Francisco and Harald Uhlig. 2014. "The Dynamics of Sovereign Debt Crises and Bailouts." Manuscript, University of Chicago.

Sanchez, Juan M., Horazio Sapriza, and Emircan Yurdagul. 2015. "Sovereign Default and Maturity Choice." Manuscript, University of Rochester.

Scarpelli, Giandomenico. 2001. *La Gestione del Debito Pubblico in Italia. Obiettivi e Tattiche di emissione dei titoli di Stato dagli anni Settanta ai giorni dell'euro*. Bancaria Editrice.

Stangebye, Zachary. 2014. "Dynamic Panics: Theory and Application to the Eurozone." Manuscript, University of Pennsylvania.

Svensson, Lars E. O. 1994. "Estimating and Interpreting Forward Rates: Sweden 1992 1994." NBER Working Paper No. 4871.

Tabellini, Guido. 1988. "Monetary and Fiscal Policy Coordination With a High Public Debt." In *High public debt: the Italian experience*, edited by Francesco Giavazzi and Luigi Spaventa. Cambridge University Press.

Tamer, Elie. 2003. "Incomplete Simultaneous Discrete Response Model with Multiple Equilibria." *Review of Economic Studies* 70:147–165.

Yue, Vivian Z. 2010. "Sovereign default and debt renegotiation." *Journal of international Economics* 80 (2):176–187.

Appendix to "Self-Fulfilling Debt Crises: A Quantitative Analysis"

by Luigi Bocola and Alessandro Dovis

A Timing, rollover risk, and crisis zone

In this section, we carefully define the crisis zone in the model. Recall that the timing within the period is as follows:

- Enter with state $\mathbf{S} = (B, \lambda, s)$;
- Taking as given the pricing schedule, $q(\mathbf{S}, B', \lambda') = \{q_n(\mathbf{S}, B', \lambda')\}_{n=1}^{\infty}$, the government chooses its new portfolio of debt, (B', λ')
- Lenders choose the price for the government bonds, $q(\mathbf{S}, B', \lambda')$, according to the no-arbitrage conditions (6).
- Finally, the government decides whether to default on its debt or not. The default decision is given by $\delta(\mathbf{S}, B', \lambda', \{q_n\}) \in \{0, 1\}$,[27] with $\delta = 1$ if

$$U\left(\tau Y(s_1) - B + \sum_{n=1}^{\infty} q_n \left[(1-\lambda')^{n-1} B' - (1-\lambda)^n B\right]\right) + \beta \mathbb{E}[V(B', \lambda', s') | \mathbf{S}] \geq \underline{V}(s_1),$$

  and $\delta = 0$ otherwise.

For notational convenience, it is useful to define the price of one unit of an arbitrary portfolio of maturity λ given that the government's portfolio is (B', λ') as

$$Q(\mathbf{S}, B', \lambda' | \lambda) = \sum_{n=1}^{\infty} (1-\lambda)^{n-1} q_n(\mathbf{S}, B', \lambda').$$

We denote by $\mathcal{S}^{max}$ the *largest* region of the state space for which a default is possible. We can think of $\mathcal{S}^{max}$ as the collection of states in which the government defaults if lenders choose the worst possible price from the government's perspective conditional on satisfying the lenders' no-arbitrage condition. The next lemma characterizes the set $\mathcal{S}^{max}$. To this end, define

[27] We allow δ to depend on arbitrary $\{q_n\}$ to have the notation to think about off-path situation. The problem in (5) is enough to determine default decision along the equilibrium path given pricing functions.

$$\Omega\left(\mathbf{S}\right) \quad \equiv \quad \max_{B',\lambda'} U\left(\tau Y\left(s_1\right) - B + \Delta^{\text{fund}}\left(\mathbf{S}, B', \lambda'\right)\right) + \beta \mathbb{E}[V\left(B', \lambda', s'\right)|\mathbf{S}] \tag{A.1}$$

$$\text{subject to} \quad \Delta^{\text{fund}}\left(\mathbf{S}, B', \lambda'\right) \leq 0$$

Lemma 1. *Given $V\left(B, \lambda, s\right)$ and $Q\left(\mathbf{S}, B', \lambda'\right)$, $\mathbf{S} \in \mathcal{S}^{max}$ if and only if*

$$\underline{V}\left(s\right) > \Omega(\mathbf{S}) \tag{A.2}$$

Proof. For the necessity part, note that if condition (A.2) does not hold, then the government will never default when the inherited state is $\mathbf{S}$ because it has the option to buy back part of the debt. Imposing the fundamental pricing function - the highest possible prices - in (A.1) is without loss of generality: because the government is buying-back debt, a lower price will only increase the value of Ω.

Consider now the sufficiency part. First note that $\mathbf{S} \in \mathcal{S}^{max}$ if for all (B', λ') such that $\Delta^{\text{fund}}\left(\mathbf{S}, B', \lambda'\right) \geq 0$ we have

$$U\left(\tau Y\left(s_1\right) - B\right) + \beta \mathbb{E}[V\left(B', \lambda', s'\right)|\mathbf{S}] < \underline{V}(s_1), \tag{A.3}$$

and for all (B', λ') such that $\Delta^{\text{fund}}\left(\mathbf{S}, B', \lambda'\right) < 0$ we have

$$U\left(\tau Y\left(s_1\right) - B + \Delta^{\text{fund}}\left(\mathbf{S}, B', \lambda'\right)\right) + \beta \mathbb{E}[V\left(B', \lambda', s'\right)|\mathbf{S}] < \underline{V}(s_1). \tag{A.4}$$

In condition (A.3) we use the fact that when net issuances are positive, $\Delta^{\text{fund}}\left(\mathbf{S}, B', \lambda'\right) \geq 0$, the worst price for the government's perspective is zero and in condition (A.4) we use the fact that when net issuances are negative, $\Delta^{\text{fund}}\left(\mathbf{S}, B', \lambda'\right) < 0$, the worst price for the government's perspective is the fundamental price. If conditions (A.3) and (A.4) are satisfied, it is then rational for lenders to expect a default and it is optimal for the government to default. We can further simplify condition (A.3) by noticing that it is sufficient to check such condition only for (B', λ') such that $\Delta^{\text{fund}}\left(\mathbf{S}, B', \lambda'\right) = 0$ because the continuation value $\mathbb{E}[V\left(B', \lambda', s'\right)|\mathbf{S}]$ is decreasing in B'. Combining this simplified condition (A.3) with condition (A.4) implies that $\mathbf{S} \in \mathcal{S}^{max}$ if (A.2) holds, proving the claim. *Q.E.D.*

We can then define the crisis zone as $\mathcal{S}^{crisis} = \mathcal{S}^{max} \setminus \mathcal{S}^{fund}$.

B Three-period model

To illustrate in the most transparent way the key trade-offs that govern the optimal maturity composition of debt we consider a three-period version of the economy. At $t = 0$ the government can issue two types of securities: a zero coupon bond maturing in period 1, $b_{01} \geq 0$, and a zero coupon bond maturing in period 2, $b_{02} \geq 0$. In period 1, the government decides whether to default or not. If there is no default, the government can issue a bond maturing in period 2, b_{12}. We allow for negative values of b_{12} and interpret these as buy-back of outstanding long-term bonds.

It is convenient to present the model starting from the last period. At $t = 2$, inheriting a state (b_{02}, b_{12}, s_2) [28] the government chooses whether to default on the previously issued debt ($\delta_2 = 0$) or not ($\delta_2 = 1$) to maximize

$$V_2\left(b_{02} + b_{12}, s_2\right) = \max_{\delta_2} \delta_2 U\left(\tau Y_2 - b_{02} - b_{12}\right) + (1 - \delta_2)\underline{V}_2.$$

At $t = 1$, inheriting a state (b_{01}, b_{02}, s_1), the government issues b_{12} and it decides whether to default ($\delta_1 = 0$). The decision problem at $t = 1$ is

$$V_1\left(b_{01}, b_{02}, s_1\right) = \max_{\delta_1, G_1, b_{12}} \delta_1 \left\{U\left(G_1\right) + \beta \mathbb{E}_1[V_2\left(b_{02} + b_{12}, Y_2\right)]\right\} + (1 - \delta_1)\underline{V}_1$$

subject to

$$G_1 + b_{01} \leq \tau Y_1 + q_{12}\left(b_{01}, b_{02}, s_1, b_{12}\right) b_{12}$$

Finally at $t = 0$ the government issues both short and long term debt to solve

$$V_0\left(s_0\right) = \max_{G_0, b_{01}, b_{02}} U\left(G_0\right) + \beta \mathbb{E}_0[V_1\left(b_{01}, b_{02}, s_1\right)]$$

subject to

$$G_0 + D_0 \leq \tau Y_0 + q_{01}\left(s_0, b_{01}, b_{02}\right) b_{01} + q_{02}\left(s_0, b_{01}, b_{02}\right) b_{02},$$

with D_0 being the debt inherited from the past. To avoid issues associated with dilution of legacy debt, we assume that the government does not inherit long-term debt. We further assume that D_0 is sufficiently small that the government does not default at $t = 0$. Price

[28] In this appendix, a subscript on s denotes time.

schedules q_{01}, q_{02}, and q_{12} must be consistent with lenders no-arbitrage condition

$$
\begin{aligned}
q_{01}\left(s_0, b_{01}, b_{02}\right) &= \mathbb{E}_0\left[m\delta_1\left(s_1, b_{01}, b_{02}\right)\right] \\
q_{02}\left(s_0, b_{01}, b_{02}\right) &= \mathbb{E}_0\left[m^2\delta_1\left(s_1, b_{01}, b_{02}\right)\delta_2\left(s_2, b_{02}+b_{01}\right)\right] \\
q_{12}\left(b_{01}, b_{02}, s_1, b_{12}\right) &= \delta_1(s_1, b_{01}, b_{02})\mathbb{E}_1\left[m\delta_2\left(s_2, b_{02}+b_{01}\right)\right]
\end{aligned}
$$

where for simplicity we assume that lenders are risk neutral: $M(s_0, s_1) = M(s_1, s_2) = m$.

B.1 Maturity choices and rollover risk

We next show that expectations of rollover crisis generate a preference for the government to issue long term bonds at $t = 0$. In the extreme case in which all default risk at $t = 0$ reflects rollover risk, the government at $t = 0$ issues only long term debt.

To illustrate the relation between maturity choices and rollover risk, we assume that a rollover crisis occurs with probability π if the government is in the crisis zone at $t = 1$.

Proposition 2. *In the three period economy, if there is only rollover risk and fundamental defaults never happen at $t = 1, 2$ then $b_{01} = 0$ and all debt is long term.*

Proof. The proof is by contradiction. Suppose that a rollover crisis can arise, in that there are states s_1 with associated output level Y_1 such that

$$U(\tau Y_1 - b_{01}) + \beta\mathbb{E}_1[V_2(b_{02}, s_2)] < \underline{V}_1, \tag{A.5}$$

hold and $b_{01} > 0$. Consider then the following variation: increase b_{02} by $\varepsilon/q_{02} > 0$, and decrease b_{01} by $\varepsilon/q_{01} > 0$ so that G_0 in unchanged at the original price.

We next show that under the assumption that there is no fundamental default risk, the variation can replicate the consumption pattern (G_1, G_2) prescribed by the original allocation conditional on not having a rollover crisis. In fact, since there is no default risk between $t = 1$ and $t = 2$, conditional not having a rollover crisis at $t = 1$, we have that $q_{12} = m$. Hence optimality implies that at the original allocation the following Euler equation is satisfied:

$$mU'(G_1) = \beta\mathbb{E}_1[U'(G_2)]. \tag{A.6}$$

Hence achieving same the G_1 and G_2 is optimal and budget feasible if the government inherits $(b_{01} - \varepsilon/q_{01}, b_{02} + \varepsilon/q_{02})$ because the government can just decrease b_{12} by ε/q_{02} and

$$Y_1 - (b_{01} - \varepsilon/q_{01}) + m(b_{12} - \varepsilon/q_{01}) = Y_1 - b_{01} + mb_{12} = G_1$$

where we used the fact that under our assumptions $q_{02} = mq_{12}$.

Finally, we turn to show that the proposed variation reduces the crisis zone and so it increases the prices of debt in period zero and in turn it increases consumption in period 0. To this end, note that under the original allocation condition (A.5) holds for some states s_1 and there is no fundamental default risk, so for all s_1

$$U(\tau Y_1 - b_{01} + mb_{12}) + \beta \mathbb{E}_1[V_2\left(b_{02} + b_{12}, Y_2\right)] \leq \underline{V}_1.$$

Condition (A.5) and the equation above imply that the b_{12} that solves (A.6) is greater than zero. This observation, (A.6), and concavity of U imply that

$$q_{12} U'\left(\tau Y_1 - b_{01}\right) > \beta \mathbb{E}_1[V_2'\left(b_{02} + b_{12}\right)] \iff \frac{1}{q_{01}} U'\left(\tau Y_1 - b_{01}\right) > \frac{1}{q_{02}} \beta \mathbb{E}_1[V_2'\left(b_{02} + b_{12}\right)]$$

where in the second relation we used the fact that $q_{12} = \frac{q_{02}}{q_{01}} = \frac{m^2 \pi \Pr(\text{crisis zone})}{m \pi \Pr(\text{crisis zone})}$. So we have that

$$\begin{aligned} U\left(\tau Y_1 - b_{01} + \varepsilon / q_{01}\right) + \beta \mathbb{E}_1[V_2\left(b_{02} + \varepsilon / q_{02}, Y_2\right)] \;\; &\approx \;\; \{U\left(\tau Y_1 - b_{01}\right) + \beta \mathbb{E}_1[V_2\left(b_{02}, Y_2\right)]\} \\ &+ \;\; \left\{\frac{1}{q_{01}} U'\left(\tau Y_1 - b_{01}\right) + \frac{1}{q_{02}} \beta \mathbb{E}_1[V_2'\left(b_{02}, Y_2\right)]\right\} \varepsilon \end{aligned}$$

and so

$$U\left(\tau Y_1 - b_{01} + \varepsilon / q_{01}\right) + \beta \mathbb{E}_1[V_2\left(b_{02} + \varepsilon / q_{02}, Y_2\right)] > U\left(\tau Y_1 - b_{01}\right) + \beta \mathbb{E}_1[V_2\left(b_{02}, Y_2\right)] \quad \text{(A.7)}$$

Since under our variation the economy is in the crisis zone if

$$U\left(\tau Y_1 - b_{01} + \varepsilon / q_{01}\right) + \beta \mathbb{E}_1[V_2\left(b_{02} + \varepsilon / q_{02}, Y_2\right)] \leq \underline{V}_1, \quad \text{(A.8)}$$

the inequality (A.7) implies that the probability of being in the crisis zone is smaller under our variation because (A.8) is satisfied for a lower output level than (A.5). Hence bond prices at $t = 0$, $q_{01} = m\pi \Pr(\text{crisis zone})$ and $q_{02} = m^2 \pi \Pr(\text{crisis zone})$, increases and the government can increase consumption in the first period. So the variation increases utility, a contradiction. Therefore we must have $b_{01} = 0$. *Q.E.D.*

B.2 Incentive channel

We now show how the incentive channel discussed in Section 3 generates a preference for the government to issue short term bonds. Consider now a situation in which there is no rollover risk, $\pi = 0$, and Y_0 and Y_1 are deterministic. Y_2 is the only source of uncertainty and the uncertainty is revealed in $t = 2$. Because output is deterministic at $t = 1$, issuing long term debt at time $t = 0$ does not entail hedging benefits. Hence, this environment isolates the incentive channel. The following proposition shows that the government at $t = 0$ issues only short term debt.

Proposition 3. *In the three period economy, if there is no rollover risk and there are no shocks in $t = 1$ then the optimal solution must have $b_{02} = 0$ if the probability of default in $t = 2$ is positive.*[29]

Proof. It is helpful to use a primal approach to solve for the equilibrium outcome. Without rollover risk and uncertainty at $t = 0$, we can consider the following programming problem:

$$\max_{b_{01},b_{02},b_{12},\delta_1,\delta_2} U(G_0) + \beta \mathbb{E}_0 \left\{ \delta_1 \left[U(G_1) + \beta \left(\delta_2 U(G_2) + (1-\delta_2)\underline{V}_2 \right) \right] + (1-\delta_1)\underline{V}_1 \right\} \tag{A.9}$$

subject to budget constraints

$$\begin{aligned} G_0 + D_0 &\leq q_{01}b_{01} + q_{02}b_{02} + \tau Y_0 \\ G_1 + b_{01} &\leq q_{12}b_{12} + \tau Y_1 \\ G_2 + b_{02} + b_{12} &\leq \tau Y_2 \end{aligned}$$

the pricing equations

$$\begin{aligned} q_{01} &= m, \quad q_{02} = m q_{12}, \\ q_{12} &= \mathbb{E}_1 \left[m \delta_2 \right], \end{aligned}$$

the "default" constraints

$$\begin{aligned} U(G_1) + \beta \mathbb{E}_1 U(G_2) &\geq \underline{V}_1 \\ U(G_2) &\geq \underline{V}_2 \end{aligned}$$

and the "issuance" constraint

$$U(G_1) + \beta \mathbb{E}_1 U(G_2) \geq V_1(b_{01}, b_{02}) \tag{A.10}$$

It is clear that an equilibrium outcome solves the above problem and the converse is also true. The default and issuance constraints capture the sources of time inconsistency. The default constraint captures the fact that the time zero government cannot choose allocations that attain a value lower than the value of default since future governments at $t = 1$ and

[29] A sufficient condition for this is that β/m is sufficiently low or D_0 sufficiently large.

$t = 2$ can always choose such option if ex-post optimal. The issuance constraint captures the fact that the time zero government cannot control debt issuances of the government in period 1. Such issuances must be optimal from the $t = 1$ government's perspective given its inherited state (b_{01}, b_{02}).

We now show that short term debt is desirable because it relaxes the issuance constraint (A.10). To this end, consider a relaxed version of (A.9) in which we drop the debt-dilution constraint (A.10). Such relaxed problem has a continuum of solutions because the split between long and short term debt issued in period zero is undetermined. Let $\{b_{01}^*, b_{02}^*, b_{12}^*, \delta_1^*, \delta_2^*\}$ be a generic solution to this relaxed programming problem. The optimality condition for b_{12} for this relaxed problem is

$$\begin{aligned} 0 \;=\;& m\frac{\partial q_{12}}{\partial b_2} b_{02}^* U'\left(G_0^*\right) + \\ & \left(q_{12}^* + \frac{\partial q_{12}}{\partial b_2} b_{12}^*\right) U'\left(\tau Y_1 - b_{01}^* + q_{12}^* b_{12}^*\right) - \int_{\mathcal{Y}_2\left(b_{02}^*, b_{12}^*\right)} U'\left(\tau Y_2 - b_{02}^* - b_{12}^*\right) d\mu_{Y2} \end{aligned} \tag{A.11}$$

where $\mathcal{Y}_2\left(b_{02}, b_{12}\right) \equiv \{Y_2 : U\left(\tau Y_2 - (b_{02} + b_{12})\right) \geq \underline{V}_2\}$ is the set of output levels Y_2 for which the government does not default in period 2.

We next show that if $b_{02}^* = 0$ then the government at $t = 0$ can achieve the value of this relaxed problem in the more constrained problem (A.9). To see this it is sufficient to check that the issuance constraint is met at $\{b_{01}^*, b_{02}^*, b_{12}^*, \delta_1^*, \delta_2^*\}$. To this end notice that starting at $\left(b_{01}^*, b_{02}^*\right)$ in period $t = 1$ the optimal b_{12} chosen by period 1 government is such that

$$0 = \left(q_{12}^* + \frac{\partial q_{12}}{\partial b_2} b_{12}\right) U'\left(\tau Y_1 - b_{01}^* + q_{12} b_{12}\right) - \int_{\mathcal{Y}_2\left(b_{02}^*, b_{12}\right)} U'\left(\tau Y_2 - b_{02}^* - b_{12}\right) d\mu_{Y2} \tag{A.12}$$

where $q_{12} = m \Pr\left(U\left(\tau Y_2 - (b_{02}^* + b_{12}^*)\right) \geq \underline{V}_2\right)$. Hence the allocation $\{b_{01}^*, b_{02}^*, b_{12}^*, \delta_1^*, \delta_2^*\}$ satisfies the issuance constraint if and only if it satisfies (A.12) with $b_{12} = b_{12}^*$ and $q_{12} = q_{12}^*$. Now, when $b_{02}^* = 0$, condition (A.11) implies that (A.12) is satisfied. Hence the solution to the relaxed problem can be implemented when $b_{02}^* = 0$.

The final step in the proof is to show that $b_{02}^* = 0$ is necessary when the solution to (A.9) is such that there are defaults in $t = 2$ in some states. Note that (A.11) and (A.12) can be jointly satisfied if and only if $\frac{\partial q_{12}}{\partial b_2} b_{02}^* = 0$ so at least one of the following conditions must be satisfied: i) $b_{02}^* = 0$, ii) no default at $t = 2$ so that $\partial q_{12}/\partial b_2 = 0$. Hence if there are defaults in $t = 2$ then it must be that $b_{02}^* = 0$. *Q.E.D.*

B.3 Insurance channel

We now turn to illustrate the insurance channel. To isolate this channel, we consider an economy in which there is no rollover risk, $\pi = 0$, and the current government can choose debt issued by future governments so that the incentive channel just described is not operative. We can think of this as studying the *best Subgame Perfect Equilibrium (SPE)* from the perspective of the government in period 0. In this case, any deviations from the prescribed path of plays is punished with a reversion to $\underline{V}_t$.

To illustrate that long term debt is a better instrument than short term debt to provide insurance absent outright default, we consider a minimalistic stochastic structure. In period $t = 1$, $s \in \{s_L, s_H\}$ with $Y_1(s_L) < Y_1(s_H)$. The output at time 2 is again distributed in a continuous fashion as in the previous example. We further assume that the realization of s does not affect the distribution of Y_2.

The best SPE outcome solves a similar problem to the one considered in the previous subsection without the issuance constraint. That is

$$\max_{b_{01}, b_{02}, b_{12}, \delta_1, \delta_2} U(G_0) + \beta \mathbb{E}_0 \left\{ \delta_1 \left[U(G_1) + \beta \left(\delta_2 U(G_2) + (1-\delta_2)\underline{V}_2 \right) \right] + (1-\delta_1)\underline{V}_1 \right\} \tag{A.13}$$

subject to budget constraints

$$\begin{aligned} G_0 + D_0 &\leq q_{01} b_{01} + q_{02} b_{02} + \tau Y_0 \\ G_1(s) + b_{01} &\leq q_{12}(s) b_{12}(s) + \tau Y_1(s) \\ G_2(s, Y_2) + b_{02} + b_{12}(s) &\leq \tau Y_2 \end{aligned}$$

the pricing equations

$$\begin{aligned} q_{01} &= m \sum_{s \in \{s_L, s_H\}} \mu(s) \delta_1(s), \quad q_{02} = m \sum_{s \in \{s_L, s_H\}} \mu(s) \delta_1(s) q_{12}(s), \\ q_{12}(s) &= \mathbb{E}_1 [m \delta_2 | s], \end{aligned}$$

and the "default" constraints

$$\begin{aligned} U(G_1(s)) + \beta \mathbb{E}_1 U(G_2|s) &\geq \underline{V}_1 \\ U(G_2|s) &\geq \underline{V}_2. \end{aligned}$$

Proposition 4. *In the three period economy described above, the best SPE is such that if in period 1 there is no default then the government at time 0 issues only long term debt and $b_{01} = 0$.*

Proof. Assuming there is no default in period $t = 1$ (and so the default constraint at $t = 1$ is slack) the solution of problem (A.13) must satisfy

$$0 = \quad U'(G_0)\, m - \beta \textstyle\sum_{s\in\{s_L,s_H\}} \mu(s)\, U'(G_1(s)) + \eta_{01} \tag{A.14}$$

$$\begin{aligned} 0 \quad &= \quad U'(G_0)\left[q_{02} + \frac{\partial q_{02}}{\partial b_{02}} b_{02}\right] + \\ &\sum_{s\in\{s_L,s_H\}} \mu(s)\left[\frac{\partial q_{12}(s)}{\partial b_{02}} b_{12}(s)\, \beta U'(G_1(s)) - \int_{\mathcal{Y}_2(b_{02},b_{12}(s))} \beta^2 U'(\tau Y_2 - b_{02} - b_{12}(s))\, d\mu_{Y2}\right] \end{aligned} \tag{A.15}$$

$$\begin{aligned} 0 \quad &= \quad m \frac{\partial q_{12}(s)}{\partial b_2} b_{02} U'(G_0) + \\ &\left(q_{12}(s) + \frac{\partial q_{12}(s)}{\partial b_2} b_{12}(s)\right) \beta U'(G_1(s)) - \int_{\mathcal{Y}_2(b_{02},b_{12}(s))} \beta^2 U'(\tau Y_2 - b_{02} - b_{12}(s))\, d\mu_{Y2} \end{aligned} \tag{A.16}$$

where η_{01} is the multiplier on the non-negativity constraint for b_{01} and we used the fact that $q_{02} = m \sum_s \mu(s)\, q_{12}(s)$. Combining (A.15) with (A.16) and using the fact that $q_{02} = m \sum_s \mu(s)\, q_{12}(s)$, we obtain

$$0 = U'(G_0)\, m - \beta \sum_{s\in\{s_L,s_H\}} \mu(s) U'(G_1(s))\, \frac{q_{12}(s)}{\sum_{s\in\{s_L,s_H\}} \mu(s) q_{12}(s)}. \tag{A.17}$$

There can be two cases: either $G_1(s)$ is not constant across s or $G_1(s)$ is constant across s. We consider the first case and later show that the second case cannot arise.

Suppose by way of contradiction that $G_1(s)$ is not constant and $b_{01} > 0$ and so the multiplier η_{01} in (A.14) equals zero. Combining (A.14) with (A.17) and using $\eta_{01} = 0$ we obtain

$$\begin{aligned} 0 \quad &= \quad \sum_{s\in\{s_L,s_H\}} \mu(s) U'(G_1(s))\left[1 - \frac{q_{12}(s)}{\sum_{s\in\{s_L,s_H\}} \mu(s) q_{12}(s)}\right] \\ &= \quad \mathrm{Cov}\left(U'(G_1(s)), \frac{q_{12}(s)}{\sum_{s\in\{s_L,s_H\}} \mu(s) q_{12}(s)}\right). \end{aligned} \tag{A.18}$$

Since $G_1(s)$ is not constant by assumption, for the covariance above to be zero, we need that

$$q_{12}(s) = m \Pr\left(U(Y_2 - b_{02} - b_{12}(s)) \geq \underline{V}_2\right)$$

does not depend on s, which is equivalent to have that $b_{12}(s)$ does not depend on s. Then, all the terms in (A.16) other than $G_1(s)$ do not depend on s. Hence, for (A.16) to hold at s_L

and s_H, it must be that also $G_1(s)$ does not depend on s. This is a contradiction.

We now turn to show that we cannot attain perfect insurance in that $G_1(s)$ cannot be constant across s. Suppose by way of contradiction that $G_1(s) = G_1$ for all s. In this case, $(b_{01}, b_{02}, b_{12}(s_L), b_{12}(s_H))$ solve the following four equations:

$$0 = U'(G_0)\, m - \beta U'(G_1)$$

where $G_1 = G_1(s)$ for all s and so

$$\tau [Y_1(s_H) - Y_1(s_L)] = q_{12}(s_L)\, b_{12}(s_L) - q_{12}(s_H)\, b_{12}(s_H)$$

and equation (A.16) for s_L and s_H, which can be written compactly as

$$0 = \Phi(b_{12}(s_L), G_0, G_1(s_L), b_{01}, b_{02}) \tag{A.19}$$

$$0 = \Phi(b_{12}(s_H), G_0, G_1(s_H), b_{01}, b_{02}), \tag{A.20}$$

where Φ is defined by the right side of (A.16). Note that if $G_1(s_L) = G_1(s_H)$ then conditions (A.19)-(A.20) imply that $b_{12}(s_L) = b_{12}(s_H)$. Using this into the budget constraint of the government in $t = 1$ is state s_L and s_H implies that $G_1(s_H) > G_1(s_L)$ and so $G_1(s)$ is not constant across states. A contradiction. Therefore, we must be in the first case in which $G_1(s_H) \neq G_1(s_L)$ and $b_{01} = 0$ and all debt issued at $t = 0$ is long term. *Q.E.D.*

C Proof of Proposition 1

Given V^* and q^*, let $\mathcal{S}^{\text{crisis}}(V^*)$ be the crisis zone associated with the fundamental equilibrium value function. Construct the policy rule $(q_{CB}, \bar{B}_{CB})$ so that: for all $\mathbf{S} \in \mathcal{S}^{\text{crisis}}(V^*)$ there exists at least one (B', λ') with $B' \leq \bar{B}_{CB}(\mathbf{S}, \lambda')$ such that if the government asks for assistance then it prefers to repay than default:

$$U\left(\tau Y - B + \sum_n q_{n,CB}(\mathbf{S}) \left[(1-\lambda')^{n-1} B' - (1-\lambda)^n B\right]\right) + \beta \mathbb{E}\left[V^*(B', \lambda', s') \,|\mathbf{S}\right] \geq \underline{V}(s_1), \tag{A.21}$$

and the fundamental equilibrium is always preferable than asking for assistance, in that for all (B', λ') such that $B' \leq \bar{B}_{CB}(\mathbf{S}, \lambda')$

$$U\left(\tau Y - B + \sum_n q_{n,CB}(\mathbf{S}) \left[(1-\lambda')^{n-1} B' - (1-\lambda)^n B\right]\right) + \beta \mathbb{E}\left[V^*(B', \lambda', s') \,|\mathbf{S}\right] \leq V^*(\mathbf{S}). \tag{A.22}$$

Clearly it is possible to find policy rules that satisfy (A.21) and (A.22). An obvious example is to set $q_{n,CB}(\mathbf{S}) = q_n^*\left(s, B^{*\prime}(\mathbf{S}), \lambda^{*\prime}(\mathbf{S})\right)$ and $\bar{B}_{CB}\left(\mathbf{S}, \lambda\right) = B^{*\prime}\left(\mathbf{S}\right)$ if $\lambda = \lambda^{*\prime}\left(\mathbf{S}\right)$ and zero otherwise.

Under (A.21) and (A.22), no self-fulfilling run is possible, the optimal B' and λ' are the same that arise in the fundamental equilibrium, and the government has no incentives to activate OMT along the equilibrium path. Hence, given a policy rule that satisfies (A.21) and (A.22), there exists a recursive equilibrium with OMT that implements the fundamental equilibrium outcome for any sunspot process $\{s_{2t}\}$. [30] *Q.E.D.*

D Data appendix

Term structure of German interest rates. Data on the term structure of ZCB for German federal government securities is obtained from the Bundesbank online database. We collect monthly data on the parameters of the Nelson and Siegel (1987) and Svensson (1994) model, and we generate nominal bond yields for all maturities between $n = 1$ and $n = 20$ quarters. We convert these monthly series at a quarterly frequency using simple averages. These series are obtained for the period 1973:Q1-2013:Q4.

Real gross domestic product. OECD Quarterly National Accounts, GDP expenditure approach, US dollars, volume estimates (reference year 2010), 1960:Q1-2013:Q4.

Debt to output ratio. Debt is outstanding debt securities of the central government obtained from OECD Quarterly Public Sector Debt, expressed in million of euros.[31] The debt to output ratio is the ratio between this series and GDP measured at current prices. The latter is obtained from OECD Quarterly National Accounts. We obtain this series for the period 1999:Q1-2013:Q4.

Interest rate spreads. Yields differentials between an Italian and a German ZCB with a residual maturity of five years. Nominal yields on Italian bonds are obtained through Bloomberg (GBTPGR5 index). Nominal yields on the corresponding German bonds is obtained from the Bundesbank online database.

[30] We cannot establish that given a policy rule that satisfies (A.21) and (A.22), the fundamental equilibrium is the unique recursive equilibrium with OMT. This is because there may be multiple fixed point of the operator that defines a recursive equilibrium. Hence the fact that (V^*, q^*) is a fixed point of such operator for an arbitrary sunspot process does not necessarily imply that there is not another fixed point (V, q).

[31] We seasonally adjust the series in two steps. We first regress it (in logs) on a linear trend and on quarterly dummies. We then subtract the seasonal dummies, and transform it in levels.

Weighted average life of outstanding government bonds. We use detailed information on outstanding bonds issued by the Italian central government to construct an indicator of debt maturity for the 2008:Q1-2013:Q4 period. We collect outstanding principal and coupon payments to bondholders for every maturity n in every quarter t $(C_{n,t})$ using data from the Italian treasury at http://www.dt.tesoro.it/en/debito_pubblico/dati_statistici/scadenze_titoli_suddivise_anno/index.html. The indicator of debt maturity that we consider is

$$\sum_{n=1}^{N} n \frac{C_{n,t}}{V_t},$$

where $V_t = \sum_{n=1}^{N} C_{n,t}$ is the ZCB-equivalent outstanding face value of the bonds issued. This indicator, the weighted average of the times of principal and coupon repayments, maps exactly to $\frac{1}{\lambda'}$ in our model.

Government deficit. ISTAT National Accounts, General Government Statistics, Quarterly non-financial account, http://dati.istat.it/?lang=en#.

Term structure of Italian interest rates. Data on the term structure of Italian government debt is obtained from Datastream. Datastream provides for each period an estimate of the yield curve of Italian government debt for maturities from 1 year to 10 years based on a fifth degree polynomial approximation of the data. The series mnemonics are GVIL05(CM01), GVIL05(CM02), ..., GVIL05(CM10). We collect these data monthly and we generate nominal bond yields for all maturities between $n = 1$ and $n = 40$ quarters.

E Rollover risk and public debt management in 1980s Italy

Our identification builds on the hypothesis that Treasury departments would respond to heightened rollover risk by actively lengthening the maturity of their debt. While the literature on this topic is limited, previous cross-country studies have shown that the maturity of new issuances in emerging markets typically shortens around default crises (Broner, Lorenzoni, and Schmukler, 2013; Arellano and Ramanarayanan, 2012), and examples of governments extending the life of their debt in turbulent times are not well documented. This section details one of these examples. Using a narrative approach, we analyze the experience of the Italian Treasury department in the early 1980s.

Two main factors at the beginning of the 1980s contributed to place the Italian government at risk of a roll-over crisis. First, the Italian government needed to refinance almost its entire debt, which was roughly 60% of GDP at the time, within the span of a year: following the chronic inflation of the 1970s, in fact, investors became discouraged from holding long duration bonds that were unprotected from inflation risk, and the average residual maturity of Italian debt went from a peak value of 9.2 years in 1972 to 1.1 years in 1980 (Pagano, 1988). Second, and in an effort to increase the independence of the central bank, a major institutional reform freed the Bank of Italy from the obligation of buying unsold public debt in auctions.[32]

The short residual life of government debt coupled with the loss of central bank financing meant that the Italian government had to use primary markets to refinance its maturing debt. However, these markets were not well developed at the time, and private demand of government bonds was weak and volatile (Campanaro and Vittas, 2004). Panel (a) of Figure A-1 reports statistics regarding the placement of Italian treasuries during the 1981-1982 period. The solid line plots the private bid-to-cover ratio for Italian treasuries. This ratio averaged only 0.65 over this period, with a standard deviation of 0.25. The dashed line reports the ratio between the quantity sold and the Treasury's target. Until July 1981, this ratio was equal to 1 because of the statutory requirement for the Central Bank to buy unsold bonds. Following the reform of the Central Bank, though, the Treasury became exposed to variation in the private demand of bonds.

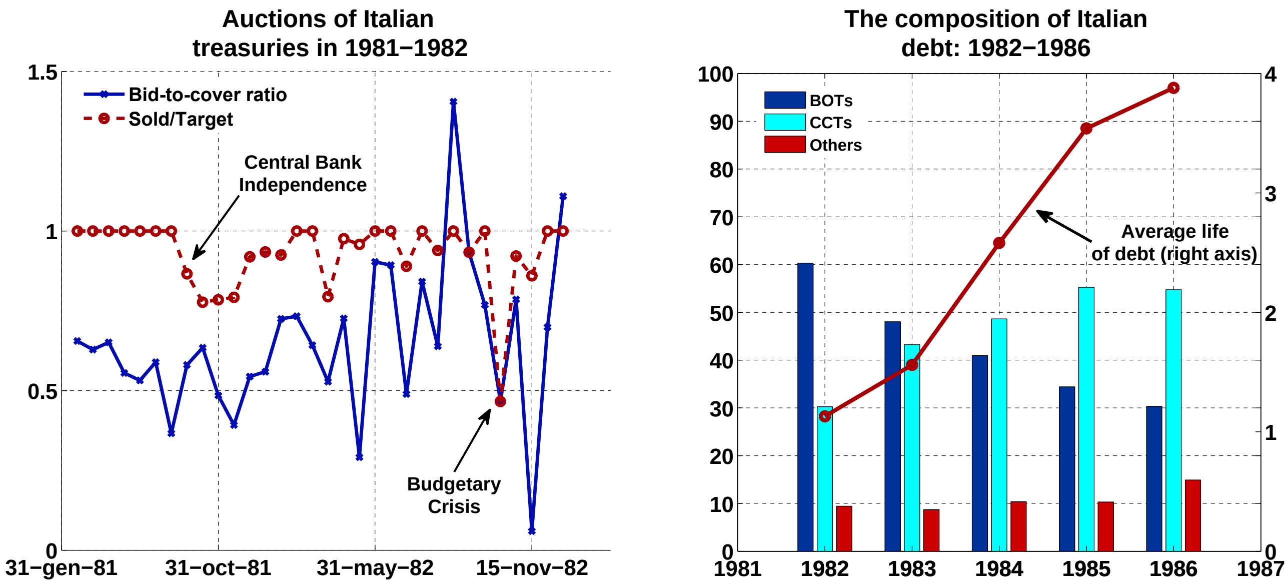
The possibility that rollover problems may eventually lead to a debt crisis became evident in the last quarter of 1982. On the auction of October 15th, private demand covered only 46% of the Treasury's needs, and the Central Bank decided not to purchase unsold bonds. The Treasury was thus forced to use the overdraft account it had with the Bank of Italy to cover its financing needs, reaching the statutory limit. This led to a budgetary crisis, which further depressed private demand of bonds out of fears of a debt restructuring.[33] While the Parliament later voted a law that allowed a temporal overshoot of the overdraft account (Scarpelli, 2001), these events exposed to policymakers the risks implicit in refinancing large amounts of debt in short periods of time.

The response of the Italian government to these events is consistent with the logic of our identification strategy. As documented in Alesina, Prati, and Tabellini (1989) and in Scarpelli (2001), the Treasury actively pursued a policy to extend the life of its public debt. Financial innovation was the main tool used for this purpose, with the introduction of new types of bonds whose interest payments were indexed to the prevailing nominal rate. These *Certificati di Credito del Tesoro* (CCT) were palatable to investors because they offered protection from

[32]Starting from 1975, the Bank of Italy was required to act as a residual buyer of all the public debt that was unsold in the auctions. This resulted in a massive increase in the share of public debt held by the Bank of Italy, reaching a maximum of 40% in 1976. See Tabellini (1988) for a discussion of the historical context underlying the "divorce" between the Bank of Italy and the Italian Treasury.

[33]These fears were not without motivations. Rino Formica, ministry of Finance at the time, publicly called for an "agreement" with bondholders that would allow the Treasury to reimburse only part of its debt. Beniamino Andreatta, ministry of the Treasury, strongly opposed this view. This controversy, known in the public debate as "lite delle comari", eventually led to the fall of the Italian government on December 1st 1982.

Figure A-1: **Rollover risk and public debt management: Italy in the early 1980s**



Notes: The statistics in Panel (a) are constructed using data from Bank of Italy, Supplements to the Statistical Bulletin-Financial Markets. The statistics in Panel (b) are constructed using data from the Italian Treasury. The bar indicates the percentage of a particular class of bonds over total outstanding debt. The line is the average life of outstanding debt (reported in years on the right axis.

inflation risk, and at the same time they had longer maturity than the *Buoni Ordinari del Tesoro* (BOT), the prevailing form of bond financing at the time.[34] Panel (b) of Figure A-1 reports the composition of the outstanding Italian debt (bars) along with its residual average life during the 1982-1986 period. We can see that the Treasury quickly replaced BOTs with CCTs as the main source of public financing. The efforts of the Treasury were successful in increasing the maturity of outstanding debt, with its residual average life more than tripling within the span of four years.

F The lenders' stochastic discount factor

We now derive some results concerning the lenders' stochastic discount factor, and describe in more details the calibration of θ_1. Let $q_t^{*,n}$ be the log price of a non-defaultable ZCB maturing in n periods. These bond prices satisfy the recursion

$$\exp\{q_t^{*,n}\} = \mathbb{E}_t[M_{t,t+1} \exp\{q_{t+1}^{*,n-1}\}],$$

[34]Indexed securities like CCT are not subject to refinancing and rollover problem but are essentially equal to short term debt for the incentive to generate ex-post inflation because any effort to generate ex-post inflation will not reduce the real value of debt. See Missale and Blanchard (1994).

where $M_{t,t+1}$ is defined in the system (13), and the initial condition is $q_t^0 = 0$. Ang and Piazzesi (2003) show that $\{q_t^{*,n}\}$ are linear functions of the state variable χ_t,

$$q_t^{*,n} = A_n + B_n \chi_t,$$

where A_n and B_n satisfy the recursion

$$\begin{aligned} B_{n+1} &= -\phi_1 + B_n \rho^*, \\ A_{n+1} &= -\phi_0 + A_n + B_n \mu^* + \frac{1}{2} B_n^2 \sigma_\chi^2, \end{aligned} \tag{A.23}$$

with $A_0 = B_0 = 0$, $\rho^* = [\rho_\chi - \sigma_\chi^2 \kappa_1]$ and $\mu^* = [\mu_\chi(1-\rho_\chi) - \sigma_\chi^2 \kappa_0]$. In order to avoid the divergence of B_n for large n, we restrict θ_1 to satisfy $|\rho^*| < 1$.

F.1 Results from the pricing model

F.1.1 The risk-free rate

By definition, log-yields on a bond maturing next quarter equal $y_t^1 = -q_t^{*,1}$. In the model, those are equal to

$$y_t^1 = \phi_0 + \phi_1 \chi_t. \tag{A.24}$$

The mean and variance of y_t^1 can then be derived as a function of the model parameters

$$\mathbb{E}[y_t^1] = \phi_0 + \phi_1 \mu_\chi \qquad \text{var}[y_t^1] = \phi_1^2 \frac{\sigma_\chi^2}{(1-\rho_\chi^2)}. \tag{A.25}$$

F.1.2 Expected excess returns

By definition, holding period excess log returns on a ZCB maturing in n periods equal $rx_{t+1}^n = q_{t+1}^{*,n-1} - q_t^{*,n} + q_t^{*,1}$. Substituting the expression for log prices, we can rewrite it as

$$rx_{t+1}^n = \underbrace{[A_{n-1} + B_{n-1}\mu_\chi(1-\rho_\chi) - A_n + A_1]}_{\tilde{A}_n} + \underbrace{[B_{n-1}\rho_\chi - B_n + B_1]}_{\tilde{B}_n} \chi_t + B_{n-1}\varepsilon_{\chi,t+1}, \tag{A.26}$$

where A_j and B_j are defined in (A.23). Taking conditional expectations on both sides, we obtain

$$\mathbb{E}_t[rx_{t+1}^n] = \tilde{A}_n + \tilde{B}_n \chi_t, \tag{A.27}$$

which verifies that expected excess log returns are linear functions of χ_t.

F.2 Calibration of θ_1

We use the data on the term structure of German's interest rates to construct time series for realized excess log returns and the log forward rates for $n = 4, 8, 12, 16, 20$. Table A-1 reports summary statistics on yields and realized excess log returns as a function of n.

Table A-1: **Summary statistics: yields and holding period returns**

	Mean	Standard deviation	Sharpe Ratio
$y_t^1 - \text{infl}_t$	2.16	1.93	
$y_t^{20} - \text{infl}_t$	2.94	1.72	
rx_{t+1}^4	0.21	2.05	0.11
rx_{t+1}^8	0.94	4.22	0.22
rx_{t+1}^{12}	1.54	6.08	0.25
rx_{t+1}^{16}	2.02	7.70	0.26
rx_{t+1}^{20}	2.40	9.14	0.26

Notes: The sample period is 1973:Q1-2013:Q4. The inflation rate is the year-on-year percentage change in the German CPI index, obtained from OECD Main Economic Indicators. Variables are reported as annualized percentages.

We can verify that the yield curve slopes up on average: yields on 5 years bonds are, on average, 80 basis points higher than yields on bonds maturing next quarter. We can also see that long term bonds earn a positive excess return on average. For example, holding a 5 year bond and selling it off next quarter earns, on average, an annualized premium of 2.40% relative to investing in a bond that matures next quarter. Excess returns on long term bonds increase monotonically with n, and so does their Sharpe ratio.

Table A-2 reports the results of the C-P regressions. The top panel reports OLS estimates of equation (17), where $\overline{rx}_{t+1}$ are realized excess log returns averaged across $n = 4, 8, 12, 16, 20$ and the vector $\mathbf{f}_t$ includes the risk-free rate and the log forward rates for these five maturities. The bottom panel reports the individual bond regressions of equation (18).

Differently from the analysis of Cochrane and Piazzesi (2005) on U.S. data, the estimated vector $\hat{\gamma}$ is not "tent" shaped. However, we confirm using German data the finding that a single linear combination of log forward rates has predictive power for excess log returns, and that the sensitivity of the latter to this factor (the estimated b_n's) increases with the maturity of the bonds.

The parameters in θ_1 are chosen as follows. First, we set ϕ_0 and ϕ_1 so that the model implied mean and standard deviation of y_t^1, defined in the equations (A.25), match the sample statistics reported in Table A-1.[35] The remaining parameters are obtained using the method

[35]This sets ϕ_0 and ϕ_1 implicitly as functions of μ_χ, ρ_χ and σ_χ.

Table A-2: **C-P regressions**

	γ_0	γ_1	γ_2	γ_3	γ_4	γ_5	γ_6	R^2
Estimates of equation (17)	-0.002	-1.65	5.00	-21.70	47.20	-45.18	16.53	0.12
	(-0.27)	(-2.89)	(2.92)	(-2.10)	(1.58)	(-1.19)	(0.95)	

		a_n	b_n	R^2
Estimates of equation (18)	4	-0.001	0.46	0.20
		(-2.06)	(5.48)	
	8	-0.000	0.77	0.13
		(-0.37)	(4.92)	
	12	0.000	1.02	0.11
		(0.14)	(4.60)	
	16	0.001	1.27	0.11
		(0.30)	(4.55)	
	20	0.001	1.48	0.11
		(0.34)	(4.56)	

Notes: The sample period is 1973:Q1-2013:Q4. Robust $t-$statistics in parenthesis.

of simulated moments. The empirical targets contains two distinct sets of moments. The first set includes the results of the second stage regression reported in Table A-2 for an $n = 20$ bond, specifically the point estimates for a_{20}, b_{20}, and the standard deviation of the OLS residuals. In the second set of moments we include the parameters of an AR(1) model estimated on the first stage factor, $x_t = \hat{\gamma}_0 + \hat{\gamma}'\mathbf{f}_t$. We choose the remaining parameters in θ_1 in order to minimize a weighted distance between these sets of moments and the corresponding statistics computed in model simulated data. The weighting matrix is diagonal, with the inverse of each sample moment (in absolute value) on the main diagonal. The model implied statistics are computed on a long simulation ($T = 5000$) from the model. In simulations, we add small measurement errors to the forward rates in order to avoid multicollinearity when estimating the Cochrane and Piazzesi (2005) first stage regression.

G Numerical solution

Before explaining the numerical solution, it is convenient to simplify the objects in the recursive equilibrium. In particular, we can drop ξ from the state vector and when a function depends on ξ we will make such dependence explicit. From now onward we let

$\mathbf{S} = [B, \lambda, y, \chi, \pi]$ and $\mathbf{s} = [y, \chi, \pi]$.

In the model set up, we argued that since bond prices depend on the current default decision, they must depend on the inherited portfolio of debt, (B, λ). Because we are only interested in characterizing the equilibrium outcome along the equilibrium path, it is convenient to restrict attention to the fundamental pricing schedule, $q_n^{\text{fund}}(\mathbf{s}, B', \lambda')$ defined in (7). This schedule solves the following functional equation:

$$q_n^{\text{fund}}(\mathbf{s}, B', \lambda') = \mathbb{E}\left\{M(s_1, s_1')\,\delta(\mathbf{S}', \xi')\,q_{n-1}^{\text{fund}}(\mathbf{s}', B'', \lambda'')\,|\mathbf{S}\right\} \text{ for } n \geq 1$$

where we again adopt the convention that $q_0^{\text{fund}} = 1$.

Moreover, to save on notation, we let $q(\mathbf{s}, B', \lambda'|\lambda)$ be the fundamental value of a portfolio of ZCB with decay parameter λ given the realization $\mathbf{s}$ for the exogenous state, and given that the government's choices for the new portfolio is $[B', \lambda']$.[36] Also, we now let B to be the face value of outstanding debt. The price of this portfolio of ZCB can be written as

$$q(\mathbf{s}, B', \lambda'|\lambda) = \mathbb{E}\left\{M(s_1, s_1')\,\delta(\mathbf{S}', \xi')\,[\lambda + (1-\lambda)q(\mathbf{s}', B'', \lambda''|\lambda)]\,|\mathbf{S}\right\}, \tag{A.28}$$

where $B'' = B'(s', B', \lambda')$ and $\lambda'' = \lambda'(s', B', \lambda')$.

With this notation, we can then rewrite the decision problem for the government using three simple sub-problems. We define the value of repaying the debt conditional on lenders rolling over the debt, $V_{\text{roll}}^R(\mathbf{S})$, as follows

$$V_{\text{roll}}^R(\mathbf{S}) = \max_{B', \lambda'}\left\{U(\tau Y - \lambda B + \Delta) + \beta \mathbb{E}[V(B', \lambda', \mathbf{s}')|\mathbf{S}]\right\}, \tag{A.29}$$

where

$$\Delta = q(\mathbf{s}, B', \lambda'|\lambda')\,B' - q(\mathbf{s}, B', \lambda'|\lambda)\,(1-\lambda)B,$$

and $Y = \exp\{y\}$. The value of repaying conditional on lenders not rolling over the debt, $V_{\text{no roll}}^R(\mathbf{S})$, is

$$V_{\text{no roll}}^R(\mathbf{S}) = \left\{U(\tau Y - \lambda B) + \beta \mathbb{E}[V(B(1-\lambda), \lambda, \mathbf{s}')|\mathbf{S}]\right\}, \tag{A.30}$$

while the value of defaulting, $V^D(y, \chi)$, is

$$V^D(y, \chi) = \left\{U(\tau Y[1 - d(Y)]) + \beta\{\psi \mathbb{E}[V(0, \overline{\lambda}, y', \chi', \pi')|\mathbf{S}] + (1-\psi)\mathbb{E}[V^D(y', \chi')|\mathbf{S}]\}\right\}. \tag{A.31}$$

Note that $V^D(.)$ does not depend on π because this process is assumed to be iid. The value

[36]Note that we need to price an arbitrary λ portfolio, given government choices $[B', \lambda']$, in order to know the market value of the portfolio repurchased by the government. See Sanchez, Sapriza, and Yurdagul (2015) for a discussion of this issue.

function of the government can then be written as

$$V(\mathbf{S},\xi) = \begin{cases} V^R_{\text{roll}}(\mathbf{S}) & \text{if } V^R_{\text{no roll}}(\mathbf{S}) \geq V^D(y,\chi) \\ V^R_{\text{roll}}(\mathbf{S}) & \text{if } V^R_{\text{no roll}}(\mathbf{S}) < V^D(y,\chi) \text{ and } \xi = 0 \\ V^D(y,\chi) & \text{if } V^R_{\text{no roll}}(\mathbf{S}) < V^D(y,\chi) \text{ and } \xi = 1 \end{cases}$$

This value function, its associated policy functions and the fundamental pricing function are enough to determine the equilibrium outcome path. This is because the equilibrium price of a bond portfolio on path is either zero - in the case of a fundamental default or a rollover crisis - or it is equal to the fundamental value defined in (A.28) if there is repayment in the current period.

The numerical solution of the model consists in approximating the fundamental pricing schedule q, and the value functions $\{V^R_{\text{roll}}(\mathbf{S}), V^R_{\text{no roll}}(\mathbf{S}), V^D(y,\chi)\}$. We approximate the value functions using a mixture of projection and discrete state space methods. The inverse of the maturity for the debt portfolio, λ, is assumed to be a discrete variable from the set $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_{N_\lambda}\}$. Moreover, we let $\mathcal{B} = \{B_1, \ldots, B_{N_B}\}$ be the set of debt levels over which we approximate the value function. The value functions are approximated using piece-wise smooth functions. Specifically, $V^R_{\text{roll}}(.)$, is approximated as follows,

$$V^R_{\text{roll}}(\lambda_j, B_k, \mathbf{s}) = {\gamma^R_{\text{roll},(\lambda_j,B_k)}}' \mathbf{T}(\mathbf{s}),$$

where $\mathbf{s} = [y, \chi, \pi] \in \mathcal{S}$ is a realization of the exogenous state variables from a set of points $\mathcal{S}$, $\gamma^R_{\text{roll},(\lambda_j,B_k)}$ is a vector of coefficients and $\mathbf{T}(.)$ is a vector collecting Chebyshev's polynomials. The value of repaying conditional on the lenders not rolling over the debt, and the value of defaulting are defined in a similar fashion, and we denote by $\gamma^R_{\text{no roll},(\lambda_j,B_k)}$ and γ^D the coefficients that parametrize those values. Note that γ^D is independent on the (λ, B) inherited by the government, and it is restricted to be a function of the exogenous state variables $\mathbf{s}$. The pricing schedule q in equation (A.28) is approximated on a grid. Specifically, let $\tilde{\mathcal{B}} = \{\tilde{B}_1, \ldots, \tilde{B}_{N_{\tilde{B}}}\}$ be the set of debt levels that the government can choose. The pricing schedule is then approximated on $\tilde{\mathcal{B}} \times \Lambda \times \Lambda \times \mathcal{S}^q$. Note that we allow the grid for the exogenous state variables, $\mathcal{S}^q$, to be different from the one used in the approximation of the value function. This degree of flexibility is helpful because pricing schedules in model of sovereign debt are highly non-linear in the state variables, while value functions tend to be smoother.

We index the numerical solution by $[\Gamma, q]$, with $\Gamma = \{[\gamma^R_{\text{roll},(\lambda_j,B_k)}, \gamma^R_{\text{no roll},(\lambda_j,B_k)}]_{j,k}, \gamma^D\}$ collecting the coefficients that parametrize the value functions. The numerical solution is obtained via value function iteration. Specifically, the algorithm is as follows:

Step 0: Defining the state space and the polynomials. Specify the set of values in $\mathcal{B}, \Lambda, \tilde{\mathcal{B}}$. Set upper and lower bounds for the exogenous state variables (y, χ, π), and construct individual grids for each exogenous state. Construct a tensor grid $\mathcal{S}$ for the exogenous state variables, and the associated Chebyshev's polynomials $\mathbf{T}(.)$. These are used for the approximation of the value functions. Construct a tensor grid $\mathcal{S}^q$ for the approximation of the pricing schedule.

Step 1: Update value functions. Start with a guess for the value and pricing functions, (Γ^c, q^c). For each point in $\mathcal{B} \times \Lambda \times \mathcal{S}$, update the value functions using the definitions in equations (A.29)-(A.31). Denote by Γ^u the updated guess, and by $[r^R_{\text{roll}}, r^R_{\text{no roll}}, r^D]$ the distance between the initial guess and its update using the sup-norm.

Step 2: Update pricing function. For each exogenous state $\mathbf{s}$ in $\mathcal{S}^q$, and for each $(B', \lambda', \lambda) \in \tilde{\mathcal{B}} \times \Lambda \times \Lambda$, evaluate the right hand side of equation (A.28) using (Γ^u, q^c). Denote by $\hat{q}^u(s, B', \lambda'|\lambda)$ this value, and by r^Q the distance between q^c and $\hat{q}^u$ under the sup norm. Update the pricing schedule as

$$q^u(.) = a\hat{q}^u(.) + (1-a)q^c(.) \qquad a \in (0,1).$$

This step is carried out once every 10 iterations.

Step 3: Iteration. If $\max\{r^R_{\text{roll}}, r^R_{\text{no roll}}, r^D\} \leq 10^{-5}$ and $r^Q \leq 10^{-3}$, stop the algorithm. If not, set (Γ^u, q^u) as the new guess, and repeat Step 1-2. □

Regarding the specifics of the algorithm, we set the upper and lower bound for the exogenous states to be equal to +/- 3 standard deviations of the stochastic processes. We then select 5 equally spaced points between these bounds for the approximation of the value function. The set $\mathcal{S}$ contains, therefore, 125 distinct points. For the approximation of the pricing function, instead, we consider 41 points on the y dimension, and 5 points on the χ and π dimension. The set $\mathcal{S}^q$ contains, therefore, 1025 distinct points. The upper and lower bound for B are $[0, 14]$. When approximating the value function, we construct $\mathcal{B}$ using 71 equally spaced points in this interval. The grid for λ contains 11 equally spaced values within the interval $[1/(4 \times 8), 1/(4 \times 5.5)]$. This interval implies a range of $+/-$ 1.25 years around an average observed maturity of 6.75 years, the Italian pre-crisis level. The grid for debt choices over which the pricing schedule is defined, $\tilde{\mathcal{B}}$, consists of 650 points in the $[0, 14]$ interval. The grid has 50 equally spaced points on the $[0, 6)$ segment, 590 points on the $[6, 12)$ segment, and 20 points on the $[12, 14]$ segment.

When iterating over the value and pricing functions, we compute expectations over future outcomes using Gauss-Hermite quadrature, with $n = 5$ points on each random variable. The smoothing parameter for the updating of the pricing schedule is set at $a = 0.20$.

G.1 Simulations

We now explain how we simulate trajectories from the model. Let $\{y_t, \chi_t, \pi_t, \xi_t\}_{t=1}^{T}$ a realization of length T for the exogenous variables of the model, and let (B_t, λ_t) be the face value of debt and the decay parameter of the debt portfolio that the government inherited. We assume that the government is not currently in a default. The simulation consists in obtaining the default decision, δ_t, the characteristics of the new debt portfolio, (B_{t+1}, λ_{t+1}), and the equilibrium price of a portfolio of type λ, $q_t(\lambda)$.

Given the state variables $\mathbf{S}_t$, we first compute $V^R_{\text{no roll}}(\mathbf{S}_t)$ and $V^D(y_t, \chi_t)$ using the numerical solution Γ. If $\xi_t \geq \pi_{t-1}$ and $V^R_{\text{no roll}}(\mathbf{S}_t) \leq V^D(\mathbf{S}_t)$, the government enters the default state and we set $\delta_t = 0$. If $V^R_{\text{no roll}}(\mathbf{S}_t) > V^D(y_t, \chi_t)$ or $\xi_t < \pi_{t-1}$, instead, we proceed as follows

Step 1: For each $B' \in \tilde{\mathcal{B}}$ and each $\lambda' \in \Lambda$, linearly interpolate the fundamental pricing schedule along the $\mathbf{s}$ dimension to obtain $\hat{q}(\mathbf{s}_t, B', \lambda'|\lambda_t)$ and $\hat{q}(\mathbf{s}_t, B', \lambda'|\lambda')$.

Step 2: Given $\hat{q}(s_t, B', \lambda'|\lambda_t)$ and $\hat{q}(s_t, B', \lambda'|\lambda')$, compute

$$U(\tau \exp\{y_t\} - B_t + \Delta(\mathbf{S}_t, B', \lambda')) + \beta \mathbb{E}[V(B', \lambda', s_{t+1})|s_t],$$

for each (B', λ') pair. Expectations over future values are evaluated using Gauss-Hermite quadrature and the numerical solution Γ.

Step 3: Maximize the above expression over (B', λ'), and let $(B^{*\prime}, \lambda^{*\prime})$ be the portfolio that obtains the maximum. If the maximum is greater than $V^D(y_t, \chi_t)$, than set $B_{t+1} = B^{*\prime}$, $\lambda_{t+1} = \lambda^{*\prime}$, $\delta_t = 1$, and $q_t(\lambda) = \hat{q}(\mathbf{s}_t, B_{t+1}, \lambda_{t+1}|\lambda)$. Else, set $\delta_t = 0$. $\square$

This procedure for simulating the model allows us to compute endogenous variables at every point in the state space, even if they are not in our grid. When computing summary statistics on model simulations, we condition on periods during which the economy is not in a default. The government reenters capital markets with a debt of zero, this implying debt trajectories that drift up for some periods. When computing summary statistics, we drop these periods in order to guarantee that the endogenous variables settle at their ergodic distribution.

H Details of the Counterfactual Experiment

This section details the counterfactual experiment of Section 5.1. First, we explain how we use the auxiliary particle filter to extract information on the state vector given data on detrended real GDP, the estimated χ_t series, interest rate spreads and the empirical counterpart of λ'. Second, we discuss how the retrieved state vector is used to generate the decomposition of Figure 5.

H.1 Particle Filtering

The model has five state variables $\mathbf{S}_t = [B_t, \lambda_t, y_t, \chi_t, \pi_t]$.[37] The vector $\mathbf{Y}_t$ collects the observables for quarter t. The state-space representation is

$$\begin{aligned} \mathbf{Y}_t &= \mathbf{g}(\mathbf{S}_t) + \eta_t \\ \mathbf{S}_t &= \mathbf{f}(\mathbf{S}_{t-1}, \varepsilon_t). \end{aligned}$$

The first equation is the measurement equation, with η_t being a vector of iid Gaussian errors with variance-covariance matrix equal to Σ. The second equation is the transition equation, describing the law of motion for the model's state variables. The vector ε_t collects the innovations to the structural shocks y_t, χ_t and π_t. The functions $g(.)$ and $f(.)$ are generated using the numerical procedure described in Appendix G.

Let $\mathbf{Y}^t = [\mathbf{Y}_1, \ldots, \mathbf{Y}_t]$, and denote by $p(\mathbf{S}_t|\mathbf{Y}^t)$ the conditional distribution of the state vector given observations up to period t. Although the conditional density of $\mathbf{Y}_t$ given $\mathbf{S}_t$ is known and Gaussian, there is no analytical expression for the density $p(\mathbf{S}_t|\mathbf{Y}^t)$. We use the auxiliary particle filter to approximate this density for each t. The approximation is done via a set of pairs $\{\mathbf{S}_t^i, \tilde{w}_t^i\}_{i=1}^N$, in the sense that

$$\frac{1}{N}\sum_{i=1}^{N} f(\mathbf{S}_t^i)\tilde{w}_t^i \overset{\text{a.s.}}{\rightarrow} \mathbb{E}[f(\mathbf{S}_t)|\mathbf{Y}^t],$$

and it is used to obtain the (mean) trajectory of the state vector over the sample. We refer to $\mathbf{S}_t^i$ as a particle and to $\tilde{w}_t^i$ as its weight. The algorithm used to approximate $\{p(\mathbf{S}_t|\mathbf{Y}^t)\}_t$ builds on Kitagawa (1996) and Pitt and Shephard (1999), and it goes as follows

Step 0: Initialization. Set $t = 1$. Initialize $\{\mathbf{S}_0^i, \tilde{w}_0^i\}_{i=1}^N$ from the ergodic distribution of the model and set $\tilde{w}_0^i = 1\ \forall i$.

[37] We can drop ξ from the state vector because we are considering paths during which a default does not occur.

Step 1: Prediction. For each $i = 1, \ldots, N$, simulate a particle $\mathbf{S}^i_{t|t-1}$ given $\mathbf{S}^i_{t-1}$ from the proposal density $g(\mathbf{S}^i_t|\mathbf{Y}^t, \mathbf{S}^i_{t-1})$ following the procedure described in Appendix G.1.

Step 2: Filtering. Assign to each particle $\mathbf{S}^i_{t|t-1}$ the particle weight

$$w^i_t = \frac{p(\mathbf{Y}_t|\mathbf{S}^i_{t|t-1})p(\mathbf{S}_t|\mathbf{S}^i_{t|t-1})}{g(\mathbf{S}_t|\mathbf{Y}^t, \mathbf{S}^i_{t|t-1})}\tilde{w}^i_{t-1}.$$

Step 3: Resampling. Rescale the particles $\{w^i_t\}$ so that they add up to unity, and denote these rescaled values by $\{\tilde{w}^i_t\}$. Sample N values for the state vector with replacement from $\{\mathbf{S}^i_{t|t-1}, \tilde{w}^i_t\}^N_{i=1}$, and denote these draws by $\{\mathbf{S}^i_t\}_i$. Set $\tilde{w}^i_t = 1\ \forall i$. If $t < T$, set $t = t + 1$ and go to Step 1. If not, stop. $\square$

Regarding the tuning of the filter, we set $N = 20000$. The choice for the proposal density $g(\mathbf{S}_t|\mathbf{Y}^t, \mathbf{S}^i_{t|t-1})$ follows Bocola (2016), see its Online Appendix. The matrix $\mathbf{\Sigma}$ is diagonal. We set the diagonal elements as follows. We compute the sample variance of the observables for the 2008:Q1-2012:Q2 period. The variance of the measurement errors for $\hat{y}_t$, $\hat{\chi}_t$, and λ_{t+1} is set to 1% of their sample variance. For interest rate spreads, the variance of the measurement errors is set to 5% of the series sample variance. By choosing larger measurement errors for the interest rate spreads series, we are implicitly asking the filter to track as close as possible the observable shocks and the debt maturity series, while allowing for deviations between the observed interest rate spreads and the ones implied by the model.

H.2 Counterfactual Experiment

We now discuss how we use the approximation to $\{p(\mathbf{S}_t|\mathbf{Y}^t)\}^{2012:Q2}_{t=2008:Q1}$ along with the structural model to generate the decomposition presented in Figure 5.

Let $\text{spr}^{\text{data}}_t = y^{20,\text{ita}}_t - y^{20,\text{ger}}_t$ be the interest rate spread at time t, and let $\hat{\text{spr}}^{\text{model}}_t$ be

$$\hat{\text{spr}}^{\text{model}}_t = \sum_{i=1}^{N} g_{\text{spr}}(\mathbf{S}^i_t)\tilde{w}^i_t,$$

where $g_{\text{spr}}(.)$ is the implicit relation between interest rate spreads and the state variables in the model. The measurement error component in Figure 5 is defined as $\text{spr}^{\text{data}}_t - \hat{\text{spr}}^{\text{model}}_t$.

In order to construct the fundamental and the non-fundamental components, we generate a counterfactual spread $\hat{\text{spr}}^{\text{fund}}_t$ by simulation. We proceed as follows. Let $t = 1$. Set $\pi^i_1 = 0$ for every i, and let $\mathbf{S}^{i,\text{fund}}_1$ be the state vector with $\pi^i_1 = 0$. For each i, feed the model with $\mathbf{S}^{i,\text{fund}}_1$, and define $s^{i,\text{fund}}_1$ to be the model implied counterfactual. We define $\mathbf{S}^{i,\text{fund}}_2$ be the updated state vector, where χ^i_2 and y^i_2 are consistent with their values in $\{\mathbf{S}^i_2, \tilde{w}^i_2\}$, $\pi^i_2 = 0$, and the endogenous state variables are the one implied by the model's law of motion $f(.)$. We then repeat this procedure for each $t = 2, \ldots, T$.

Given $\{\text{spr}_t^{i,\text{fund}}\}_{i \in N, t \in T}$, we next construct, for each t

$$\hat{\text{spr}}_t^{\text{fund}} = \frac{1}{N} \sum_{i=1}^{N} \text{spr}_t^{i,\text{fund}} \tilde{w}_t^i \approx \mathbb{E}[\text{spr}_t^{\text{fund}} | \mathbf{Y}^t].$$

This is the (average) interest rate spread implied by the model under the assumption that π_t was identically zero over the 2008:Q1-2012:Q2 period, and it is the fundamental component of interest rate spreads in Figure 5. The non-fundamental component of the spreads is then defined as $\hat{\text{spr}}_t^{\text{model}} - \hat{\text{spr}}_t^{\text{fund}}$ for each t.