What are extreme points of $\{u:[0,B]^n \rightarrow \mathbb{R} \mid u(\mathbf{0})=0,\ u \text{ convex},\ \nabla u(x) \in [0,1]^n\ \forall x\}$?

For n=1: $u_c(x) = \max\{x-c, 0\}\ \forall x$, so "$\nabla u_c(x) \in \{0,1\}^n\ \forall x$."

Manelli & Vincent, JET 137:153-185 (2007). *Are extreme points dense in the set?*

Given n, B>0, $X \subseteq [0,B]^n$ finite, $\mathbf{0} \in X$, and $f \in \Delta(X)$.

Primal problem: (with $U(x) = \sum_{i\in\{1,\ldots,n\}} x_i P_i(x) - T(x)$)

maximize $\sum_{x\in X} (\sum_{i\in\{1,\ldots,n\}} x_i P_i(x) - U(x)) f(x) + U(\mathbf{0})$

over $P_i(x) \geq 0$, $U(x)$, $\forall i \in \{1,\ldots,n\}$, $\forall x \in X$

subject to: $P_i(x) \leq 1$, $\forall i, \forall x$; $\quad \beta_i(x)$

$(U(y) + \sum_i (x_i - y_i) P_i(y)) - U(x) \leq 0$, $\forall x, \forall y \in X$. $\quad \alpha(y|x)$

Dual problem

minimize $\sum_{x\in X} \sum_i \beta_i(x)$

over $\beta_i(x) \geq 0$, $\alpha(y|x) \geq 0$, $\forall x, \forall y, \forall i$;

subject to $\sum_{y\in X} \alpha(y|x) - \sum_{y\in X} \alpha(x|y) = f(x) - \mathbb{1}_{\{x=0\}}$, $\forall x$; $\quad U(x)$

$\beta_i(x) + \sum_{y\in X} \alpha(x|y)(y_i - x_i) \geq f(x) x_i$, $\forall i, \forall x$. $\quad P_i(x)$

So $\beta_i(x) = \max\{0, f(x)x_i + \sum_{y\in X} \alpha(x|y)(x_i - y_i)\}$

$= \max\{0, \sum_{y\in X} \alpha(y|x) x_i - \sum_{y\in X} \alpha(x|y) y_i\}$.

$f(x)x_i + \sum_y \alpha(x|y)(x_i - y_i) > 0 \Rightarrow P_i(x) = 1$. $f(x)x_i + \sum_y \alpha(x|y)(x_i - y_i) < 0 \Rightarrow P_i(x) = 0$.