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Abstract

Economic activities such as crowdfunding often involve sequential interactions, observational learning, and project implementation contingent on achieving certain thresholds of support. We incorporate endogenous all-or-nothing thresholds in a classic model of information cascade. We find that early supporters tap the wisdom of a later “gate-keeper” and effectively delegate their decisions, leading to uni-directional cascades and preventing agents’ herding on rejections. Consequently, entrepreneurs or project proposers can charge supporters higher fees, and proposal feasibility, project selection, and information production all improve, even when agents have the option to wait. Novel to the literature, equilibrium outcomes depend on the crowd size, and in the limit, efficient project implementation and full information aggregation ensue.

JEL Classification: D81, D83, G12, G14, L26

Keywords: All-or-nothing, Capital Markets, Crowd-funding, Dynamic Learning, Entrepreneurship, FinTech Platform, Information Aggregation, Information Cascade.

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1 Introduction

Financing projects and aggregating information are arguably the most important functions of financial markets. Yet numerous economic and financial settings involve sequential actions from privately informed agents, and are prone to information cascades (Banerjee, 1992; Bikhchandani, Hirshleifer, and Welch, 1992). Once a cascade starts, all subsequent individuals act regardless of their private signals, leading to incomplete information aggregation and suboptimal financing. Standard models tend to focus on the case of pure informational externalities by assuming that one agent’s payoff is independent of the action of others. They leave out an important feature commonly observed in real-life: many projects or proposals are only implemented with a sufficient level of support — an “all-or-nothing” (AoN) threshold.¹ We show that such a small difference in the mechanism can drastically alter the informational environments and economic outcomes.

While a recent literature on sequential voting (Dekel and Piccione, 2000; Ali and Kartik, 2006) sheds light on how herding and the interdependence of voter payoffs interact, most other settings are still under-explored. First, unlike the case of voting wherein an elected candidate or bill passed affect all agents regardless whether they voted in favor or against, in many situations such as crowdfunding, venture investment, and campaign contributions intended to buy favors, the implementation of the project only affects agents taking a particular action. Second, AoN thresholds in extant models are typically taken as exogenous, yet entrepreneurs or campaign leaders frequently set contribution amounts and target thresholds for implementation. Third, the Internet and latest technologies democratize many opportunities for participation, especially in entrepreneurial finance through initial coin offerings, crowdfunding, and online IPO auctions (Ritter, 2013). We understand very little about the social efficiency of information aggregation and financing given the large number and decentralized nature of the agent base.

To better understand these issues, we introduce AoN design in a standard framework

¹AoN threshold is predominant on crowdfunding platforms and in venture financing; super-majority rule or q-rule is a common practice in many voting procedures; assurance contract or crowdfaction in public goods provision is also characterized by sequential decisions and implementation thresholds (e.g., Bagnoli and Lipman, 1989); charitable projects need a minimum level of funding-raised to proceed (e.g., Andreoni, 1998). Other examples abound.

of information cascade à la Bikhchandani, Hirshleifer, and Welch (1992), allowing both endogenous pricing commonly observed in financial markets (Welch, 1992) and AoN threshold implementation. A project proposer sequentially approaches N agents who choose to support or reject the project. Each supporter pays a pre-determined price, and then gets a payoff normalized to one if the project is good. All agents are risk-neutral and have a common prior on the project's quality. They each receive a private, informative signal, and observe the actions of preceding agents, before deciding whether to make a contribution/support. Deviating from the literature, supporters only pay the price and receive the project payoff if the number of supporters reaches an AoN threshold, potentially pre-specified by the proposer.

AoN thresholds lead to uni-directional cascades in which agents never rationally ignore positive private signals to reject the project (DOWN cascade), but may rationally ignore negative private signals to support the project (UP cascade). Information production also becomes more efficient, especially with a large crowd of agents, leading to more successes of good projects and weeding out bad projects. When the implementation threshold and price for supporting are endogenous, the proposer no longer under-price the issuance as seen in Welch (1992). Consequently, proposal feasibility, project selection, and information aggregation improve. In particular, when the number of agents grows large, equilibrium project implementation and information aggregation approach full efficiency, in stark contrast to the literature's previous findings (Banerjee, 1992; Lee, 1993; Bikhchandani, Hirshleifer, and Welch, 1998; Ali and Kartik, 2012).

To derive these, we first take the AoN threshold and price as given, and show that before reaching the threshold, the aggregation of private information only stops upon an UP cascade. The intuition follows from that the threshold links an agent's payoff to subsequent agents' actions, making her partially internalize the informational externalities of her action.

Interestingly, such forward-looking considerations lead to asymmetric outcomes: agents with positive private signals always support because they essentially delegate their decisions to a subsequent "gate-keeping" agent whose supporting decision brings the total support to the threshold. Delegation hedges against supporting a bad project because the "gate-keeping" agent, having observed a longer history of actions by the time she makes the decision, evaluates the value of supporting with better information than previous agents.

Meanwhile, before an UP cascade, agents with negative private signals are reluctant to support before the threshold is reached, because in equilibrium their supporting actions may mislead subsequent agents and cause either a too-early UP cascade or the support's reaching the AoN threshold without enough number of positive signals, both implying a negative expected payoff for her. Therefore, DOWN cascades are always interrupted by agents with positive signals before the threshold is reached. Agents including and following the “gate-keeper” know that the project would be implemented for sure when they support, and the situation returns to the standard information-cascade setting.

The proposer considers the choices of the AoN threshold for a given price to maximize the proceeds or the amount of support in non-financial scenarios. A higher AoN threshold delays potential DOWN cascades (after AoN threshold being reached) but is also less likely to be reached. The proposer wants to rule out DOWN cascades as in Welch (1992). The difference is that he utilizes both the price and the minimum implementation threshold to achieve this. In other words, he trades off increasing price to increase the proceed from every supporter with lowering price to boost the probability of reaching the AoN threshold and winning more supporters. Consequently, in equilibrium there is no DOWN cascade except for a special scenario in which a DOWN cascade starts at the last agent and the project would not be implemented anyway even if all private signals become public. Good projects do not fail due to rational herding, but bad projects may still be implemented in an UP cascade.

Endogenous AoN thresholds and uni-directional cascades have three important implications. First, they allow projects of high quality but with costly production to be supported. In standard cascade models with endogenous pricing (Welch, 1992), a high production cost suggests a high price, which would trigger an immediate DOWN cascade and thus a proposal failure. An AoN threshold provides the proposer an additional tool to avoid DOWN cascades and expands the feasible pricing range to potentially finance all positive NPV projects no matter what the production cost is.

Second, the exclusion of DOWN cascades also affects project selection. In standard models of financial markets with information cascades, the severe underpricing excludes information production by triggering UP cascades from the very beginning. Both good

and bad projects are financed. The introduction of AoN threshold restores information production by allowing a higher price with a larger investor base. The probability that a bad project is weeded out and an UP cascade happens to a good project is increasing in the agent base. Overall, AoN thresholds allows a better harnessing of the wisdom of the crowd to distinguish good projects from bad ones.

Third, AoN thresholds produce more information whose benefits go beyond project implementation. For example, crowdfunding adds an option value to experimentation, which can facilitate entrepreneurial entry and innovation (Manso, 2016), and produces useful information for future decision-making (Chemla and Tinn, 2016; Xu, 2017). Whereas in Welch (1992) there is no information production, in our model the proposer learns about the project’s quality before an UP cascade starts. A proposer facing a large number of potential agents can charge a higher price for issuance to delay potential UP cascades, and in doing so, aggregate more information even when the project is not implemented.

While outcomes in standard models of information cascades typically are independent of the size of agent base, the case with AoN thresholds differs: the errors in mis-supporting or mis-rejecting decrease with the crowd size, as does the convergence of the endogenous price to the level at which the proposer extracts full surplus. In the limit, projects are implemented if and only if they are of high quality. The public knowledge about the project’s true type also becomes perfect. We therefore obtain socially efficient project implementation and full information aggregation with a large crowd, hitherto unachievable in most models of information cascades.

Finally, we demonstrate that our key insights apply even when agents have the option to postpone their decisions, and are thus less subject to the usual critiques on information-cascade models. We also discuss how AoN thresholds induce in sequential interactions strategic complementarity of agents’ actions, a phenomenon novel to models of information cascades. We analyze all resulting equilibria and show that under weak equilibrium refinements, their limiting behaviors converge in terms of project implementation and information aggregation to the aforementioned equilibrium outcomes.

Our theory is mainly motivated by and applies to entrepreneurial finance, particularly,

crowdfunding.² Decentralized individuals often chance upon a project for example through social media, but lack the expertise to fully evaluate a startup’s prospect or a product’s quality (due diligence is too costly when their investment is limited), e.g., in recent blockchain-based initial coin offerings, leading to high uncertainty and collective-action problems (Ritter, 2013). Yet the observation of funding targets and supports up-to-date allow them to learn and act in a Bayesian manner (Agrawal, Catalini, and Goldfarb, 2011; Zhang and Liu, 2012; Burtch, Ghose, and Wattal, 2013). Another example is venture financing: In an angel or A round of financing, entrepreneurs seek financing from multiple agents who face strategic risk: the firm can only implement its project with sufficient funding from them (Halac, Kremer, and Winter, 2018). Investors approached later often learn which others indicate support for the project, and many condition their contributions on the fundraising reaching the threshold for implementing the project.³ Our findings highlight the importance in these applications of designing AoN thresholds and utilizing new technologies to reach out to a broader base of potential supporters, not only for the entrepreneurs but also for welfare considerations.

Literature — Our paper foremost contributes to the large literature on information cascades, social learning, and rational herding (Banerjee, 1992; Bikhchandani, Hirshleifer, and Welch, 1998; Chamley, 2004). Our model largely builds on Bikhchandani, Hirshleifer, and Welch (1992) which discusses informational cascade as a general phenomenon. Welch (1992) relates information cascade to IPO underpricing. Guarino, Harmgart, and Huck (2011) and Herrera and Hörner (2013) consider information cascades when only one of the binary actions is observable and agents’ positions are unknown, and find that welfare could improve

²Since its inception in the arts and creativity-based industries (e.g., recorded music, film, video games), crowdfunding has quickly become a mainstream source of capital for early startups, and typically features AoN threshold. The Jumpstart Our Business Startups (JOBS) Act indicates that AoN features will likely be mandated and intermediaries need to “ensure that all offering proceeds are only provided to the issuer when the aggregate capital raised from all agents is equal to or greater than a threshold offering amount” (Sec. 4A.a.7). See <http://beta.congress.gov/bill/112th-congress/senate-bill/2190/text>.

³For example, the blockchain startup String Labs approached multiple agents such as IDG capital and Zhenfund sequentially, many of whom decided to invest after observing Amino Capital’s investment decision, and conditioned the pledge on the founders’ “successfully fundraising” in the round (meeting the AoN threshold). Syndicates involving both incumbent agents from earlier rounds and new agents are also common. One related example involves initial public offerings (IPOs): late investors learn from observing the behavior of early investors, and IPOs with high institutional demand in the first days of book-building also see high levels of bids from retail investors in later days (e.g., Welch, 1992; Amihud, Hauser, and Kirsh, 2003). The issuer faces an unknown demand for its stock and aggregates information from sequential agents about the demand curve (e.g., Ritter and Welch, 2002), therefore the issuer may choose to withdraw the offering if the market reaction is lukewarm.

and cascades can be uni-directional. Studies such as Anderson and Holt (1997), Çelen and Kariv (2004), and Hung and Plott (2001) provide experimental evidence for information cascades. We add by incorporating threshold implementation as a common manifestation of payoff interdependence, and show that it mitigates inefficiencies typically associated with information cascades. Moreover, we also contribute by characterizing multiple equilibria with information cascades and agents’ strategic complementarity.

Related to information cascades is information aggregation — an important function of financial markets and a central question in information economics (Wilson, 1977; Pesendorfer and Swinkels, 1997; Kremer, 2002). In the context of sequential voting, Ali and Kartik (2012) explore the optimality of collective choice problems; Dekel and Piccione (2000) extend Feddersen and Pesendorfer (1997) and demonstrate the equivalence for simultaneous and sequential elections; by restricting off-equilibrium beliefs, Wit (1997) and Fey (2000) show that a signaling motive can always halt cascade; Callander (2007) introduces voters’ desire to conform to show the sequential nature matters and cascades occur with probability one in the limit of large voter crowd.

In our setting, sequential action matters because it reveals more information than is contained in the event that the voter is pivotal. More fundamentally, the payoff structure in our setting is closer to the classic models of information cascades in that it not only depends on whether the project is implemented, but also depends on whether the agent supports the project. This distinguishes our model from extant ones which model either economic agents consuming their choice regardless of the choices of others or voters consuming the group selection independent of their own choice. Therefore, our model describes a new set of phenomena with equilibrium behavior (especially before an AoN is reached) not described by earlier studies such as Ali and Kartik (2006). Moreover, we obtain perfect information aggregation in large markets, which is typically unachievable in settings with information cascades (Ali and Kartik, 2012).

The paper also adds to an emerging literature on AoN design in the context of crowdfunding. Strausz (2017) and Ellman and Hurkens (2015) find that AoN is crucial for mitigating moral hazard and price discrimination. Chemla and Tinn (2016) demonstrate that the AoN design Pareto-dominates the alternative “keep-it-all” mechanism. Chang (2016) shows that

under common-value assumptions AoN generates more profit by making the expected payments positively correlated with values. As a cautionary tale, Brown and Davies (2017) show in a static setting that an exogenous AoN threshold can reduce the financing efficiency. Hakenes and Schlegel (2014) argue that endogenous loan rates and AoN thresholds encourage information acquisition by individual households in lending-based crowdfunding. Instead of introducing moral hazard or financial constraint, or deriving static optimal designs, we focus on pricing and learning under *endogenous* AoN thresholds in a *dynamic* environment.

The rest of the paper is organized as follows: Section 2 sets up the model and derives agents’ belief dynamics; Section 3 characterizes the equilibrium, starting with exogenous AoN threshold and issuance price to highlight the main mechanism of uni-directional cascades, before endogenizing them; Section 4 discusses model implications and demonstrates how AoN better utilizes the wisdom of the crowd to improve proposal feasibility, project selection, and information production; Section 5 analyzes the dependence of project implementation and information aggregation on the size of the crowd; Section 6 discusses agents’ option to wait and characterizes all other equilibria, before Section 7 concludes. The appendix contains all the proofs.

2 A Model of Directional Cascades

2.1 Setup

Consider a project (or proposal or type of behavior) presented to a sequence of agents $i = 1, 2, \dots, N$ who can either support (adopt) or reject it. The action of agent i is $a_i \in \{-1, 1\}$, where $a_i = 1$ indicates supporting and $a_i = -1$, rejecting.⁴ If the proposal is implemented eventually, then the proposer charges every supporting agent a pre-specified “price” p , and each agent receives a payoff V from the project, which is either 0 or 1.⁵ In fund-raising

⁴While in crowdfunding they may choose the quantity of investment, under risk-neutrality it suffices to consider the case where each supporter can only make a unit contribution: If an agent finds the project to be positive NPV, then she scales to full investment capacity. What matters is the information conveyed by her action. In practice, crowdfunders often observe both the total capital raised and the number of supporters thus far (Vismara, 2016).

⁵In practice, p is typically pre-determined by the proposer. There is a separate literature studying herding and financial markets that allows asset price to dynamically change and focuses on asset pricing implications (Avery and Zemsky, 1998; Brunnermeier, 2001; Vives, 2010; Park and Sabourian, 2011). We

activities such as crowdfunding or venture financing rounds, p is the amount of money that each supporting agent contributes and is returned if the project is not implemented. In other activities such as political petitions, p can be interpreted as the supporting effort or reputation cost if the petition goes through and becomes public.

We depart from prior literature by incorporating an “all-or-nothing” (AoN) threshold: the proposer receives “all” support/contribution if the campaign succeeds in reaching a pre-specified threshold number of supporters, or “nothing” if it fails to do so. In other words, the project is implemented if and only if at least T agents support, where T could be exogenous in the case of legal legacy, or endogenous in the case of IPO issuance or crowdfunding. We also remind the readers that supporters only incur p if the project is implemented.

By deviating from the canonical models of information cascade only in allowing AoN thresholds, we aim to make the fundamental theoretical point in a transparent way that threshold implementation significantly affects financing and information aggregation. While various extensions have been made about the classic models, doing them here does not alter or add to our general economic insight, but reduces the model’s analytical tractability. We therefore leave them for future studies on specific applications.

Agents’ Information and Decision

All agents including the proposer are rational, risk-neutral, and share the same prior that the project pays $V = 0$ and $V = 1$ with equal probability.⁶ Each agent i observes one conditionally independent private signal $x_i \in \{1, -1\}$, which is informative:

$$Pr(x_i = 1|V = 1) = Pr(x_i = -1|V = 0) = q \in \left(\frac{1}{2}, 1\right); \quad (1)$$

$$Pr(x_i = -1|V = 1) = Pr(x_i = 1|V = 0) = 1 - q \in \left(0, \frac{1}{2}\right). \quad (2)$$

follow the standard cascade models to fix the price for taking an action ex ante, which more closely matches applications such as those in entrepreneurial finance.

⁶Idiosyncratic preferences or private valuations would not change the intuition of the economic mechanism. Our specification is fitting for equity-based crowdfunding. Even in reward-based crowdfunding whereby agents have private valuations and preferences, there is a common value corresponding to the basic quality of the product. Our assumption of common value also allows us to make unambiguous welfare comparisons concerning the financing and information aggregation efficiency (Fey, 1996; Wit, 1997).

We represent the sequence of private signals by $x = (x_1, \dots, x_N)$ and the set of all such sequences by $X = \{1, -1\}^N$.⁷

The order of agents' decision-making is exogenous and known to all.⁸ When agent i makes her decision, she observes x_i and the history of actions $\mathcal{H}_{i-1} \equiv (a_1, a_2, \dots, a_{i-1}) \in \{-1, 1\}^{i-1}$. Her strategy can thus be represented as $a_i(\cdot, \cdot) : \{1, -1\} \times \{-1, 1\}^{i-1} \rightarrow \{-1, 1\}$. To simplify exposition, we define $A_i = \sum_{j=1}^i a_j \mathbb{1}_{\{a_j=1\}}$, for $1 \leq i \leq N$. When $1 \leq i' < i \leq N$ and $\mathcal{H}_{i'}$ has the same first i' elements as \mathcal{H}_i does, we say $\mathcal{H}_i \in \{-1, 1\}^i$ *nests* $\mathcal{H}_{i'} \in \{-1, 1\}^{i'}$ and write $\mathcal{H}_{i'} \prec \mathcal{H}_i$. Agent i 's optimization is:

$$\max_{a_i \in \{-1, 1\}} \mathbb{1}_{\{a_i=1\}} \mathbb{E} \left[(V - p) \mathbb{1}_{\{A_N \geq T\}} \mid x_i, \mathcal{H}_{i-1}, a_i = 1, a_{-i}^* \right], \quad (3)$$

where $\mathbb{1}_{\{A_N \geq T\}}$ is the indicator function for project implementation, and a_{-i}^* are equilibrium strategies of other agents as defined later in Definition 2. Agent i gets zero payoff from rejecting ($a_i = -1$) and gets $(V - p) \mathbb{1}_{\{A_N \geq T\}}$ from supporting ($a_i = 1$) the proposal.

Finally, we introduce a tie-breaking rule for agents.

Assumption 1 (Tie-breaking). *When indifferent between supporting and rejecting, an agent supports if the AoN threshold can be reached with all remaining agents supporting regardless of their private signals.*

As discussed in Section 6, the strategic complementarity of agents' actions becomes important with AoN thresholds. What Assumption 1 rules out are the trivial equilibria where everyone believes that there would not be enough supporting agents and therefore rejects. It can thus be viewed as an equilibrium refinement to avoid discussing such extreme forms of coordination. The assumption is also natural in that when implementation is not completely infeasible, the proposer can always provide an infinitesimal subsidy contingent on implementation to break agents' indifference to induce more support.

⁷The binary information and action structure here are the canonical focus in both the information cascade literature (Bikhchandani, Hirshleifer, and Welch, 1992) and the voting literature (Feddersen and Pesendorfer, 1996; McLennan, 1998).

⁸While real world examples such as crowdfunding may involve endogenous orders of agents, our setup allows us to relate and compare to the large literature on information cascades which typically has exogenous orders of agents. We show in Section 6.1 that our fundamental result is robust when agents have the option to wait. Related is Liu (2018) that studies how AoN affects the timing of investor moves.

Proposer’s Optimization

Let $0 \leq \nu < 1$ be the per supporter cost for the proposer. In the context of reward-based crowdfunding, ν can be the production cost of each product. In the equity-based crowdfunding or IPO process, ν can be interpreted as the issuer’s share reservation value. In essence, to a social planner varying ν is equivalent to varying the prior on the project’s NPV.⁹ Given the campaign length N , the proposer chooses price p and AoN threshold T to solve the following problem:

$$\max_{p,T} \pi(p, T, N) = E \left[(p - \nu) A_N \mathbb{1}_{\{A_N \geq T\}} \mid \{a_i^*\}_{i=1,2,\dots,N} \right], \quad (4)$$

Again, $\{a_i^*\}_{i=1,2,\dots,N}$ are investor agents’ equilibrium strategies. For the remainder of the paper, we drop a_i^* and a_{-i}^* in (3) and (4) for notational simplicity. In fund-raising scenarios, the proposer maximizes his expected profit. In non-financial scenarios, the proposer solicits the maximum amount of support, given that p can be interpreted as the amount of support from each agent.

2.2 Belief Dynamics and Information Cascade

We first analyze the dynamics of the common posterior belief after observing the action history. In particular, suppose the agents update their beliefs according to Bayes’ rule, then the common posteriors, $\mathbb{E}[V|\mathcal{H}_i]$, only takes values from a countable set.

Lemma 1. (a) For every $1 \leq i \leq N$ and every history of actions $\mathcal{H}_i \in \{-1, 1\}^i$, there exists an integer $k(\mathcal{H}_i)$ such that $\mathbb{E}[V|\mathcal{H}_i] = V_{k(\mathcal{H}_i)}$, where

$$V_k = \frac{q^k}{q^k + (1 - q)^k} \quad k \in \mathbb{Z} \quad (5)$$

(b) If $\mathcal{H}_i = (\mathcal{H}_{i-1}, a_i)$ for some $a_i \in \{-1, 1\}$, then $k(\mathcal{H}_i) - k(\mathcal{H}_{i-1}) \in \{-1, 0, 1\}$, depending on whether agent i ’s action is informative in equilibrium (playing separating strategy), and if it is, whether she supports or rejects.

⁹Having a fixed cost does not add additional insight, and is left out for clearer exposition and intuition.

k here is the posterior valuation of the project from agent i 's perspective right after her decision. Lemma 1 states that the posterior belief on project type only depends on the difference between numbers of inferred high and low signals so far, a convenient property also in Bikhchandani, Hirshleifer, and Welch (1992). Given Lemma 1, it is easy to verify that agent i 's expected project value conditional on \mathcal{H}_{i-1} and her private signal x_i is

$$\mathbb{E}_i(V|\mathcal{H}_{i-1}, x_i, a_{-i}^*) = V_{k(\mathcal{H}_{i-1})+x_i} \quad (6)$$

It should be understood that the expectation is on an agent i 's information set (her own signal and actions up till her decision-making, given action strategies of other agents). But for notational simplicity, we drop the subscript i in the expectation \mathbb{E}_i for the remainder of the paper unless otherwise stated.

When an agent's action does not reflect her private signal, the market fails to aggregate dispersed information. Our notion of informational cascade is standard (Bikhchandani, Hirshleifer, and Welch, 1992):

Definition 1 (Information Cascade). *An UP cascade occurs following a history of actions \mathcal{H}_n ($1 \leq n < N$) if along the equilibrium path, all subsequent agents support the proposal, regardless of their private signal, while agent n herself is not part of any cascade. We denote the set of such histories by \mathbb{H}^U . A DOWN cascade is similarly defined, replacing "support" with "reject," and \mathbb{H}^U with \mathbb{H}^D .*

Standard models feature both UP and DOWN cascades. If a few early agents observe high signals, their support may push the posterior so high that the project remains attractive even with a private low signal. Similarly, a series of low signals may doom the offering. An early preponderance towards supporting or rejecting causes all subsequent individuals to ignore their private signals, which are then never reflected in the public pool of knowledge.

2.3 Equilibrium Definition

We use the concept of perfect Bayesian Nash equilibrium (PBNE).

Definition 2 (Equilibrium). *An equilibrium consists of the proposer's proposal choice $\{p^*, T^*\}$, agents' action strategies $\{a_i^*(x_i, \mathcal{H}_{i-1}, p^*, T^*)\}_{i=1,2,\dots,N}$, and their beliefs such that:*

1. For each agent i , given the price p^* , implementation threshold T^* , and other agents' strategies $a_{-i}^* = \{a_j^*\}_{j=1,2,\dots,i-1,i+1,\dots,N}$, a_i^* solves her optimization problem in (3).
2. Given agents' strategies $\{a_i^*\}_{i=1,2,\dots,N}$, p^* and T^* solve the proposer's optimization problem in (4).
3. Agents' belief dynamics are formed according to Bayes rule strategies whenever an action history is reached with positive probability in equilibrium.

In our baseline model we focus on equilibria in which all actions are informative outside a cascade, as formalized here:

Definition 3 (Informer Equilibrium). *An equilibrium is called an “informer equilibrium” if for every i and history $\mathcal{H}_{i-1} \in \{-1, 1\}^{i-1}$ that does not nest any history in \mathbb{H}^U or \mathbb{H}^D , agent i 's action differs for different x_i , i.e. $a_i(1, \mathcal{H}_{i-1}) \neq a_i(-1, \mathcal{H}_{i-1})$.*

In other words, agents' actions are informative before an information cascade, making them “informers.” Subsequent agents Bayesian-update their beliefs. Informer equilibria are natural and clearly illustrate our economic mechanisms and intuition. We analyze all other PBNE in Section 6.2 and show that they can be viewed as variants of the “informer equilibrium” and asymptotically converge ($N \rightarrow \infty$) to the “informer equilibrium” in terms of pricing, project implementation, and information aggregation.

2.4 Benchmark without Threshold Implementation

Before solving the equilibrium, we first consider the benchmark case without threshold implementation (or equivalently, $T = 1$). Every agent making the decision knows that the project is implemented for sure if she supports. Hence her payoff does not depend on subsequent agents' actions, and our model reduces to those in Bikhchandani, Hirshleifer, and Welch (1992) and Welch (1992). Here each agent i chooses to support if and only if

$$\mathbb{E}[V|x_i, \mathcal{H}_{i-1}] \geq p. \tag{7}$$

For an exogenous p , in equilibrium both UP and DOWN cascades can occur, which stops

public information production. When p is endogenous, imprecise private signals can cause “underpricing”:

Lemma 2. *The proposer always charges $p \leq q$. In particular, when $\nu = 0$ and $q \leq \frac{3}{4} + \frac{1}{4} \left(3^{\frac{1}{3}} - 3^{\frac{2}{3}} \right)$, the optimal price is $p^* = 1 - q < \frac{1}{2} = \mathbb{E}[V]$.*

The lemma basically restates and generalizes the underpricing results in Welch (1992) (in which $\nu = 0$), particularly Theorem 5. The general pricing upper bound q comes from the concern for DOWN cascades from the start. If $p > q$, then even with a positive signal $x_1 = 1$, the first agent rejects and so does every subsequent agent, yielding zero payoff for the proposer.

The second part of the lemma concerns the optimal pricing when agents’ signals are not very precise. UP and DOWN cascades affect the proposer’s payoff asymmetrically even though they both reduce the information aggregation. While the proposer benefits from UP cascades by attracting support from future agents with negative signals, he is concerned with DOWN cascades since a few early rejections may doom the offering. Consequently, he charges a low price $p = 1 - q < \frac{1}{2}$ to ensure an UP cascade at the very first agent (even when the agent has bad signal). When private signals are imprecise, underpricing ($1 - q$) is less costly and can be optimal for the proposer.

3 Equilibrium Characterization

We now solve the equilibrium in several steps. First, we take the price p and AoN threshold T as exogenous, and examine agents’ supporting/rejection decisions. We then derive the proposer’s endogenous pricing and AoN-threshold setting, and compare the equilibrium outcomes to the benchmark outcomes without implementation thresholds.

3.1 Exogenous Price and AoN Threshold

The first main result in our paper is that given any p and T , only UP cascades may exist before the AoN threshold is reached.

Proposition 1. For a given price $p \in (0, 1)$, define $\bar{k}(p)$ as the smallest integer such that

$$p \leq \frac{q^{\bar{k}(p)}}{q^{\bar{k}(p)} + (1 - q)^{\bar{k}(p)}} \quad (8)$$

Then, for any given pair of (p, T) , there exists an essentially unique informer equilibrium with $a_i^*(x_i, \mathcal{H}_{i-1}) \equiv a^*(x_i, k(\mathcal{H}_{i-1}), A_{i-1}(\mathcal{H}_{i-1}))$ and posteriors $P(V = 1 | \mathcal{H}_i) = V_{k^*(\mathcal{H}_i)}$, where:

$$a_i^*(x_i, k_{i-1}, A_{i-1}) = \begin{cases} x_i & \text{if } A_{i-1} < T - 1 \ \& \ k_{i-1} \leq \bar{k}(p) \\ 1 & \text{if } k_{i-1} > \bar{k}(p) \\ -1 & \text{if } A_{i-1} \geq T - 1 \ \& \ k_{i-1} < \bar{k}(p) - 1 \\ x_i & \text{if } A_{i-1} \geq T - 1 \ \& \ k_{i-1} \in \{\bar{k}(p), \bar{k}(p) - 1\} \end{cases} \quad (9)$$

$$k_i^*(\mathcal{H}_i) \equiv k^*(a_i, k_{i-1}, A_{i-1}) = \begin{cases} k_{i-1} + a_i & \text{if } A_{i-1} < T - 1 \ \& \ k_{i-1} \leq \bar{k}(p) \\ k_{i-1} & \text{if } k_{i-1} > \bar{k}(p) \\ k_{i-1} & \text{if } A_{i-1} \geq T - 1 \ \& \ k_{i-1} < \bar{k}(p) - 1 \\ k_{i-1} + a_i & \text{if } A_{i-1} \geq T - 1 \ \& \ k_{i-1} \in \{\bar{k}(p), \bar{k}(p) - 1\} \end{cases} \quad (10)$$

where $k_0 = 0$ and $A_0 = 0$.

This equilibrium is essentially unique in the sense that all informer equilibria are equivalent in terms of agents' payoff and project implementation. Proposition 1 describes agents' corresponding strategies. When it is impossible to reach the AoN target, the action strategies could differ, but are irrelevant for agent payoffs and project implementation outcome.

Proposition 1 shows that there is no DOWN cascade before approaching the AoN threshold ($A_{i-1} < T - 1$) as the first line of (9) captures. An UP cascade starts once the posterior belief $k_i(\mathcal{H}_i)$ exceeds $\bar{k}(p)$, the threshold difference between high signals and low signals inferred from past actions, as the second line of (9) describes. In other words, the valuation is sufficiently high that even a subsequent agent with low signal should always support. In equilibrium agents with high signals always support regardless of the history they observe

while agents with bad signals support only when there is an UP-cascade. Once $A_{i-1} = T - 1$, agent i and subsequent ones face exactly the same decision as in standard cascade model, as lines 2-4 of (9) delineate. Beliefs are updated correspondingly, as (10) summarizes.

The intuition for the uni-directional cascade is that an agent with a high signal is protected from supporting losses because she does not pay if the project turns out to be bad. The agent observing $T - 1$ preceding supporters would be the “gate-keeper” for her because their interests are aligned yet the gate-keeper observes a longer history and makes a more informed decision.

Observing a longer history is helpful only when actions reflect private information. So to complete the argument, we need to show that when there is no UP cascade yet and before the AoN threshold is approached, agents with bad signals reject the proposal. If an agent with a bad signal deviates and supports, then all subsequent agents would misinterpret her action and form wrong posterior beliefs. The over-optimistic belief implies that subsequent agents either start an UP cascade too early or reach the AoN threshold when the true posterior is not high enough. Taking that into account, agents with bad signals find deviations unattractive.

The proof for Proposition 1 informs both the possibility and arrival time of cascades, as summarized in the following corollary.

Corollary 1. *An UP cascade starts whenever the history has $\bar{k}(p)+1$ more agents supporting rather than rejecting; a DOWN cascade starts whenever the history has $\bar{k}(p) - 2$ more agents supporting rather than rejecting and there are at least $T - 1$ preceding supporters.*

3.2 Endogenous Price and AoN Threshold

In real life, especially in financial settings such as crowdfunding and venture financing, the proposer endogenously sets the price and the AoN threshold, which we now model.

To avoid discussing the case in which the proposal trivially fails due to the production cost exceeding the highest possible valuation, we focus on $\nu \leq V_N$ for the remainder of the paper. With the AoN threshold, there exists an informer equilibrium such that DOWN cascade is impossible except for some special scenarios.

Proposition 2. *Given agents’ equilibrium strategies specified in Proposition 1, there exists*

a pair of price and AoN target (p^*, T^*) that maximizes the proposer's expected revenue in (4). Specifically, there exists $k^* \in \{-1, 0, \dots, N\}$ such that $p^* = V_{k^*}$ and $T^* = \lfloor \frac{N+k^*}{2} \rfloor$.

Proposition 2 characterizes agents' strategies and the proposer's endogenous proposal design in the equilibrium. Since for any $p \in (V_{k-1}, V_k]$, all agents make the same supporting decisions, the proposer can always charge $p = V_k$ and receives a higher profit. We therefore can focus our analysis on $p \in \{V_{-1}, V_0, \dots, V_N\}$. We exclude $k < -1$ because $V_{-1} = 1 - q$ is already sufficiently low to induce an UP cascade from the very beginning.

In the equilibrium the proposer chooses the optimal level of AoN threshold jointly with price to exclude DOWN cascades. Recall that there is no DOWN cascade before approaching the threshold. A higher AoN threshold reduces the burden of using underpricing to exclude DOWN cascade once the threshold is reached. Yet a higher threshold itself is more difficult to reach. The proposer considers the tradeoff and for a given p chooses the lowest level of AoN threshold to exclude DOWN cascades except for some very special cases described in the next corollary.

Corollary 2 (Uni-directional Cascades). *Denote the $(T^* - 1)^{th}$ supporting agent by i , then there is no DOWN cascade unless the following holds simultaneously:*

1. $N + k^*$ is odd;
2. There is no UP cascade;
3. $N - 3 \leq i \leq N - 1$;
4. $x_j = -1, \forall i < j \leq N - 1$.

Consequently, any DOWN cascade entails a start at $i = N$ and implementation failure, i.e., there is no DOWN cascade with more than one agent herding.

For all practical purposes, DOWN cascades are of no concern here because they rarely happen and only start from the last agent. Moreover, the project would not be implemented anyway in those scenarios even if all private signals were aggregated. Consequently, a DOWN cascade, if exists, does not affect project implementation and has almost no impact on information aggregation.

Next, we examine the properties of UP cascades and their impact on pricing. To characterize the distribution of UP-cascade arrival time, let $\varphi_{k+1,i}$ denote the probability that an UP cascade starts after agent i . Since each agent privately observes either $x_i = 1$ or $x_i = -1$ and her decision perfectly reveals her private signal before an UP cascade starts, the arrival of an UP cascade is equivalent to the first passage time of a one-dimension biased random walk. Using the well-established result on hitting times (Van der Hofstad and Keane, 2008), we can compute $\varphi_{k+1,i}$:

Lemma 3. *Suppose $\bar{k}(p) = k$ (recall $\bar{k}(p)$ is defined in Proposition 1), then we have*

(a) *The probability that an UP cascade starts after agent i is*

$$\varphi_{k+1,i} = \frac{k+1}{i} \binom{i}{\frac{i+k+1}{2}} [q(1-q)]^{\frac{i-k-1}{2}} \frac{(1-q)^{k+1} + q^{k+1}}{2}, \quad (11)$$

where

$$\binom{i}{\frac{i+j}{2}} = \begin{cases} \frac{i!}{\frac{i+j}{2}! \frac{i-j}{2}!} & \text{if } i \geq j \text{ and } j+i \text{ even;} \\ 0 & \text{otherwise.} \end{cases} \quad (12)$$

(b) *For a given pair of price and threshold (p, T) , the probability of reaching the threshold at agent i without a prior UP cascade is $\frac{q^k + (1-q)^k}{q^{k+1} + (1-q)^{k+1}} \varphi_{k+1,i+1}$.*

We note that a project is eventually implemented once an UP cascade starts. But for any agent $i \leq N - 2$, if the UP cascade has not started yet, there is a strictly positive probability that the project is not implemented. Therefore, for a project to be implemented, either (i) there is an UP cascade, or (ii) the total number of supporting agents is exactly T . We illustrate the two scenarios in Figure 1, which plots the difference between supporting agents and rejecting agents when n agents have arrived. The figure also includes a sample path that leads to an implementation failure because AoN threshold is not reached.

Now for optimal pricing, a higher price allows the proposal to extract more rent from each supporter, but at the same time reduces the number of supporters and probabilities of implementation. We can show an interior optimal exists and is weakly increasing in N , as the next proposition describes.

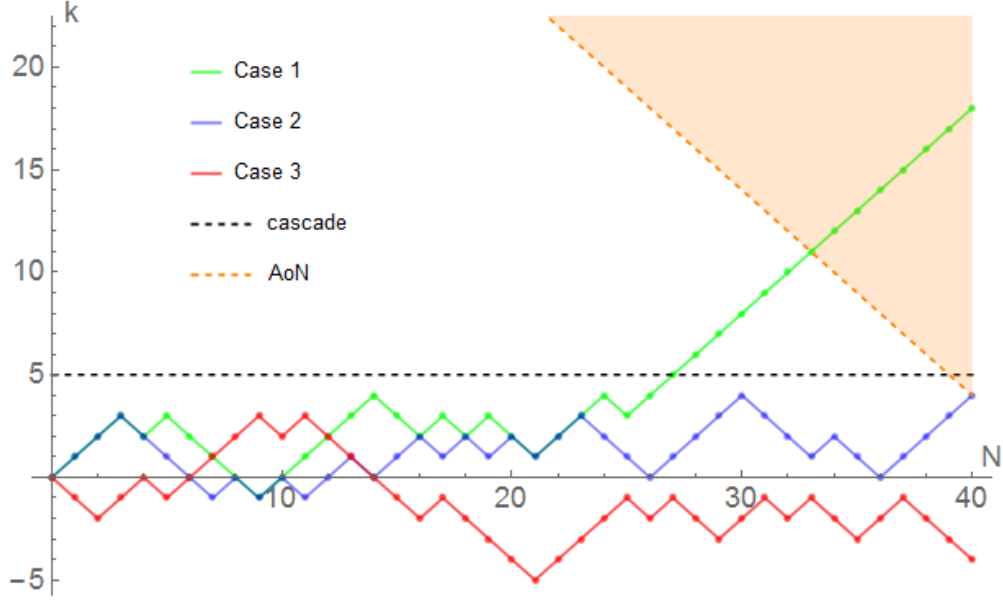


Figure 1: Evolution of support-reject differential

Simulated paths for $N = 40$, $q = 0.7$, $p^* = V_4 = 0.9673$, and AoN threshold $T^*(N) = 22$. Case 1 indicates a path that crosses the cascade trigger $\bar{k}(p) + 1 = 5$ at the 26th agent and all subsequent agents support regardless of their private signal; case 2 indicates a path with no cascade, but the project is still funded by the end of the fundraising; case 3 indicates a path where AoN threshold is not reached and the project is not funded. The orange shaded region above the AoN line indicates that the project is funded.

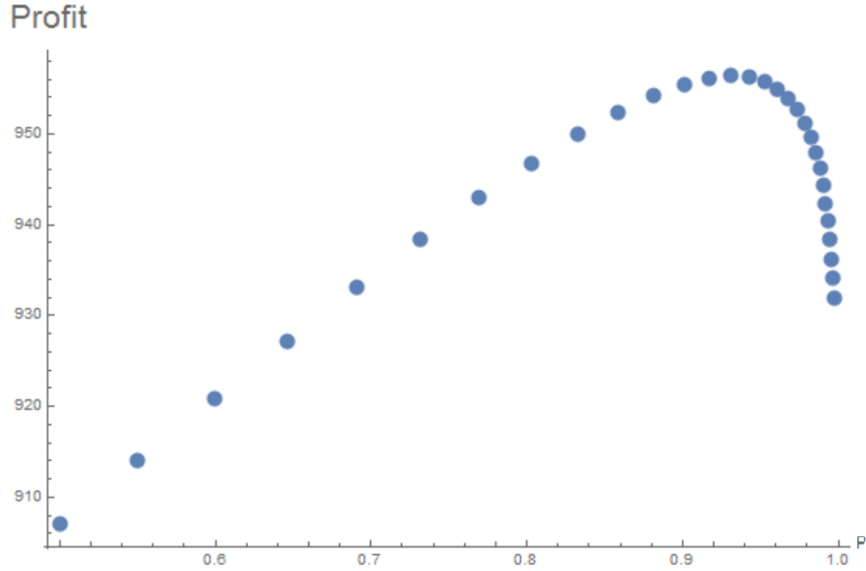


Figure 2: Proposal profit as a function of price with $N = 2000$, $\nu = 0$ and $q = 0.55$.

Proposition 3. Let $\bar{k}(\nu) \in \{0, 1, 2, \dots\}$ be the smallest integer satisfying $V_{\bar{k}(\nu)} \geq \nu$. For each $k \in \{\bar{k}(\nu), \bar{k}(\nu) + 1, \bar{k}(\nu) + 2, \dots\}$, there exists a finite positive integer $\underline{N}(k)$ such that for

$\forall N \geq \underline{N}(k)$, $\pi(V_k, T^*(V_k), N) > \pi(V_{k-1}, T^*(V_{k-1}), N)$, where $T^*(V_k)$ is defined as $\lfloor \frac{N+k}{2} \rfloor$.

In the proof of Proposition 3 we derive an explicit characterization of the proposer’s expected profit as a function of price $p = V_k$, associated optimal threshold $T^*(V_k)$ and number of potential agents N . Figure 2 provides an illustration of an interior optimal price.

Our findings on pricing are important because the underpricing or overpricing of securities or products may affect the success or failure of a project proposal, and thus impact the real economy. IPOs with limited distribution channels of investment banks (Welch, 1992) constitute a salient example. We discuss these model implications next.

4 Implications of Threshold Implementation

We examine the immediate implications of threshold implementation for proposal feasibility, project selection, and information aggregation, which are the key functionalities of markets and platforms. The results primarily pertain to the general equilibrium with endogenous p and T , although results derived from the sub-game equilibrium solution apply to situations where p and T are exogenous as well.

4.1 Feasibility of Proposal

From Lemma 2, there is a pricing upper bound in standard cascade models above which the proposal is infeasible. For example, good projects with production cost $\nu > q$ cannot be supported because even the break-even price triggers DOWN cascades. With threshold implementation, however, the proposer can charge a price $p > q$ and still implement the projects.

Proposition 4 (Proposal Feasibility). *Without AoN thresholds, no project with $\nu > q$ can be implemented; with endogenous AoN thresholds, all projects with $\nu \leq V_N$ have a positive probability to be implemented.*

The proposition follows directly from that charging $p \geq \nu$ does not trigger a DOWN cascade if T is set to be sufficiently high. As a result, crowdfunding and the like can enable financing of projects of higher production costs for which the proposer cannot avoid DOWN cascades

in traditional settings without AoN thresholds. This is consistent with Mollick and Nanda (2015) which empirically documents that crowdfunding is more likely to finance a project that a group of experts would not finance under traditional settings.

4.2 Wisdom of the Crowd for Project Selection

A support-gathering process produces little information in most extant models of information cascade. As soon as the public pool becomes slightly more informative than the signal of a single individual, individuals defer to the actions of predecessors and a cascade begins. When the pricing is endogenous, as in Welch (1992), cascades always start from the very beginning, and all projects are implemented, leading to poor project selection.

With AoN thresholds, DOWN cascades do not occur before reaching the implementation threshold, and UP cascades do not start from the beginning. Good projects thus have a higher chance of reaching the target threshold due to the information production before an UP cascade starts. We denote the probabilities of missing a good project (Type I error) and financing a bad project (Type II error) by $\mathcal{P}^I = 1 - Pr(A_N \geq T|V = 1)$ and $\mathcal{P}^{II} = Pr(A_N \geq T|V = 0)$ respectively. The probability that a good project is implemented is simply $1 - \mathcal{P}^I$, which is greater than \mathcal{P}^{II} , the probability of a bad project getting implemented. Project selection therefore improves.

While UP cascades do lead to some bad projects being financed, such Type I errors are not as frequent as in Welch (1992), in which all bad projects are financed and the probability of the cascade being correct is $\frac{1}{2}$. AoN thresholds reduce underpricing, which in turn delays cascade and increases the probability of correct cascades (UP cascade when $V = 1$) given by

$$Pr(V = 1|p) = \frac{q^{\bar{k}(p)+1}}{q^{\bar{k}(p)+1} + (1 - q)^{\bar{k}(p)+1}} \quad (13)$$

Because $\bar{k}(p)$ is weakly increasing in p (Proposition 1) and the optimal pricing is weakly increasing in N (Proposition 3), the following proposition ensues.

Proposition 5 (Project Selection). *Good projects are more likely to be implemented than bad projects. Moreover, the probability of a cascade being correct is larger than $\frac{1}{2}$, increasing in q , weakly increasing in p (thus in N when p is endogenous), and weakly increasing in T .*

Uni-directional cascade and threshold implementation also mean that offerings in our setting can fail whereas offerings never fail in Welch (1992). Our model thus helps explain why some offerings fail occasionally and/or are withdrawn, without invoking insider information as Welch (1992) does.

More importantly, whereas N does not matter in standard cascade models, threshold implementation links the timing and correctness of cascades to the size of the crowd. We elaborate on the two error probabilities further in Section 5.1 and show that with a large N , as is the case for Internet-based crowdfunding, the concern that a good project may fail to be implemented goes away. The proposer and supporters fully harness the wisdom of the crowd to distinguish good projects from the bad ones.

4.3 Information Aggregation and Proposer’s Real Option

In many crowdfunded projects, although entrepreneurs commit to threshold implementation, they still retain discretion over their own effort provision and the project’s future commercialization—a form of real option whose exercise utilizes the information aggregated in fundraising.¹⁰ Viotto da Cruz (2016) and Xu (2017) provide empirical evidence that entrepreneurs indeed benefit from the information aggregated from crowdfunding platforms when making real decisions. For example, Xu (2017) documents in a survey of 262 unfunded Kickstarter entrepreneurs that after failing, 33% continued as planned.

More generally, proposers learn about projects’ prospect (true V in our model) from the support-gathering process, for future decisions. The level of information aggregation is reflected in how the posterior distribution of V relates to the true V . Given agents’ risk-neutrality, a simple metric is $\mathbb{E}[|\mathbb{E}[V|\mathcal{H}_N] - V|]$.

Proposition 6. $\mathbb{E}[V|\mathcal{H}_N, A_N < T^*]$ is weakly increasing in A_N . Moreover, $\mathbb{E}[|\mathbb{E}[V|\mathcal{H}_N] - V|]$ is decreasing in N .

Compared with models without AoN threshold, information aggregation improves for

¹⁰For example, V can be interpreted as a transformation of the aggregate demand, which could be high ($V = 1$) or low ($V = 0$). Suppose that after the crowdfunding, an entrepreneur considers commercialization or abandoning the project (upon crowdfunding failure), and for simplicity the commercialization or continuation decision pays V (after normalization), but incurs an effort or reputation or monetary cost represented in reduced-form by I . Then the entrepreneur’s expected payoff for the real option is $\max\{\mathbb{E}[V - I|\mathcal{H}_N], 0\}$.

two reasons. First, DOWN cascade is absent in equilibrium (except the last agent in special scenarios) with an endogenous AoN threshold. Rejections are thus informative. Second, the endogenous price p is sufficiently high that UP cascades do not arrive immediately, making action history informative.

Different from standard cascade models with DOWN cascade, conditional on failing to reach the AoN threshold, the proposer updates the belief more positively with more supporting agents. Our model further implies that the belief updates on V based on incremental support is smaller conditional on project implementation because it likely involves an UP cascade and information aggregation is more limited. This is consistent with Xu (2017), which finds that conditional on fundraising success, a 50% increase in pledged amount leads to a 9% increase in the probability of commercialization outside the crowdfunding platform — a small sensitivity of the update on project prospective to the level of support.

5 Equilibrium with Large Crowds

Given that the equilibrium outcomes and implications depend on the size of the crowd, N , several questions naturally arise: How big is the efficiency gain from having access to a large crowd? Is there full information aggregation in the limit? How is the surplus split between the proposer and the supporters?

In this section, we answer these questions by first examining the properties of the error probabilities and prices as N grows to be large. We then discuss how in the limit, we obtain efficient project implementation and full information aggregation.

5.1 Errors and Prices with Large Crowds

We find that the probability that a good project fails to implement (type I error) declines exponentially and approaches zero for a sequence of exogenous prices and implementation thresholds that weakly increase in N . That said, bad projects are implemented with positive probability, i.e., type II error persists. We then show that type II errors vanish in the limit once we endogenize the price and AoN target.

Exogenous Price and AoN Targets

Lemma 4. Consider a fixed price $p > 0$ and an increasing sequence of AoN targets $\{T_N\}_{N=1}^{\infty}$, where $T_N \leq \lfloor \frac{N+\bar{k}(p)}{2} \rfloor$ for every N , and $\liminf_{N \rightarrow \infty} \frac{T_N}{N} \in (0, \frac{1}{2}]$. Let $\mathcal{P}_N^I = 1 - \Pr(A_N \geq T_N | V = 1)$ and $\mathcal{P}_N^{II} = \Pr(A_N \geq T_N | V = 0)$ denote the probabilities of missing a good project (Type I error) and financing a bad project (Type II error) respectively. Then, for any $\beta > \frac{q}{2q-1}$, there exist positive numbers M and $x < 1$, independent from the choice of p , such that if $T_N > \beta(\bar{k}(p) + 1)$:

$$\begin{aligned} \mathcal{P}_N^I &< Mx^{T_N - \beta\bar{k}(p)} \quad \text{and} \\ \left(1 - Mx^{T_N - \beta\bar{k}(p)}\right) \left(\frac{1-q}{q}\right)^{\bar{k}(p)+1} &< \mathcal{P}_N^{II} < \left(\frac{1-q}{q}\right)^{\bar{k}(p)+1}. \end{aligned} \tag{14}$$

Lemma 4 shows that when the price is fixed, the probability of reaching the AoN target increases, as the probability of having an UP cascade increases. While fewer good projects are missed as N increases (type I error decreases), more bad projects are also implemented (type II error increases).

The proof for Lemma 4 also shows that in the limit, all good projects are implemented.

Corollary 3. For fixed p and a sequence $\{T(N)\}_{N=1}^{\infty}$, if $\liminf_{N \rightarrow \infty} \frac{T(N)}{N} \in (0, 1)$, then a good project with $V = 1$ is implemented almost surely with an UP cascade, as $N \rightarrow \infty$.

A large agent base in a sense improves the implementation of good projects. The intuition is that when $V = 1$ and T_N is sufficiently large, the law of large numbers implies that with a high probability, the number of agents with favorable signals among the first T_N agents exceeds the ones with negative signals by a large amount, which in turn guarantees implementation through an UP cascade.¹¹ Therefore, all good projects are implemented as N becomes arbitrarily large. However, if $V = 0$ and few agents with favorable signals happen to be concentrated at the beginning of the queue, then an UP cascade may start too early. In this case, the agent ignore the negative private signals obscured by the UP cascade and support a bad project.

¹¹It follows from the assumption that $T_N \leq \lfloor \frac{N+\bar{k}(p)}{2} \rfloor$.

Endogenous Price and AoN Target

The next lemma shows that when the proposer chooses p and T optimally, p approaches the highest possible value.

Lemma 5. *Suppose $\{p_N^*, T_N^*\}_{N=1}^\infty$ is the sequence of optimal prices and AoN targets. Then there exist $\gamma_1, \gamma_2 > 0$ such that $N^{\gamma_1}(1 - p_N^*) < \gamma_2$ for every N and $T_N^* = \lfloor \frac{k_N^* + N}{2} \rfloor$. Therefore, $\lim_{N \rightarrow \infty} p_N^* = 1$.*

Unlike Lemma 2, Lemma 5 implies that the optimal price depends on the number of potential agents N . A financial technology (Internet-based platforms) that allows reaching a greater N thus has a fundamental impact.¹²

Figure 3 shows the optimal pricing for different values of N , with the left panel plots the associated k and right panel plots the absolute price level. With more agents, the proposer can charge a higher price, which can appear “overpriced,” ex ante, i.e., $p > \mathbb{E}[V]$.

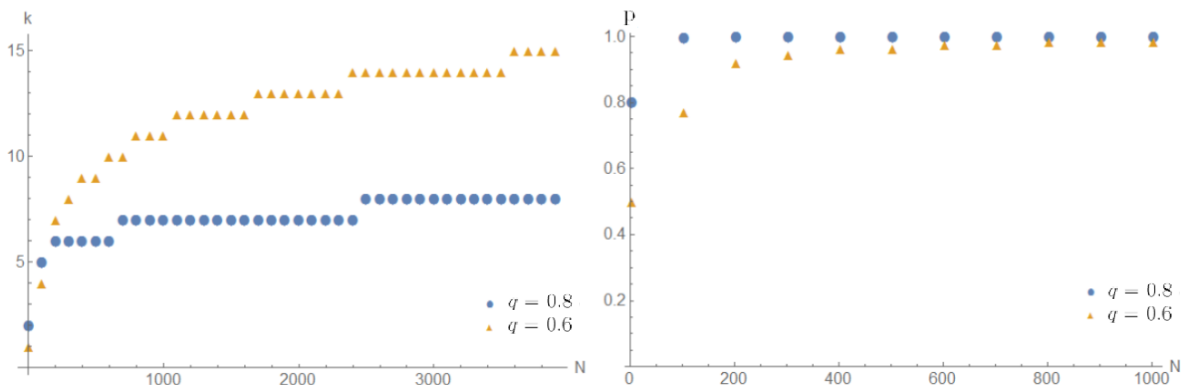


Figure 3: Cascades and optimal prices as N increases

The following corollary tells us that a large size of agent base not only implies a higher price but also ensures a certain probability of implementing good projects.

Corollary 4. *Let $N^{\min}(y, \nu)$ be the lowest number for which there exists price $p \geq \nu$ and AoN target T such that conditional on $V = 1$, the project is implemented with a probability*

¹²In the standard cascades models, a DOWN cascade hurts the proposer significantly because subsequent agents all reject. The concern for DOWN cascades pushes down the optimal price, and can cause immediate start of an UP cascade, independent of the number of agents because the decisions of later agents have no impact on the first agent’s payoffs (Welch, 1992).

higher than $y \in (0, 1)$, i.e.

$$\Pr(A_N \geq T_N | p, T, V = 1) \geq y.$$

Then, $N^{\min}(y, \nu)$ has the following upper bound:

$$N^{\min}(y, \nu) \leq (2\beta - 1)\bar{k}(\nu) + \frac{\ln(1 - y) - \ln M}{\ln x} \quad (15)$$

To ensure a good project is implemented with a probability higher than y , it is sufficient to have an agent base of $(2\beta - 1)\bar{k}(\nu) + \frac{\ln(1 - y) - \ln M}{\ln x}$.

5.2 Efficient Project Implementation

A larger crowd implies a higher optimal price, which in turn delays the arrival of UP cascades and reduces the probability of type II error. In the limit, both type I and II errors vanish and the project implementation is fully efficient.

Proposition 7. $\lim_{N \rightarrow \infty} \mathcal{P}_N^I = 0$ and $\lim_{N \rightarrow \infty} \mathcal{P}_N^{II} = 0$.

Proposition 7 implies that when the proposer has access to a large crowd, endogenous threshold implementation achieves the social optimal. As for the allocation of surplus, the investors' share vanishes in the limit because the price approaches the true value of a good project, and the proposer eventually gets all the surplus from the project implementation.

5.3 Full Information Aggregation

Note that the equilibrium characterization provided in Propositions 1 and 2 imply that when (p_N^*, T_N^*) are endogenous, all private signals become public and efficiently aggregated before an UP cascade starts. Therefore, the number of aggregated signals depends on the distribution of times at which an UP cascade starts. Proposition 7 shows that the proposer optimally increases the price with N , which perpetually delays the arrival of UP cascades.

Proposition 8 (Full Information Aggregation). *As $N \rightarrow \infty$, there is full information aggregation about the project quality, that is, $\mathbb{E}[V | \mathcal{H}_N] \xrightarrow{a.s.} V$ under the endogenously chosen (p_N^*, T_N^*) .*

We thus obtain perfect information aggregation in large markets, which is unachievable in settings with information cascades (Ali and Kartik, 2012) unless the action set maps to each agent’s posterior belief one-to-one (Lee, 1993) — a condition we do not require.¹³

Financing projects and aggregating information are paramount functions of modern financial markets. Propositions 7 and 8 demonstrate that threshold implementations have profound implications on them and should be an important consideration in market design.

6 Discussion and Extensions

In this section, we characterize equilibrium outcomes when agents have options to wait, or when they do not necessarily play the informer equilibrium. Our goal is two-fold: (i) to demonstrate that our findings about the impact of AoN on project implementation and information aggregation are robust; (ii) to demonstrate how the AoN feature leads to strategic considerations and equilibrium multiplicity that are absent in most extant information cascade models, which are of game theoretical interests.

6.1 Option to Wait

One common critique on information cascade models concerns the exogenous ordering of agents’ decision-making. In some cases, agents may choose to wait to observe more information, which alters many classic results.¹⁴ One particular feature of threshold implementation is that the findings are robust to the option to wait.

To analyze such a situation, for agent i who first arrives in period i , we denote her action in each period $t \geq i$ by $a_i^t \in \{-1, 0, 1\}$, where 0 means that agent i delays her decision in period t to the next period, and is a feasible action only when $i = t$ or $a_i^{t-1} = 0$, i.e., she has not supported or rejected the project yet. In any period t , after agent t ’s decision, all agents already waiting from earlier periods make decisions one by one (ordered by their first arrival

¹³We follow Milgrom (1979) and Pesendorfer and Swinkels (1997) for the convergence concept regarding information aggregation.

¹⁴That said, potential contributors in crowdfunding often do not wait because of high attention cost. Moreover, in many cases the shares or products sold are often in limited supply, and waiting may cause an agent to miss out the opportunity.

time). For ease of exposition, if agent i chooses not to wait at time t , $a_i^t \neq 0$, then we write $a_i^l = a_i^t, \forall l > t$.

With the option to wait, for agent i at period $t \geq i$, the history can be summarized as

$$\mathcal{H}_i^t = \begin{cases} (a_1^1, a_2^2, a_1^2, a_3^3, a_1^3, a_2^3, \dots, a_{t-2}^{t-1}, a_t^t) & \text{if } i = t \\ (a_1^1, a_2^2, a_1^2, a_3^3, a_1^3, a_2^3, \dots, a_{t-2}^{t-1}, a_t^t, a_1^t, a_2^t, \dots, a_i^t) & \text{if } i < t \end{cases} \quad (16)$$

and A_i^t can be defined as

$$A_i^t = \begin{cases} \sum_{1 \leq j \leq t-1} a_j^{t-1} \mathbb{1}_{a_j^{t-1}=1} + a_t^t \mathbb{1}_{a_t^t=1} & \text{if } i = t \\ \sum_{i+1 \leq j \leq t-1} a_j^{t-1} \mathbb{1}_{a_j^{t-1}=1} + \sum_{1 \leq j \leq i} a_j^t \mathbb{1}_{a_j^t=1} + a_i^t \mathbb{1}_{a_i^t=1} & \text{if } i < t \end{cases} \quad (17)$$

The option to wait results in multiple equilibria due to the coordination problem on waiting decisions and off-equilibrium beliefs. Nevertheless, in terms of project implementation and information aggregation, there exists an equilibrium that is essentially the same as the one characterized in Proposition 1.

Proposition 9. *For any given pair of (p, T) , there exists an equilibrium with strategy profile a_i^{j*} and posteriors $P(V = 1 | \mathcal{H}_{i-1}^i) = V_{k^*(\mathcal{H}_{i-1}^i)}$, where:*

$$a_i^{j*} = \begin{cases} \mathbb{1}_{\{x_i=1\}} & \text{if } A_{i-1}^{i-1} < T - 1, k_{i-1} \leq \bar{k}(p), i < N \\ 1 & \text{if } k_{i-1} > \bar{k}(p) \\ 0 & \text{if } A_{i-1}^{i-1} \geq T - 1, k_{i-1} < \bar{k}(p) - 1 \\ \mathbb{1}_{\{x_t=1\}} & \text{if } A_{i-1}^{i-1} \geq T - 1, k_{i-1} \in \{\bar{k}(p), \bar{k}(p) - 1\} \end{cases} \quad (18)$$

and for $j > i$,

$$a_i^{j*} = \begin{cases} 2\mathbb{1}_{\{k_N \geq \bar{k}(p)\}} - 1 & \text{if } j = N \ \& \ a_i^{N-1} = 0 \\ a_i^{i*} & \text{otherwise} \end{cases} \quad (19)$$

$$k_i^*(\mathcal{H}_{i-1}^i) = \begin{cases} k_{i-1} + (2a_i^i - 1) & \text{if } A_{i-1}^{i-1} < T - 1, \quad k_{i-1} \leq \bar{k}(p), \quad i < N \\ k_{i-1} & \text{if } k_{i-1} > \bar{k}(p) \\ k_{i-1} & \text{if } A_{i-1}^{i-1} \geq T - 1, \quad k_{i-1} < \bar{k}(p) - 1 \\ k_{i-1} + (2a_i^i - 1) & \text{if } A_{i-1}^{i-1} \geq T - 1, \quad k_{i-1} \in \{\bar{k}(p), \bar{k}(p) - 1\} \end{cases} \quad (20)$$

The strategies differ from Proposition 1 only in that those agents who reject without the option to wait now wait upon their first decision-making and continue doing so till their last chance of taking an action (the case of $j = N$). Those who support upon their first decision-making are exactly those supporting agents in Proposition 1.

To see this, if there is already an UP cascade, then no one wants to deviate to wait. Now suppose there is no cascade yet, then for agents with positive signals, supporting always dominates rejection and thus there is no need to wait. For agents with negative signals, waiting till the end weakly dominates rejection and they wait.

Observational learning remains equivalent as before because agents with different signals choose different actions. In equilibrium, before the arrival of an UP cascade, all agents infer supporting actions as good news and waiting as bad news, yielding exactly the same information aggregation as in the baseline model.¹⁵

In terms of proposal feasibility, this equilibrium is qualitatively the same as the one in Proposition 2. If $\nu > q$, a proposal with $p > \nu$ would have a strictly positive success probability when the proposer commits to an AoN threshold. Our finding on project implementation with large crowds is also robust to option to wait, as the next proposition summarizes.

Proposition 10. *When $N \rightarrow \infty$, the optimal price goes to 1 even when agents have the option to wait. Good projects and only good projects are implemented.*

The intuition is similar to that for Propositions 7 and 8. The absence of DOWN cascades helps us avoid missing good projects and the high price screens out bad projects whose valuation cannot be sufficiently high as information gradually gets aggregated.

¹⁵The option to wait may affect the optimal price p^* because agents with negative signal can still contribute if the posterior valuation after the information aggregation is good.

6.2 Free-Rider Equilibria

Next we examine all PBNE, which are not necessarily informer equilibria. We first show that all possible equilibria involve a group of “informers” and a group of “free-riders” whose actions before a cascade are ignored in equilibrium. We call the type of equilibrium with free-riders “free-rider equilibrium.”

Definition 4. *For an equilibrium strategy profile $\mathcal{A}(\cdot; \mathcal{H}_{i-1}) : \{-1, 1\} \rightarrow \{-1, 1\}$, we call agent i a “free-rider,” if $\mathbb{E}[V|\mathcal{H}_{i-1}] = \mathbb{E}[V|\mathcal{H}_i] < \bar{k}(p) + 1$. In other words, agent i becomes free-rider following sub-history \mathcal{H}_{i-1} if everyone knows that subsequent agents would not update their beliefs based on agent i ’s action, even though an UP cascade has not started yet.*

Although in both cascades and the case of free-riders an agent’s action is uninformative, agents still take informative actions after the free-rider’s move, and information aggregation continues until a cascade starts or the game ends. This is not the case once information cascades start.

Lemma 6. *A PBNE is either an informer equilibrium or a free-rider equilibrium.*

Free-rider equilibria can be viewed as derivatives of the equilibrium characterized in Proposition 2 in the sense that on each equilibrium path, if one excludes all free-riders, then sub-game dynamics are exactly the same as the one described in Proposition 1.

In a free-rider equilibrium, who become free-riders is generally path-dependent. Those agents essentially delegate their investment decision to the gate-keeper and this is common knowledge. In other words, they free-ride on information aggregation from subsequent investors. Similar to the informer equilibrium, a free-rider equilibrium differs from the equilibria in most information cascade models because coordination issues manifest themselves. Whether an agent becomes a free-rider depends on subsequent agents’ beliefs and his beliefs on their beliefs, etc. Such phenomenon is absent in conventional models because the agent’s expected payoff at the time of decision-making is independent of subsequent agents’ actions.

To give an example, suppose $\nu < \frac{1}{2}$, $p = \frac{1}{2} = \frac{q^0}{q^0 + (1-q)^0}$ and the target is $T = N$. Then there is a sub-game free-rider equilibrium in which all agents but the N th ones support regardless of their private signal, and the N th agent supports if and only if $X_N = 1$. The next lemma

Lemma 7. *If $p \in \{V_k, K = -1, 0, \dots, N\}$, then all free-rider sub-game equilibria are weakly Pareto-dominated by the informer sub-game equilibrium described in Proposition 1. Moreover, all free-rider sub-game equilibria involving at least two free-riders are strictly Pareto dominated by the informer sub-game equilibrium.*

Recall free-rider equilibria can be path-dependent, and are specific to realizations of the sequence of signals. The lemma states that given any history of actions, if agents decide to play a free-rider sub-game equilibrium that results in at least two free-riders, then agents are better off playing an informer sub-game equilibrium.

In the proof we argue that a free-rider would not reject in any equilibrium because she would not do so with a positive signal and the definition of free-riding implies that she also supports with a negative signal. The intuition behind Lemma 7 then is that every free-rider sub-game equilibrium resembles a subset of possible realization paths of an informer equilibrium with UP cascade early on, but “shifting” the agents after the UP cascade to the front instead. Investors prefer informer sub-game equilibrium because it induces more information aggregation and thus a higher chance to finance a good project.

The remainder of the discussion focuses on *Pareto-undominated* sub-game equilibria. Lemma 7 then implies that whenever $p \in \{V_k, K = -1, 0, \dots, N\}$, we only need to consider informer equilibria and free-rider equilibria with only one free-rider. This is equivalent to a standard equilibrium selection based on payoff dominance (Harsanyi, Selten, et al., 1988), and can be easily motivated by some communication among agents before they draw the signals. This weak refinement merely rules out nuisance equilibria such as the one given before the lemma where investors coordinate on Pareto inferior outcomes, but still allows the large class of free-rider equilibria for general $p \notin \{V_k, K = -1, 0, \dots, N\}$.

Note that nuisance free-rider equilibria can also be ruled out by the option to wait. This is straightforward because for every agent observing signal $x_i = -1$, she can be better off waiting. So no matter what, free-rider equilibria cannot emerge if the proposer’s payoff is dominated by that in the informer equilibrium when he sets $p \in \{V_k, K = -1, 0, \dots, N\}$. This proves to be useful when analyzing the limiting behavior because the proposer can always resort to $p \in \{V_k, K = -1, 0, \dots, N\}$ to bound his payoff in the large crowd limit.

The next proposition and corollaries show that in the limit, Pareto-undominated free-

rider equilibria deliver qualitatively the same results as informer equilibria do.

Proposition 11. *In any sequence of endogenous proposal designs $\{p_N, T_N\}_{N=1}^\infty$ and Pareto-undominated sub-game equilibria, $p_N \rightarrow 1$ as N goes to infinity, and good projects and only good projects are implemented.*

The proposition implies that no matter which equilibrium we select, in the limit the proposer charges a high enough price to avoid DOWN cascade and ensure that a good project is always financed and all bad projects are denied.

Corollary 5. *Let the number of informers in a sub-game equilibrium E be $Z_N^E(p_N, T_N)$ when the proposer's endogenous design is (p_N, T_N) , then for any positive integer l , as N goes to infinity, $\Pr(Z_N^E(p_N, T_N) < l) \rightarrow 0$.*

Even in a free-rider equilibrium, the number of informers is unbounded as N goes up. This means for large N , the information aggregation improves relative to that in standard information cascade settings. Public information becomes arbitrarily informative as N goes to infinity.

Corollary 6. *In any free-rider equilibrium with endogenous $\{p_N, T_N\}_{N=1}^\infty$ and a sub-game equilibrium that is Pareto-undominated, $\lim_{N \rightarrow \infty} \mathcal{P}_N^I = 0$, $\lim_{N \rightarrow \infty} \mathcal{P}_N^{II} = 0$, and $\mathbb{E}[V | \mathcal{H}_N] \xrightarrow{a.s.} V$ as $N \rightarrow \infty$.*

Finally, Corollary 6 extends Propositions 7 and 8, and reveals that our earlier findings are robust to considering these free-rider equilibria: PBNEs feature efficient project implementation and full information aggregation in the limit of large crowds. Given that financing projects and aggregating information are arguably the most important functions of financial markets, the impact of threshold implementation cannot be overstated.

7 Conclusion

We incorporate AoN thresholds into a classic model of information cascade, and find that agents' payoff interdependence results in uni-directional cascades in which agents rationally ignore private signals and imitate preceding agents only if the preceding agents decide to

support. Information production, proposal feasibility, and project selection all improve. In particular, when the number of agents grows large, equilibrium project implementation and information aggregation achieve the socially efficient levels, even under information cascades.

An important application of our model is that financial technologies such as Internet-based funding platforms can help entrepreneurs reach out to a larger agent base to better harness the wisdom of the crowd, as envisioned by the regulatory authorities. We highlight that specific features and designs such as endogenous AoN thresholds are crucial in capitalizing on these potential benefits, especially for sequential sales in the presence of informational frictions. For parsimony and generality, we have left out some details specific to individual applications. For example, third-party certification has significant impacts in equity crowdfunding (Knyazeva and Ivanov, 2017), and private values are equally important as product quality in reward-based crowdfunding. A project proposer may also price discriminate or control the information flow to potential investors. Specific applications taking into consideration these features as well as the joint information and mechanism design (using strategies beyond threshold implementation) definitely constitute useful future studies.

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A Online Appendix: Derivations and Proofs

A.1 Proof of Lemma 1

Proof. We prove the lemma by induction on the length of the history $l \in \{0, 1, \dots, N\}$, where a zero-length corresponds to the null history. For $i = 0$, clearly $\mathbb{E}[V|\mathcal{H}_0] = \frac{1}{2} = V_0$. Now suppose the statement is true for all histories with length weakly smaller than $l < N$, i.e. for every \mathcal{H}_i , $i \leq l$, we have $\mathbb{E}[V|\mathcal{H}_i] = V_{k(\mathcal{H}_i)} = \frac{q^{k(\mathcal{H}_i)}}{q^{k(\mathcal{H}_i)} + (1-q)^{k(\mathcal{H}_i)}}$, where $k(\mathcal{H}_i) \in \mathbb{Z}$.

Let us consider a history \mathcal{H}_{l+1} . Bayes' rule implies:

$$\mathbb{E}[V|\mathcal{H}_{l+1}] = \mathbb{E}[V|a_{l+1}, \mathcal{H}_l] = \frac{Pr(a_{l+1}|V=1)V_{k(\mathcal{H}_l)}}{Pr(a_{l+1}|V=1)V_{k(\mathcal{H}_l)} + Pr(a_{l+1}|V=0)(1 - V_{k(\mathcal{H}_l)})} \quad (21)$$

If agent $l+1$ has a “pooling” strategy after history \mathcal{H}_l , then $P(a_{l+1}|V=1) = P(a_{l+1}|V=0) = 1$, and consequently, $\mathbb{E}[V|\mathcal{H}_{l+1}] = \mathbb{E}[V|\mathcal{H}_l] = V_{k(\mathcal{H}_l)}$. If her strategy is “separating,” then $P(a_{l+1}|V=1) \in \{q, 1-q\}$ and we know $P(a_{l+1}|V=1) + P(a_{l+1}|V=0) = 1$. It is easy to verify that if $P(a_{l+1}|V=1) = q$, then $\mathbb{E}[V|\mathcal{H}_{l+1}] = V_{k(\mathcal{H}_l)+1}$ and if $P(a_{l+1}|V=1) = 1-q$, then $\mathbb{E}[V|\mathcal{H}_{l+1}] = V_{k(\mathcal{H}_l)-1}$. Both parts of the lemma follow. \square

A.2 Proof of Lemma 2

Proof. For Agent 1, her posterior belief after observing $x_1 = 1$ is $\mathbb{E}[V|x_1 = 1] = q$. If $p > q$, then agent 1 chooses rejection regardless of her private signal and a DOWN cascade starts from the beginning for sure. We thus have the first part of the Lemma.

Similarly, $p = 1 - q = \mathbb{E}[V|x_1 = -1]$ induces an UP cascade starting from the beginning for sure, the entrepreneur or proposer has no incentive to charge $p < 1 - q$. Therefore, $p \in [1 - q, q]$. For each $p \in (V_{k-1}, V_k]$, $p = V_k$ induces exactly the same number of supporting agents, so in the equilibrium proposer always charges $p^* = V_k$ for some $k \in \{-1, 0, 1, \dots, N\}$. Consequently, only three prices are possible: $p_{-1} = 1 - q$, $p_0 = \frac{1}{2}$ and $p_1 = q$. Let $\Pi(p, N)$ be the expected profit when the price is p and there are $N \geq 2$ potential agents. Without AoN thresholds, $\Pi(p, N)$ is obviously increasing in N . Next, we examine the three possible optimal prices and show $p = 1 - q$ dominates.

- $p = 1 - q$: The total profit is $(1 - q)N$ when $\nu = 0$;
- $p = \frac{1}{2}$: After the first two observations, $(x_1, x_2) = (-1, -1)$ induces a DOWN cascade, $(1, -1)$ and $(1, 1)$ both induce an UP cascade at agent 1 due to the tie-breaking assumption, and $(-1, 1)$ does not change the belief. $\Pi(p, N) = \frac{q+(1-q)}{2} \frac{1}{2} N + \frac{q(1-q)+(1-q)q}{2} (\frac{1}{2} + \Pi(p, N - 2)) \leq$

$\frac{1}{4}N + (1 - q)q (\frac{1}{2} + \Pi(p, N))$. Thus $p = \frac{1}{2}$ is dominated by $p = 1 - q$ if:

$$\Pi(p, N) \leq \frac{\frac{N}{4} + \frac{(1-q)q}{2}}{1 - (1-q)q} \leq (1 - q)N \text{ for } N = 2, 3, \dots, \quad (22)$$

which holds for $q \in \left(\frac{1}{2}, \frac{3}{4} + \frac{1}{4}(3^{\frac{1}{3}} - 3^{\frac{2}{3}})\right]$;

- $p = q$: After the first two observations, $(1, 1)$ induces an UP cascade, $(-1, -1)$ and $(-1, 1)$ both induce a DOWN cascade after agent 1, and $(1, -1)$ does not change the belief. The expected profit is $\Pi(p, N) = \frac{(1-q)^2 + q^2}{2}qN + \frac{q(1-q) + (1-q)q}{2}(q + \Pi(p, N - 2)) \leq \frac{(1-q)^2 + q^2}{2}qN + (1 - q)q(q + \Pi(p, N))$. Thus $p = q$ is dominated by $p = 1 - q$ if:

$$\Pi(p, N) \leq \frac{\frac{(1-q)^2 + q^2}{2}qN + q^2(1 - q)}{1 - (1 - q)q} \leq (1 - q)N \text{ for } N = 2, 3, \dots \quad (23)$$

One can verify that the inequality holds for $q \in \left(\frac{1}{2}, \frac{3}{4} + \frac{1}{4}(3^{\frac{1}{3}} - 3^{\frac{2}{3}})\right]$.

□

A.3 Proof of Proposition 1

Proof. To prove that the characterization in (9) and (10) constitute a PBNE, we first state and prove Lemma A.1 — the expected value of the adoption is bounded above by $V_{\bar{k}(p)+1}$. In fact, an UP cascades starts once a strong expectation is formed and it blocks further learning by subsequent agents. Anticipating this behavior from subsequent agents, early agents with negative signal do not support the proposal if an UP cascade has not started yet. It makes the support of the agents with positive signal informative for the subsequent agents. For the histories that either the AoN target or a cascade has been reached, the proof is trivial.

Lemma A.1. *Suppose $\bar{k}(p) \geq -1$. Then, in every equilibrium, when it is still possible to reach the AoN target T , the following relation holds for every $2 \leq i \leq N$:*

$$\mathbb{E}[V|\mathcal{H}_{i-1}] \leq V_{\bar{k}(p)+1} \quad (24)$$

In other words, there is an upper-bound on the expected value of the project as a function of p .

Proof. Suppose the contrary. Lemma 1(b) then implies that there exists an agent u with $\mathbb{E}[V|\mathcal{H}_u] = V_{\bar{k}(p)+2}$ and $\mathbb{E}[V|\mathcal{H}_{u-1}] = V_{\bar{k}(p)+1}$. Now given $\mathbb{E}[V|\mathcal{H}_{u-1}] = V_{\bar{k}(p)+1}$, however, u would accept the

proposal regardless of her private signal because

$$\mathbb{E}[V|x_u, \mathcal{H}_{u-1}, A_N \geq T] \geq \mathbb{E}[V|x_u, \mathcal{H}_{u-1}] \geq \mathbb{E}[V|x_u = -1, \mathcal{H}_{u-1}] = V_{\bar{k}(p)} \geq p. \quad (25)$$

We remind the readers that all expectations are conditional on equilibrium strategies of other agents, which is not explicitly written for notational simplicity. The first inequality (from left) in (25) follows from the fact that the gate-keeper makes her decision based on an information set that fully nests \mathcal{H}_{u-1} . Therefore, her support positive updates agent u 's belief.

Equation (25) shows that agent u 's action is not informative for subsequent agents. Thus $\mathbb{E}[V|\mathcal{H}_u] = \mathbb{E}[V|\mathcal{H}_{u-1}] = V_{\bar{k}(p)+1}$, a contradiction. \square

Now, we are ready to prove the equilibrium characterization. For notational ease, we replace $\bar{k}(p)$ by \bar{k} . We proceed by examining the optimal strategy for different histories:

$A_{i-1} < T - 1$ and $k_{i-1} \leq \bar{k}$

According to (3), the agent chooses to support if and only if $\mathbb{E}[V|x_i, \mathcal{H}_{i-1}, A_N \geq T] \geq p$. We examine two cases $x_i = 1$ and $x_i = -1$ separately:

- $x_i = 1$: Suppose agent i supports and consider a history $\mathcal{H}_N \succ \mathcal{H}_i = (\mathcal{H}_{i-1}, 1)$ in which the proposal is accepted. Denote the gate-keeper by g , namely g is the smallest integer that $A_g = T$. Since the subsequent agents perfectly infer $x_i = 1$ from the support of agent i , then agent g 's information set fully nests that of agent i . We now have:

$$\mathbb{E}[V|x_i, \mathcal{H}_{i-1}, A_N \geq T] = \mathbb{E}[V|x_i, \mathcal{H}_{i-1}, \mathbb{E}[V|x_g, \mathcal{H}_{g-1}] \geq p] = \mathbb{E}[V|\mathbb{E}[V|x_g, \mathcal{H}_{g-1}] \geq p] \geq p$$

In other words, it is optimal for agent i to support.

- $x_i = -1$: In this case, in the equilibrium, agent i 's support would be misinterpreted by follow agents as a positive signal. In other words, $k(\mathcal{H}_i) = k(\mathcal{H}_{i-1}) + 1$, while the correct posterior should be $k(\mathcal{H}_i) = k(\mathcal{H}_{i-1}) - 1 = k(\mathcal{H}_{i-1}) + 1 - 2$. Moreover, if the proposal is accepted, Lemma A.1 implies $\mathbb{E}^g[V|x_g, \mathcal{H}_{g-1}] \in \{V_{\bar{k}}, V_{\bar{k}+1}\}$. Therefore, agent i , knowing that her signal is incorrectly inferred by her action, assigns an expected value bounded above by $V_{\bar{k}-1} < p$ conditional on her signal and project implementation. She thus optimally chooses $a_i = -1$, lest she gets negative expected payoff.

Therefore, when $k_{i-1} \leq \bar{k}$ and $A_{i-1} < T - 1$, agent i follows her private signal and the subsequent agents update their beliefs accordingly, as specified in (9) and (10).

$A_{i-1} \geq T - 1$ and $k_{i-1} \leq \bar{k}$

In this case, agent i supports iff $\mathbb{E}[V|x_i, \mathcal{H}_{i-1}] = V_{k_{i-1}+x_i} \geq p$. The only case that the strategy is separating is when $k_{i-1} \in \{\bar{k}, \bar{k}-1\}$. Given this observation, it is easy to check that the strategies specified in (9) are optimal in this case.

Finally, we show that the equilibrium strategy profile is essentially unique among the set of all informer equilibria. Note that an informer equilibrium is an equilibrium that all strategies are separating before reaching a cascade and the AoN threshold is possible to reach. Therefore, we only need to show that a cascade cannot occur when $A_{i-1} < T - 1$ and $k_{i-1} \leq \bar{k}$ (and the AoN threshold is possible to reach).¹⁶

Suppose the contrary that such a cascade exists in an equilibrium. Then, the agents should choose the same action after such a history. First, it is not possible that all agents support if this leads to project implementation, because the gate-keeper would certainly reject if she has a negative signal, and it contradicts the assumption that the gate-keeper is part of the cascade. Second, it is not possible that all reject because this would lead to the eventual rejection of the proposal. In this case, all agents are indifferent between supporting and not supporting, and their rejections violate Assumption 1. The contradiction implies that the equilibrium is essentially unique.

$$\mathbf{k}_{i-1} > \bar{\mathbf{k}}$$

Clearly $\mathbb{E}[V|x_i, \mathcal{H}_{i-1}] \geq V_{\bar{k}} \geq p$. In other words, she would lose at least $V_{\bar{k}} - p$ in expectation if the AoN target is reached. Therefore, it is not profitable to reject regardless of agent i 's type, which proves the optimality in this case.

Corollary 1 also follows directly. □

A.4 Proof of Proposition 2

Proof. We prove the proposition through multiple steps and lemmas. In Step 1, we show a profit maximizer proposer optimally chooses a price in $\{V_{-1}, V_0, \dots, V_N\}$. Therefore, the optimal pair (p^*, T^*) belongs to set $\{(p, T) | p \in \{V_{-1}, \dots, V_N\}, T \in \{1, \dots, N\}\}$ with finite number of elements, which ensures the existence of a solution to the proposer's optimization problem. Then, we show $T^* = \lfloor \frac{N+k^*}{2} \rfloor$, by showing T^* is a dominant choice compared to all $T > T^*$ and $T < T^*$, respectively in Steps 2 and 3, when $k^* > 0$. For $k^* = 0$, we prove the optimality of T^* for $q(1-q) > \frac{1}{6}$ and $q(1-q) \leq \frac{1}{6}$, respectively in Steps 4 and 5. In step 6, We finish the proof by showing T^* is optimal when $k^* = -1$.

Step 1: Existence of the Solution

¹⁶When $A_{i-1} \geq T - 1$, the subsequent actions are irrelevant for agent i 's payoff and her action only depends on the price and the expected value given x_i and \mathcal{H}_{i-1} . Therefore, the optimal strategy is the same across all equilibria.

First, we show $p^* \in \{V_{-1}, V_0, \dots, V_N\}$. Note that all $p < V_{-1}$ are suboptimal since an UP cascade starts from the very first agent and all agents would support if $p \leq V_{-1}$. Moreover, clearly the posterior of the agents never exceeds V_N , therefore all $p > V_N$ are suboptimal, as well. Thus we focus on the case $p \in [V_{-1}, V_N]$.

For any $p \in (V_{k-1}, V_k]$, $k \in \{0, 1, \dots, N\}$, in the sub-game the agents follow the same equilibrium strategy profile specified in (9), as it only depends on $\bar{k}(p)$ and T . It implies any choice of $p \in (V_{k-1}, V_k)$ induce the same $\bar{k}(p)$ and is dominated by $p = V_k$, for $k \in \{0, 1 \dots N\}$. Consequently, $p^* \in \{V_{-1}, V_0, \dots, V_N\}$. Notice that the set of T^* is also finite as $T \in \{1, 2, \dots, N\}$. Then the proposer only needs to choose from a finite set of pairs (p, T) , which guarantees the existence of solution.

Some Helpful Definitions and Results

For the rest of the proof, suppose the optimal pair is (p^*, T^*) , where $p^* = V_{k^*}$. Moreover, we say a sequence of signals $x \in X$ is “ T -supported” for some $1 \leq T \leq N$, if the proposal is accepted for pair (V_{k^*}, T) .

The following two lemmas are useful for our analysis.

Lemma A.2. *Suppose sequence $x \in X$ is $(T-1)$ -supported and not T -supported, for some $T \leq T^*$. Then, there are at least $T-1$ positive signals in x . Furthermore, if h_{T-1} is the agent that has the $(T-1)$ th positive signal in the queue, then $h_{T-1} \leq N-2$ and $(x_{h_{T-1}}, x_{h_{T-1}+1}, x_{h_{T-1}+2}) = (1, -1, -1)$.*

Proof. First, we show that if a sequence $x' \in X$ induces an UP cascade, then more than T^* agents support the proposal. To see this, suppose an UP cascade starts after agent $r < N$, which implies $\sum_{i=1}^r x_i = k^* + 1$. Therefore, since all the subsequent agents support the proposal, the total number of supporters is

$$\frac{r + k^* + 1}{2} + N - r > \frac{N + k^* + 1}{2} > T^*$$

Given this result, since x is not T -supported, the AoN target $T-1$ cannot be reached by an UP cascade, and the existence of at least $T-1$ positive signals is necessary.

To see $h_{T-1} \leq N-2$, note that when the AoN target is $T-1$, the support of h_{T-1} requires that the following condition holds:

$$T-1 - (h_{T-1} - (T-1)) \geq k^* \Rightarrow h_{T-1} \leq 2(T^* - 1) - k^* \leq N-2$$

The only claim is left to show is that $(x_{h_{T-1}}, x_{h_{T-1}+1}, x_{h_{T-1}+2}) = (1, -1, -1)$. It is resulted from the assumption that x is not T -supported. Therefore, it implies that both $\sum_{i=1}^{h_{T-1}+1} x_i$ and

$\sum_{i=1}^{h_{T-1}+2} x_i$ should be strictly less than k^* . The result is straightforward from the observation that $\sum_{i=1}^{h_{T-1}} x_i = k^*$. \square

Lemma A.3. *Suppose sequence $x \in X$ is $(T-1)$ -supported, for some $T \leq T^*$. Then, there exists an injective function $x'(x)$ that maps each sequence x to a distinct sequence $x'(x)$ such that $x'(x)$ is T -supported. The number of supporting agents in $x'(x)$ for T is weakly higher than the one in x for $T-1$.*

Proof. The proof of Lemma A.2 shows that if a sequence is $(T-1)$ -supported and induces an UP cascade, then it is also T -supported and also induces an UP cascade. Consider $X'(x) = x$, the number of supporting agents for both cases then are the same. If a sequence x is both $(T-1)$ -supported and T -supported and there is no UP cascade, then at least one of agents $h_{T-1} + 1$ and $h_{T-1} + 2$ observes a positive signal and chooses to support. Consider again $x'(x) = x$, then after the first $h_{T-1} + 2$ agents, there are the same number of supports ($\geq T$) and the posterior is the same, so for any such sequence x (and $x'(x) = x$) the number of supporting agents for both cases are the same. Suppose \tilde{X}_{T-1} is the set of all sequence of signals that they are $(T-1)$ -supported, but not T -supported. Similarly, suppose \tilde{X}_T is the set of sequence of signals that are T -supported, but not $(T-1)$ -supported. For any sequence $x = (\underbrace{\dots}_{h_{T-1}-1}, 1, -1, -1, \dots) \in \tilde{X}_{T-1}$, there exists a corresponding sequence $x' = (\underbrace{\dots}_{h_{T-1}-1}, -1, 1, 1, \dots) \in \tilde{X}_T$, in which only the three middle signals are reversed. There are exactly $T-1$ supporters in x (since a DOWN cascade starts at agent $h_{T-1} + 2$), while there are at least T supporters in $x'(x)$. By construction, each x has a distinct image $x'(x)$, so $x'(x)$ is an injective function. \square

Over the next couple of steps, we show next the optimal AoN Threshold is $T^* = \lfloor \frac{N+k^*}{2} \rfloor$ if $k^* > 0$. In the final step, we prove the optimality of T^* , even for $k^* \in \{-1, 0\}$.

Step 2: The Proof of $\pi(p^*, T) < \pi(p^*, T^*)$, for $T > T^*$ and $k^* > 0$

We simply show that if a sequence of signals is T -supported for some $T > T^*$, then it is T^* -supported, as well. Finally, we show that there exists at least one sequence that is T^* -supported but not T -supported for any $T > T^*$.

To prove the first claim, suppose $x \in X$ is a T -supported sequence for some $T > T^*$. Denote s_j the agent that makes the j 'th support. There are two possibilities. If s_{T^*} is part of an UP cascade, namely she supports regardless of her private signals, then the formation of the UP cascade is not affected by reducing the AoN target to T^* . Therefore, x is a T^* -supported sequence, as well.

If s_{T^*} is not part of an UP cascade, then all the first T^* supporters have a positive private signal. If the AoN target is reduced to T^* , it does not affect the decision of all agents $i \leq s_{T^*-1}$.

Agent s_{T^*} , as the gate-keeper, supports only if the total number of positive signals exceed the negative ones by at least k^* , i.e. $\sum_{i=1}^{s_{T^*}} x_i \geq k^*$. It is the case, because:

$$\sum_{i=1}^{s_{T^*}} x_i = 2T^* - s_{T^*} \geq 2T^* - N + (T - T^*) \geq 2T^* - N + 1 \geq k^*$$

The first inequality comes from the fact that $s_{T^*} + (T - T^*) \leq N$. Therefore, x is T^* -supported too. Moreover, it is easy to check that the following sequence is T^* -supported and not T -supported for any $T > T^*$: If $N + k^*$ is even, consider a sequence that its first $\frac{N-k^*}{2}$ elements are -1 and the last $\frac{N+k^*}{2}$ elements are 1 . If $N + k^*$ is odd, then consider a sequence that its first $\frac{N-k^*-1}{2}$ elements and the last element are -1 and the remaining elements are 1 .

Step 3: The Proof of $\pi(p^*, T) < \pi(p^*, T^*)$, for $T < T^*$ and $k^* > 0$

As follows, we first show the probability of reaching the AoN target ($P(A_N \geq T|T)$) strictly increases with T for $k^* \leq T \leq T^*$. Then, we show the expected number of supporters conditional on reaching the AoN target ($\mathbb{E}[A_N|A_N \geq T, T]$) is also increasing with T . These two results are enough to conclude that the proposer's expected profit is increasing in T for $k^* \leq T \leq T^*$.

First, we show $P(A_N \geq T|T) \geq P(A_N \geq T - 1|T - 1)$, for $T \leq T^*$. We aim to show $Pr(\tilde{X}_T) \geq Pr(\tilde{X}_{T-1})$.

Following the proof of Lemma A.3, for every sequence of signals $x = (\underbrace{\dots}_{h_{T-1}-1}, 1, -1, -1, \dots) \in \tilde{X}_{T-1}$, there exists a corresponding sequence $x'(x) = (\underbrace{\dots}_{h_{T-1}-1}, -1, 1, 1, \dots) \in \tilde{X}_T$, in which only the three middle signals are reversed. $x'(x)$ is an injective function, therefore,

$$\begin{aligned} Pr(\tilde{X}_T) &= \frac{1}{2} \left[Pr(\tilde{X}_T|V=1) + Pr(\tilde{X}_T|V=0) \right] \geq \frac{1}{2} \left[\frac{q}{1-q} Pr(\tilde{X}_{T-1}|V=1) + \frac{1-q}{q} Pr(\tilde{X}_{T-1}|V=0) \right] \\ &\Rightarrow Pr(\tilde{X}_T) - Pr(\tilde{X}_{T-1}) \geq \frac{2q-1}{2} \left[\frac{1}{1-q} Pr(\tilde{X}_{T-1}|V=1) - \frac{1}{q} Pr(\tilde{X}_{T-1}|V=0) \right] \end{aligned}$$

The first inequality comes from the fact that $x'(x)$ is an injection but not necessarily a bijection. Consequently, we only need to show $\frac{Pr(\tilde{X}_{T-1}|V=1)}{Pr(\tilde{X}_{T-1}|V=0)} \geq \frac{1-q}{q}$. To see this, note that

$$\mathbb{E} \left[V|x \in \tilde{X}_{T-1} \right] = \mathbb{E} \left[V \mid \sum_{i=1}^{h_{T-1}+2} x_i \right] = V_{k^*-2} \Rightarrow \frac{Pr(\tilde{X}_{T-1}|V=1)}{Pr(\tilde{X}_{T-1}|V=0)} = \frac{V_{k^*-2}}{1 - V_{k^*-2}} \geq \frac{V_{-1}}{1 - V_{-1}} = \frac{1-q}{q}$$

We thus have $Pr(\tilde{X}_T) \geq Pr(\tilde{X}_{T-1})$ for $T \leq T^*$ and $k^* > 0$.

We next discuss the number of supporters conditional on reaching the AoN target. Based on the proof of Lemma A.3, we know that for each $x \in X$ that is $(T - 1)$ -supported, there exists

a distinct sequence $x_T \in X$ that is T -supported and has at least the same number of supporting agents. Moreover, for $x \in \tilde{X}_{T-1}$, the corresponding x_T has strictly higher number of supporters.

As a result, the probability of reaching the AoN target is greater for T than $T - 1$, and the expected number of supporters conditional on reaching the target is strictly greater for T than $T - 1$. Therefore, $\pi(p^*, T) > \pi(p^*, T - 1)$ for $k^* \leq T \leq T^*$. By combining the results of Steps 2 and 3, we conclude $T^* = T^*$ if $k^* > 0$.

Step 4: Optimality of T^* when $k^* = 0$, $q(1 - q) > \frac{1}{6}$

For $k^* = 0$ ($p^* = \frac{1}{2}$), the proof of step 2 also applies here, so $\pi(\frac{1}{2}, T) < \pi(\frac{1}{2}, T^*)$, for $T > T^*$. Therefore, we only need to show $\pi(\frac{1}{2}, T) < \pi(\frac{1}{2}, T^*)$ for $T < T^*$. To be more specific, we will show that any strategy $(p = \frac{1}{2}, T - 1)$, $2 \leq T \leq T^*$, is dominated by $(\frac{1}{2}, T)$. Given $p = \frac{1}{2}$, for any sequence x that has agent $2T$ as part of an UP cascade, it is both $(T - 1)$ -supported and T -supported, and the number of supporters are the same for $T - 1$ and T . So the proposer is indifferent between $(\frac{1}{2}, T - 1)$ and $(\frac{1}{2}, T)$ when agent $2T$ is part of an UP cascade.

Define $\mathbb{Q}_m = \{x | \sum_{i=1}^j x_i \leq 0, \forall j \leq m, \sum_{i=1}^m x_i = 0\}$. For any $x \in \mathbb{Q}_m$, agent m is not part of an UP cascade. Then we can characterize \tilde{X}_{T-1} and \tilde{X}_T :

$$\tilde{X}_{T-1} = \{x | x \in \mathbb{Q}_{2T-2}, x_{2T-1} = x_{2T} = -1\}$$

$$\tilde{X}_T = \mathbb{Q}_{2T} / \{x | x \in \mathbb{Q}_{2T-2}, x_{2T-1} = -1, x_{2T} = 1\}$$

By lemma 3 and using the reflection principle, we find the probability of \mathbb{Q}_m :

$$Pr(\mathbb{Q}_m) = \frac{1}{2}Pr(\mathbb{Q}_m | V = 1) + \frac{1}{2}Pr(\mathbb{Q}_m | V = 0) = \frac{1}{2} \frac{Pr(\mathbb{U}_{m+1} | V = 1)}{q} + \frac{1}{2} \frac{Pr(\mathbb{U}_{m+1} | V = 0)}{1 - q}$$

Let $\pi(p, T, \mathbb{Z})$ be the expected revenue on event set \mathbb{Z} , given strategy (p, T) . Then:

$$\begin{aligned} \frac{\pi(\frac{1}{2}, T, \tilde{X}_T)}{\pi(\frac{1}{2}, T - 1, \tilde{X}_{T-1})} &\geq \frac{Pr(\tilde{X}_T)}{Pr(\tilde{X}_{T-1})} \frac{T}{T - 1} = \frac{Pr(\mathbb{Q}_{2T}) - Pr(\mathbb{Q}_{2T-2})q(1 - q)}{Pr(\mathbb{Q}_{2T-2}) \frac{(1-q)^2 + q^2}{2}} \frac{T}{T - 1} \\ &= \frac{\frac{1}{2} \frac{Pr(\mathbb{U}_{2T+1} | V=1)}{q} + \frac{1}{2} \frac{Pr(\mathbb{U}_{2T+1} | V=0)}{1-q} - \frac{1}{2} \frac{Pr(\mathbb{U}_{2T-1} | V=1)q(1-q)}{q} - \frac{1}{2} \frac{Pr(\mathbb{U}_{2T-1} | V=0)q(1-q)}{1-q}}{\frac{1}{2} \frac{Pr(\mathbb{U}_{2T-1} | V=1)q^2}{q} + \frac{1}{2} \frac{Pr(\mathbb{U}_{2T-1} | V=0)(1-q)^2}{1-q}} \\ &= \frac{6T}{T + 1} \frac{q(1 - q)}{(1 - q)^2 + q^2} \geq \frac{4q(1 - q)}{1 - 2q(1 - q)} \end{aligned}$$

where, the first inequality comes from the fact that any $x \in \tilde{X}_{T-1}$ has exactly $T - 1$ supporting agents while any $x \in \tilde{X}_T$ has at least T supporters, and the last inequality applies the fact that $T \geq 2$. When $q(1 - q) > \frac{1}{6}$, $\pi(\frac{1}{2}, T, \tilde{X}_T) > \pi(\frac{1}{2}, T - 1, \tilde{X}_{T-1})$, thus $(\frac{1}{2}, T - 1)$ is dominated by

$(\frac{1}{2}, T)$.

Step 5: Optimality of T^* when $k^* = 0$, $q(1 - q) \leq \frac{1}{6}$

Similar to step 4, we only need to show for any $2 \leq T \leq T^*$, $(\frac{1}{2}, T - 1)$ is a dominated strategy. To achieve this, we decompose all possible implementation histories under strategy $(\frac{1}{2}, T - 1)$ into several sets, and shows that for each set, there is a corresponding distinct set of implementation histories under strategy (q, T) associated with more profit.

When $(1 - q)q \leq \frac{1}{6}$, we have $q \geq \frac{1}{2} + \frac{\sqrt{3}}{6} > \frac{3}{4}$. We now show that in this scenario the strategy $T - 1$ and $p^* = \frac{1}{2}$ is strictly dominated by alternative strategy $p^* = q$ (so $k^* = 1$) and AoN threshold T . For $p^* = \frac{1}{2}$ and AoN threshold $T - 1$, we have shown earlier in Lemma A.3 that the project would be implemented either when there is already an UP cascade by agent $2T - 2$ or when there is no UP cascade by agent $2T - 2$ and the $2T - 2$ th agent is the $T - 1$ th supporting agent. It suffices to show that in each scenario, the alternative strategy fares better for the proposer.

1. When there is already an UP cascade by agent $2T - 2$, let \mathbb{H}_i^U be the set of histories that result in an UP cascade starting after agent $i \leq 2T - 2$. Given any $\mathcal{H}_i \in \mathbb{H}_i^U$, $\mathbb{E}[V|k(\mathcal{H}_i)] = q$, and denote the number of supporters by $A_N(\mathbb{H}_i^U)$. If $x_{i+1} = 1$, then there would be an UP cascade starting after agent $i + 1$ for the strategy $(p^* = q, T)$, and the number of supporting agents is also $A_N(\mathbb{H}_i^U)$. Let $\pi(p, T|\mathbb{H}_i^U)$ be the expected payoffs for the proposer conditional on strategy (p, T) and event \mathbb{H}_i^U . $(p^* = q, T)$ dominates $(\frac{1}{2}, T - 1)$ conditional on \mathbb{H}_i^U because

$$\begin{aligned} \pi(q, T|\mathbb{H}_i^U) &\geq (q - \nu)A_N(\mathbb{H}_i^U)[Pr(V = 1|\mathbb{H}_i^U)q + Pr(V = 0|\mathbb{H}_i^U)(1 - q)] \\ &= A_N(\mathbb{H}_i^U)(q - \nu)[(1 - q)^2 + q^2] > A_N(\mathbb{H}_i^U) \times \left(\frac{3}{4} - \nu\right) \times (1 - 2(1 - q)q) \\ &\geq A_N(\mathbb{H}_i^U) \times \left(\frac{3}{4} - \nu\right) \times \frac{2}{3} \geq \left(\frac{1}{2} - \nu\right) A_N(\mathbb{H}_i^U) = \pi\left(\frac{1}{2}, T - 1|\mathbb{H}_i^U\right) \end{aligned}$$

2. When there is no UP cascade by agent $2T - 2$ but the $(2T - 2)$ th agent is the $(T - 1)$ th supporting agent, $x \in \mathbb{Q}_{2T-2}$. Consider the following two sets of histories for strategy (q, T) :

(a) $\mathbb{Q}_{2T-1}^A = \{x|x \in \mathbb{Q}_{2T-2}, x_{2T-1} = 1\}$:

Obviously the threshold T is met. Since given any $\mathcal{H}_{2T-2} \in \mathbb{Q}_{2T-2}$, there are equal number of positive and negative signals by agent $2T - 2$, we have

$$Pr(\mathbb{Q}_{2T-1}^A) = Pr(\mathbb{Q}_{2T-2})[Pr(V = 1|\mathbb{Q}_{2T-2})q + Pr(V = 0|\mathbb{Q}_{2T-2})(1 - q)] = \frac{1}{2}Pr(\mathbb{Q}_{2T-2})$$

We then discuss the expected number of supporting agents:

For event \mathbb{Q}_{2T-2} under strategy $(\frac{1}{2}, T-1)$, when $x_{2T-1} = 1$, the UP cascade starts after agent $2T-1$ and the number of supporting agents is $N-T+1$, the maximum conditional on \mathbb{Q}_{2T-2} . The associated conditional probability is $Pr(x_{2T-1} = 1|\mathbb{Q}_{2T-2}) = \frac{1}{2}$. For event \mathbb{Q}_{2T-1}^A under strategy (q, T) , if $x_{2T} = 1$, the UP cascade starts after agent $2T$ and the number of supporting agents is also $N-T+1$. The associated conditional probability is $Pr(x_{2T} = 1|\mathbb{Q}_{2T-1}^A) = q^2 + (1-q)^2 \geq \frac{2}{3} > \frac{1}{2}$.

For event \mathbb{Q}_{2T-2} under strategy $(\frac{1}{2}, T-1)$, $Pr(x_{2T-1} = -1|\mathbb{Q}_{2T-2}) = \frac{1}{2}$. On the other hand, for event \mathbb{Q}_{2T-1}^A , under strategy (q, T) , $Pr(x_{2T} = -1|\mathbb{Q}_{2T-1}^A) = 2q(1-q) < \frac{1}{2}$. For each possible subsequence $z = \{x_{2T}, x_{2T+1}, \dots, x_N\}$ of a sequence $x \in \mathbb{Q}_{2T-2}$, and $x_{2T-1} = -1$, let $A_N(\frac{1}{2}, T-1|\mathbb{Q}_{2T-2}, x_{2T-1} = -1, z)$ be the associated number of supporting agents under strategy $(\frac{1}{2}, T-1)$. The corresponding subsequence $z' = \{x_{2T}, x_{2T+1}, \dots, x_{N-1}\}$ satisfies $A_N(q, T|\mathbb{Q}_{2T-1}^A, x_{2T} = -1, z') > A_N(\frac{1}{2}, T-1|\mathbb{Q}_{2T-2}, x_{2T-1} = -1, z)$. This inequality holds because in each scenario, right before the corresponding subsequence $z(z')$ starts, the the posterior is $\bar{k}(p) - 1$, the project would be implemented for sure, and there are T supporters in \mathbb{Q}_{2T-1}^A before z' but only $T-1$ supporting decisions in \mathbb{Q}_{2T-2} before z . So $\mathbb{E}[A_N(q, T)|\mathbb{Q}_{2T-1}^A, x_{2T} = -1] \geq \mathbb{E}[A_N(\frac{1}{2}, T-1)|\mathbb{Q}_{2T-2}, x_{2T-1} = -1]$.

Then:

$$\begin{aligned}
\mathbb{E}[A_N(q, T)|\mathbb{Q}_{2T-1}^A] &= \sum_{-1,1} Pr(x_{2T} = i|\mathbb{Q}_{2T-1}^A) \mathbb{E}[A_N(q, T)|\mathbb{Q}_{2T-1}^A, x_{2T} = i] \\
&= (q^2 + (1-q)^2)(N-T-1) + 2q(1-q) \mathbb{E}[A_N(q, T)|\mathbb{Q}_{2T-1}^A, x_{2T} = -1] \\
&> \frac{1}{2}(N-T-1) + \frac{1}{2} [A_N(q, T)|\mathbb{Q}_{2T-1}^A, x_{2T} = -1] \\
&> \frac{1}{2}(N-T-1) + \frac{1}{2} [A_N(\frac{1}{2}, T-1)|\mathbb{Q}_{2T-2}, x_{2T-1} = -1] \\
&= \sum_{-1,1} Pr(x_{2T-1} = i|\mathbb{Q}_{2T-2}) \mathbb{E}\left[A_N(\frac{1}{2}, T-1)|\mathbb{Q}_{2T-2}, x_{2T-1} = i\right] \\
&= \mathbb{E}\left[A_N(\frac{1}{2}, T-1)|\mathbb{Q}_{2T-2}\right],
\end{aligned}$$

where the first inequality comes from the fact that $N-T-1$ is the maximum number of possible supporters conditional on \mathbb{Q}_{2T-2} .

- (b) $\mathbb{Q}_{2T-1}^B = \{x | \sum_{i=1}^j x_i \leq 1, \forall 1 \leq j \leq 2T-1, \sum_{i=1}^{2T-3} x_i = 1, x_{2T-2} = -1, x_{2T-1} = 1\}$:

\mathbb{Q}_{2T-1}^B is the event that there is no UP cascade (with respect to $p^* = q$) by agent $2T-2$, and $k(\mathcal{H}_{2T-3}) = 1$, $x_{2T-2} = -1$ and $x_{2T-1} = 1$. Obviously the threshold T is met. Notice that for strategy (q, T) , histories in this set are distinct from those we have

discussed (in case 1 we only cover UP cascades for strategy (q, T)). For any sequence $x \in \mathbb{Q}_{2T-1}^B$, there is a mapping $x^A(x) = \{x_{2T-2}, x_1, x_2, \dots, x_{2T-3}, x_{2T-1}, \dots\}$. $x^A(x)$ is a bijection that establishes a one-to-one mapping between finite sets \mathbb{Q}_{2T-1}^B and \mathbb{Q}_{2T-1}^A . Following the discussion in part (a), we have $Pr(\mathbb{Q}_{2T-1}^B) = Pr(\mathbb{Q}_{2T-1}^A) = \frac{1}{2}Pr(\mathbb{Q}_{2T-2})$, and the expected number of supporters conditional on event \mathbb{Q}_{2T-1}^B and strategy (q, T) is higher than that of event \mathbb{Q}_{2T-2} and strategy $(\frac{1}{2}, T-1)$.

Since $Pr(\mathbb{Q}_{2T-1}^B) + Pr(\mathbb{Q}_{2T-1}^A) = Pr(\mathbb{Q}_{2T-2})$, and in either case there are more supporting agents paying higher price $q > \frac{1}{2}$. So $(p^* = q, T)$ dominates $(\frac{1}{2}, T-1)$ when there is no cascade and $(1-q)q \leq \frac{1}{6}$.

Step 6: Optimality of T^* when $k^* = -1$

Note that for $k^* = -1$ ($p^* = 1 - q$), an UP cascade is reached from the first agent. Therefore, regardless of the choice of T , all the agents support the proposal. Therefore, $T = T^*$ is an optimal choice.

In conclusion, $T = T(p^*)$ is the proposer's weakly dominating strategy, and it is a strictly dominating strategy whenever different T choices may lead to different equilibrium outcomes. □

A.5 Proof of Corollary 2

Proof. First of all, if there is an UP cascade, then there would be no DOWN cascade. Second, if there are less than $T^* - 1$ supporting agents, then from Proposition 1 there would be no DOWN cascade. Third, if there is no UP cascade by the T^* th supporting agent i , then by the construction of $T^* = \lfloor \frac{N + \bar{k}(p^*)}{2} \rfloor$, $i = 2T^* - \bar{k}(p^*) \geq N - 1$, and there would be no DOWN cascade.

The only remaining case is when there is no UP cascade by the $T^* - 1$ th supporting agent i , since there is no UP cascade yet then by the construction of T^* , $i \geq 2(T^* - 1) - \bar{k}(p^*) \geq N - 3$.

To be specific:

1. If $N + \bar{k}(p^*)$ is even, then $i \geq N - 2$. When $i = N$, then then from Proposition 1 there would be no DOWN cascade. When $i \in \{N - 2, N - 1\}$, $k(\mathcal{H}_i) = \bar{k}(p^*) + N - 2 - i$, and there would be no DOWN cascade (a DOWN cascade starts after $k = \bar{k}(p^*) - 2$).
2. If $N + \bar{k}(p^*)$ is odd, then $i \geq N - 3$. When $i = N$, then then from Proposition 1 there would be no DOWN cascade. When $i \in \{N - 3, N - 2, N - 1\}$, $k(\mathcal{H}_i) = \bar{k}(p^*) + N - 3 - i$, and there would a DOWN cascade after agent $N - 1$ if all agents $i < j \leq N - 1$ observes negative signals. □

A.6 Proof of Lemma 3

Proof. The following lemma from Van der Hofstad and Keane (2008) is useful for our analysis later.

Lemma A.4 (Hitting Time Theorem). *For a random walk starting at $k_0 \geq 1$ with i.i.d. steps $\{Y_i\}_{i=1}^\infty$ satisfying $Y_i \geq -1$ almost surely, the distribution of the stopping time $\tau_0 = \inf\{n : S_n = k_0 + \sum_{i=1}^n Y_i\}$ is given by*

$$Pr(\tau_0 = n) = \frac{k_0}{n} Pr(S_n = 0). \quad (26)$$

Part (a). Note that an UP cascade starts after agent $i < T$ if $\sum_{j=1}^i x_j = k + 1$. It implies that there are $\frac{i-k-1}{2}$ negative (-1) signals and $\frac{i+k+1}{2}$ positive ($+1$) signals among the first i signals. Furthermore, we need to have the following conditions:

$$\sum_{j=1}^{i'} x_j \leq k, \quad 1 \leq i' \leq i-1; \quad \sum_{j=1}^{i-1} x_j = k; \quad x_i = 1 \quad (27)$$

We can find the probability of the set of such sequences that satisfy (27) by using Lemma A.4 and the reflection principle:

$$\varphi_{k+1,i} = \frac{k+1}{i} \binom{i}{\frac{i+k+1}{2}} [q(1-q)]^{\frac{i-k-1}{2}} \frac{(1-q)^{k+1} + q^{k+1}}{2}$$

Part (b). If the threshold is reached at agent i and agent i is not part of an UP cascade, then the posterior satisfies $k(\mathcal{H}_i) = \bar{k}(p)$. There would be an UP cascade starts after agent $i+1$ if and only if agent $i+1$ observes a positive signal. Then the probability of reaching T at agent i without an UP cascade is

$$\begin{aligned} \frac{1}{2} \frac{Pr(\mathbb{H}_{i+1}^U | V=1)}{q} + \frac{1}{2} \frac{Pr(\mathbb{H}_{i+1}^U | V=0)}{q} &= \frac{\bar{k}(p)+1}{i} \binom{i}{\frac{i+\bar{k}(p)+1}{2}} [q(1-q)]^{\frac{i-\bar{k}(p)-1}{2}} \frac{(1-q)^{\bar{k}(p)} + q^{\bar{k}(p)}}{2} \\ &= \frac{(1-q)^{\bar{k}(p)} + q^{\bar{k}(p)}}{(1-q)^{\bar{k}(p)+1} + q^{\bar{k}(p)+1}} \varphi_{\bar{k}(p)+1,i+1} \end{aligned}$$

□

A.7 Proof of Proposition 3

Proof. To show the existence of $\underline{N}(k)$, we first prove the existence of $\underline{N}(0)$, then proceed to the $k \geq 1$ case. From the standard Gambler's Ruin problem we know that as $N \rightarrow \infty$, for a given V_k , the conditional probability that an UP cascade occurs at a finite time is 1 if $V = 1$, and $\frac{(1-q)^{k+1}}{q^{k+1}}$ if $V = 0$ (Feller, 1968, Page 347 Eq. 2.8).

Note that $\pi(V_{-1}, N) = (1 - q - \nu)N$. Furthermore, for $p = V_0 = \frac{1}{2}$, an UP cascade starts if $\sum_{j=1}^i x_j = 1$ for some $1 \leq j < 2T$, where T is the AoN target. Because $(1 - q)q < \frac{1}{4}$, we have:

$$\begin{aligned} \lim_{N \rightarrow \infty} \frac{\pi(V_0, T_N^*, N)}{N} &= (V_0 - \nu) \left(Pr(V = 1) + Pr(V = 0) \frac{1 - q}{q} \right) \\ &= \left(\frac{1}{2} - \nu \right) \left(\frac{1}{2} + \frac{1 - q}{2q} \right) > \left(\frac{1}{2} - \nu \right) 2(1 - q) \\ &> 1 - q - \nu = V_{-1} - \nu, \end{aligned}$$

where $T_N^* = \lfloor \frac{N}{2} \rfloor$. Since $\varphi_{1,i}$ is strictly positive, there exists a strictly positive integer $N_1(0)$ such that:

$$(V_0 - \nu) \sum_{i=1}^{N_1(0)} \varphi_{1,i} > 1 - q - \nu$$

Let $D = (V_0 - \nu) \sum_{i=1}^{N_1(0)} \varphi_{1,i} - (1 - q - \nu) > 0$, $Q = (V_0 - \nu) \sum_{i=1}^{N_1(0)} \varphi_{1,i} \frac{i-1}{2}$, and $\underline{N}(0)$ be the smallest integer that is larger than $\max\{N_1(0), \frac{Q}{D}\}$. Then for any $N \geq \underline{N}(0)$:

$$\begin{aligned} \pi(V_0, T_N^*, N) &\geq (V_0 - \nu) \sum_{i=1}^{\underline{N}(0)} \varphi_{1,i} \left(N - \frac{i-1}{2} \right) \geq N(V_0 - \nu) \sum_{i=1}^{N_1(0)} \varphi_{1,i} - Q \\ &\geq \frac{Q}{D} D + (1 - q - \nu)N - Q = (1 - q - \nu)N. \end{aligned}$$

Now, we prove the existence of $\underline{N}(k)$ for $k > 0$. For each $k \geq 1$, and the time i arrival rate $\varphi_{k+1,i+1}$, there exists a corresponding $\varphi_{k,i}$ for price V_{k-1} . For each i , we have:

$$\begin{aligned} \frac{(V_k - \nu)\varphi_{k+1,i+1}}{(V_{k-1} - \nu)\varphi_{k,i}} &\geq \frac{V_k \varphi_{k+1,i+1}}{V_{k-1} \varphi_{k,i}} = \frac{V_k \frac{k+1}{i+1} \frac{(i+1)!}{i+k+2!} \frac{i-k!}{i-k!} [(1-q)q]^{\frac{i-k}{2}} \frac{(1-q)^{k+1} + q^{k+1}}{2}}{V_{k-1} \frac{k}{i} \frac{i!}{i+k!} \frac{i-k!}{i-k!} [(1-q)q]^{\frac{i-k}{2}} \frac{(1-q)^k + q^k}{2}} \\ &= q \frac{k+1}{k} \frac{i}{\frac{i+k}{2} + 1} \left(1 + \frac{[q(1-q)]^{k-1} (q - (1-q))^2}{((1-q)^k + q^k)^2} \right). \end{aligned}$$

Since $\lim_{i \rightarrow \infty} q \frac{i}{\frac{i+k}{2} + 1} = 2q > 1$, for each k , the ratio $\frac{V_k \varphi_{k+1,i+1}}{V_{k-1} \varphi_{k,i}}$ is monotonically increasing in i for large enough values of i , and consequently, there exists an integer N_1 that $\frac{V_k \varphi_{k+1,i+1}}{V_{k-1} \varphi_{k,i}} \geq 1$ whenever $i \geq N_1$. We then have

$$\begin{aligned}
& \lim_{N \rightarrow \infty} (V_k - \nu) \sum_{i=1}^{N-1} \varphi_{k+1, i+1} = (V_k - \nu) \left(\frac{1}{2} + \frac{(1-q)^{k+1}}{2} \right) \\
&= \frac{V_k - \nu}{V_k} \frac{1}{2} \frac{q^k}{q^k + (1-q)^k} \frac{(1-q)^{k+1} + q^{k+1}}{q^{k+1}} = \frac{V_k - \nu}{V_k} \frac{1}{2q} \frac{(1-q)^{k+1} + q^{k+1}}{(1-q)^k + q^k} \\
&> \frac{V_k - \nu}{V_k} \frac{1}{2q} \frac{(1-q)^k + q^k}{(1-q)^{k-1} + q^{k-1}} = \frac{V_k - \nu}{V_k} V_{k-1} \left(\frac{1}{2} + \frac{(1-q)^k}{2} \right) \\
&\geq (V_{k-1} - \nu) \left(\frac{1}{2} + \frac{(1-q)^k}{2} \right) = \lim_{N \rightarrow \infty} (V_{k-1} - \nu) \sum_{i=1}^N \varphi_{k, i},
\end{aligned}$$

where we have used Cauchy-Schwarz inequality to derive $(q^{k+1} + (1-q)^{k+1})(q^{k-1} + (1-q)^{k-1}) > (q^k + (1-q)^k)^2$.

Given that $\lim_{N \rightarrow \infty} \sum_{i=1}^N \varphi_{k+1, i+1}$ converges to a finite number, there exists an integer $N_2 \geq N_1$ such that:

$$D \equiv (V_k - \nu) \sum_{i=1}^{N_2-1} \varphi_{k+1, i+1} - (V_{k-1} - \nu) \sup_{N \geq N_2} \left\{ \sum_{i=1}^N \varphi_{k, i} + \frac{q^{k-1} + (1-q)^{k-1}}{(1-q)^k + q^k} \varphi_{k, N+1} \right\} > 0,$$

where $\frac{q^{k-1} + (1-q)^{k-1}}{(1-q)^k + q^k} \varphi_{k, N+1}$ is the probability that there is no UP cascade and agent N is the T th supporting agent given price V_{k-1} . Let $Q \equiv (V_k - \nu) \sum_{i=1}^{N_2-1} \varphi_{k+1, i+1} \frac{i-k}{2}$. Then for each k , let $\underline{N}(k)$ be the smallest integer that is larger than $\max\{N_2, \frac{Q}{D}\}$. Then for any $N \geq \underline{N}(k)$:

$$\pi \left(V_k, \left\lfloor \frac{N+k}{2} \right\rfloor, N \right) - \pi \left(V_{k-1}, \left\lfloor \frac{N+k-1}{2} \right\rfloor, N \right) > ND - Q > \underline{N}(k)D - Q \geq 0.$$

The proposition follows. \square

A.8 Proof of Proposition 4

Proof. According to Lemma 2, the proposer is never able to cover the cost when no threshold is set (equivalently, when $T = 1$). For the second part, it is enough to consider the case $p = V_N$ and $T = N$. With a positive probability, all agents receive a positive signal, in which case all projects with a cost not exceeding V_N are financed. Since the agents' posterior cannot exceed V_N after any history of actions, projects with $\nu > V_N$ cannot be financed through the threshold implementation. \square

A.9 Proof of Proposition 5

Proof. To prove the first part, i.e. $Pr(A_N \geq T|V = 1) \geq Pr(A_N \geq T|V = 0)$, we only need to show

$$\frac{Pr(V = 1|A_N \geq T)}{Pr(V = 0|A_N \geq T)} \geq 1 \quad \forall p \in \mathbb{R}_{++}, T \in \{1, \dots, N\} \quad (28)$$

since good projects and bad projects are equally likely. For $p \leq V_{-1}$, An UP cascade starts from the first agent, regardless of the sequence of the private signals and the AoN target is certainly reached. Hence, Inequality (28) holds for $p \leq V_{-1} = 1 - q$. For $p > V_{-1}$ and a sequence $x \in X$ for which the target is reached, let $s_T(x)$ be the T 'th supporter. Therefore, the law of iterative expectations implies:

$$\frac{Pr(V = 1|A_N(x) \geq T)}{Pr(V = 1|A_N(x) \geq T) + Pr(V = 0|A_N(x) \geq T)} = \mathbb{E}[V|A_N(x) \geq T] = \mathbb{E}[V|\mathcal{H}_{s_T(x)}] \geq V_{\bar{k}(p)} \geq \frac{1}{2}$$

Inequality (28) clearly follows and the proof for the first part is complete. The remaining claims can be easily verified using (13). □

A.10 Proof of Proposition 6

Proof. First, we prove $\mathbb{E}[V|A_N]$ is weakly increasing in $A_N < \lfloor \frac{N+\bar{k}(p)}{2} \rfloor$.

1. $A_N < T^* - 1$: According to (9), all agents act based on their private signals when $A_{i-1} < T - 1$ and $k_{i-1} \leq \bar{k}(p)$. Moreover, we know that if an UP cascade starts after some agent i in the queue, then the total support is at least $\frac{i+\bar{k}(p)+1}{2} + N - i > \lfloor \frac{N+\bar{k}(p)}{2} \rfloor \geq T$. Therefore, certainly no UP cascade starts in this case. These observations imply that there are exactly A_N positive signals in x , or in other words $\sum_{i=1}^N x_i = 2A_N - N$. Therefore, $\mathbb{E}[V|A_N] = V_{2A_N - N}$ for $A_N < T - 1$. It is clear that the function is increasing in A_N .
2. $A_N = T^* - 1$: In this case, with an argument similar to the previous case, we can show that there are at least $T - 1$ positive signals. However, it is possible that a DOWN cascade starts after the $T - 1$ 'th supporter, which implies the expected value stays at a value not exceeding $V_{\bar{k}(p)-2}$ for the agents in the cascade, including the last agent. Since $\bar{k}(p) - 2 \geq 2(T - 1) - N$, the expected value in this case is bounded by $\mathbb{E}[V|\mathcal{H}_N] \in (V_{2(T-1)-N}, V_{\bar{k}(p)-2})$.

By comparing the posteriors in the cases mentioned above, we can simply verify that $\mathbb{E}[V|A_N]$ is weakly increasing in A_N .

We then move to the $\mathbb{E}[|\mathbb{E}[V|\mathcal{H}_N] - V|]$. We first show that this metric for information aggregation always decreases with one more signal, and then we show that in the equilibrium, as N

increases, the number of informative actions is increasing.

Suppose given the history \mathcal{H}_N the posterior is now V_k . Bayesian updating implies that $\omega V_{k+1} + \omega V_{k-1} = V_k$, where ω is the probability that the next signal is 1 conditional \mathcal{H}_N . If the true $V = 1$, then

$$\begin{aligned} \mathbb{E}[|\mathbb{E}[V|\mathcal{H}_{N+1}] - 1|] &> \omega|V_{k+1} - 1| + (1 - \omega)|V_{k-1} - 1| = 1 - \omega V_{k+1} - \omega V_{k-1} \\ &= |V_k - 1| = \mathbb{E}[|\mathbb{E}[V|\mathcal{H}_N] - 1|] \end{aligned}$$

, where the first inequality comes from the fact that conditional on $V = 1$, the conditional probability that the next signal is 1 is higher than ω . Similarly, one can show that the same result holds for $V = 0$ as well.

We next show that for $N_1 \geq N + 1$, the number of informative actions is always increasing. Suppose $k(\mathcal{H}_N) = \bar{k}(p) + 1$, then for any history $\mathcal{H}_N \prec \mathcal{H}_{N_1}$, if $p^*(N_1) = p^*(N)$, the posterior is the same. If $p^*(N_1) > p^*(N)$, then $\mathbb{E}[V|\mathcal{H}_{N+1}]$ incorporates more signals. Suppose $k(\mathcal{H}_N) = \bar{k}(p) - 2$, from Corollary 2, and the fact that $p^*(N_1) \geq p^*(N)$, $T^*(N_1) > T^*(N)$ and thus the $\mathbb{E}[V|\mathcal{H}_{N+1}]$ incorporates at least agent N 's private signal as well. Suppose $k(\mathcal{H}_N) \in \{\bar{k}(p) - 1, \bar{k}(p)\}$, then $\mathbb{E}[V|\mathcal{H}_{N+1}]$ incorporates at least agent $N + 1$'s private signal as well. □

A.11 Proof of Lemma 4

Proof. To derive the inequalities in (14), first we show that reaching the AoN target, i.e. $Pr(A_N \geq T_N | V = j)$, $j \in \{0, 1\}$, can be approximated with the probability of reaching an UP cascade before or at the same time of reaching the target. We denote this probability by η_N^j :

$$\eta_N^j = \bigcup_{n=\bar{k}(p)+1}^{2T_N-\bar{k}(p)-1} Pr\left(\sum_{i=1}^n x_i = \bar{k}(p) + 1 | V = j\right) \quad j \in \{0, 1\} \quad (29)$$

In particular, we show $\lim_{N \rightarrow \infty} |\mathcal{P}_N^I - (1 - \eta_N^1)| = \lim_{N \rightarrow \infty} |\mathcal{P}_N^{II} - \eta_N^0| = 0$. Then, we find the expressions for η_N^0 and η_N^1 by appealing to the reflection principle. Finally, we show that the sequence of $\{\eta_N^j\}_{N=1}^\infty$, $j \in \{0, 1\}$ are convergent and bounded by some algebraic functions.

First, let $\varphi_{\bar{k}(p)+1,i}^j$ be the probability of starting a cascade after agent i when $V = j$, for $j \in \{0, 1\}$. With an argument similar to the proof of Lemma 3, one can show:

$$\varphi_{k+1,i}^1 = \frac{k+1}{i} \binom{i}{\frac{i+k+1}{2}} q^{\frac{i+k+1}{2}} (1-q)^{\frac{i-k-1}{2}} \quad \varphi_{k+1,i}^0 = \left(\frac{1-q}{q}\right)^{k+1} \varphi_{k+1,i}^1 \quad (30)$$

The following lemma shows $\varphi_{\bar{k}(p)+1,i}^j$ decreases in i for $i > \lceil (2\beta - 1)(\bar{k}(p) + 1) \rceil$. We use this

lemma to show the error probabilities also converge to their limiting value with T_N .

Lemma A.5. *There exists $x \in (0, 1)$ such that for every $p \geq 0$ and $i > \lceil (2\beta - 1)(\bar{k}(p) + 1) \rceil$, we have:*

$$\frac{\varphi_{\bar{k}(p)+1,i}^j}{\varphi_{\bar{k}(p)+1,i-2}^j} < x \quad j \in \{0, 1\} \quad (31)$$

Proof. By using (30), we have:

$$\begin{aligned} \frac{\varphi_{\bar{k}(p)+1,i}^j}{\varphi_{\bar{k}(p)+1,i-2}^j} &= [4q(1-q)]^2 \frac{(i-1)(i-2)}{[(i)^2 - (\bar{k}(p) + 1)^2][(i-1)^2 - (\bar{k}(p) + 1)^2]} \\ &< \left[4q(1-q) \frac{i}{i^2 - (\bar{k}(p) + 1)^2} \right]^2 < \left[4q(1-q) \frac{(2\beta-1)^2}{(2\beta-1)^2 - 1} \frac{1}{i-1} \right]^2 < 1 \end{aligned} \quad (32)$$

Hence, inequality (31) holds by setting $x = 4q(1-q) \frac{(2\beta-1)}{(2\beta-1)^2 - 1}$. \square

Corollary A.1. *For a fixed p and conditional on either $V = 1$ or $V = 0$, the probability of reaching the AoN target without an UP cascade converges to zero as T_N and N become arbitrarily large.*

Proof. According to (30) and with a similar argument to the proof of Lemma 3(b), the probability of reaching the AoN target without an UP cascade is $(q^j(1-q)^{1-j})^{-1} \varphi_{\bar{k}(p)+1, 2T_N - \bar{k}(p)+1}^j$. Note that to find the probability, we used the fact that the target in this case should be reached exactly at agent $s_{T_N} = 2T_N - \bar{k}(p)$. Now, by appealing to Lemma A.5, we have:

$$\lim_{T_N \rightarrow \infty} \frac{\varphi_{\bar{k}(p)+1, 2T_N - \bar{k}(p)+1}^j}{(q^j(1-q)^{1-j})^{-1}} \leq \lim_{T_N \rightarrow \infty} x^{2T_N - \bar{k}(p)+1 - \lceil (2\beta-1)(\bar{k}(p)+1) \rceil} \frac{\varphi_{\bar{k}(p)+1, \lceil (2\beta-1)(\bar{k}(p)+1) \rceil}^j}{(q^j(1-q)^{1-j})^{-1}} = 0$$

\square

Corollary A.1 shows that as N and T_N go to infinity, the AoN target is reached by an UP cascade with probability one. In other words, we proved our first claim that the following asymptotic results hold:

$$\lim_{N \rightarrow \infty} |\mathcal{P}_N^I - (1 - \eta_N^1)| = \lim_{N \rightarrow \infty} |\mathcal{P}_N^{II} - \eta_N^0| = 0$$

We use these results to find the limiting probability of errors in the following lemma.

Lemma A.6. *For a fixed price $p \geq \frac{1}{2}$, if N and T_N go to infinity, then we have:*

$$\lim_{N \rightarrow \infty} \mathcal{P}_N^I = 0 \quad \lim_{N \rightarrow \infty} \mathcal{P}_N^{II} = \left(\frac{1-q}{q} \right)^{\bar{k}(p)+1} \quad (33)$$

Proof. Note that when $V = 1$, we can see that, according to the law of large numbers, the probability of not starting an UP cascade by the $T_N - 1$ 'th supporter goes to zero, because:

$$\begin{aligned} \lim_{T_N \rightarrow \infty} P\left(\sum_{i=1}^{s_{T_N-1}} x_i \leq \bar{k}(p)\right) &= \lim_{T_N \rightarrow \infty} P\left(\frac{1}{s_{T_N-1}} \sum_{i=1}^{s_{T_N-1}} x_i \leq \frac{\bar{k}(p)}{s_{T_N-1}}\right) \\ &\leq \lim_{T_N \rightarrow \infty} P\left(\left|\frac{1}{s_{T_N-1}} \sum_{i=1}^{s_{T_N-1}} x_i - \mathbb{E}[x_i|V=1]\right| \geq \mathbb{E}[x_i|V=1] - \frac{\bar{k}(p)}{s_{T_N-1}}\right) = 0 \end{aligned}$$

where $s_{T_N-1} = 2(T_N - 1) - (\bar{k}(p) + 1)$ is the agent after whom an UP cascade starts if she is the $T_N - 1$ 'th supporter and no UP cascade starts before her. Moreover, in the last inequality, we used that fact that $\mathbb{E}[x_i|V=1] = 2q - 1 > 0$, hence $\lim_{T_N \rightarrow \infty} \mathbb{E}[x_i|V=1] - \frac{\bar{k}(p)}{s_{T_N-1}} = 2q - 1 > 0$. This result shows that when $V = 1$ the AoN target is reached with probability one as the target is set arbitrarily large, i.e. $\lim_{N \rightarrow \infty} \mathcal{P}_N^I = 0$.

As a result, we should have:

$$\sum_{i=0}^{\infty} \varphi_{\bar{k}(p)+1, \bar{k}(p)+2i+1}^1 = 1 \Rightarrow \sum_{i=0}^{\infty} \varphi_{\bar{k}(p)+1, \bar{k}(p)+2i+1}^0 = \left(\frac{1-q}{q}\right)^{\bar{k}(p)+1} \quad (34)$$

Equation (34) shows the probability of reaching an UP cascade when $V = 0$ and the target is set arbitrarily large. Since earlier we showed that as N and T_N go to infinity, the AoN target is reached by an UP cascade with probability one, then the expression in (34) gives the probability of type-two error for arbitrary large values of N , which proves the second equation in Lemma A.6. \square

Now, we are ready to prove the inequalities in (14). For the first inequality, note that:

$$\mathcal{P}_N^I < 1 - \sum_{i=\bar{k}(p)+1}^{s_{T_N}} \varphi_{\bar{k}(p)+1, i}^1$$

where $s_{T_N} = 2T_N - (\bar{k}(p) + 1)$. It is easy to check that if an UP cascade starts by the time that the target is reached, then the UP cascade should start at agent s_{T_N} at the latest. Lemma A.5 and equation (34) imply the following provided $s_{T_N} > (2\beta - 1)(\bar{k}(p) + 1)$:

$$\begin{aligned} 1 - \sum_{i=\bar{k}(p)+1}^{s_{T_N}} \varphi_{\bar{k}(p)+1, i}^1 &= \sum_{j=1}^{\infty} \varphi_{\bar{k}(p)+1, 2j+s_{T_N}}^1 < \sum_{j=1}^{\infty} x^{0.5(s_{T_N} - (2\beta-1)(\bar{k}(p)+1)) + j} \varphi_{\bar{k}(p)+1, [(2\beta-1)(\bar{k}(p)+1)] + \varepsilon_{\bar{k}(p)}}^1 \\ &< \frac{x^{0.5(s_{T_N} - (2\beta-1)(\bar{k}(p)+1))}}{1-x} \varphi_{\bar{k}(p)+1, [(2\beta-1)(\bar{k}(p)+1)] + \varepsilon_{\bar{k}(p)}}^1 = \frac{x^{T_N - \beta(\bar{k}(p)+1)}}{1-x} \varphi_{\bar{k}(p)+1, [(2\beta-1)(\bar{k}(p)+1)] + \varepsilon_{\bar{k}(p)}}^1 \end{aligned}$$

where $\varepsilon_{\bar{k}(p)}$ is chosen either zero or one such that makes $[(2\beta-1)(\bar{k}(p)+1)] + \varepsilon_{\bar{k}(p)} + \bar{k}(p) + 1$ an even

number. Now, suppose $M \equiv \sup_{k \geq 0} \frac{x^{-\beta}}{1-x} \varphi_{k+1, [(2\beta-1)(k+1)] + \varepsilon_k}^1$. Note that (34) ensures the existence of M and imply $M \leq 1$. Hence, we have

$$\mathcal{P}_N^I < 1 - \sum_{i=\bar{k}(p)+1}^{s_{T_N}} \varphi_{\bar{k}(p)+1, i}^1 < Mx^{T_N - \beta \bar{k}(p)} \quad (35)$$

which is the first inequality in (14). The second inequality can be similarly shown by using the second equation in (33). \square

The following corollary is helpful for the proof of next propositions.

Corollary A.2. *The proposer's profit $\frac{1}{N}\pi(p_N^*, T_N^*, N)$ converges to the full-surplus extraction profit $\frac{1}{2}(1 - \nu)$ from each supporting agent as N becomes arbitrarily large.*

Proof. First of all, since for each agent, the total surplus of efficient investment is $\frac{1}{2}(1 - \nu)$. If the profit for the proposer were bigger than that, then at least the first supporting agent is in expectation losing money, and thus she would choose rejection, a contradiction. Then

$$\lim_{k \rightarrow \infty, N \rightarrow \infty} \frac{1}{N}\pi(p_N^*, T_N^*, N) \leq \frac{1}{2}(1 - \nu) \quad (36)$$

Suppose $T_N^k = \lfloor \frac{N+k}{2} \rfloor$. Clearly, $\pi(p_N^*, T_N^*, N) \geq \pi(V_k, T_N^k, N)$, for every k . Therefore, we only need to show

$$\lim_{k \rightarrow \infty, N \rightarrow \infty} \frac{1}{N}\pi(p_N^*, T_N^*, N) \geq \frac{1}{2}(1 - \nu) \quad (37)$$

Given the expressions provide in Lemma 4, we have:

$$\lim_{N \rightarrow \infty} \frac{1}{N}\pi(V_k, T_N^k, N) = \frac{1}{2}(V_k - \nu)(1 + (\frac{1-q}{q})^{k+1}) \Rightarrow \lim_{k \rightarrow \infty} \lim_{N \rightarrow \infty} \frac{1}{N}\pi(V_k, T_N^k, N) = \frac{1}{2}(1 - \nu)$$

which completes the proof. \square

A.12 Proof of Lemma 5

Proof. Suppose A_N^k is the number of supporters when there are N agents, $p = V_k$ and $T = \lfloor \frac{N+k}{2} \rfloor$. Note that A_N^k is a random variable depending on the sequence of signals. Furthermore, the proposer's profit is $\pi(V_k, \lfloor \frac{N+k}{2} \rfloor, N) = (V_k - \nu)\mathbb{E}[A_N^k \mathbb{1}_{\{A_N^k \geq \lfloor \frac{N+k}{2} \rfloor\}}]$, where $\mathbb{E}[A_N^k]$ is the expected number of supporters when $p = V_k$. For the rest of the proof, let $S_N^k \equiv A_N^k \mathbb{1}_{\{A_N^k \geq \lfloor \frac{N+k}{2} \rfloor\}}$.

Now, we want to provide an upper bound for $\mathbb{E}[S_N^k - S_N^{k+1}]$, the expected number of supporters lost due to the price increase from V_k to V_{k+1} . According to Lemma 4, Lemma A.6 and Corollary A.1, if $\frac{N}{2} > (\beta - \frac{1}{2})k$ and N and k are sufficiently large, then the probability of not reaching

the target with an UP cascade for $p = V_{k+1}$, while reaching the target with $p = V_k$ is at most $1 - Mx^{\frac{N}{2} - (\beta - \frac{1}{2})(k+1)}$ when $V = 1$ and $(\frac{1-q}{q})^{k+1}(1 - Mx^{\frac{N}{2} - (\beta - \frac{1}{2})(k+1)})$ when $V = 0$. Therefore:

$$\mathbb{E}[S_N^k - S_N^{k+1}] \leq (1 - \delta)\mathbb{E}\left[S_N^k - S_N^{k+1} \mid S_N^{k+1} \geq \left\lfloor \frac{N+k+1}{2} \right\rfloor\right] + \delta N, \quad (38)$$

where $\delta = Mx^{\frac{N}{2} - (\beta - \frac{1}{2})k}$. Note that in (38), we use the fact that $\frac{1}{2}(1 + (\frac{1-q}{q})^{k+1}) < 1$. In fact, we provide the bound conditional on $V = 1$. The proof for $V = 0$ is similar.

In Corollary A.1, we showed that the probability of reaching the target without reaching an UP cascade goes to zero for arbitrarily large values of k and N . Furthermore, it is clear that if an UP cascade is reached for $p = V_{k+1}$, it should follow an UP cascade for $p = V_k$. Now, in Lemma A.7, we find an upper bound for the expected number of supporters lost due to the price increase.

Lemma A.7. *Suppose C_i is the probability that the UP cascade when $V = 1$ for $p = V_{k+1}$ starts $2i + 1$ periods after that for $p = V_k$, then*

$$(a) C_i \leq \frac{1}{2^{i+1}} q^{i+1} (1-q)^i \binom{2i+1}{i+1}; \quad (b) \sum_{i=0}^{\infty} i C_i < \frac{q}{1-4q(1-q)}.$$

Proof. (a) Following the lemma 3, suppose an UP cascade starts at $u \leq \lfloor \frac{N+k}{2} \rfloor$ when $p = V_k$. The probability that an UP cascade starts at $u + 2i + 1$ for $p = V_{k+1}$ conditional on u (and $u + 2i + 1 \leq \lfloor \frac{N+k+1}{2} \rfloor$) is the probability of reaching an UP cascade when $k = 0$. Moreover, $\frac{1}{2^{i+1}} q^{i+1} (1-q)^i \binom{2i+1}{i+1}$ is the probability of such cascade conditional on $V = 1$, which is higher than the one conditional on $V = 0$.

(b) The inequality can be easily derived by noting that $\binom{2i+1}{i+1} < 2^{2i+1}$, $\frac{i}{2^{i+1}} < \frac{1}{2}$ and using the inequality obtained in part (a). \square

Note that $\sum_{i=0}^{\infty} i C_i$ is an upper bound on the expected number of supporters lost, conditional on eventually reaching an UP cascade. Therefore, we can modify (38) as follows:

$$\mathbb{E}[S_N^k - S_N^{k+1}] < B(1 - \delta) + \delta N, \quad (39)$$

where $B \equiv \frac{q}{1-4q(1-q)}$. We next use this result to find an optimality condition for k_N^* . Note that for the optimal price $p_N^* = V_{k_N^*}$, we should have:

$$\begin{aligned} V_{k_N^*} \mathbb{E}[S_N^{k_N^*}] - V_{k_N^*+1} \mathbb{E}[S_N^{k_N^*+1}] &\geq 0 \Rightarrow V_{k_N^*} (\mathbb{E}[S_N^{k_N^*} - S_N^{k_N^*+1}]) > (V_{k_N^*+1} - V_{k_N^*}) \mathbb{E}[S_N^{k_N^*+1}] \\ &\Rightarrow V_{k_N^*} (B(1 - \delta^*) + \delta^* N) > \frac{N + k_N^* + 1}{2} (1 - \delta^*) (V_{k_N^*+1} - V_{k_N^*}) \\ &\Rightarrow \frac{V_{k_N^*}}{V_{k_N^*+1} - V_{k_N^*}} > \frac{(1 - \delta^*) \frac{N + k_N^* + 1}{2}}{B(1 - \delta^*) + \delta^* N} \Rightarrow \frac{2q}{2q - 1} \left(\frac{q}{1-q}\right)^{k_N^*} > \frac{(1 - \delta^*) \frac{N + k_N^* + 1}{2}}{B(1 - \delta^*) + \delta^* N}, \end{aligned}$$

where $\delta^* \equiv Mx^{\frac{N}{2} - (\beta - \frac{1}{2})k_N^*}$. Note that if $\frac{k_N^*}{\ln N} \rightarrow 0$, then $\delta^* N$ converges to zero as N goes to infinity. Therefore, for large enough values of N , we should have:

$$\frac{2q}{2q-1} \left(\frac{q}{1-q}\right)^{k_N^*} > \frac{1}{2B} N \Rightarrow \ln \frac{2q}{2q-1} + k_N^* \ln \frac{q}{1-q} > \ln \frac{1}{2B} + \ln N \Rightarrow \liminf_{N \rightarrow \infty} \frac{k_N^*}{\ln N} \geq \frac{1}{\ln \frac{q}{1-q}},$$

which is a contradiction with $\frac{k_N^*}{\ln N} \rightarrow 0$. There thus exists $a > 0$ such that $\liminf_{N \rightarrow \infty} \frac{k_N^*}{\ln N} \geq a > 0$. It implies that there exist a, b such that $k_N^* > a \ln N + b$ for every N . We then have,

$$\begin{aligned} 1 - p_N^* &= \frac{(1-q)^{k_N^*}}{(1-q)^{k_N^*} + q^{k_N^*}} < \left(\frac{1-q}{q}\right)^{k_N^*} < \left(\frac{1-q}{q}\right)^{a \ln N + b} = N^{a \ln \frac{1-q}{q} + b \frac{\ln \frac{1-q}{q}}{\ln N}} \\ &\Rightarrow N^{a \ln \frac{q}{1-q}} (1 - p_N^*) < N^{\frac{b \ln \frac{1-q}{q}}{\ln N}} = e^{b \ln \frac{1-q}{q}} \Rightarrow N^{\gamma_1} (1 - p_N^*) < \gamma_2, \end{aligned}$$

where $\gamma_1 = a \ln \frac{q}{1-q} > 0$ and $\gamma_2 = e^{b \ln \frac{1-q}{q}} > 0$. This completes the proof. \square

A.13 Proof of Corollary 4

Proof. Inequality (15) can be derived using the first inequality provided in Lemma 4 for $p = \nu$. \square

A.14 Proof of Proposition 7

Proof. We first show $\lim_{N \rightarrow \infty} \mathcal{P}_N^{II} = 0$. Note that in expectation, the investors at least break even. To be more concrete, we know $\mathbb{E}[V - p_N^* | A_N \geq T_N^*] \geq 0$ for the optimal choice of (p_N^*, T_N^*) . It implies for any N , the following relation for the probability of errors should hold:

$$\frac{1}{2}(1 - \mathcal{P}_N^I)(1 - p_N^*) - \frac{1}{2}\mathcal{P}_N^{II} p_N^* \geq 0 \quad (40)$$

We know $p_N^* \rightarrow 1$ as N goes to infinity. Therefore, (40) implies $\mathcal{P}_N^{II} \rightarrow 0$. Now, we show $\lim_{N \rightarrow \infty} \mathcal{P}_N^I = 0$. To see this, first note that the following inequality holds for every N

$$\frac{1}{N} \pi(p_N^*, T_N^*, N) < \frac{1}{2}(1 - \mathcal{P}_N^I + \mathcal{P}_N^{II})(p_N^* - \nu) \quad (41)$$

In Corollary A.2, we show that the left hand side in (41) goes to $\frac{1}{2}(1 - \nu)$ as N becomes arbitrarily large. It implies:

$$\lim_{N \rightarrow \infty} 1 - \mathcal{P}_N^I + \mathcal{P}_N^{II} \geq 1 \Rightarrow \lim_{N \rightarrow \infty} \mathcal{P}_N^I \leq \lim_{N \rightarrow \infty} \mathcal{P}_N^{II} = 0 \Rightarrow \lim_{N \rightarrow \infty} \mathcal{P}_N^I = 0$$

\square

A.15 Proof of Proposition 8

Proof. In Proposition 7 we show both probability of errors go to zero as N goes to infinity. Therefore:

$$\begin{aligned} \lim_{N \rightarrow \infty} Pr(A_N \geq T_N^* | V = 0) &= \lim_{N \rightarrow \infty} Pr(A_N < T_N^* | V = 1) = 0 \\ \Rightarrow \lim_{N \rightarrow \infty} \mathbb{E}[V | A_N < T_N^*] &= 0, \lim_{N \rightarrow \infty} \mathbb{E}[V | A_N \geq T_N^*] = 1 \end{aligned}$$

Since \mathcal{H}_N includes event $\mathbb{1}\{A_N \geq T_N^*\}$, we can simply conclude that for any $\epsilon > 0$, $\lim_{N \rightarrow \infty} Pr(|\mathbb{E}[V | \mathcal{H}_N()] - V| > \epsilon) = 0$, which proves the statement. \square

A.16 Proof of Proposition 9

Proof. First, suppose agent i observes $x_i = 1$, she has no incentive to deviate. If she chooses rejection or waiting, then all follow agents misinterpret her action and update their beliefs as if $x_i = -1$. This results in failures for some project that should be financed if i correctly reveals her information.

If agent i observes $x_i = -1$, as we discussed in the baseline model, if there is an UP cascade she chooses to invest. When there is no UP cascade yet, she has no incentive to invest, and waiting is a weakly dominating strategy since she can always reject latter. Thus her first action of waiting still reveals her information. \square

A.17 Proof of Proposition 10

Proof. When agents have options to wait, investors with negative signals invest if they are part of an UP cascade and the project would be implemented. From the proof for Proposition 3, we have

$$\lim_{N \rightarrow \infty} (V_k - \nu) \sum_{i=1}^{N-1} \varphi_{k+1, i+1} \geq \lim_{N \rightarrow \infty} (V_{k-1} - \nu) \sum_{i=1}^N \varphi_{k, i}.$$

When N goes to infinity, since the probability that the project is implemented (reaching AoN threshold) without an UP cascade goes to 0, the optimal choice of k goes to infinity. That is to say, p goes to 1.

Because $Pr(x_i = 1 | V = 1) = q > 1 - q$, Feller (1968), page 347 equation 2.8 shows that the probability that an UP cascade takes place from some finite agent is 1 when $V = 1$, and $\frac{(1-q)^{k(p)+1}}{q^{k(p)+1}}$ when $V = 0$. So all good project would be implemented almost surely when N goes to infinity, and bad projects would be abandoned almost surely as p goes to 1 when N goes to infinity. \square

A.18 Proof of Lemma 6

Proof. Any equilibrium involves a sub-game equilibrium following the proposer's decision on p and T . We only need to show that any sub-game equilibrium is either the informative one characterized in Proposition 1 or one involves a group of free-riders whose actions before a cascade are ignored in equilibrium.

For any agent observing a high signal, it is her dominating strategy to contribute when there is a positive probability to reach the AoN threshold (and her action would be irrelevant if the project would not be implemented for sure). For an agent observing a low signal, if she knows that in the equilibrium the subsequent agents update their beliefs based on her action, then she always rejects as discussed in the proof for Proposition 1. However, if she knows her supporting the proposal does not positively update the subsequent agents' posteriors, then for any rational off-equilibrium belief (that is to say, if subsequent agent observes a rejection instead of a support action, they do not positively update their beliefs), then supporting becomes a dominant action, since it allows her to free-ride on the gate-keeper's decision. Hence, she would support the proposal. \square

A.19 Proof for Lemma 7

Proof. We first show that if $p \in \{V_k, K = -1, 0, \dots, N\}$, then in any free-rider sub-game equilibrium, the project would be implemented only if there is an UP cascade. Suppose agent i is a free-rider (given the history \mathcal{H}_{i-1}) in a free-rider sub-game equilibrium, then $a_i = 1$. Otherwise, suppose the contrary that there exists a free-rider sub-game equilibrium in which agent i always rejects the project, then the project would only be implemented when the public posterior after the T th supporting agent is at least p . However, if agent i 's private signal is $x_i = 1$, then her conditional posterior is strictly higher than p , suggesting that she has incentive to deviate.

If agent i always supports the project, then she must have no incentive to deviate when her private signal is $x_i = -1$. Given $x_i = -1$, if the public posterior after the T th supporting agent is $V_{\bar{k}(p)+1}$, then agent i 's conditional posterior is $V_{\bar{k}(p)} = p$. If the public posterior after the T th supporting agent is $V_{\bar{k}(p)}$, then agent i 's conditional posterior is $V_{\bar{k}(p)-1} < p$. For agent i , the following inequality must hold for her individual rationality of investing:

$$\varphi(V_{\bar{k}(p)} - p) + Q(V_{\bar{k}(p)-1} - p) = Q(V_{\bar{k}(p)-1} - p) \geq 0, \quad (42)$$

where φ is the probability that the public posterior after the T th supporting agent is $V_{\bar{k}(p)+1}$ and Q is the probability that the public posterior after the T th supporting agent is $V_{\bar{k}(p)} = p$, conditional on the history \mathcal{H}_{i-1} and agent i 's private observation $x_i = -1$. In other words, conditional on $x_i = -1$, agent i breaks even when the public posterior after the T th supporting agent is $V_{\bar{k}(p)+1}$

and loses money when the public posterior after the T th supporting agent is $V_{\bar{k}(p)}$. For the agent to be rationally free riding by always supporting, it must be $Q = 0$, that is, the project is implemented with an UP cascade.

We next show that for every investor, the informer sub-game equilibrium weakly Pareto dominates all free-rider sub-game equilibria. In a free-rider sub-game equilibrium, for each realization path $x \in X$ that results in the project implementation, let h_T be the T th supporting agent. Consider the a corresponding $x'(x)$: if there exists a sequence of free-riders $\{j_1, j_2, \dots\}$, then move the sequence of signals $\{x_{j_1}, x_{j_2}, \dots\}$ to the right of signal x_{h_T} . For each free-rider sub-game equilibrium, $x'(x)$ is an injective function that maps each sequence x to a distinct sequence $x'(x)$ such that $x'(x)$ is a realization path that results in an Up cascade (and thus project implementation) in the informer sub-game equilibrium. Now consider agent i in the free-rider sub-game equilibrium, if she observes $x_i = -1$, she breaks even if she is a free-rider, and also receives 0 when she is either an informer or in an UP cascade. Now suppose agent i observes $x_i = 1$, for each realization path x that results in the project implementation, she always chooses the same action and receives the same payoff on the corresponding realization path $x'(x)$ in the informer sub-game equilibrium. Since x and $x'(x)$ are equally likely to happen, agent i is weakly better off in the informer sub-game equilibrium.

Finally, suppose a free-rider sub-game equilibrium involves at least two free-riders. Then with positive probability that there exists a sequence x with at least two free-riders all observing positive private signals and the project is implemented. Similar to the discussion above, on the realization path x , right after the $(T - 1)$ th supporting agent the public posterior is $V_{\bar{k}(p)}$. Also notice that the $(T - 1)$ th supporting agent must be the $(N - 1)$ th agent, for otherwise $Q > 0$. In the free-rider sub-game equilibrium, consider the following realization path $\hat{x}(x)$: the first $N - 1$ signals are the same as the ones in sequence x , while $x_N = -1$. Given the sequence $\hat{x}(x)$, the project would not be implemented in the free-rider sub-game equilibrium, but would be implemented in the informer sub-game equilibrium. Moreover, given at least two more positive signals, all investing agents receive positive expected profit on the sequence $\hat{x}(x)$. Therefore, this free-rider equilibrium is strictly dominated by the informer equilibrium. The proposition ensues. \square

A.20 Proof for Proposition 11

Proof. Clearly, for equilibria with only one free-rider, one can simply view the case to be an informer equilibrium with $N - 1$ agents. As $N \rightarrow \infty$, the resulting equilibria converge. Therefore, we only need to focus on free-rider equilibria with at least two free-riders.

As discussed in the proof of Lemma 7, in particular Eq.(42), the unique equilibrium is an informer equilibrium when $p \in \mathbb{Z} \cup \{-1\}$. Then, the proposer's per-investor profit $\frac{1}{N}\pi(p_N, T_N, N)$

should be at least $\frac{1}{N}\pi(p_N^*, T_N^*, N)$, which is the profit when the proposer is restricted to choose the price from $\mathbb{Z} \cup \{-1\}$.

Therefore, Corollary A.2 in Appendix A.11 implies that any optimal path of $\{(p_N, T_N)\}_{N=1}^\infty$ should satisfy the following condition

$$\lim_{N \rightarrow \infty} \frac{1}{N}\pi(p_N, T_N, N) = \frac{1}{2}(1 - \nu).$$

In other words, the first-best social investment efficiency is reached and the entrepreneur or proposer extracts all the surplus as N goes to infinity. This condition requires the price p_N converges to one and both error probabilities go to zero, as shown in Proposition 7. \square

A.21 Proof for Corollary 5

Proof. Again, we only need to consider free-rider equilibria with at least two free-riders. For a given N , the corresponding proposal (p_N, T_N) , sub-game equilibrium E , and a sequence of signals x , define $Z_N^E(x; (p_N, T_N))$ as the total number of informers for sequence $x \in X$. Hence, we need to show $Pr(Z_N^E(x; (p_N, T_N)) < l) \rightarrow 0$, where the probability is taken over $x \in X$.

Consider the contrary and suppose for some $\varepsilon > 0$, $Pr(Z^E(x; (p_N, T_N)) < l) > \varepsilon$ for infinite values of N . For such an N , we have:

$$\frac{\pi^E(p_N, T_N)}{N} < \frac{1}{2Np_N}\pi_1^E(p_N, T_N) < \frac{1}{2Np_N}(1 - \varepsilon(1 - q)^l)Np_N(1 - \nu) = \frac{1 - \varepsilon(1 - q)^l}{2}(1 - \nu),$$

where $\pi_1^E(p_N, T_N)$ denotes the proposer's profit when the project is good. The first inequality holds because an agent should assign at least probability p_N on the project being good to be willing to pay p_N . Since it holds for all the agents and after all the histories, the gross proposer's revenue when $V = 0$ cannot exceed $\frac{1-p_N}{p_N}$ times that of when $V = 1$. Therefore, $\frac{1-p_N}{2Np_N}(\pi_1^E(p_N, T_N) + \nu) \geq \frac{1}{2N}(\pi_0^E(p_N, T_N) + \nu)$, which further can be simplified and get the first inequality, since $\nu \geq 0$.

The second inequality follows from the fact that the proposal is not accepted when all the informers receive a low signal. But in the proof of Proposition 11, we showed that $\frac{\pi^{E_N}(p_N, T_N)}{N}$ goes to $\frac{1}{2}(1 - \nu)$ in this sequence of numbers, which is a contradiction. \square

A.22 Proof for Corollary 6

Proof. Again, we only need to consider free-rider equilibria with at least two free-riders. In Proposition 11 we have shown that both error probabilities go to zero. The almost-sure convergence of $\mathbb{E}[V|\mathcal{H}_N]$ can be shown similar to the proof of Proposition 8. \square