Approximating Markov Equilibria with Heterogeneous Agents and Incomplete Markets

Elisabeth Pröhl
University of Geneva and Swiss Finance Institute

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Motivation

- Heterogeneity due to different idiosyncratic risks $\Rightarrow$ Incompleteness
- Labor income risk: Aiyagari-Bewley growth model with aggregate risk as in Krusell and Smith (1998)
- ~~Closed-form solutions~~ $\Rightarrow$ Numerical algorithms
- Challenges for numerical algorithms:
  - Cross-sectional distribution as an element of the state space
  - Distribution varies over time due to aggregate shocks: Law of motion?
  - Most existing algorithms assume bounded rationality and use a two-step procedure
  - Convergence Results?

Methodology

- Goal: Solve for DGMM-recursive equilibrium, i.e. $k' = h_1^*(z', k, \mu^{z^{id}, k})$ and $c' = h_2^*(z', k, \mu^{z^{id}, k})$
- Note that the Euler equation for the infinite horizon growth model results from a series of two-period constrained convex OP:

$$
\begin{aligned}
\min_{\{h_1, h_2\}} \quad & -u(h_2) \\
& - \sum_{z'' \in \mathcal{Z}} p^{z''|z'} \beta u \left(I \left(z''; \quad \mathbf{T}\mu \quad \right) + \left[1 + R \left(z^{ag''}; \quad \mathbf{T}\mu \quad \right) - \delta \right] h_1 - \quad h_1^{*'} \right) \\
\text{s.t.} \quad & 0 = I \left(z'; \quad \mu \quad \right) + \left[1 + R \left(z^{ag'}; \quad \mu \quad \right) - \delta \right] k - h_1 - h_2 \\
& 0 \geq -h_1 \\
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Solving the Issues

- μ : Focus on one distribution at a time, i.e. compute the optimal response dynamically. Use the stationary distribution at initial time and fix the distributions at subsequent times via the law of motion.
- $\mathbf{T}\mu$: Consistent law of motion

  $$\mu^{(z^{id'},k')}(x) = \mathbf{T}\mu^{(z^{id},k)} = \mathbb{P}^{\mu^{(z^{id},k)}}\left(h_1^*(z', k, \mu^{(z^{id},k)}) \leq x\right)$$
- $h_1^{*'}$: Approximate the next-period optimal response via a Taylor approximation of h_1 around the current distribution $\Rightarrow$ Calculus for measure spaces

$\Rightarrow$ Show that the Proximal Point Algorithm applies to the OP with consistent law of motion and obtain convergence results from it.

Comparison to the Krusell-Smith Algorithm

- Bounded rationality assumption:
  $k' = h(z', k, K)$ vs. $k' = h(z', k, \mu^{z^{id}, k})$
- Log-linear parametric form of the law of motion of the mean:
  $K' = \exp(b_1 + b_2 \log(K))$ vs. $\mu(x) = \mathbb{P}^{\mu}\left(h(z', k, \mu) \leq x\right)$
- Component-wise fixed-point iteration for $H = [h, \mu]$:

$$\mathbf{S}_{KS} H = \begin{bmatrix} \mathbf{T}^{h}_{KS} H \\ \mathbf{T}^{\mu}_{KS} H \\ \mathbf{D}_{KS} H \end{bmatrix} \quad \text{vs.} \quad \mathbf{S}_{DGMM} H = \begin{bmatrix} \mathbf{T}^{h}_{DGMM} H \\ \mathbf{T}^{\mu}_{DGMM} H \\ \mathbf{D}_{DGMM} H \end{bmatrix}$$

⇒ Two-step procedure leads to problems illustrated by considering the limit of the OLS estimator in the simulation step:

$$\lim_{M,T \to \infty} b_2|_{z^{ag}} = \frac{\mathbb{C}o\mathbb{V}^{\mu^{* z^{ag}}}\left[\log\left(\bar{K}|_{z^{ag}}\right), \log\left(\bar{K}'|_{z^{ag}}\right)\right]}{\mathbb{V}^{\mu^{* z^{ag}}}\left[\log\left(\bar{K}|_{z^{ag}}\right)\right]}$$

Numerical Experiment on the limiting Krusell-Smith Algorithm

	Initialization		Convergence	# Iterations
$h(z', k, K)$	$b^{z^{bad}}$	$b^{z^{good}}$		
$0.98k$	$[\log(0.98), 1]$	$[\log(0.98), 1]$	No	1
$0.9k$	$[0, 1]$	$[0, 1]$	No	5
$0.9k$	$[0.085, 0.965]$	$[0.095, 0.962]$	No	1
k	$[0, 1]$	$[0, 1]$	No	5

Table: The Krusell-Smith algorithm using the limiting law of motion. The number of iterations indicates after how many iterations on all components, i.e. on the policy and the law of motion, the algorithm was aborted.

Conclusions and Outlook

- Three major differences to existing algorithms:
  - No additional assumption on the model like bounded rationality
  - The stationary distribution can be computed separately, it is the joint distribution of aggregate, idiosyncratic and endogenous variables
  - An integrated approach rather than a two-step procedure for the law of motion
- A drawback is computation time
  - Discontinuities in the c.d.f.
  - Taylor approximation of the next-period policy is expensive
- Outlook: Discretizing the space of distributions via stochastic Galerkin methods would eliminate the need for the Taylor approximation

Thank you for your attention!