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Abstract

We quantify the fiscal multipliers in response to the American Recovery and Reinvestment Act (ARRA) of 2009. We extend the benchmark Smets-Wouters (Smets and Wouters, 2007) New Keynesian model, allowing for credit-constrained households, the zero lower bound, government capital and distortionary taxation. The posterior yields modestly positive short-run multipliers around 0.52 and modestly negative long-run multipliers around -0.42. The multiplier is sensitive to the fraction of transfers given to credit-constrained households, the duration of the zero lower bound and the capital. The stimulus results in negative welfare effects for unconstrained agents. The constrained agents gain, if they discount the future substantially.

Keywords: Fiscal Stimulus, New Keynesian model, liquidity trap, zero lower bound, fiscal multiplier

JEL codes: E62, E63, E65, H20, H62

"Fiscal Stimulus", the size of "fiscal multipliers" and the impact of discretionary fiscal spending on GDP and unemployment, has once again become central to policy debates in wake of the financial crisis of 2008 and fiscal policy responses in a number of countries. In this paper, we therefore seek to quantify the size, uncertainty and sensitivity of fiscal multipliers in response to a "fiscal stimulus" as in the American Recovery and Reinvestment Act (ARRA) of 2009 in the United States, using an extension of a benchmark New Keynesian model.

From a purists' perspective, this may be the wrong question to ask. Policy should care about welfare, rather than derivative measures such as GDP or unemployment. Moreover, it should seek to solve a Mirrlees-Ramsey problem, and use the best combinations of available tools and taxes to maximize welfare, subject to constraints imposed by markets and the asymmetry of information. We do not disagree. Indeed, there is a considerable literature on these topics. We address welfare issues in section 3.9, but they are not the main focus of this paper.

Indeed, many public debates focus on the effects of fiscal spending on GDP and unemployment. Economists have the tools to answer these questions, and therefore, perhaps they should. Several recent papers have addressed these issues. This paper seeks to make a contribution to this emerging literature. In essence, we seek to understand how much of the rather negative perspective on long-run multipliers in Uhlig (2010b), due to distortionary taxation in a neoclassical growth model, survives in a model that takes a very Keynesian perspective. In a nutshell, the answer is: while the benchmark long-run multiplier is now modestly negative rather than substantially negative and while the precise answer is sensitive to some key assumptions and uncertain parameters, much survives indeed.

We view the following elements as important. First, "fiscal stimulus" takes time in practice, despite calls for immediate actions as in e.g. Spilimbergo et al. (2008). The American Recovery and Reinvestment Act or ARRA (2009) therefore serves as a useful benchmark and example for the speed at which fiscal policy tools can be deployed, as emphasized by Cogan et al.

(2010). Second, government expenditures are financed eventually with distortionary taxes, creating costly disincentive effects, a point emphasized by Uhlig (2010b). Third, monetary policy and its restrictions due to the zero lower bound (ZLB) on interest rates can matter substantially for the effectiveness of "fiscal stimulus", as emphasized by Eggertsson (2010) as well as Christiano et al. (2009), in particular if there are sticky prices and wages. Fourth, transfers are a substantial part of the ARRA and similar programs: the degree to which they are given to credit-constrained households may matter considerably, see Coenen et al. (2010). Finally, model coefficients are uncertain and results are sensitive to specific assumptions. For that reason, we use a reasonably tractable "small-scale" model rather than a larger "black box", employing Bayesian estimation techniques as well as sensitivity analysis to quantify the uncertainty in our answers. As Leeper et al. (2011) have pointed out, the New Keynesian model employed here together with its prior already are already an important determinant of our answers. This is desirable: the model assumptions should be crucial. The Bayesian estimation serves to quantify the results more sharply and to inform us about the overall posterior uncertainty.

The analysis here has much in common and is inspired by Cogan et al. (2010), but there are a number of important differences. Like them, we start from the benchmark Smets-Wouters Smets and Wouters (2007) New Keynesian model and analyze the impact of the ARRA. In contrast to these authors, we allow for a government raising revenues with distortionary taxation, and we introduce credit-constrained consumers in our benchmark model.

This analysis postulates the presence of the ZLB, either with a deterministic or endogenized duration. However, Correia et al. (2010) have shown that when consumption tax rates are a policy instrument, adjusting tax rates can substitute for adjusting interest rates, thereby circumventing the ZLB. Since we only consider various kinds of government spending as policy instruments and treat taxes as determined by different feedback rules and exogenous shocks, we neglect this potentially important channel of fiscal policy here.

We distinguish between short-run and long-run multipliers. For a benchmark parameterization, we find modestly positive short-run multipliers with a posterior mean of 0.51 and modestly negative long-run multipliers centered around -0.42. The multiplier is particularly sensitive to the fraction of transfers given to rule-of-thumb consumers, is sensitive to the anticipated length of the zero lower bound, is sensitive to the capital share and is nonlinear in the degree of price and wage stickiness. Reasonable specifications are consistent with substantially negative short-run multipliers within a short time frame.

We compute the welfare effects of the policy intervention separately for both types of agents. The effects on unconstrained agents are significantly negative but small as they are close to their unconstrained optimum. As credit-constrained agents exhibit a higher rate of time-preference, we consider a range of rates of time preference, up to 30% higher than that of unconstrained agents on an annual basis. If agents are not too impatient, the welfare gains through higher short-run consumption are more than offset by the disutility of hours worked and lower consumption in the transition back to the balanced growth path. However, starting at rates of time preference about 20% higher than that of unconstrained agents, the welfare effects can become significantly positive for constrained agents.

These models have also been criticized considerably for the lack of a financial sector, a feature likely for understanding the events of 2008 (see Uhlig, 2010a; Krugman, 2009; Buiter, 2009). We agree with this critique and therefore feature a financial friction per the "short cut" of allowing for time-varying wedges between the central bank interest rate, government bond rates and the return to private capital, following Hall (2010). Our estimates show that these wedges are indeed the key to understanding the recession of 2007 to 2009. Understanding their nature more deeply should therefore be high on the research agenda, but is not the focus of this paper and beyond its scope. An interesting explanation has been forwarded by Ilut and Schneider (2011): increases in ambiguity in markets may result in increased wedges between safe and risky assets.

Aside from the contributions cited above, the analysis here is related to a number of additional important contributions, notably Erceg and Linde (2010) as well as Leeper and various co-authors (Davig and Leeper, 2009; Leeper et al., 2010, 2009). In a model which also features distortionary taxes, rule of thumb consumers, and financial frictions, Erceg and Linde (2010) point out that the marginal multiplier differs from the average multiplier: If the stimulus is successful, the economy leaves the binding ZLB earlier and the effect of additional spending is reduced. We address this issue by endogenizing the duration of the ZLB in robustness tests. A key difference is their focus on the short-run when the effects of adjusting distortionary taxes instead of transfers matter less. Leeper et al. (2010) allow future government consumption and transfers to adjust in order to rebalance the government budget, and find that adjusting spending and component in addition to taxes raises the multiplier. Leeper et al. (2009) point out the importance of productive government investment and government capital, Davig and Leeper (2009) allow for fiscal policy to switch between passive and active regimes in a New Keynesian model. Interestingly, they find the largest difference in multipliers due to switches in the monetary policy regime, which we address by varying the ZLB duration.

The paper is structured as follows. Section 1 gives an overview of the model. Section 2 discusses the estimation and calibration procedure. It provides a decomposition of the shocks driving the 2007-2009 recession, and shows that financial frictions have been key, in stark contrast to the full-sample variance decomposition. Both sections are complemented by a detailed technical appendix which provides all model details as well as code for replicating our results or calculating other fiscal experiments. Section 3 presents the main results on the fiscal multiplier. It provides a sensitivity analysis which highlights the main driving forces behind our results. In addition, it provides a discussion of the welfare effects of the stimulus package. Section 4 concludes.

1 The model

The model is an extension of Smets and Wouters (2007), and we shall refer the reader to that paper as well as to the technical appendix for the complete details. Here we shall provide a brief overview as well as describe the extensions.

The Smets and Wouters (2007) model is a New Keynesian model, set in discrete time. There is a continuum of households. Workers supply homogeneous labor in monopolistic competition. Unions differentiate the labor supplied by households and set wages for each type of labor. Wages are Calvo-sticky. There is a continuum of intermediate good firms. They supply intermediate goods in monopolistic competition. They set prices. Prices are Calvo-sticky. Final goods use intermediate goods. Final goods are produced in perfect competition. Households have preferences for final goods, allowing for habit formation, as well as leisure. Capital is produced with investment in the form of the final good, but there are adjustment costs to investment: given installed capital and previous-period investment, the marginal product of investment for producing new capital is decreasing. There is variable capital utilization.

We extend the model with several features. Briefly, we constrain the interest rate set by the central bank to be nonnegative. We let the government raise revenues with distortionary taxation. We introduce credit-constrained consumers. We feature government capital. We introduce a wedge between various returns, as a stand-in for financial frictions. We adopt the notation convention that variables indexed t are known in period t.

1.1 The zero lower bound

More precisely, the monetary authority follows a Taylor-type rule, but interest rates may be held constant for a deterministic period of time or are modelled to be bounded below by a constant slightly above zero. It is easier to describe these scenarios it in their log-linearized form: for the original version, the reader is referred to the technical appendix.

In our benchmark scenario, the central bank keeps the interest rate at its historical level of 2008:4 for k quarters. Households fully anticipate this policy. Let $\hat{R}_t^{TR}$ denote the log-deviation of the shadow Taylor Rule return, given by:

$$\begin{aligned}\hat{R}_t^{TR} &= \psi_1(1-\rho_R)\hat{\pi}_t + \psi_2(1-\rho_r)(\hat{y}_t - \hat{y}_t^f) \\ &\quad + \psi_3\Delta(\hat{y}_t - \hat{y}_t^f) + \rho_R \hat{R}_{t-1}^{TR} + ms_t\end{aligned}$$

where $\hat{\pi}_t$ is the log-deviation for inflation, $\hat{y}_t$ is the log-deviation for output, $\hat{y}_t^f$ is the log-deviation in the flexible-price version of the economy and ms_t is a shock to the interest rate set by the central bank.

The effective interest rate in our benchmark scenario is then given by:

$$\hat{R}_t^{FFR} = (1 - ZLB_t)\hat{R}_t^{TR} + ZLB_t \hat{R}_0^{FFR},$$

where ZLB_t is an indicator function modelling which takes the value of one while the ZLB lasts and zero otherwise. During the ZLB, the central bank return equals its historical starting value, $\hat{R}_0^{FFR}$.

When endogenizing the ZLB duration, the central bank sets the log-deviation of the central bank return to

$$\hat{R}_t^{FFR} = \max\{-(1 - \bar{R}^{FFR}) + \bar{\epsilon}, \hat{R}_t^{TR}\}$$

where $\bar{R}^{FFR}$ is the steady state nominal return, $\bar{\epsilon} > 0$ is a constant set slightly above zero for technical reasons (and set to $\bar{\epsilon} = \frac{0.25}{400}$ in the numerical calculations, implying a lower bound of 25 basis points for the central bank interest rate).

1.2 Households, distortionary taxation and financial frictions.

A fraction $1 - \phi$ of the household is unconstrained and solves an infinite-horizon maximization problem. The preferences of such a household j are given by

$$U = \mathbb{E}\left[\sum_{t=0}^{\infty} \beta^s \left(\frac{1}{1-\sigma}\left(c_t(j) - h\, c_{t-1}^{aggr}\right)^{1-\sigma}\right) \exp\left(\frac{\sigma-1}{1+\nu}\, n_t(j)^{1+\nu}\right)\right] \quad (1.1)$$

where $c_t(j)$ is consumption of household j, $n_t(j)$ is its labor supply and c_t^{aggr} is aggregate consumption. $h \in [0,1)$ captures external habit formation, σ denotes the inverse of the intertemporal elasticity of substitution, and ν equals the inverse of the labor supply elasticity. Households discount the future by $\beta \in (0,1)$.

Following Trabandt and Uhlig (2010), we assume that the government provides transfers and collects linear taxes on labor income, capital income net of depreciation as well as consumption, adapted to the model here. The budget constraint of household j is therefore given by

$$\begin{aligned}
&(1+\tau^c)c_t(j) + x_t(j) + \frac{B_t^n(j)}{R_t^{gov} P_t} \\
&\leq\ s_t^{\text{unconstr}} + \frac{B_{t-1}^n(j)}{P_t} + (1-\tau_t^n)\frac{W_t}{P_t}\left(n_t(j) + \lambda_{w,t} n_t^{(\text{aggr})}\right) + \\
&\quad + \left((1-\tau^k)\left(\frac{R_t^k u_t(j)}{P_t} - a(u_t(j))\right) + \delta\tau^k\right)\left((1-\omega_{t-1}^k)k_{t-1}^p(j) + \omega_{t-1}^k k_{t-1}^{p,aggr}\right) + \frac{\Pi_t^p}{P_t},
\end{aligned}$$

and the capital accumulation constraint is given by

$$k_t^p(j) \;=\; \frac{(1-\delta)}{\mu} k_{t-1}^p(j) + q_{t+s}^x \left(1 - \xi\left(\frac{x_t(j)}{x_{t-1}(j)}\right)\right) x_t(j),$$

where $c_t(j)$ is consumption, $x_t(j)$ is investment, $B_{t-1}^n(j)$ are nominal government bond holdings, $n_t(j)$ is labor, $k_{t-1}^p(j)$ is private capital, and $u_t(j)$ is capacity utilization, all of household j and chosen by household j. R_t^{gov} is the nominal return for the one-period government bond from t to $t+1$ set at date t, $n_t^{(\text{aggr})}$ is aggregate labor, P_t is the aggregate price level, W_t is aggregate wages, $\lambda_{w,t}$ is the aggregate mark-up from union-determined wages, R_t^k is the undistorted return on capital and ω_t^k is a friction or wedge on private capital markets. In the budget constraint, note that it enters as a variable known at date $t-1$, so that the distortions to future capital returns impacts on investment in the current period. Also note that the individual losses due to this wedge are redistributed in the aggregate, so that the wedge distorts investment decisions, but does not destroy aggregate resources directly. Π_t^p are nominal firm profits, q_{t+s}^x is an investment-specific technology parameter, $\xi(\cdot)$ are adjustment costs, satisfying $\xi(\mu) = \xi'(\mu) = 0$, $\xi'' > 0$, τ^c, τ^n, τ^k are taxes and s_t^{unconstr} are real transfers to unconstrained households, all taken as given by household j, and $a(\cdot)$ represents the strictly increasing and strictly convex cost function of varying capacity utilization. In particular, note that taxing capital net of depreciation implies deducting a depreciation rate that depends on capacity utilization. Furthermore, the household receives labor income both directly from working as well as indirectly from the surplus that unions charge on labor: both sources of labor income are taxed.

We assume that the interest rate R_t^{gov} on government bonds, which unconstrained households can freely trade, equals the federal funds rate R_t^{FFR} up to an exogenous friction or wedge ω_t^{gov}:

$$R_t^{gov} = (1 + \omega_t^{gov}) R_t^{FFR}.$$

In difference to Smets and Wouters (2007), the discount factor β of the households is not subject to shocks. Rather, we focus on the wedges ω_t^k and ω_t^{gov} on financial markets as part of the household budget constraint.

We assume that a fraction $\phi \in (0, 0.5)$ of the households is credit-constrained. In their version of the budget constraint, $B_{t-1}^n(j) = 0$, $x_t(j) = 0$ and $k_{t-1}^p(j) = 0$, i.e. these households do not save or borrow. They do receive profit income from intermediate producers (which equals zero in the steady state). Put differently, the budget constraint of a credit-constrained household j is

$$\begin{aligned} &(1+\tau^c) c_t(j) \\ &\quad \leq \; s_t^{\text{constr}} + (1-\tau_t^n) \frac{W_t}{P_t} \left(n_t(j) + \lambda_{w,t} n_t^{(\text{aggr})} \right) + \frac{\Pi_t^p}{P_t} \end{aligned}$$

where s_t^{constr} are the transfers to credit-constrained agents. As a justification, one may suppose that credit-constrained discount the future substantially more steeply, and are thus uninterested in accumulating government bonds or private capital, unless their returns are extraordinarily high. Conversely, these households find it easy to default on any loans, and are therefore not able to borrow. We hold the identity of credit-constrained households and thereby their fraction of the total population constant. Note that we allow the transfers s_t^{constr} to constrained households to differ from the transfers s_t^{unconstr} to the unconstrained households.

Wages are set by unions on behalf of the households, recognizing that each differentiated wage is Calvo-sticky. Since workers of the unconstrained households represent the majority in these unions, wages are set according to their preferences. Firms hire workers randomly from both types of households, so that labor supplied by both types of households is the same in equilibrium.

1.3 Government capital and policy feedback rules

As the ARRA contains a government investment, we wish to feature government capital as productive input. We also wish to keep the final goods production function to have constant returns to scale on the firm level, in order to maintain the assumption of perfect competition there. We therefore assume that government capital K_{t-1}^g enters private production as an externality for the individual intermediate-goods firm, similar to the model in Barro and Sala-i Martin (1992). In order to obtain an aggregate constant-returns-to-scale production function before fixed costs, we assume that the externality of K_{t-1}^g at the firm level is relative to aggregate output, before fixed costs.

Specifically, we assume that the technology of intermediate firm i is given by

$$Y_t(i) = \tilde{\epsilon}_t^a \left(\frac{K_{t-1}^g}{\int_0^1 Y_t(\iota)d\iota + \Phi\mu^t} \right)^{\frac{\zeta}{1-\zeta}} (K_t^{\text{eff.}}(i))^\alpha (\mu^t n_t(i))^{1-\alpha} - \mu^t \Phi,$$

where Φ are fixed costs, $K_t^{\text{eff.}}$ is effective capital used by firm i, created from aggregate private capital,

$$K_t^{\text{eff.}} = u_t k_{t-1}^p (1-\phi)$$

(assuming symmetric choices for the unconstrained households), where ϵ_t^a is an exogenous, stochastic component of TFP, and where the services of government capital K_{t-1}^g are subject to congestion: what matters is the ratio of government capital to average gross output, i.e. inclusive of the fixed costs. As a result, the aggregate production function in the absence of price dispersion is given by

$$Y_t = \epsilon_t^a {K_{t-1}^g}^\zeta K_t^{s\alpha(1-\zeta)} (\mu^t n_t)^{(1-\alpha)(1-\zeta)} - \mu^t \Phi, \quad \epsilon_t^a \equiv (\tilde{\epsilon}_t^a)^{1-\zeta}.$$

where TFP in terms of the private factors of production is

$$TFP = \epsilon_t^a {K_{t-1}^g}^\zeta \mu^{(1-\alpha)(1-\zeta)t}$$

We assume that the accumulation of government capital is symmetric to the accumulation of private capital, i.e., is subject to a similar technology,

$$k_t^g = \frac{(1-\delta)}{\mu} k_{t-1}^g + q_t^g \left(1 - S_g \left(\frac{x_t^g + \epsilon_t^{x,g}}{x_{t-1}^g + \epsilon_{t-1}^{x,g}} \right) \right) (x_t^g + \epsilon_t^{x,g})$$

where $S_g(\mu) = S_g'(\mu) = 0, S_g''(\cdot) > 0$ represent adjustment costs, $q_t^{x,g}$ is a shock to the government-investment-specific technology parameter, and $\epsilon_t^{x,g}$ is additional, exogenous government investment. We assume that the capacity utilization of government capital and therefore its depreciation is constant. We assume that the government chooses investment to maximize the present discounted value of output net of investment costs, except for a discretionary fiscal stimulus, denoted by $\epsilon_t^{x,g}$ and set to zero at steady state. Put differently, the first-order condition of the government determines optimal government investment, while actual government investment may be higher by some amount chosen along the stimulus path. To enforce the expansion of government investment, we stipulate that the government cannot undo the stimulus investment for the first twelve periods, but has to provide at least replacement for the depreciated ARRA investment – otherwise, the deviation from the optimality condition would imply complete crowding out.

We assume a feedback rule for labor tax rates as follows (for the full detail, see the technical appendix), following Uhlig (2010b). Break the period-by-period government budget constraint in two parts. On the "right side", there is a "deficit" d_t, prior to new debt and labor taxes

$$\begin{aligned} d_t \quad = \quad & \text{gov.spend.+subs.}_t + \text{old debt repaym.}_t \\ & -\text{cons.tax rev.,cap.tax rev.}_t - \bar{\tau}^l \, \text{lab.income}_t \end{aligned}$$

which needs to be financed on the "left side" with labor tax revenues and new debt,

$$\tau_t^l \, \text{lab.income}_t + \text{new debt}_t = d_t$$

Along the balanced growth path, there is a path for the debt level as well as the deficit $\bar{d}_t$. The labor tax rate is then assumed to solve

$$(\tau_t^l - \bar{\tau}^l) \, \text{lab.income}_t = \psi_\tau (d_t - \bar{d}_t) + \epsilon_{\tau,l}$$

where $\epsilon_{\tau,l}$ is a labor tax shock.

1.4 Shocks

We assume that there are ten stochastic processes driving the economy. Unless stated otherwise, the processes follow independent AR(1)'s in logs: (1) Technology $\tilde{\epsilon}_t^a$, (2) Gov.bond wedge ω_t^{gov}: financial friction wedge between

FFR and gov't bonds, (3) Priv. bond wedge ω_t^k: financial friction wedge between gov't bond returns and a component of the returns to private capital, (4) Gov. spending plus net export. Co-varies with technology, (5) Investment specific technology q_t^x (rel. price), (6) Gov. investment specific technology q_t^g (rel. price), (7) Monetary policy ms_t, (8) Labor tax rates $\epsilon_{\tau,l}$, (9) Mark-up for prices: ARMA(1,1), and (10) Mark-up: wages: ARMA(1,1).

For the stimulus plan, we use three series, capturing the changes in transfers, government consumption and government investment. We followed the strategy of Cogan et al. (2010), but our decomposition of government spending into consumption and investment as well as the need to pay particular attention to transfers meant that we needed to reclassify the various spending categories, according to the American Recovery and Reinvestment Act (ARRA). As source, we have used the estimates by the Congressional Budget Office (CBO, 2009) for the effects of the ARRA by budget title. The annual time path for these expenditures is directly taken from the CBO, whereas the distribution within each year is proportional to the Cogan et al. (2010) path within each year. The details on the components are contained in appendix 5.1. A graphical overview on the time path is presented in figure 1. Essentially, we decomposed their government spending path into a separate consumption and investment path, and furthermore included transfers. Most importantly, much of the transfers are "front-loaded", i.e. occur earlier than government spending, while the "stimulus" government investment occurs later.

Furthermore, we assume that the central bank will leave the federal funds rate unchanged at near zero for eight quarters, and that this is fully anticipated, as of the first quarter of 2009. For the numerical calculations, the relaxation algorithm proposed by Juillard (1996) and implemented in Dynare is particularly convenient for the type of forward-simulation (rather than estimation) performed here. By solving a potentially time-varying system of equations backward from terminal conditions, it allows to incorporate anticipated shocks even when they interact coefficients for example to "switch off" the interest rate rule temporarily. In Drautzburg and Uhlig (2011) we investigate whether the particular modelling and solution method we employ here to account for the ZLB may play a role for our results.

2 Estimation and Analysis

2.1 Data and Estimation

We solve the model, using a log-linear approximation and Dynare. The first-order conditions and their log-linearized versions are in a technical appendix, available up on request. We estimate the model, using the following ten time series: (1) Output: Chained 2005 real GDP, growth rates, (2) Consumption: Private consumption expenditure, growth rates, (3) Investment: private fixed investment, growth rates, (4) Government investment: growth rates, (5) Hours worked: Civilian employment index × average nonfarm business weekly hours worked index, demeaned log, (6) Inflation: GDP deflator, quarterly growth rates, (7) Wages: Nonfarm Business, hourly compensation index. Growth rates, (8) FFR: Converted to quarterly rates, (9) Corporate-Treasury bond yield spread: Moody's Baa index – 10 yr Treasury bond at quarterly rates, demeaned, (10) Dallas Fed gross federal debt series at par value, demeaned log.

Sources and details for the data are described in appendix 5.1. We use an updated version of the Smets-Wouters dataset, for the range 1947:2-2009:4, using quarterly data and four periods for the start-up. In difference to the original dataset, we classify consumer durables as investment expenditure. The estimation of the model uses data from 1948:2 up to 2008:4, with the additional four quarters for comparison of the model prediction to the actual evolution and the first four quarters used to presample. We choose the longer sample, as it includes the Korean war as well as the Vietnam war build-up, in contrast to the shorter Smets-Wouters sample from 1967 onwards. Figure 2 shows the additional evidence from the larger fluctuation in fiscal expenditures available in this larger sample.

We fixed ("calibrated") several parameters a priori. For tax rates and the

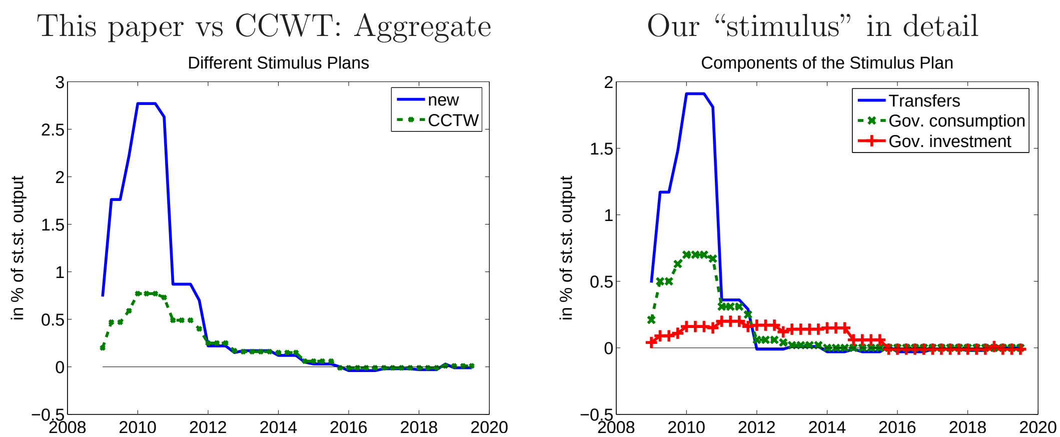

Figure 1: *Our three stimulus components and their comparison to Cogan et al. (2010). Essentially, we decomposed their government spending path into a separate consumption and investment path, and furthermore included transfers.*

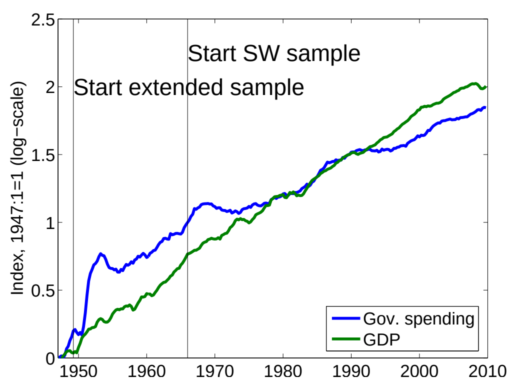

Figure 2: *Comparing our extended sample to the original Smets-Wouters data set. Notice the additional variation in government spending in the larger sample.*

debt-GDP ratio, we relied on Trabandt and Uhlig (2010). Time averages of government spending components were obtained from the NIPA, Table 3.1 (quarterly), lines 35 (investment), 16 (consumption), transfers (17). Government consumption includes net exports (line 2 minus line 14 in Table 4.1). To obtain ratios relative to GDP, GDP data from line 1, Table 1.1.5 was used. Following Smets and Wouters (2007), the Kimball curvature parameter is taken from Eichenbaum and Fisher (2007), who set it to roughly match it to their data on the empirical frequency of price adjustment. Following Cooley and Prescott (1995), the depreciation rate is derived from the law of motion for capital and their observation of $\frac{\bar{x}}{\bar{k}} = 0.0076$ at quarterly frequency. The complete list of calibrated parameters, and their comparison to the corresponding parameters in Smets and Wouters (2007), if available, is in table 1. We estimate our model, using Dynare and a fairly standard Bayesian prior. Details on the estimation can be found in appendix 5.2. The estimates largely agree with those found by Smets and Wouters (2007), leaning somewhat more to more endogenous persistence: the habit parameter is slightly higher, as are estimates of price and wage stickiness, for example. Like these authors, our estimates also yield a rather small capital share: our posterior mean is 0.24, while they found 0.19. This is at odds with calibrated values in the literature, see e.g. Cooley and Prescott (1995), and may play a substantial role in calculating the long-horizon impact of distortionary taxation. We shall investigate this issue in our sensitivity analysis. The calibrated government investment-to-GDP ratio as well as the estimated growth trend $\mu \approx 1.005$ implies a government share in production of $\zeta \approx 2.30$ percent.

2.2 Decomposing the 2007-2009 recession

The model allows the decomposition of movements in our ten macroeconomic time series into the ten shocks that caused them. The first-order conditions of the households imply:

$$1 \;=\; \beta E_t \left[\frac{u_{c,t+1}}{u_{c,t}} \frac{R_t^{gov}}{\pi_{t+1}} \right] = \beta E_t \left[\frac{u_{c,t+1}}{u_{c,t}} (1 + \omega_t^{gov}) \frac{R_t^{FFR}}{\pi_{t+1}} \right]$$

Table 1: *Calibrated parameters.*

	SW (1966:1–2004:4)	This paper (1948:2–2008:4)
Depreciation δ	0.025	0.0145
Wage mark-up λ_w	0.5	0.5
Kimball curvature goods mkt. $\hat{\eta}_p$	10	10
Kimball curvature labor mkt. $\hat{\eta}_w$	10	10
Capital tax τ^k	n/a	0.36
Consumption tax τ^c	n/a	0.05
Labor tax τ^n	n/a	0.28
Share credit constrained ϕ	n/a	0.25
Gov. spending, net exports-GDP $\frac{\bar{g}}{\bar{y}}$	0.18	0.153
Gov. investment-GDP $\frac{\bar{x}^g}{\bar{y}}$	n/a	0.04
Debt-GDP $\frac{\bar{b}}{\bar{y}}$	n/a	$4\times$ 0.63

$$= \beta E_t \left[\frac{u_{c,t+1}}{u_{c,t}} \left((1-\omega_t^k)((1-\tau^k)(r_{t+1}^k u_{t+1} - a(u_{t+1})) + \delta\tau^k) + (1-\delta)\frac{Q_{t+1}}{Q_t} \right) \right]$$

where ω_t^{gov} is due to government bond shocks and creates a wedge between between the FFR and government bonds, while ω_t^k is due to private bond shocks, creating a wedge between government bonds and private capital. Q_t is the price of capital. It is instructive to simplify the above expression by assuming a constant price of capital Q_t and constant capacity utilization as well as ignoring uncertainty. Then the first line can be substituted in the second to yield:

$$1 = \frac{1}{(1+\omega_t^{gov})} \frac{\pi_{t+1}}{R_t^{FFR}} \Big((1-\omega_t^k)(r_{t+1}^k - \tau^k(r_{t+1}^k - \delta)) + (1-\delta) \Big).$$

This equation shows that, up to a first order approximation, the wedges ω_t^k (after re-scaling) and ω_t^b both add up to the total wedge between the return on private capital net of taxes and the Federal Funds Rate R_t^{FFR}. These wedges are stand-ins for financial frictions. It is therefore interesting to examine their role for the 2007-2009 recession.

As figure 2.2 as well as table 2 document, shocks to these wedges indeed played a large role in understanding the recent recession, accounting alone for over 100% of the decline in output, in stark contrast to their small contribution to the full-sample variance of output as well as other included time series. Figure 4 provides the impulse response to a one-standard deviation shock to these two wedges. As one can see, the government bond shock depresses output, consumption and private as well as government investment, whereas the shock to the spread between private bonds and government bonds leads to a decline in consumption only with some delay and actually increases government investments. These shocks furthermore result in a modest decline in the federal funds rate (not shown).

Since not only GDP growth but also unemployment is at the center of many public debates, we back out a predicted change in the unemployment rate from the model. To that end we regress the quarterly unemployment rate on the hours worked measure used to estimate model and use the implied OLS estimate to infer the effect on the unemployment rate. The fit is reasonable with an R^2 of 0.77. We neglect the additional parameter uncertainty introduced because of the uncertain estimates of the regression coefficients.[1]

3 Results

Armed with our posterior estimates as well as the specification of the stimulus path, we shall now proceed to calculate the implied effects. We provide confidence bands, covering 90 percent or 67 percent of the posterior probability.

Our main focus is on the fiscal multiplier, i.e. the ratio of output changes to the total stimulus-planned change in spending and transfers. Note that due to the eventual balancing of the government budget, there will also be an induced movement in tax rates as a "secondary" effect. As is customary, we shall not include these secondary movements in the denominator, i.e. in quantifying the stimulus-planned changes. As this is a dynamic model, the horizon plays a role. Following Uhlig (2010b), we use the net present value fiscal multiplier φ_t, dividing the net present value of output changes up to

[1] Details of the estimation are available in table 12 and figure 20 in the technical appendix

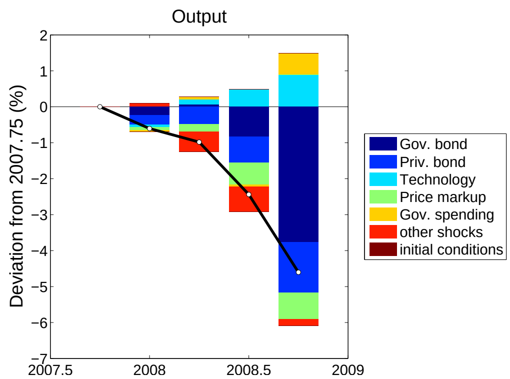

Figure 3: *Historical Shock Decomposition: Output. Results are at the posterior median. 2007:4 is the NBER recession date.*

Table 2: *Historical decomposition of recent recession and overall variance decomposition for output. All numbers are at the Bayesian posterior mean.*

	2008:4 vs. 2007:4		Total Sample
	Historical decomposition		Variance decomposition
Shock	total	percent	percent
Gov. bond	-3.76	81.69	5.11
Priv. bond	-1.41	30.63	1.38
Technology	0.89	-19.44	19.23
Price markup	-0.74	16.14	6.68
Gov. spending	0.60	-12.95	3.49
Priv. inv.	-0.30	6.57	14.04
Labor tax	-0.26	5.60	19.63
Monetary pol.	0.22	-4.69	17.37
Wage Markup	0.14	-3.11	8.38
Gov. inv.	0.03	-0.65	4.59
Initial Values	-0.01	0.22	n/a
Sum	-4.60	100.00	100.00

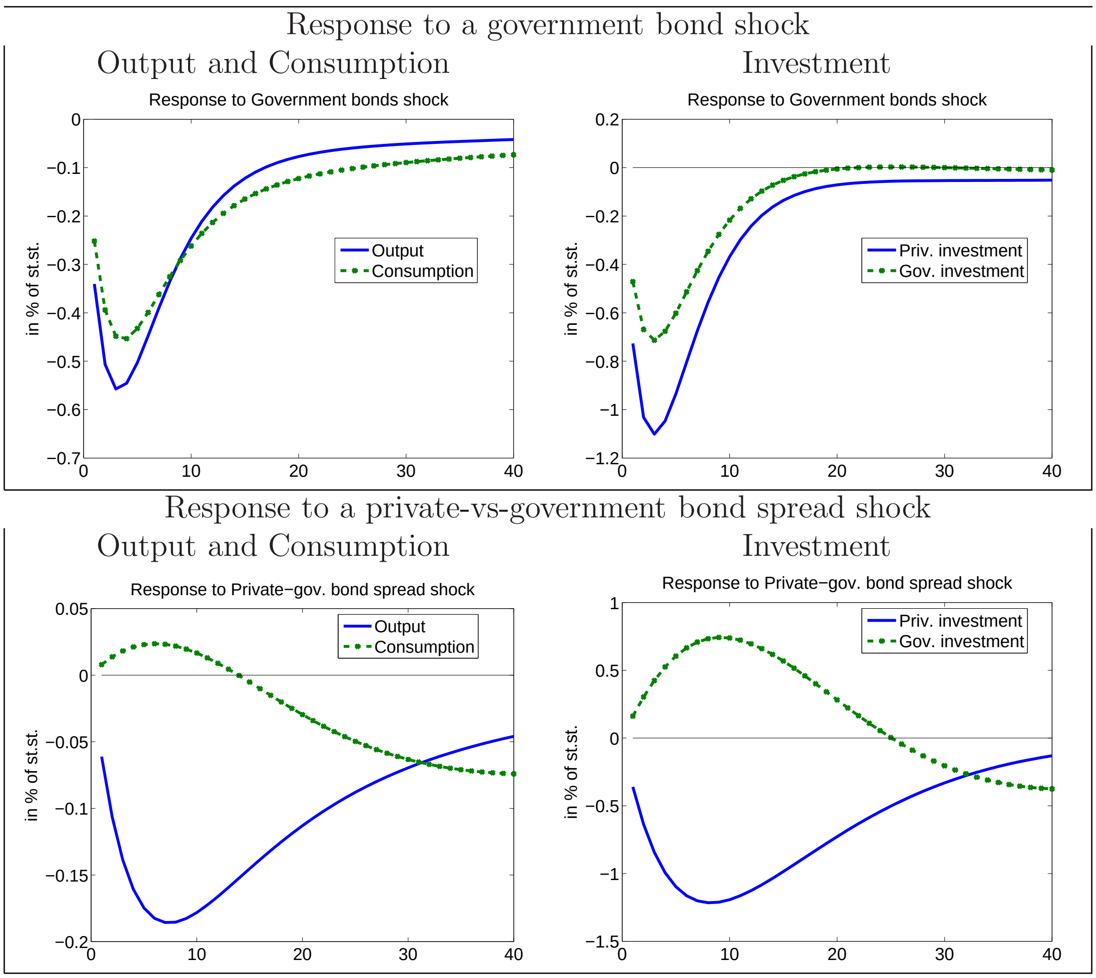

Figure 4: *Response to the bond shocks*

some horizon t by the change in government spending and transfers until the same time. I.e., we shall use

$$\varphi_t = \sum_{s=1}^{t} \left(\mu^s \prod_{j=1}^{s} R_{j,ARRA}^{-1} \right) \hat{y}_s / \sum_{s=1}^{t} \left(\mu^s \prod_{j=1}^{s} R_{j,ARRA}^{-1} \right) \hat{g}_s \tag{3.1}$$

where φ_t: horizon-t multiplier, $R_{j,ARRA}$ is the government bond return, from $j-1$ to j, $\hat{y}_s$ is the output change at date s due to ARRA in percent of the balanced-growth GDP path and $\hat{g}_s$: ARRA spending at date s in percent of the balanced-growth GDP path.

3.1 Benchmark results

Figure 5 contains our benchmark results for output, the unemployment rate, the federal funds rate, inflation, government debt, and consumption.[2] These graphs are perhaps reminiscent of the information shown in the official White House piece by Bernstein and Romer (2009). However, we include an important piece of information, which is missing there. The short-run debt dynamics shown here induce a long-run debt-and-tax dynamics, shown in figure 6. The increase in labor tax rates long after the fiscal stimulus phase has finished induces the decline of output for many years to come.

The resulting fiscal multiplier will therefore decline with the horizon. The fiscal multipliers for the shorter horizon, shown in the left panel of figure 7 can therefore be quite misleading in terms of assessing the long-term costs of fiscal stimulus. Indeed, the long-run multipliers are considerably smaller or negative, compared to the short-run multipliers, as show in the right panel of figure 7. These results are qualitatively in line with Uhlig (2010b), though the results are quantitatively rather different: the long-run fiscal multipliers are negative there and here, but considerably more negative there. One may be tempted to read the difference as “relief” compared to the pessimistic scenario in Uhlig (2010b). Note, however, that the model here is heavily

[2] Results for the consumption of both types of agents, real wages, tax rates, and investment are shown in Figure 15.

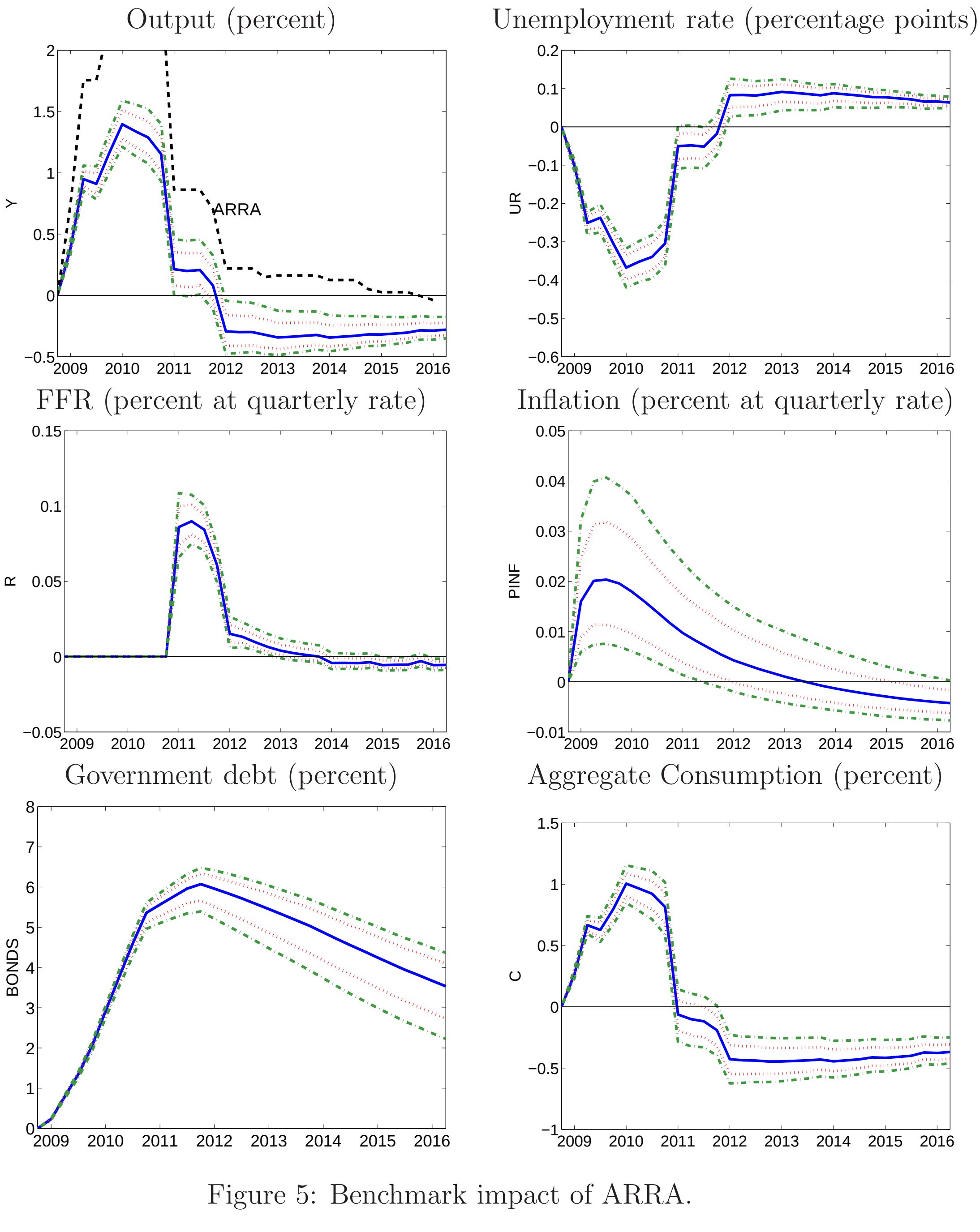

Figure 5: Benchmark impact of ARRA.

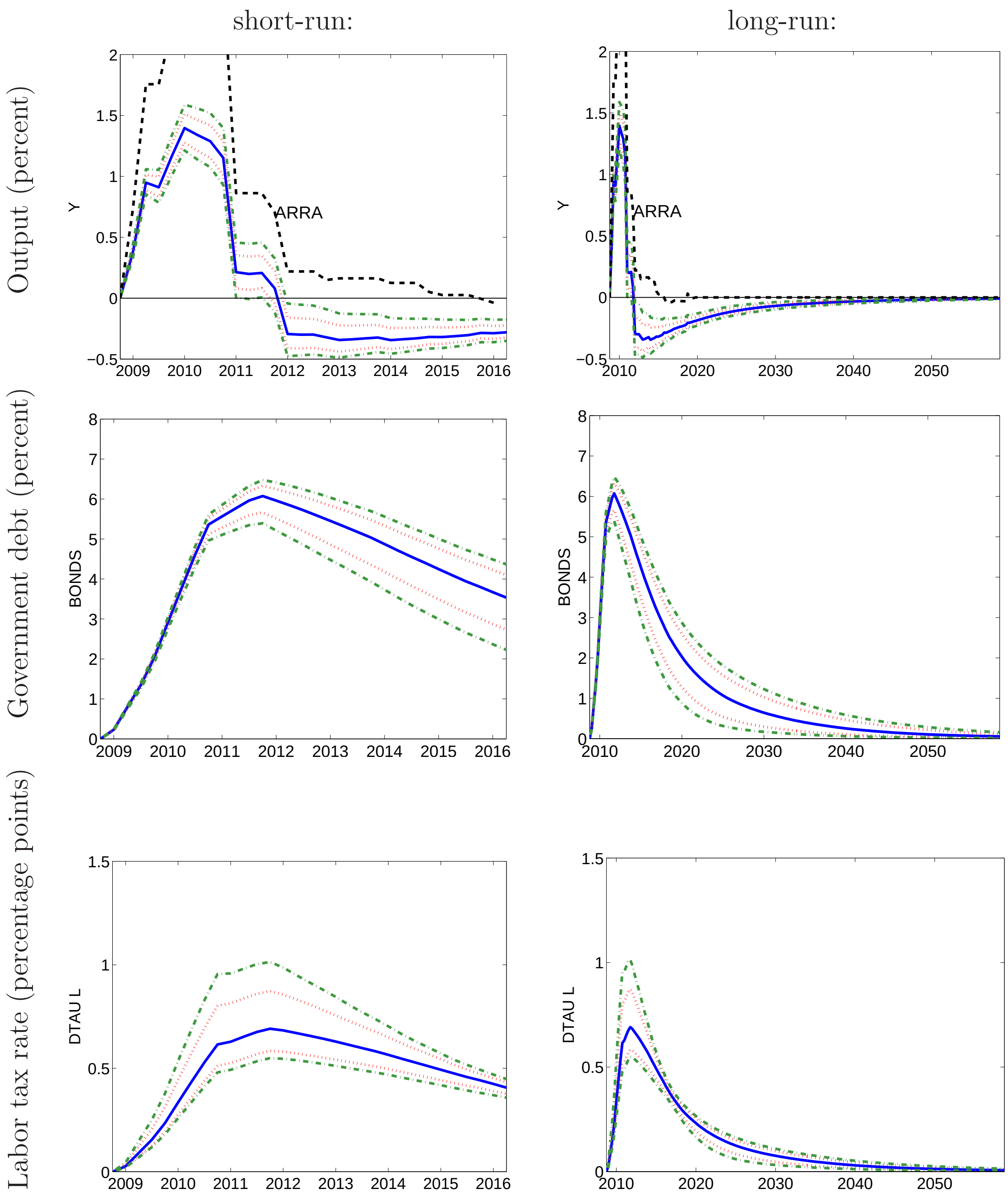

Figure 6: Short- and long-run impact of ARRA.

tilted towards a model in which fiscal stimulus is often thought to work well: we therefore believe that the negative long-run effects of fiscal stimulus should give pause to arguments in its favor. Even at the short horizon, the benchmark multiplier is just around 0.5.

3.2 Sensitivity analysis: overview

Which features of the model contribute to the size of the fiscal multipliers, which are particularly important? Where does the difference to Uhlig (2010b) come from? Understanding the differences and understanding the sensitivity of the benchmark results to key assumptions is important. Figure 8 as well as table 3 and table 4 provide an overview of our sensitivity analysis. The next subsections provide the details.

3.3 Sensitivity to distortionary taxation

Along with Uhlig (2010b), we emphasize the importance of assuming distortionary rather than lump-sum taxes in this analysis. Figure 9 provides a comparison. As should be clear, distortionary rather than lump-sum taxation makes a considerable difference and creates significantly lower long-run multipliers, whereas the short-run multipliers are not significantly different. Adjusting consumption taxes only yields a slightly higher multiplier than adjusting labor tax rates.

Note that the dramatic difference due to distortionary taxation is not an artefact of the stimulus being spread out over time. To illustrate this, we consider the case when the entire stimulus is spent uniformly over the first four quarters and compute the multiplier for two cases: when lump-sum transfers are adjusted and when distortionary labor taxes are adjusted. Figure 3.3 shows a large difference. When transfers are adjusted, the multiplier is large and in excess of one, whereas the median multiplier with distortionary taxes declines to almost minus one in the long run.

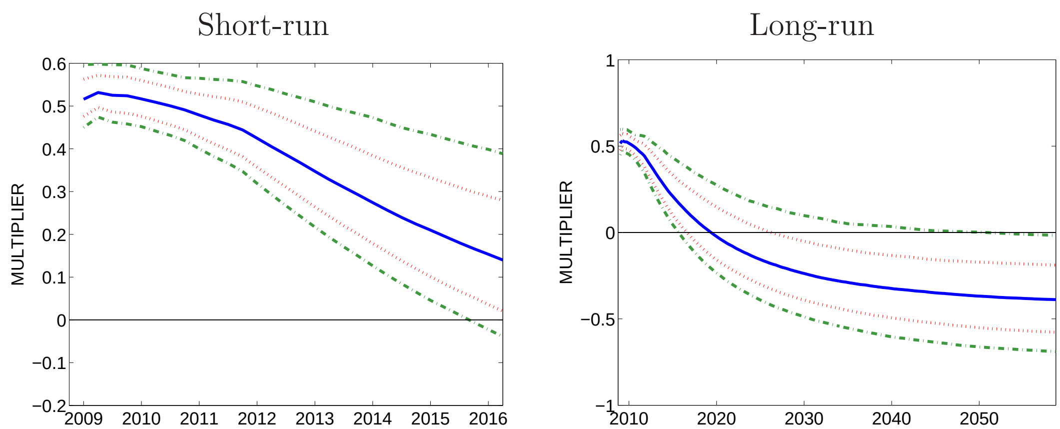

Figure 7: *Short-run and long-run fiscal multipliers in the benchmark parameterization.*

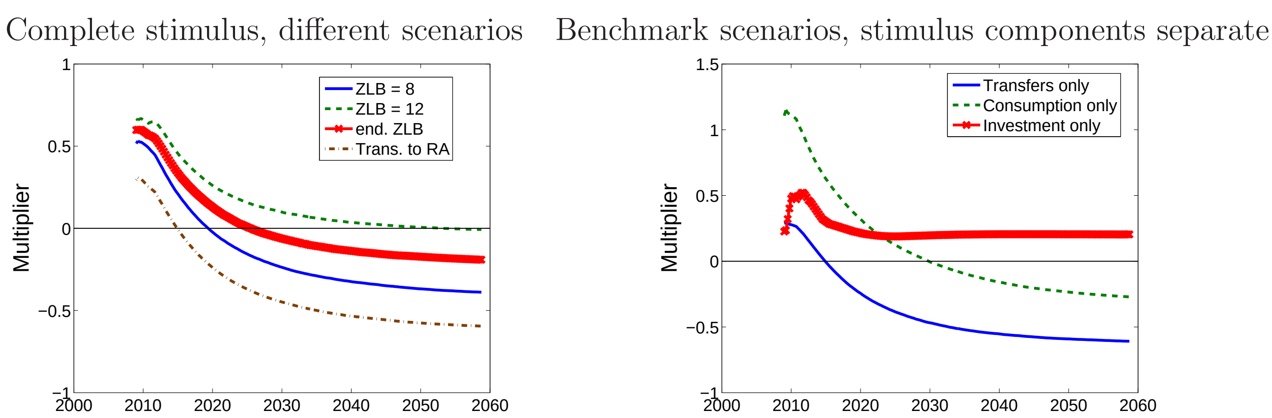

Figure 8: *Comparison of long-run multipliers: medians of posterior distributions.*

Table 3: Long run fiscal multipliers as $t \to \infty$: sensitivity

Scenario	5 percent	16.5 percent	median	83.5 percent	95 percent
Benchmark	-0.72	-0.61	-0.42	-0.22	-0.04
lump-sum taxes	0.34	0.44	0.60	0.78	0.94
consumption taxes	-0.48	-0.38	-0.20	-0.02	0.14
ZLB: 0 Quart.	-1.30	-1.18	-1.03	-0.87	-0.73
ZLB: 12 Quart.	-0.45	-0.31	-0.03	0.27	0.52
ZLB: Endogenous	-0.56	-0.43	-0.19	0.14	0.57
RoT=0.15	-0.91	-0.79	-0.63	-0.43	-0.26
RoT=0.35	-0.59	-0.44	-0.24	-0.04	0.18
Share transfers to RoT= 0.00	-0.86	-0.77	-0.65	-0.52	-0.42
Share transfers to RoT= 0.50	-0.64	-0.50	-0.24	0.03	0.29
Share transfers to RoT= 1.00	-0.50	-0.28	0.16	0.64	1.05
Priv. capital share=0.35	-1.13	-0.98	-0.76	-0.51	-0.27
price/wage-stickiness=0.10 × estim.	-0.96	-0.87	-0.75	-0.62	-0.52
price/wage-stickiness=0.50 × estim.	-0.78	-0.69	-0.58	-0.46	-0.37
price/wage-stickiness=1.15 × estim.	-0.91	-0.76	-0.56	-0.33	-0.12
Budget balance: $\psi_\tau = 0.025$	-0.70	-0.58	-0.40	-0.21	-0.04
Budget balance: $\psi_\tau = 0.05$	-0.77	-0.66	-0.49	-0.30	-0.13

Table 4: One-year fiscal multipliers: sensitivity

Scenario	5 percent	16.5 percent	median	83.5 percent	95 percent
Scenario	5 %	16.5 %	median	83.5 %	95 %
Benchmark	0.46	0.48	0.52	0.57	0.60
lump-sum taxes	0.55	0.57	0.61	0.66	0.70
consumption taxes	0.48	0.50	0.54	0.58	0.61
ZLB: 0 Quart.	0.17	0.20	0.23	0.27	0.30
ZLB: 12 Quart.	0.75	0.78	0.84	0.93	1.02
ZLB: Endogenous	0.51	0.54	0.60	0.69	0.78
RoT=0.15	0.39	0.42	0.46	0.49	0.52
RoT=0.35	0.47	0.54	0.59	0.64	0.69
Share transfers to RoT= 0.00	0.25	0.26	0.29	0.31	0.33
Share transfers to RoT= 0.50	0.65	0.69	0.75	0.81	0.85
Share transfers to RoT= 1.00	1.05	1.11	1.21	1.32	1.39
Priv. capital share=0.35	0.44	0.47	0.52	0.57	0.61
price/wage-stickiness=0.10 × estim.	0.05	0.07	0.11	0.14	0.16
price/wage-stickiness=0.50 × estim.	0.35	0.38	0.42	0.47	0.50
price/wage-stickiness=1.15 × estim.	0.44	0.46	0.50	0.53	0.56
Budget balance: $\psi_\tau = 0.025$	0.48	0.51	0.54	0.58	0.61
Budget balance: $\psi_\tau = 0.05$	0.43	0.46	0.49	0.53	0.56

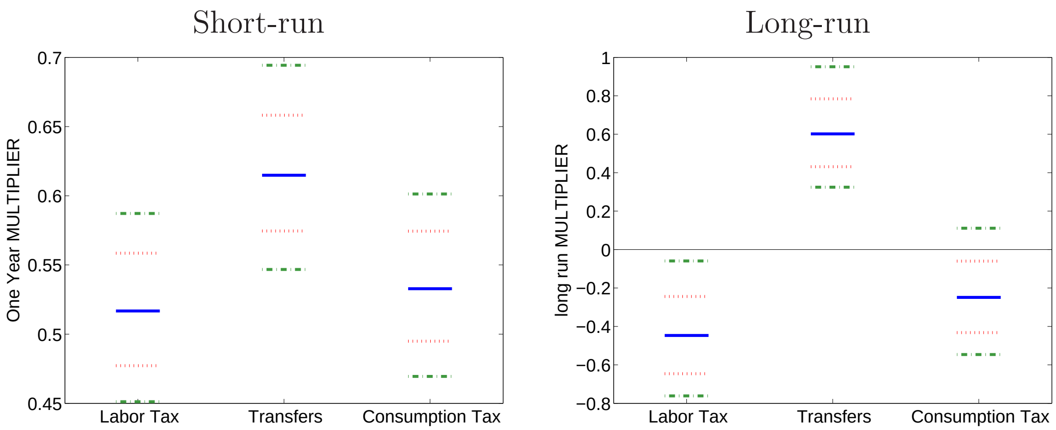

Figure 9: *Fiscal multipliers. Comparing distortionary labor taxes (benchmark) to consumption and lump-sum taxation.*

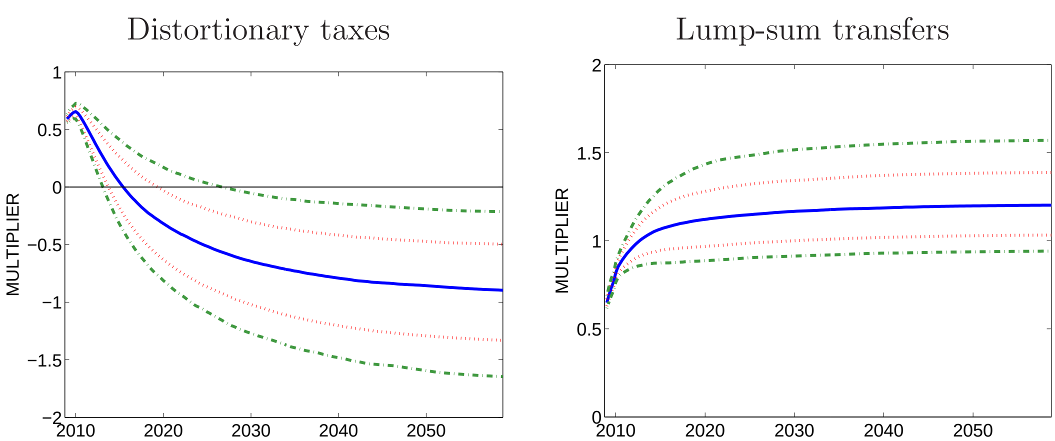

Figure 10: *Fiscal multipliers. Stimulus spend uniformly over first four quarters. Comparing distortionary labor taxes (benchmark) lump-sum taxation.*

Within the range of stable parameter values, increasing the speed at which the budget is balanced ψ_τ leads to lower multipliers as shown in tables 3 on page 25 and 4 on page 25.[3]

3.4 Sensitivity to the length zero lower bound

The literature has emphasized the sensitivity of fiscal multipliers to the zero lower bound, and to generating "fiscal stimulus", while the central bank is not changing its interest rates, see Eggertsson (2010) as well as Christiano et al. (2009). Our benchmark has been set to 8 quarters, implying that at the beginning of 2009, households anticipated the zero lower bound constraint to no longer bind at the beginning of 2011. That time horizon seems to have been extended meanwhile. However, it is hard to argue that this was anticipated two years ago. Nonetheless, we provide some experimentation here. Figure 11 provides that sensitivity analysis. It shows that when we endogenize the ZLB, the resulting multipliers are comparable since a successful stimulus shortens the ZLB and thereby reduces its effectiveness, even though the expected duration is longer. With an endogenous ZLB or a deterministic duration of twelve quarters, the long-run multipliers are centered at -0.19 and -0.03. [4]

3.5 Sensitivity to credit-constrained households

The "credit-contrained" or "rule-of-thumb" households are important in two respects. First, there is a sizeable portion of the population which violates Ricardian equivalence. Second, the split of transfers between these households and the unconstrained households leads to distributional and thereby aggregate consequences. It turns out that the second effect is more important than the first.

[3] Note that the habit formation prevents us from examining significantly higher speeds of budget balance. In the absence of habit formation, $\psi_\tau = 1$ is consistent with a locally unique equilibrium.

[4] Figure 17 in the technical appendix shows that with an endogenous ZLB only about 10% of all simulations results in an ZLB exceeding three years.

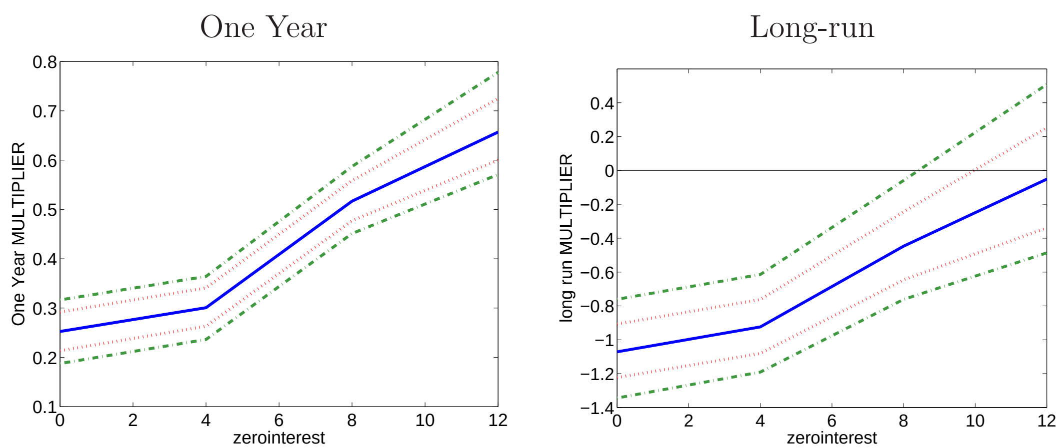

Figure 11: *Short-run and long-run fiscal multipliers: sensitivity to the length of the zero lower bound.*

The first row of table 5 shows the change in the fiscal multipliers, when we change the share of the population which is credit-constrained. In this experiment, the transfers are equally distributed across the population, i.e., the share of the transfers to the credit-constrained population equals the share of that population. This confounds two effects, however. The first is the mere rise in the share of credit-constrained households, but leaving their share of transfer receipts the same: this is shown in the second row of table 5. The second is the share of transfers received by the credit-constrained households. The third row of table 5 therefore varies the share of transfers received by these households, but keeping their share of the population constant at the benchmark value of 25 percent. While the second experiment has a rather modest impact on the short-run multiplier, the last experiment has a larger impact there. The long-run multipliers move considerably for both experiments. For example and for the last experiment, the median estimate, the long-run fiscal multiplier changes from -0.51 to 0.29, as that fraction is varied from zero to 100 percent.

One may wish to conclude from this that “fiscal stimulus” in the form of transfers to constrained agents may be quite effective in increasing output. That may be so. However, the modeling of the credit-constrained agents is done here with the simple short-cut of assuming that these agents do not keep savings and cannot borrow. For a more sophisticated exercise, the bounds to borrowing and savings should be endogenized, and may actually depend on the size of the government transfers. Furthermore, micro data can potentially be informative about the degree to which households are credit-constrained or refrain from saving. A deeper investigation into the details is called for, if "fiscal stimulus" programs in the future are to focus on this particular group.

3.6 Sensitivity to the composition of the stimulus

We departed from the original Smets-Wouters model in order to model the fiscal stimulus in more detail by being able to distinguish between money spend on government transfers, consumption, and investment. In our model, each component has a different impact on the economy. As discussed above, who receives the transfers is an important question. Since constrained households spend all their income, transfers to them are closer to direct government spending. Discretionary government investment increases private sector productivity, but may also crowd out optimal government investment, thereby effectively lowering the size of the long-term debt burden faced by households. The right panel in Figure 8 shows that in our benchmark model, the government investment component contributes to a positive multiplier, whereas the government consumption and transfer components lower the overall multiplier below zero.

3.7 Sensitivity to the capital share

The estimated capital share is around 0.24 rather than 0.35, as often used in the calibration literature, see Cooley and Prescott (1995). The comparisons in figure 12 reveal, that the results are quite sensitive to this parameter, which in our model crucially also governs the tax base for labor taxes.

Table 5: *Short-run and long-run fiscal multipliers: sensitivity to credit-constrained fraction of the population and their share of transfers. First line: all households receive the same amount of transfers, i.e. fraction of constrained households and total transfers rise together. Second line, only the fraction of constrained household rises. Third line: only the share of transfers going to constrained households rises.*

	one year multiplier			long-run multiplier		
Transfers = RoT fraction =	0.10	0.25	0.40	0.10	0.25	0.40
Const. transfers/household:	0.33	0.54	0.82	-0.62	-0.31	0.12
Transfers =0.25, RoT fraction =	0.10	0.25	0.40	0.10	0.25	0.40
Fixed absolute transfers	0.45	0.54	0.66	-0.53	-0.31	-0.03
RoT Share =0.25, Transfers =	0	0.25	1.00	0	0.25	1.00
Fixed population share	0.31	0.54	1.23	-0.51	-0.31	0.29

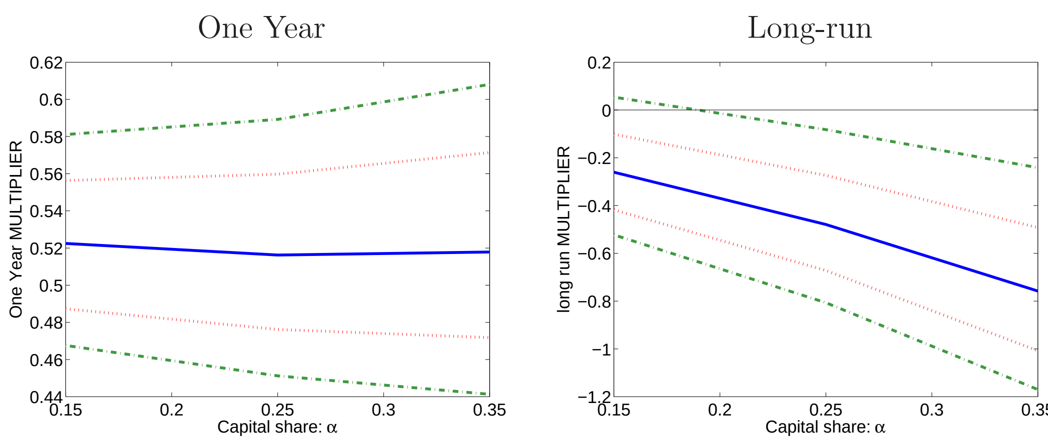

Figure 12: *Short-run and long-run fiscal multipliers: sensitivity to the capital share.*

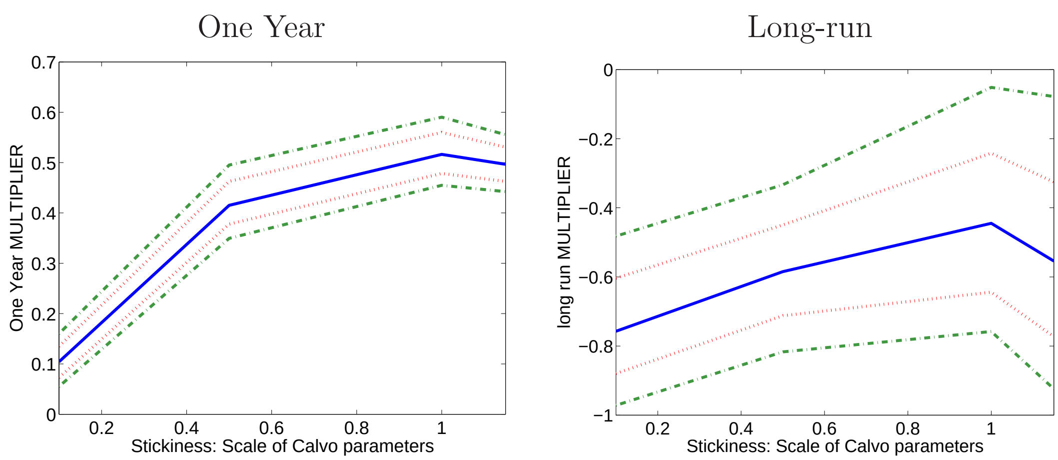

Figure 13: *Short-run and long-run fiscal multipliers: sensitivity to price and wage stickiness.*

3.8 Sensitivity to price and wage stickiness

Finally, it may be interesting to document the impact of the price and wage stickiness on the fiscal multipliers: this is done in figure 13. Note that the median estimates are $\zeta_p = 0.81$ and $\zeta_w = 0.83$ for the Calvo parameter for prices and wages. In the figure we consider values of 10% to 115% of these median estimates, scaling both parameters proportionately. While the figure mostly shows an increase in the multiplier with increasingly sticky prices and wages, this is no longer true when prices and wages get very sticky. Essentially, at that point, future inflation due to the zero lower bound no longer induces upward pressure on prices and wages, thereby lessening the impact of fiscal stimulus. Reducing the overall stickiness leads to much larger inflation responses (cf. Figure 18 in the technical appendix) and may therefore be more realistic than the estimated stickiness parameters.

3.9 Welfare effects

Both the long-run and short-run multiplier are silent on welfare implications of the stimulus package. If the output increase is driven by a disproportionate increase in hours worked, consumers are likely to be worse off even if the multiplier is large and positive.

Given perfect foresight of the stimulus plan, we can calculate the compensating variation in lifetime consumption along the balanced growth path which makes consumers indifferent between ARRA and the modified historic growth path. Let $\Gamma_i \times 100$ be the percentage of consumption without the stimulus, which consumers of type $i, i \in \{RA, RoT\}$ would be willing to give up each period to have the ARRA in place. We provide an explicit formula in a technical appendix. The expressions amount to calculating the net present value of future utility changes. The discount rate for each consumer type enters here in a crucial manner.

Two caveats complicate the welfare calculation. First, the calculation is numerically challenging because at our estimates the effective discount factor $\beta_{RA}\mu^{1-\sigma}$ is close to unity so that convergence is slow. Numerical error is important to address because we are relating the cost of an intervention over about ten years to lifetime consumption so that errors of a small magnitude might be important for the results. Second, our parameter estimates are only directly applicable to unconstrained households, whereas social welfare depends on both types of households. If constrained households are sufficiently impatient and receive a high weight in the social welfare function, the results presented above could be overturned because constrained agent might value the initial consumption increase enough. The calibration of the discount rate for the constrained households is a challenge, however. Lawrance (1991) finds that rates of time preference vary by 7 percent on an annual basis across rich and poor households. Using data on individual choices between lump-sum and annuity payments, Warner and Pleeter (2001) find differences in annual rates of time preference of up to 30 percent, varying by various characteristics. We therefore consider two discount factors for the RoT agents per adding 7% as well as 30% to the annual discount factor of the unconstrained agents, i.e.

$$1/\beta_{RoT} \in \{1/\beta_{RA} + 0.07/4, 1/\beta_{RA} + 0.3/4\}$$

noting that our model is for quarterly data. We also vary over a wider range.

For the unconstrained households, the welfare effects are small but significantly negative according to our calculations in Table 6. The median effect on constrained agents is -0.02 percent, independent of the length of the ZLB with the 90 percent posterior confidence intervals ranging from -0.04 percent to -0.01 percent. The small magnitude is not surprising given that small deviations from the optimum have small effects on welfare of unconstrained agents. Unconstrained agents suffer from an increase in hours worked and for most parameter values considered also from a drop in consumption, explaining the negative sign.

The effect on constrained agents is ambiguous, as lines two and three in table 6 show. If the discount factgor of the RoT agents is just 7% higher than that of the unconstrained agents, the welfare effect is negative, but it is positive, if their discount factor is 30% higher. Figure 14 shows the results for a range of discount factor increases, compared to the unconstrained agents. Beyond the threshold of adding 10%, a higher rate of time preference leads to a more positive evaluation of the stimulus.

4 Conclusions

We have quantified the size, uncertainty and sensitivity of fiscal multipliers in response to the American Recovery and Reinvestment Act (ARRA) of 2009. To that end, we have extended the benchmark Smets and Wouters (2007) New Keynesian model, allowing for credit-constrained households, a central bank constrained by the zero lower bound, government capital and a government raising taxes with distortionary taxation. We have distinguished

Table 6: *Welfare effects ($\Gamma \times 100$) of stimulus: Lifetime-consumption equivalent of compensating variation for stimulus. Posterior median (90% posterior confidence interval).*

Scenario	8 quarters ZLB	0 quarters ZLB	12 quarters ZLB
Unconstrained agents	-0.02(-0.04,-0.02)	-0.02(-0.04,-0.01)	-0.02(-0.03,-0.02)
RoT, 7% higher annual DF	-0.08(-0.14,-0.02)	-0.15(-0.22,-0.09)	-0.09(-0.17,0.01)
RoT, 30% higher annual DF	0.59(0.35,0.91)	0.44(0.21,0.63)	0.54(0.20,0.92)

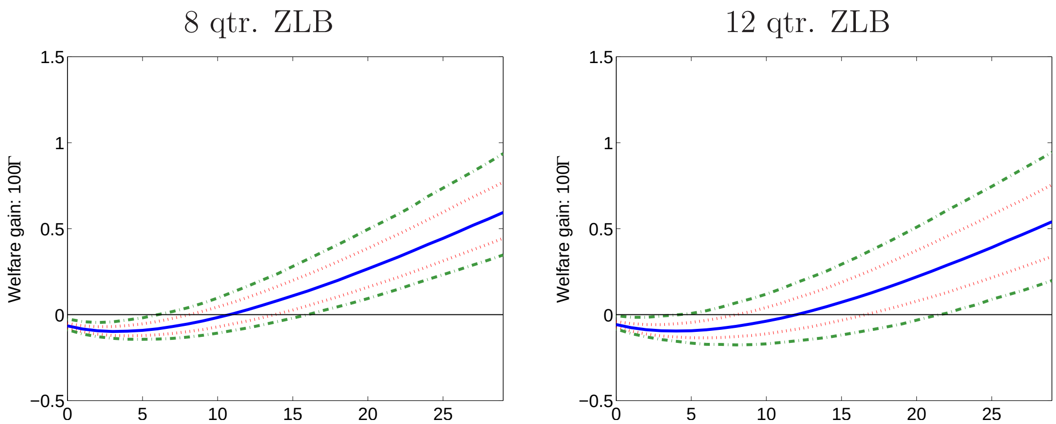

Figure 14: *Long-run welfare gains from stimulus: 8 and 12 qtr. ZLB, varying annual rate of time preference compared to unconstrained agents.*

between short-run and long-run multipliers. For a benchmark parameterization, we find modestly positive short-run multipliers with a posterior mean of 0.52 and modestly negative long-run multipliers centered around -0.42. The multiplier is particularly sensitive to the type of taxes used to finance the ARRA, is sensitive to the fraction of transfers given to credit-constrained households, is sensitive to the anticipated length of the zero lower bound, is sensitive to the capital share and is nonlinear in the degree of price and wage stickiness. Reasonable specifications are consistent with substantially negative short-run multipliers within a short time frame. Furthermore, the policy intervention may lower the welfare of agents in the economy: unconstrained agents would have a higher lifetime utility without the ARRA and even impatient constrained agents may be better off without the intervention because of the disutility of hours worked during the expansion and lower consumption in the transition to the long-run offset short-run gains from higher consumption.

References

Barro, Robert J. and Xavier Sala-i Martin (1992). "Public Finance in Models of Economic Growth". *The Review of Economic Studies*, 59(4):645–661.

Bernstein, Jared and Christina Romer (2009). "The Job Impact of the American Recovery and Reinvestment Act". *press release, January 9, Office of the Vice President Elect*.

Buiter, Willem (2009). "The unfortunate uselessness of most 'state of the art' academic monetary economics". *www.voxeu.org, March 6, 2009*.

Christiano, Lawrence, Martin Eichenbaum, and Sergio Rebelo (2009). "When is the government spending multiplier large?" *draft, Northwestern University*.

Coenen, G., C. J. Erceg, C. Freedman, D. Furceri, M. Kumhof, R. Lalonde, J. Linde, D. Muir, S. Mursula, A. Mourougane, J. Roberts, W. Roeger, S. Snudden, M. Trabandt, and J. in't Veld (2010). "Effects of Fiscal Stimulus in Structural Models". IMF Working Papers 10/73, International Monetary Fund.

Cogan, John F., Tobias Cwik, John B. Taylor, and Volker Wieland (2010). "New Keynesian versus old Keynesian government spending multipliers". *Journal of Economic Dynamics and Control*, 34(3):281 – 295.

Congressional Budget Office (2009). "Cost estimate for conference agreement for H.R.1." *http://www.cbo.gov/ftpdocs/99xx/doc9989/hr1conference.pdf*.

Cooley, Thomas F. and Edward C. Prescott (1995). "Economic Growth and Business Cycles". In Cooley, Thomas F. (ed.) "Frontiers of Business Cycle Research", chap. 1, 1–38. Princeton University Press.

Correia, Isabel, Emmanuel Fahri, Juan Pablo Nicolini, and Pedro Teles (2010). "Unconventional Fiscal Policy at the Zero Bound". *draft, Banco de Portugal*.

Davig, Troy and Eric M. Leeper (2009). "Monetary-Fiscal Policy Interactions and Fiscal Stimulus". NBER Working Papers 15133, National Bureau of Economic Research.

Drautzburg, Thorsten and Harald Uhlig (2011). "Zero Lower Bound Dynamics". *draft, University of Chicago*.

Eggertsson, Gauti B. (2010). "The Paradox of Toil". *draft, Federal Reserve Bank of New York*.

Eichenbaum, Martin and Jonas D.M. Fisher (2007). "Estimating the frequency of price re-optimization in Calvo-style models". *Journal of Monetary Economics*, 54(7):2032–2047.

Erceg, Christopher J. and Jesper Linde (2010). "Is There a Fiscal Free Lunch in a Liquidity Trap?" CEPR Discussion Paper 7624.

Hall, Robert (2010). "The Long Slump". *draft, Stanford University*.

Ilut, Cosmin and Martin Schneider (2011). "Ambiguous Business Cycles". *draft, Duke University*.

Juillard, Michel (1996). "Dynare: A program for the resolution and simulation of dynamic models with forward variables through the use of a relaxation algorithm". *CEPREMAP, Couverture Orange 9602*.

Krugman, Paul (2009). "How Did Economists Get It So Wrong?" *New York Times, September 2, 2009*.

Lawrance, Emily C (1991). "Poverty and the Rate of Time Preference: Evidence from Panel Data". *Journal of Political Economy*, 99(1):54–77.

Leeper, Eric M., Michael Plante, and Nora Traum (2010). "Dynamics of fiscal financing in the United States". *Journal of Econometrics*, 156(2):304–321.

Leeper, Eric M., Nora Traum, and Todd B. Walker (2011). "Clearing up the Fiscal Multiplier Morass: A Bayesian Perspective". *Draft, University of Indiana*.

Leeper, Eric M., Todd B. Walker, and Shu-Chun Susan Yang (2009). "Government Investment and Fiscal Stimulus in the Short and Long Runs". NBER Working Papers 15153, National Bureau of Economic Research.

Smets, Frank and Raf Wouters (2003). "An Estimated Dynamic Stochastic General Equilibrium Model of the Euro Area". *Journal of the European Economic Association*, 1(5):1123–1175.

——— (2007). "Shocks and Frictions in US Business Cycles: A Bayesian DSGE Approach". *The American Economic Review*, 97(3):586–606.

Spilimbergo, Antonio, Steve Symansky, Olivier Blanchard, and Carlo Cottarelli (2008). "Fiscal Policy for the Crisis". *IMF Staff Position Note SPN/08/01*.

Trabandt, Mathias and Harald Uhlig (2010). "How far are we from the slippery slope? The Laffer Curve revisited." *draft, University of Chicago*.

Uhlig, Harald (2010a). "Economics and Reality". Working Paper 16416, National Bureau of Economic Research.

——— (2010b). "Some Fiscal Calculus". *American Economic Review, Papers and Proceedings*, 100:30–34.

Warner, John T. and Saul Pleeter (2001). "The Personal Discount Rate: Evidence from Military Downsizing Programs". *American Economic Review*, 91(1):33–53.

5 Appendix

5.1 Data

The different series come from the NIPA tables, the FRED 2 database and the Bureau of Labor Statistics (BLS) database. Federal debt data is taken from Dallas Fed database. Nominal series for wages, consumption, government and private investment deflated with general GDP deflator.

Generally we follow Smets and Wouters (2007) when creating our dataset with the following exemptions: we use civilian non-institutionalized population throughout, although the series is not seasonally adjusted before 1976. The base year for real GDP is 2005 instead of 1996. We include durables consumption in investment instead of consumption. Using the same definition, all series but real wages exhibit a correlation of almost 100 percent across the two datasets. For the change in real wages, the correlation is 0.9. Including durables consumption in investment causes the correlation for the investment series to drop to 0.70 and for consumption to 0.78.

Since no data for the Corporate-Treasury bond yield spread is available before 1953:1 we set it to zero for the missing periods. We use the secondary market rate for 3-month TBill before 1954:3 as the FFR is not available.

The categorization of the various stimulus components is shown in detail in tables 9, 10 and 11 in the technical appendix. As source, we have used Congressional Budget Office (2009), specifically "Table 2: Estimated cost of the conference agreement for H.R. 1, the American Recovery and Reinvestment Act of 2009, as posted on the website of the House Committee on Rules." The annual time path for these expenditures is taken from Congressional Budget Office (2009) and the annual sum for each component is split across quarters in proportion to the aggregate series in Cogan et al. (2010).

5.2 Estimation

Tables 7 and 8 contain the results from estimating our model, using Dynare and a Bayesian prior.

Table 7: *Estimation, part 1. The calibrated government investment-to-GDP ratio as well as the estimated growth trend μ implies a government share in production of $\zeta = 2.30$ percent.*

	Prior	Prior mean (s.d.)	SW Model 66:1-08:4	Our Model 49:2-08:4
Adj. cost $S''(\mu)$	norm	4.000 (1.500)	5.93 (1.1)	4.51 (0.78)
Risk aversion σ	norm	1.500 (0.375)	1.42 (0.11)	1.17 (0.08)
Habit h	beta	0.700 (0.100)	0.7 (0.04)	0.85 (0.02)
Calvo wage ζ_w	beta	0.500 (0.100)	0.77 (0.05)	0.83 (0.03)
Inv. labor sup. ela. ν	norm	2.000 (0.750)	1.96 (0.54)	2.16 (0.51)
Calvo prices ζ_p	beta	0.500 (0.100)	0.69 (0.05)	0.81 (0.03)
Wage indexation ι_w	beta	0.500 (0.150)	0.62 (0.1)	0.41 (0.08)
Price indexation ι_p	beta	0.500 (0.150)	0.26 (0.08)	0.28 (0.07)
Capacity util.	beta	0.500 (0.150)	0.59 (0.1)	0.43 (0.07)
$1+\frac{\text{Fix. cost}}{Y} = 1 + \lambda_p$	norm	1.250 (0.125)	1.64 (0.08)	1.94 (0.05)
Taylor rule infl. ψ_1	norm	1.500 (0.250)	2 (0.17)	1.63 (0.18)
same, smoothing ρ_R	beta	0.750 (0.100)	0.82 (0.02)	0.92 (0.02)
same, LR gap ψ_2	norm	0.125 (0.050)	0.09 (0.02)	0.13 (0.03)
same, SR gap ψ_3	norm	0.125 (0.050)	0.24 (0.03)	0.2 (0.02)
Mean inflation (data)	gamm	0.625 (0.100)	0.76 (0.09)	0.58 (0.08)
100×time pref.	gamm	0.250 (0.100)	0.16 (0.05)	0.12 (0.04)
Mean hours (data)	norm	0.000 (2.000)	1.07 (0.95)	0.04 (0.69)
Trend $(\mu - 1) * 100$	norm	0.400 (0.100)	0.43 (0.02)	0.48 (0.01)
Capital share α	norm	0.300 (0.050)	0.19 (0.02)	0.24 (0.01)
Gov. adj. cost $S''_g(\mu)$	norm	0.000 (0.500)	n/a	7.11 (1.09)
Budget bal speed $\frac{\psi_\tau - 0.025}{0.175}$	beta	0.25 (0.1637)	n/a	0.05 (0.04)
Mean gov. debt	norm	0.000 (0.500)	n/a	-0.16 (0.51)
Mean bond spread	gamm	0.500 (0.100)	n/a	0.47 (0.04)

Table 8: *Estimation, part 2*

	Prior	Prior mean (s.d.)	SW Model 66:1-08:4	Our Model 49:2-08:4
s.d. tech.	invg	0.100 (2.000)	0.46 (0.03)	0.47 (0.02)
AR(1) tech.	beta	0.500 (0.200)	0.95 (0.01)	0.95 (0.01)
s.d. bond	invg	0.100 (2.000)	0.24 (0.03)	0.95 (0.04)
AR(1) bond ρ_q	beta	0.500 (0.200)	0.27 (0.1)	0.67 (0.03)
s.d. gov't	invg	0.100 (2.000)	0.54 (0.03)	0.36 (0.02)
AR(1) gov't	beta	0.500 (0.200)	0.98 (0.01)	0.98 (0.01)
Cov(gov't, tech.)	norm	0.500 (0.250)	0.53 (0.09)	0.3 (0.04)
s.d. inv. price	invg	0.100 (2.000)	0.43 (0.04)	1.25 (0.1)
AR(1) inv. price	beta	0.500 (0.200)	0.73 (0.06)	0.56 (0.06)
s.d. mon. pol.	invg	0.100 (2.000)	0.24 (0.02)	0.22 (0.01)
AR(1) mon. pol.	beta	0.500 (0.200)	0.16 (0.07)	0.22 (0.05)
s.d. goods m-up	invg	0.100 (2.000)	0.14 (0.01)	0.32 (0.02)
AR(1) goods m-up	beta	0.500 (0.200)	0.89 (0.04)	0.91 (0.05)
MA(1) goods m-up	beta	0.500 (0.200)	0.73 (0.08)	0.96 (0.02)
s.d. wage m-up	invg	0.100 (2.000)	0.26 (0.02)	0.23 (0.02)
AR(1) wage m-up	beta	0.500 (0.200)	0.97 (0.01)	0.97 (0.01)
MA(1) wage m-up	beta	0.500 (0.200)	0.91 (0.03)	0.92 (0.02)
s.d. Tax shock	invg	0.100 (2.000)	n/a	1.44 (0.08)
AR(1) tax shock	beta	0.500 (0.200)	n/a	0.98 (0.01)
s.d. gov. inv. price	invg	0.100 (2.000)	n/a	0.79 (0.08)
AR(1) gov. inv. price	beta	0.500 (0.200)	n/a	0.97 (0.01)
s.d. bond spread	invg	0.100 (2.000)	n/a	0.08 (0)
AR(1) bond spread	beta	0.500 (0.200)	n/a	0.91 (0.02)

TECHNICAL APPENDIX – NOT FOR PUBLICATION

6 Additional results

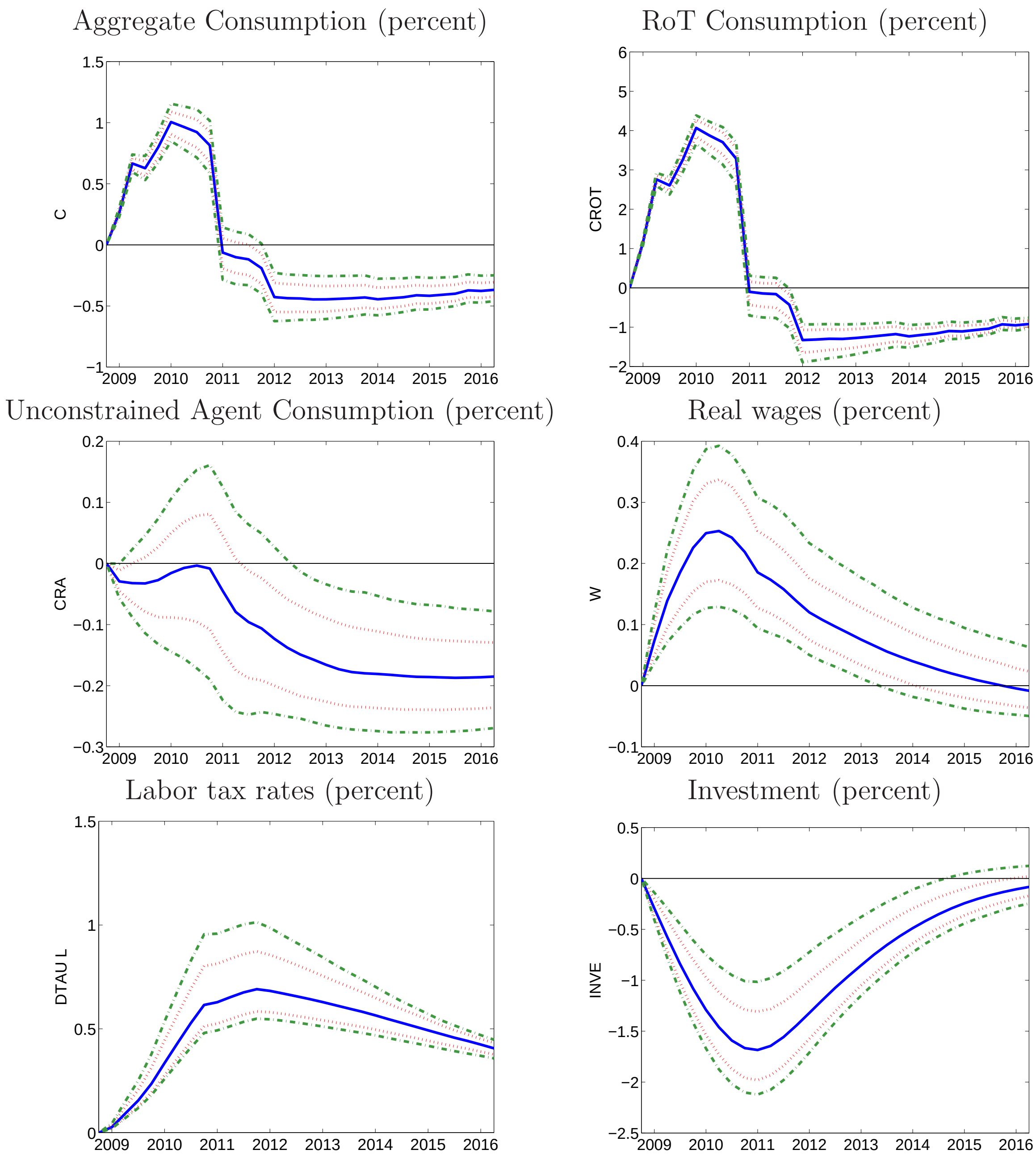

Figure 15: Benchmark impact of ARRA: Consumption, Investment, Tax rates, and Real Wages.

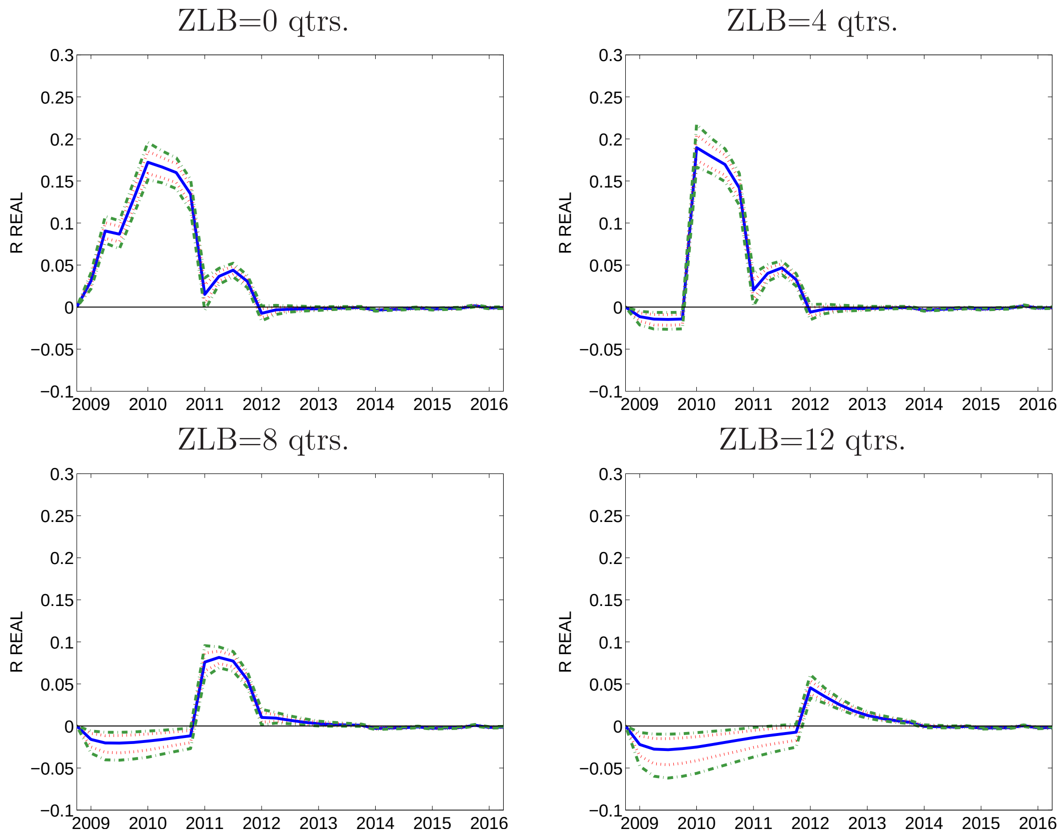

Figure 16: Impact of ARRA on real interest rates for varying ZLB length.

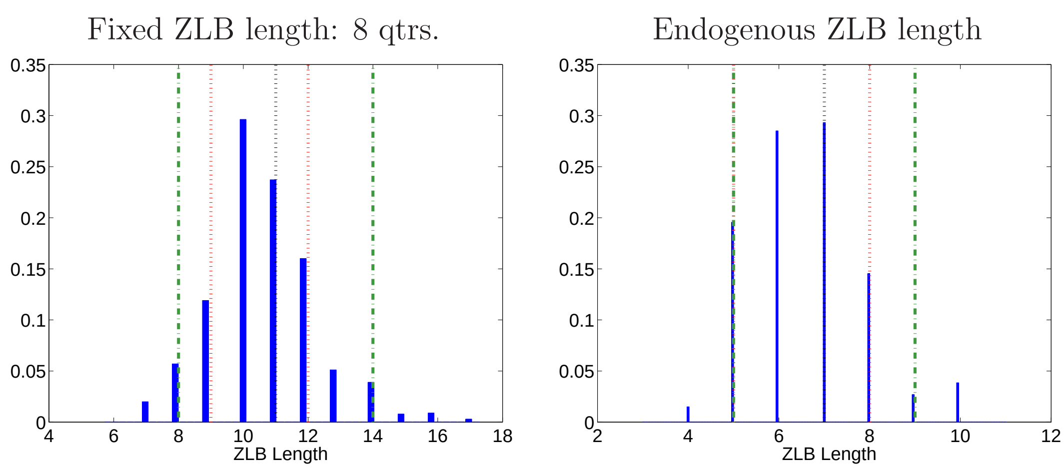

Figure 17: *ZLB duration implied by Taylor rule.*

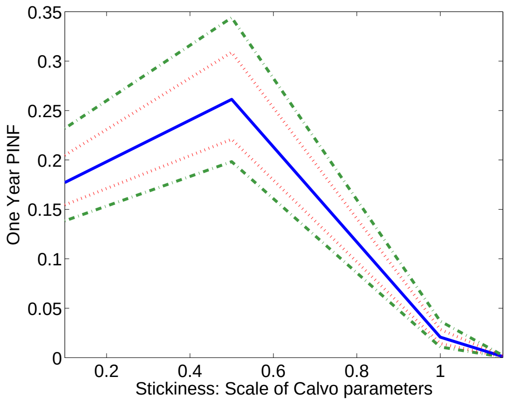

Figure 18: *Inflation response: sensitivity to price and wage stickiness.*

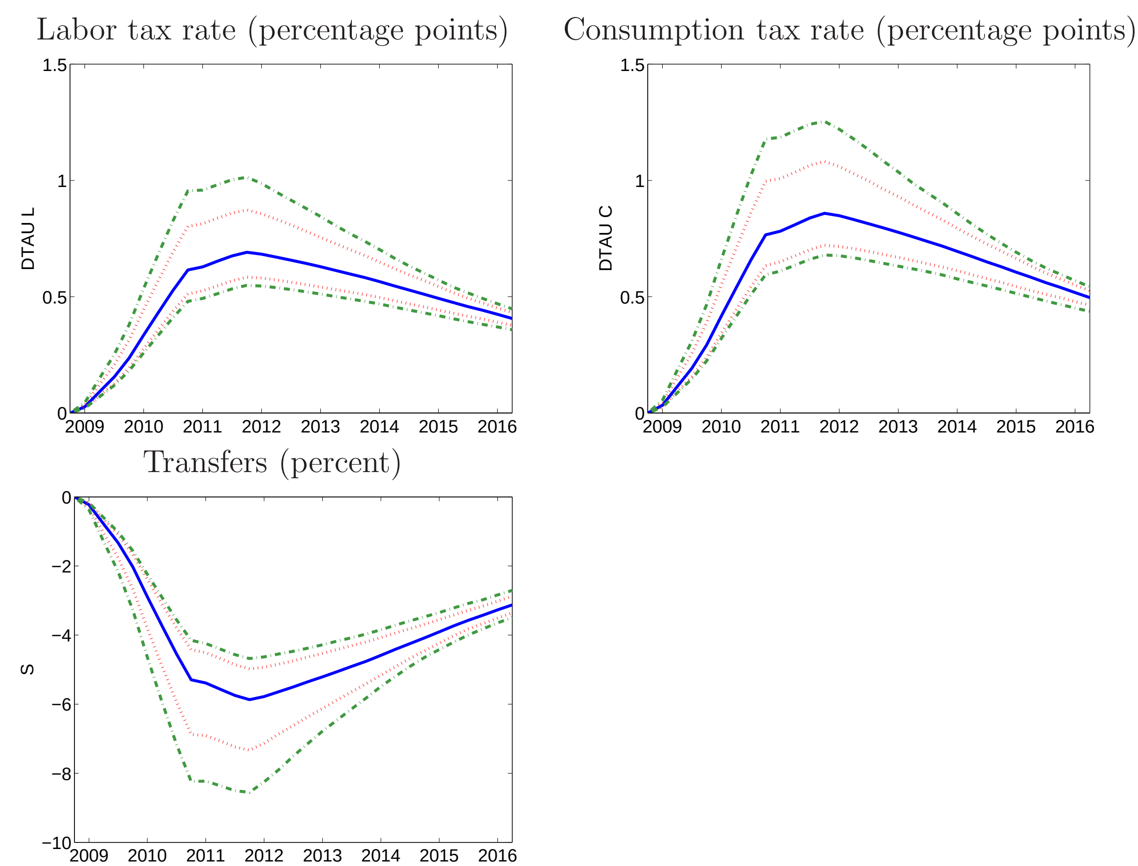

Figure 19: Changes in tax rates and lump-sum transfers due to stimulus.

7 Categorizing stimulus spending

Table 9: Categorizing the stimulus – Government Consumption

Item	Amount (bn USD)	Share
Dept. of Defense	4.53	0.59
Employment and Training	4.31	0.56
Legislative Branch	0.03	0
National Coordinator for Health Information Technology	1.98	0.26
National Institute of Health	9.74	1.26
Other Agriculture, Food, FDA	3.94	0.51
Other Commerce, Justice, Science	5.36	0.69
Other Dpt. of Education	2.12	0.28
Other Dpt. of Health and Human Services	9.81	1.27
Other Financial Services and gen. Govt	1.31	0.17
Other Interior and Environment	4.76	0.62
Special education	12.2	1.58
State and local law enforcement	2.77	0.36
State Fiscal Relief	90.04	11.68
State fiscal stabilization fund	53.6	6.95
State, foreign operations, and related programs	0.6	0.08
Other	2.55	0.33
Consumption	209.64	27.2

Table 10: Categorizing the stimulus – Government Investment

Item	Amount (bn USD)	Share
Broadband Technology opportunities program	4.7	0.61
Clean Water and Drinking Water State Revolving Fund	5.79	0.75
Corps of Engineers	4.6	0.6
Distance Learning, Telemedicine, and Broadband Program	1.93	0.25
Energy Efficiency and Renewable Energy	16.7	2.17
Federal Buildings Fund	5.4	0.7
Health Information Technology	17.56	2.28
Highway construction	27.5	3.57
Innovative Technology Loan Guarantee	6	0.78
NSF	2.99	0.39
Other Energy	22.38	2.9
Other transportation	20.56	2.67
Investment	136.09	17.66

Table 11: Categorizing the stimulus – Transfers

Item	Amount (bn USD)	Share
Assistance for the unemployed	0.88	0.11
Economic Recovery Programs, TANF, Child support	18.04	2.34
Health Insurance Assistance	25.07	3.25
Health Insurance Assistance	-0.39	-0.05
Low Income Housing Program	0.14	0.02
Military Construction and Veteran Affairs	4.25	0.55
Other housing assistance	9	1.17
Other Tax Provisions	4.81	0.62
Public housing capital fund	4	0.52
Refundable Tax Credits	68.96	8.95
Student financial assistance	16.56	2.15
Supplemental Nutrition Assistance Program	19.99	2.59
Tax Provisions	214.56	27.84
Unemployment Compensation	39.23	5.09
Transfers and Tax cuts	425.09	55.15

8 Backing out the unemployment rate

To back out the model implications for the unemployment rate, we regress the time series for hours worked used for the model estimation on the average quarterly unemployment rate. Table 12 shows the regression results. Figure 20 displays the actual and fitted unemployment rate. Multiplying hours worked on the OLS regression coefficient gives the implied change in the unemployment rate.

Table 12: *OLS regression estimates of unemployment rate on the model-implied employment measure.*

	Constant	Employment (lab_t)	R^2
Unemployment Rate (UR_t)	5.60	-0.46	0.77
	(5.51, 5.69)	(-0.49, -0.43)	

Sample period: 1948:1 – 2008:4. Unemployment rate is the arithmetic mean over the quarter. 95 percent confidence intervals in parentheses. Labor input in the model is measured as $\text{lab}_t \equiv \log \frac{\text{Avg. hours}_t \times \text{Employment}_t}{\text{Population}_t} - \text{mean}$. 95 percent OLS confidence intervals in parentheses.

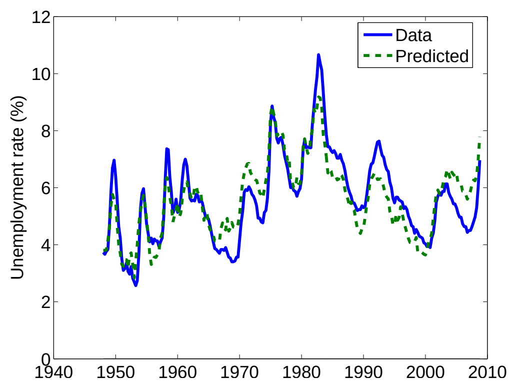

Figure 20: *Regression of quarterly unemployment rate on the model-implied employment measure: Actual vs. predicted unemployment rate.*

9 Model Appendix

Apart from the model extensions due to the introduction of government capital, rule of thumb consumers, and distortionary taxation, the following model appendix follows mostly the appendix of Smets and Wouters (2007), with minor changes to unify the notation.

9.1 Production

Final goods are produced in a competitive final goods sector which uses differentiated intermediate inputs, supplied by monopolistic intermediate producers.

9.1.1 Final goods producers

The representative final goods producer maximizes profits by choosing intermediate inputs $Y_t(i), i \in [0,1]$, subject to a production technology which generalizes a CES production function: Objective:

$$\max_{Y_t, Y_t(i)} P_t Y_t - \int_0^1 P_t(i) Y_t(i) di \quad \text{s.t.} \quad \int_0^1 G\left(\frac{Y_t(i)}{Y_t}; \tilde{\epsilon}_t^{\lambda,p}\right) di = 1. \tag{9.1}$$

$G(\cdot)$ is the **?** aggregator, which generalizes CES demand by allowing the elasticity of demand to increase with relative prices: $G' > 0, \quad G'' < 0, \quad G(1; \tilde{\epsilon}_t^{\lambda,p}) = 1$. $\tilde{\epsilon}_t^{\lambda,p}$ is a shock to the production technology which changes the elasticity of substitution.

Denote the Lagrange multiplier on the constraint by Ξ_t^f. If a positive solution to equation (9.1) exists it satisfies the following conditions

$$[Y_t] \quad P_t = \Xi_t^f \frac{1}{Y_t} \int_0^1 G'\left(\frac{Y_t(i)}{Y_t}; \tilde{\epsilon}_t^{\lambda,p}\right) \frac{Y_t(i)}{Y_t} di,$$

$$[Y_t(i)] \quad P_t(i) = \Xi_t^f \frac{1}{Y_t} G'\left(\frac{Y_t(i)}{Y_t}; \tilde{\epsilon}_t^{\lambda,p}\right).$$

From these two equations, we obtain an expression for the aggregate price index and intermediate inputs. The price index is given by:

$$P_t = \int_0^1 \frac{Y_t(i)}{Y_t} P_t(i) di. \tag{9.2}$$

Solving for intermediate input demands:

$$Y_t(i) = Y_t G'^{-1}\left(\frac{P_t(i)Y_t}{\Xi_t^f}\right) = Y_t G'^{-1}\left(\frac{P_t(i)}{P_t}\int_0^1 G'\left(\frac{Y_t(j)}{Y_t}; \tilde{\epsilon}_t^{\lambda,p}\right)\frac{Y_t(j)}{Y_t}dj\right). \tag{9.3}$$

For future reference, note that the relative demand curves $\mathrm{y}_t(i) \equiv \frac{Y_t(i)}{Y_t}$ are downward-sloping in the relative price $\frac{P_t(i)}{P_t}$ with an decreasing elasticity as the relative quantity increases. For simplicity, the dependence of the $G(\cdot)$ aggregator on the shock $\tilde{\epsilon}_t^{\lambda,p}$ is suppressed:

$$\eta_p(\mathrm{y}_t(i)) \equiv -\frac{P_t(i)}{Y_t(i)}\frac{d\mathrm{y}_t(i)}{dP_t(i)}\Big|_{dY_t = d\Xi_t^f = 0} = -\frac{G'(\mathrm{y}_t(i))}{\mathrm{y}_t(i)G''(\mathrm{y}_t(i))} \tag{9.4}$$

$$\begin{aligned}
\hat{\eta}_p(\mathrm{y}_t(i)) &\equiv \frac{P_t(i)}{\eta_p(\mathrm{y}_t(i))}\frac{d\eta_p(\mathrm{y}_t(i))}{dP_t(i)} = 1 + \eta_p + \eta_p \frac{G'''(\mathrm{y}_t(i))}{G''(\mathrm{y}_t(i))}\mathrm{y}_t(i) \\
&= 1 + \eta_p(\mathrm{y}_t(i))\left(2 + \frac{G'''(\mathrm{y}_t(i))}{G''(\mathrm{y}_t(i))}\mathrm{y}_t(i) - 1\right) \\
&= 1 + \eta_p(\mathrm{y}_t(i))\left(\frac{2 + \frac{G'''(\mathrm{y}_t(i))}{G''(\mathrm{y}_t(i))}\mathrm{y}_t(i)}{1 - \eta_p(\mathrm{y}_t(i))^{-1}}(1 - \eta_p(\mathrm{y}_t(i))^{-1}) - 1\right) \\
&\equiv 1 + \frac{1 + \lambda^p(\mathrm{y}_t(i))}{\lambda^p(\mathrm{y}_t(i))}\left(\frac{1}{[1 + \lambda^p(\mathrm{y}_t(i))]A_p(\mathrm{y}_t(i))} - 1\right),
\end{aligned} \tag{9.5}$$

where the last line defines the mark-up $\lambda_t^p(\mathrm{y}_t(i)) \equiv \frac{1}{\eta_p(\mathrm{y}_t(i)) - 1}$ and $A_p(\mathrm{y}_t(i)) \equiv \frac{\lambda^p(\mathrm{y}_t(i))}{2 + \frac{G'''(\mathrm{y}_t(i))}{G''(\mathrm{y}_t(i))}\mathrm{y}_t(i)}$. The model will be parameterized in terms of $\hat{\epsilon}(1)$, the change in the own price elasticity of demand along the balanced growth path. To that end, it is convenient to solve for A_p in terms of the mark-up and the $\hat{\epsilon}$:

$$A_p(\mathrm{y}) = \frac{1}{\lambda^p(\mathrm{y})\hat{\eta}_p(\mathrm{y}) + 1}. \tag{9.6}$$

Finally, note that in the Dixit-Stiglitz case $G(\mathrm{y}) = \mathrm{y}^{\frac{1}{1+\lambda^p}}$ so that the elasticity of demand is constant at $\eta_p(\mathrm{y}) = \frac{1}{\lambda^p} + 1 \forall \mathrm{y}$ and consequently $\hat{\eta}_p = 0$.

9.1.2 Intermediate goods producers

There is a unit mass of intermediate producers, indexed by $i \in [0, 1]$. Each producer is the monopolistic supplier of good i. They rent capital services K_t^{eff} and hire labor n_t to maximize profits intertemporally, taking as given rental rates R_t^k and wages W_t. Given a Calvo-style pricing friction, their profit-maximization problem is dynamic.

Production is subject to a fixed cost and the gross product is produced using a Cobb-Douglas technology at the firm level. Government capital K_t^g increases total factor productivity in each firm, but is subject to a congestion effect as overall production increases, similar to the congestion effects in the AK model in **?**. Firms fail to internalize the effect of their decisions on public sector productivity. Net output is therefore given by:

$$Y_t(i) = \tilde{\epsilon}_t^a \left(\frac{K_{t-1}^g}{\int_0^1 Y_t(j)dj + \Phi\mu^t} \right)^{\frac{\zeta}{1-\zeta}} K_t^{eff}(i)^{\alpha} [\mu^t n_t(i)]^{1-\alpha} - \mu^t \Phi, \qquad (9.7)$$

where $\Phi\mu^t$ represent fixed costs which grow at the rate of labor augmenting technical progress and $K_t(i)^{eff}$ denotes the capital services rented by firm i. $\tilde{\epsilon}_t^a$ denotes a stationary TFP process.

To see the implications of the congestion costs, consider the symmetric case that $Y_t(i) = Y_t, K_t^{eff}(i) = K_t^{eff} \forall i$, which is the case along the symmetric balanced growth path and in the flexible economy. We then obtain the following aggregate production function:

$$Y_t = \epsilon_t^a {K_{t-1}^g}^{\zeta} {K_t^{eff}}^{\alpha(1-\zeta)} [\mu^t n_t]^{(1-\alpha)(1-\zeta)} - \mu^t \Phi, \quad \epsilon_t^a \equiv (\tilde{\epsilon}_t^a)^{1-\zeta}. \qquad (9.8)$$

Choose units such that $\bar{\epsilon}^a \equiv 1$.

To solve a firm's profit maximization problem, note that it is equivalent to minimizing costs (conditional on operating) and then choosing the quantity optimally. Consider the cost-minimization problem first:

$$\min_{K_t(i),n_t(i)} W_t n_t(i) + R_t^k K_t(i) \text{ s.t. } (9.7).$$

Denote the Lagrange multiplier on the production function by MC_t – producing a marginal unit more raises costs marginally by MC_t. The static FOC are necessary and sufficient, given $Y_t(i)$:

$$[n_t(i)] \quad MC_t(i)(1-\alpha)\frac{Y_t(i)+\mu^t\Phi}{n_t(i)} = W_t,$$
$$[K_t(i)] \quad MC_t(i)\alpha\frac{Y_t(i)+\mu^t\Phi}{K_t(i)} = R_t^k.$$

The FOC can be used to solve for the optimal capital-labor ratio in production and marginal costs:

$$\frac{k_t(i)}{n_t(i)} = \frac{\alpha}{1-\alpha}\frac{w_t}{r_t^k}, \tag{9.9}$$

$$MC_t = \alpha^{-\alpha}(1-\alpha)^{-(1-\alpha)}\frac{W_t^{1-\alpha}(R_t^k)^\alpha\mu^{-(1-\alpha)t}}{\left(\frac{K_{t-1}^g}{Y_t+\mu^t\Phi}\right)^{\frac{\zeta}{1-\zeta}}\epsilon_t^a}, \tag{9.10}$$

$$mc_t = \alpha^{-\alpha}(1-\alpha)^{-(1-\alpha)}\frac{w_t^{1-\alpha}(r_t^k)^\alpha}{\left(\frac{\mu k_{t-1}^g}{y_t+\Phi}\right)^{\frac{\zeta}{1-\zeta}}\epsilon_t^a},$$

$$mc_t = \alpha^{-\alpha}(1-\alpha)^{-(1-\alpha)}\frac{w_t^{1-\alpha}(r_t^k)^\alpha}{\left(\frac{\mu k_{t-1}^g}{y_t+\Phi}\right)^{\frac{\zeta}{1-\zeta}}\epsilon_t^a}, \tag{9.11}$$

where lower case letters denote detrended, real variables as applicable:

$$k_t \equiv K_t\mu^{-t}, y_t \equiv Y_t\mu^{-t}, w_t \equiv \frac{W_t}{\mu^t P_t}, r_t^k \equiv \frac{R_t^k}{P_t}, mc_t \equiv \frac{MC_t}{P_t}.$$

For future reference, it is useful to detrend the FOC:

$$w_t = mc_t(i)(1-\alpha)\frac{y_t(i)+\Phi}{n_t(i)}, \tag{9.12a}$$

$$r_t^k = mc_t(i)\alpha\frac{y_t(i)+\Phi}{k_t(i)}. \tag{9.12b}$$

Given the solution to the static cost-minimization problem, the firm maximizes the present discounted value of its profits by choosing quantities optimally, taking as given its demand function (9.3), the marginal costs of production (9.10), and respecting the Calvo-style price setting friction. The Calvo-friction implies that a firm can re-set its price in each period with probability $1-\zeta_p$ and otherwise indexes its price to an average of current and past inflation $\prod_{l=1}^{s}\pi_{t+l-1}^{\iota_p}\bar{\pi}^{1-\iota_p}$. In each period t that the firm can change its prices it chooses:

$$P_t^*(i) = \arg\max_{\tilde{P}_t(i)} \mathbb{E}_t \sum_{s=0}^{\infty} \zeta_p^s \frac{\bar{\beta}^s \xi_{t+s} P_t}{\xi_t P_{t+s}} \Big[\tilde{P}_t(i)\Big(\prod_{l=1}^{s}\pi_{t+l-1}^{\iota_p}\bar{\pi}^{1-\iota_p}\Big) - MC_{t+s}(i)\Big] Y_{t+s}(i),$$

subject to (9.3) and (9.10). $\frac{\bar{\beta}^s\xi_{t+s}}{\xi_t}$ denotes the (non-credit constrained) representative household's stochastic discount factor and $\pi_t \equiv \frac{P_t}{P_{t-1}}$ denotes period t inflation.

To solve the problem, it is useful to define $\chi_{t,t+s}$ such that in the absence of further price adjustments prices evolve as $P_{t+s}(i) = \chi_{t,t+s}P_t^*(i)$:

$$\chi_{t,t+s} = \begin{cases} 1 & s = 0, \\ \prod_{l=1}^{s}\pi_{t+l-1}^{\iota_p}\bar{\pi}^{1-\iota_p} & s = 1,\ldots,\infty. \end{cases}$$

Therefore and using the definition $\mathrm{y}_{t+s}(i) = \frac{Y_{t+s}(i)}{Y_{t+s}}$:

$$\frac{d(Y_{t+s}(i)[P_{t+s}(i) - MC_{t+s}(i)])}{d\tilde{P}_t(i)} = \mathrm{y}_{t+s}(i)Y_{t+s}\left(\chi_{t,t+s}[1-\eta_p(\mathrm{y}_{t+s}(i))] + \eta_p\frac{MC_{t+s}(i)}{P_t(i)}\right).$$

The first order condition is then given by:

$$\mathbb{E}_t \sum_{s=0}^{\infty} \zeta_p^s \frac{\bar{\beta}^s\xi_{t+s}P_t}{\xi_t P_{t+s}} \mathrm{y}_{t+s}(i)Y_{t+s}\left([1-\eta_p(\mathrm{y}_{t+s}(i))]\chi_{t,t+s} + \eta_p\frac{MC_{t+s}(i)}{P_t(i)}\right) = 0 \tag{9.13}$$

For future reference, it is useful to re-write the FOC as follows:

$$\frac{P_t^*(i)}{P_t} = \frac{\mathbb{E}_t \sum_{s=0}^{\infty} (\mu\bar{\beta}\zeta_p)^s \frac{\xi_{t+s}}{\lambda_p(\mathrm{y}_{t,t+s}(i))\xi_t} y_{t,t+s}(i) \frac{\eta_p(\mathrm{y}_{t,t+s}(i))}{\eta_p(\mathrm{y}_{t,t+s}(i))-1} mc_{t+s}(i)}{\mathbb{E}_t \sum_{s=0}^{\infty} (\mu\bar{\beta}\zeta_p)^s \frac{\xi_{t+s}}{\lambda_p(\mathrm{y}_{t,t+s}(i))\xi_t} \frac{\chi_{t,t+s}}{\prod_{l=1}^{s} \pi_{t+s}} y_{t,t+s}(i)} \tag{9.14}$$

where $\mathrm{y}_{t,t+s}(i) = G'^{-1}\left(\frac{P_t^* \chi_{t,t+s} Y_{t+s}}{\Xi_{t+s}^f}\right)$, $Y_{t,t+s}(i) = \mathrm{y}_{t,t+s}(i) Y_{t+s}$.

Noting that measure $1 - \zeta_p$ of firms changes prices in each period and each firm faces a symmetric problem, the expression for the aggregate price index (9.2) can be expressed recursively as a weighted average of adjusted and indexed prices:

$$P_t = (1-\zeta_p) P_t^* G'^{-1}\left(\frac{P_t^* Y_t}{\Xi_t^f}\right) + \zeta_p \pi_{t-1}^{\iota_p} \bar{\pi}^{1-\iota_p} P_{t-1} G'^{-1}\left(\frac{\pi_{t-1}^{\iota_p} \bar{\pi}^{1-\iota_p} P_{t-1} Y_t}{\Xi_t^f}\right), \tag{9.15}$$

using that price distribution of non-adjusting firms at t is the same as that of all firms at time $t-1$, adjusted for the shrinking mass due price adjustments. Along the deterministic balanced growth path the optimal price equals the average price, which is normalized to unity:

$$\bar{P}^* = \bar{P} = 1.$$

Similarly, along the deterministic growth path the price is a constant mark-up over marginal cost:

$$\frac{\bar{P}^*}{\bar{P}} = \frac{\eta_p}{\eta_p - 1} \overline{mc} = (1 + \bar{\lambda}_p) \overline{mc} = 1 \tag{9.16}$$

Finally, the assumption of monopolistic competition in the presence of free entry requires zero profits along the balanced growth path. Real and detrended profits of intermediate producer i are given by:

$$\Pi_t^p(i) = \frac{P_t(i)}{P_t} y_t(i) - w_t n_t(i) - r_t^k k_t(i) = \frac{P_t(i)}{P_t} y_t(i) - mc_t(i)[y_t(i) + \mu^t \Phi]$$

Integrating over all $i \in [0, 1]$ and using the definition of the price index (9.2) yields:

$$\Pi_t^p = y_t - w_t \int_0^1 n_t(i)di - r_t^k \int_0^1 k_t(i)di \tag{9.17a}$$

$$= y_t - mc_t\left(\int_0^1 y_t(i)di + \Phi\right) = y_t - mc_t\left(y_t \int_0^1 \frac{P_t(i)}{P_t}di + \Phi\right) \tag{9.17b}$$

Using the expression for the steady state markup, equation (9.16), the zero profit condition (9.17b) implies that along the symmetric balanced growth path:

$$0 = \bar{\Pi}^p = \bar{y} - \frac{\bar{y}\int_0^1 \frac{P(i)}{P}di + \Phi}{1+\bar{\lambda}_p} = \bar{y} - \frac{\bar{y}+\Phi}{1+\bar{\lambda}_p} \quad \Rightarrow \frac{\Phi}{\bar{y}} = \bar{\lambda}_p. \tag{9.18}$$

9.1.3 Labor packers

Intermediate producers use a bundel of differentiated labor inputs, $\ell \in [0,1]$, purchased from labor packers. Labor packers aggregagte, or pack, differentiated labor which they purchase from unions. They are perfectly competitive and face an analogous problem to final goods producers:

$$\max_{n_t, n_t(\ell)} W_t n_t - \int_0^1 W_t(\ell) n_t(\ell) d\ell \quad \text{s.t.} \quad \int_0^1 H\left(\frac{n_t(\ell)}{n_t}; \tilde{\epsilon}_t^{\lambda,w}\right) d\ell = 1, \tag{9.19}$$

where $H(\cdot)$ has the same properties as $G(\cdot)$: $H' > 0, H'' < 0, H(1) = 1$.

The FOC yield differentiated labor demand, analogous to intermediate goods demand (9.3):

$$n_t(\ell) = n_t H'^{-1}\left(\frac{W_t(\ell) n_t}{\Xi_t^n}\right) = n_t H'^{-1}\left(\frac{W_t(\ell)}{W_t}\int_0^1 H'\left(\frac{n_t(l)}{n_t}; \tilde{\epsilon}_t^{\lambda,w}\right)\frac{n_t(l)}{n_t}dl\right). \tag{9.20}$$

Given the aggregate nominal wage $W_t = \int_0^1 \frac{n_t(\ell)}{n_t} w_t(\ell) d\ell$, labor packers are willing to supply any amount of packed labor n_t. Labor demand elasticity behaves analogously to the intermediate goods elasticity:

$$\eta_w(\mathrm{n}_t(\ell)) \equiv -\frac{W_t(\ell)}{n_t(\ell)} \frac{d\mathrm{n}_t(\ell)}{dW_t(\ell)}\bigg|_{dn_t = d\Xi_t^l = 0} = -\frac{H'(\mathrm{n}_t(\ell))}{\mathrm{n}_t(\ell) H''(\mathrm{n}_t(\ell))} \tag{9.21}$$

$$\hat{\eta}_w(\mathrm{n}_t(\ell)) \equiv \frac{W_t(\ell)}{\eta_w(\mathrm{n}_t(\ell))}\frac{d\eta_w(\mathrm{n}_t(\ell))}{dW_t(\ell)} = 1 + \frac{1+\lambda^w(\mathrm{n}_t(\ell))}{\lambda^w(\mathrm{n}_t(\ell))}\left(\frac{1}{[1+\lambda^w(\mathrm{n}_t(\ell))]A_w(\mathrm{n}_t(\ell))} - 1\right), \tag{9.22}$$

where $\mathrm{n}_t(\ell) \equiv \frac{n_t(\ell)}{n_t}$ and the mark-up is defined as $\lambda_t^n(\mathrm{n}_t(\ell)) \equiv \frac{1}{\eta_w(\mathrm{n}_t(\ell))-1}$. $A_w(\mathrm{n}_t(\ell)) \equiv \frac{\lambda^w(\mathrm{n}_t(\ell))}{2+\frac{H'''(\mathrm{n}_t(\ell))}{H''(\mathrm{n}_t(\ell))}\mathrm{n}_t(\ell)}$ can be equivalently expressed as:

$$A_w(\mathrm{n}) = \frac{1}{\lambda^w(\mathrm{n})\hat{\eta}_w(\mathrm{n})+1}.. \tag{9.23}$$

9.2 Households

There is a measure one of households in the economy, indexed by $j \in [0,1]$, endowed with a unit of labor each. Households are distributed uniformly over the real line, i.e. the measure of households is the Lebesgue measure Λ. We distinguish two types of households – intertemporally optimizing households $j \in [0, 1-\phi]$ and "rule-of-thumb" households $j \in (1-\phi, 1]$, so that they have measures $\Lambda([0,1-\phi]) = 1-\phi$ and $\Lambda([0,\phi]) = \phi$, respectively.

Households' preferences over consumption and hours worked streams $\{C_{t+s}(j), n_{t+s}(j)\}_{s=0}^{\infty}$ are represented by the life-time utility function U_t:

$$U_t = \mathbb{E}_t \sum_{s=0}^{\infty} \beta^s \left[\frac{1}{1-\sigma}\left(C_{t+s}(j) - hC_{t+s-1}\right)^{1-\sigma}\right] \exp\left[\frac{\sigma-1}{1+\nu} n_{t+s}(j)^{1+\nu}\right]. \tag{9.24}$$

Here $h \in [0,1)$ captures external habit formation, σ denotes the inverse of the intertemporal elasticity of substitution, and ν equals the inverse of the labor supply elasticity. Households discount the future by $\beta \in (0,1)$, where β varies by household type.

The fraction $1-\phi$ of the labor force who are not credit constrained, maximizes their life-time utility subject to a lifetime budget constraint and a capital accumulation technology. The remainder of the labor force, i.e. a fraction ϕ is credit constrained (or "rule-of-thumb"): they cannot save or borrow.

9.2.1 Intertemporally optimizing households

The intertemporally optimizing households choose consumption $\{C_{t+s}(j)\}$, investment in physical capital $\{X_{t+s}(j)\}$, physical capital $\{K^p_{t+s}(j)\}$, a capacity utilization rate $\{u_{t+s}(j)\}$, nominal government bond holdings $B^n_{t+s}(j)$, and labor supply $\{n_{t+s}(j)\}$ to maximize (9.24) subject to a sequence of budget constraints (9.25), the law of motion for physical capital (9.26), and a no-Ponzi constraint. Households take prices $\{P_{t+s}\}$, nominal returns on government bonds $\{q^b_{t+s}R_{t+s}\}$, the nominal rental rate of capital $\{R^k_{t+s}\}$, and nominal wages $\{W_{t+s}\}$ as given.

The budget constraint for period $t+s$ is given by:

$$(1+\tau^c_{t+s})C_{t+s}(j) + X_{t+s}(j) + \frac{B^n_{t+s}(j)}{R^{gov}_{t+s}P_{t+s}} \leq$$
$$S_{t+s} + \frac{B^n_{t+s-1}(j)}{P_{t+s}} + (1-\tau^n_{t+s})\frac{[W^h_{t+s}n_{t+s}(j) + \lambda_{w,t+s}n_{t+s}W^h_{t+s}]}{P_{t+s}} +$$
$$+\left[(1-\tau^k_{t+s})\left(\frac{R^k_{t+s}u_{t+s}(j)}{P_{t+s}} - a(u_{t+s}(j))\right) + \delta\tau^k_{t+s}\right][(1-\omega^k_{t+s-1})K^p_{t+s-1}(j) + \omega^k_{t+s-1}K^{p,agg}_{t+s-1}] + \frac{\Pi^p_{t+s}\mu^{t+s}}{P_{t+s}}, \quad (9.2$$

where $(\tau^c_{t+s}, \tau^k_{t+s}, \tau^n_{t+s})$ represent taxes on consumption expenditure, capital income, and labor income, respectively. The wage received by households differs from the one charged to labor packers because of union profits – union profits $\lambda_{w,t+s}n_{t+s}W^h_{t+s}$ are taken as given by households. Households also receive nominal lump-sum transfers $\{S_{t+s}\}$. $a(\cdot)$ represents the strictly increasing and strictly convex cost function of varying capacity utilization, whose first derivative in the case of unit capacity utilization is normalized as $a'(1) = \bar{r}^k$.[5] At unit capacity utilization, there is no additional cost: $a(1) = 0$. $\Pi^p_{t+s}\mu^{t+s}$ are nominal profits which households also take as given.

There is a financial market frictions present in the budget constraint. $\omega^k_{t+s} \neq 0$ represents a wedge between between the returns on private and government bonds and is a pure financial market friction – if $\omega^k_{t+s} > 0$ then

[5] $\bar{r}^k$ represents the real steady state return on capital services.

households obtain less than one dollar for each dollar of after tax capital income they receive, representing agency costs. Agency costs are reimbursed directly to unconstrained households, so that the friction has no effect on aggregate resources. This financial market friction is similar to a shock in Smets and Wouters (2003) who introduce it ad hoc in the investment Euler equation and motivate it as a short-cut to model informational frictions which disappear at the steady state.

Physical capital evolves according to the following law of motion:

$$K_{t+s}^{p}(j) = (1-\delta)K_{t+s-1}^{p}(j) + q_{t+s}^{x}\left[1 - S\left(\frac{X_{t+s}(j)}{X_{t+s-1}(j)}\right)\right] X_{t+s}(j), \quad (9.26)$$

where new investment is subject to adjustment costs described by $S(\dot{})$. These costs satisfy $S(\mu) = S'(\mu) = 0, S'' > 0$. The relative price of investment changes over time, as captured by the exogenous $\{q_{t+s}^{x}\}$ process. Physical capital depreciates at rate δ.

For future reference, note that the effective capital stock is given by the product of capacity utilization and physical capital stock:

$$K_{t+s}^{eff}(j) = K_{t+s-1}^{p}(j)u_{t+s}(j). \quad (9.27)$$

To obtain the aggregate capital stock, multiply the above quantity by $(1-\phi)$.

The solution to the household's problem is characterized completely by the law of motion for physical capital (9.26) and the following necessary and sufficient first order conditions. To derive these conditions, denote the Lagrange multipliers on the budget constraint (9.25) and the law of motion (9.26) by $\beta^t(\Xi_t, \Xi_t^k)$ – replacing the household index j by a superscript RA.

$$[C_t] \quad \Xi_t(1+\tau_t^c) = \exp\left(\frac{\sigma-1}{1+\nu}(n_t^{RA})^{1+\nu}\right)[C_t^{RA} - hC_{t-1}^{RA}]^{-\sigma}$$

$$[n_t] \quad \Xi_t(1-\tau_t^n)\frac{W_t^h}{P_t} = \exp\left(\frac{\sigma-1}{1+\nu}(n_t^{RA})^{1+\nu}\right)(n_t^{RA})^{\nu}[C_t^{RA} - hC_{t-1}^{RA}]^{1-\sigma}$$

$$[B_t] \quad \Xi_t = \beta q_t^b R_t \mathbb{E}_t \left(\frac{\Xi_{t+1}}{P_{t+1}/P_t} \right)$$

$$[K_t^p] \quad \Xi_t^k = \beta \mathbb{E}_t \Bigg(\Xi_{t+1} \Bigg[\tilde{q}_t^k \Bigg((1 - \tau_{t+s}^k) \Bigg[\frac{R_{t+1}^k}{P_{t+1}} u_{t+1} - a(u_{t+1}) + \delta \tau_{t+1}^k \Bigg] + (1-\delta) \frac{\Xi_{t+1}^k}{\Xi_{t+1}} \Bigg] \Bigg)$$

$$[X_t] \quad \Xi_t = \Xi_t^k q_t^x \left(1 - S\Big(\frac{X_t^{RA}}{X_{t-1}^{RA}} \Big) - S'\Big(\frac{X_t^{RA}}{X_{t-1}^{RA}} \Big) \Big(\frac{X_t^{RA}}{X_{t-1}^{RA}} \Big) \right) + \beta \mathbb{E}_t \left(\frac{\Xi_{t+1}^k}{\Xi_t} q_{t+1}^x S'\Big(\frac{X_{t+1}^{RA}}{X_t^{RA}} \Big) \Big(\frac{X_{t+1}^{RA}}{X_t^{RA}} \Big)^2 \right)$$

$$[u_t] \quad \frac{R_{t+1}^k}{P_t} = a'(u_{t+1}^{RA}).$$

By setting $a'(1) \equiv \bar{r}^k$ we normalize steady state capacity utilization to unity: $\bar{u} \equiv 1$.

For what follows, it is useful to detrend these first order conditions and the law of motion for capital. To that end, use lower case letters to denote detrended and real variables as exemplified in the following definitions:

$$k_t^{RA} \equiv \frac{K_t^{RA}}{\mu^t}, w_t \equiv \frac{W_t}{P_t \mu^t}, w_t^h \equiv \frac{W_t^h}{P_t \mu^t}, r_t^k \equiv \frac{R_t^k}{P_t}, \xi_t \equiv \Xi_t \mu^{\sigma t}, Q_t \equiv \frac{\Xi_t^k}{\Xi_t}, \bar{\beta} = \beta \mu^{-\sigma}.$$

μ denotes the gross trend growth rate of the economy. For future reference, note that government expenditure is normalized differently: $g_t = \frac{G_t}{Y \mu^t}$. Substituting in for the normalized variables yields:

$$\xi_t (1 + \tau_t^c) = \exp\left(\frac{\sigma - 1}{1 + \nu} (n_t^{RA})^{1+\nu} \right) [c_t^{RA} - (h/\mu) c_{t-1}^{RA}]^{-\sigma} \tag{9.29a}$$

$$\xi_t (1 - \tau_t^n) w_t^h = \exp\left(\frac{\sigma - 1}{1 + \nu} (n_t^{RA})^{1+\nu} \right) (n_t^{RA})^{\nu} [c_t^{RA} - (h/\mu) c_{t-1}^{RA}]^{1-\sigma} \tag{9.29b}$$

$$\xi_t = \bar{\beta} R_t^{gov} \mathbb{E}_t \left(\frac{\xi_{t+1}}{P_{t+1}/P_t} \right) \tag{9.29c}$$

$$Q_t = \bar{\beta} \mathbb{E}_t \Bigg(\frac{\xi_{t+1}}{\xi_t} \Big[\tilde{q}_t^k \big((1 - \tau_{t+1}^k) [r_{t+1}^k u_{t+1} - a(u_{t+1})] + \delta \tau_{t+1}^k \big) + (1-\delta) Q_{t+1} \Big] \Bigg) \tag{9.29d}$$

$$\begin{aligned} 1 = {} & Q_t q_t^x \left(1 - S\Big(\frac{x_t^{RA} \mu}{x_{t-1}^{RA}} \Big) - S'\Big(\frac{x_t^{RA} \mu}{x_{t-1}^{RA}} \Big) \Big(\frac{x_t^{RA} \mu}{x_{t-1}^{RA}} \Big) \right) \\ & + \bar{\beta} \mathbb{E}_t \left(\frac{\xi_{t+1}}{\xi_t} Q_{t+1} q_{t+1}^x S'\Big(\frac{x_{t+1}^{RA} \mu}{x_t^{RA}} \Big) \Big(\frac{x_{t+1}^{RA} \mu}{x_t^{RA}} \Big)^2 \right) \end{aligned} \tag{9.29e}$$

$$r_{t+1}^k = a'(u_{t+1}^{RA}). \tag{9.29f}$$

The detrended law of motion for physical capital is given by

$$k_t^{p,RA} = \frac{(1-\delta)}{\mu} k_{t-1}^{p,RA} + q_t^x \left[1 - S\left(\frac{x_t^{RA}}{x_{t-1}^{RA}}\mu\right)\right] x_t^{RA}. \tag{9.30}$$

Combining the FOC for consumption and hours worked, gives the static optimality condition for households:

$$\frac{1-\tau_t^n}{1+\tau_t^c} w_t^h = (n_t^{RA})^\nu [c_t^{RA} - (h/\mu) c_{t-1}^{RA}]. \tag{9.31}$$

Combining (9.29a) for two consecutive periods and using (9.29c) gives the consumption Euler equation:

$$\mathbb{E}_t \left(\frac{\xi_{t+1}}{\xi_t}\right) = \mathbb{E}_t \left(\exp\left(\frac{\sigma-1}{1+\nu}\left(\frac{n_{t+1}^{RA}}{n_t^{RA}}\right)^{1+\nu}\right) \left[\frac{c_{t+1}^{RA} - (h/\mu)c_t^{RA}}{c_t^{RA} - (h/\mu)c_{t-1}^{RA}}\right]^{-\sigma}\right). \tag{9.32}$$

Equation (9.29d) is the investment Euler equation. The FOC for capital (9.29e) can be used to compute the shadow price of physical capital Q_t.

Using the investment Euler equation shows that along the deterministic balanced growth path the value of capital equals unity (since $S'(\mu) = S(\mu) = 0$ and $\bar{q}^x = 1$). From the consumption Euler equation and $\bar{q}^b = 1$ we obtain the interest rate paid on government bonds under balanced growth. Finally, the pricing equation for capital and the investment Euler equation pin down the rental rate on capital. Summarizing:

$$\bar{Q} = 1, \tag{9.33a}$$

$$\bar{R} = \bar{\beta}^{-1}\bar{\pi}, \tag{9.33b}$$

$$1 = \bar{\beta}[(1-\bar{\tau}^k)\bar{r}^k + \delta\bar{\tau}^k + (1-\delta)],$$

$$\Leftrightarrow \quad \bar{r}^k = \frac{\bar{\beta}^{-1} - 1 + \delta(1-\bar{\tau}^k)}{1-\bar{\tau}^k}. \tag{9.33c}$$

The bond premium shock q_t^b differs from a discount factor shock, although it results in an observationally equivalent consumption Euler equation – if time preference was time-varying, the period utility function would become:

$$\left[\frac{1}{1-\sigma}\left(C_{t+s}(j)-hC_{t+s-1}\right)^{1-\sigma}\right]\exp\left[\frac{\sigma-1}{1+\nu}n_{t+s}(j)^{1+\nu}\right]\prod_{l=1}^{s}\check{q}^{b}_{t+l-1},$$

so that the ratio $\frac{\check{\xi}_{t+1}}{\xi_t}$ would be proportional to $\check{q}^b_t$, so that the consumption Euler equation conditions is unchanged. The effects differ, however, insofar that the present formulation on basis of the government discount factor also affects the investment Euler equation and the government budget constraint.

For measurement purposes, it is useful to re-write the linearized FOC for capital, after substituting out for the discount factor. It shows that the private bond shock represents the premium paid for private bonds over government bonds holding the rental rate on capital fixed:

$$\frac{\bar{r}^k(1-\bar{\tau}^k)\mathbb{E}_t(\hat{r}^k_{t+1})+(1-\delta)\mathbb{E}_t(\hat{Q}_{t+1})}{\bar{r}^k(1-\bar{\tau}^k)+\delta\bar{\tau}^k+1-\delta}-\hat{Q}_t=\left(\hat{R}_t-\mathbb{E}_t[\pi_t]\right)+\hat{q}^b_t+\hat{q}^k_t.$$

Note: the shock $\tilde{q}^k_t$ in the budget constraint has been rescaled here. $\hat{q}^k_t$ is the deviation of the rescaled shock from its steady state value.

9.2.2 Credit-constrained or "rule of thumb" households

A fraction $\phi\in(0,0.5)$ of the households is assumed to be credit-constrained. As a justification, one may suppose that credit-constrained discount the future substantially more steeply, and are thus uninterested in accumulating government bonds or private capital, unless their returns are extraordinarily high. Conversely, these households find it easy to default on any loans, and are therefore not able to borrow. We hold the identity of credit-constrained households and thereby their fraction of the total population constant.

"Rule of thumb" households face a static budget constraint in each period and are assumed to supply the same amount of labor as intertemporally optimizing households. Given

$$n^{RoT}_{t+s}(j)=n^{RA}_{t+s}=n_{t+s},$$

consumption follows from the budget constraint in each period:

$$(1+\tau^c_{t+s})C^{RoT}_{t+s}(j) \leq S^{RoT}_{t+s}+(1-\tau^n_{t+s})\frac{W^h_{t+s}n^{RoT}_{t+s}(j)+\lambda_{w,t+s}W^h_{t+s}n_{t+s}}{P_{t+s}}+\Pi^p_{t+s}\mu^{t+s}. \tag{9.34}$$

Rule-of-thumb households receive transfers, labor income including union profits, and profits made by intermediate goods producing firms.

Removing the trend from the budget constraint (9.34), omitting the j index, and solving for (detrended) consumption:

$$c^{RoT}_{t+s} = \frac{1}{(1+\tau^c_{t+s})}\left(s^{RoT}_{t+s} + (1-\tau^n_{t+s})[w^h_{t+s}n^{RoT}_{t+s} + \lambda_{w,t+s}w^h_{t+s}n_{t+s}] + \Pi^p_{t+s}\right). \tag{9.35}$$

From the budget constraint (9.34), the following steady state relationship holds:

$$\bar{c}^{RoT} = \frac{\bar{s}^{RoT} + (1-\bar{\tau}^n_t)\bar{w}\bar{n}}{1+\bar{\tau}^c}. \tag{9.36}$$

We assume that:

$$\bar{s}^{RoT} = \bar{s}. \tag{9.37}$$

9.2.3 Households: labor supply, wage setting

Households supply homogeneous labor to unions which differentiate labor into varieties indexed by $\ell \in [0,1]$ and sell it to labor packers. In doing so, unions take aggregate quantities, i.e. households' cost of supplying labor and aggregate labor demand and wages, as given. Unions maximize the expected present discounted value of net of tax wage income earned in excess of the cost of supplying labor. In the presence of rule-of-thumb households unions act as if they were maximizing surplus for the intertemporally optimizing households only. If the mass of rule-of-thumb households is less than the mass of intertemporally optimizing households, i.e. $\phi < 0.5$ which is satisfied in the parameterizations used, a median-voter decision rule justifies this assumption.

The labor unions problem is analogous to that of price-setting firms, with the marginal rate of substitution between consumption and leisure in the representative household taking the role of marginal costs in firms' problems. From the FOC $[C_t]$ and $[n_t]$ the marginal rate of substitution is given by $\frac{U_{n,t+s}}{\Xi_{t+s}} = (n_t^{RA})^{\nu}[C_t^{RA} - hC_{t-1}^{RA}](1+\tau_t^c)$. Whenever a union has the chance to reset the wage it charges, it chooses $W_t^*(\ell)$:

$$W_t^*(\ell) = \arg\max_{\tilde{W}_t(\ell)} \mathbb{E}_t \sum_{s=0}^{\infty} (\zeta_w)^s \frac{\bar{\beta}^s \xi_{t+s}}{\xi_t} \left[(1-\tau_{t+s}^n) \frac{W_{t+s}(\ell)}{P_{t+s}} + \frac{U_{n,t+s}}{\Xi_{t+s}} \right] n_{t+s}(\ell), \tag{9.38}$$

subject to the labor demand equation (9.20). $1-\zeta_w$ denotes the probability that a union can reset its wage. If it cannot adjust, wages are adjusted according to a moving average of past and steady state inflation and labor productivity growth:

$$W_{t+s}(\ell) = {W^*}_t(\ell) \prod_{v=1}^{s} \mu(\pi_{t+v-1})^{\iota_w} \bar{\pi}^{1-\iota_w} \equiv W_t^*(\ell)\chi_{t,t+s}^w.$$

Using that $n_t = n_t^{RA}$, the first order condition is given by

$$0 = \mathbb{E}_t \sum_{s=0}^{\infty} \zeta_p^s \frac{\bar{\beta}^s \xi_{t+s}}{\xi_t \lambda^w(\mathrm{n}_{t,t+s}(\ell))} \frac{n_{t+s}(\ell)}{W_t^*(\ell)} \Bigg((1-\tau_{t+s}^n) \frac{W_t^*(\ell)\chi_{t,t+s}^w(\ell)}{P_{t+s}} - [1+\lambda^w(\mathrm{n}_{t+s}(\ell))](1+\tau_{t+s}^c) n_{t+s}^{\nu} [C_{t+s}^{RA} - hC_{t+s-1}^{RA}] \Bigg) \tag{9.39}$$

and can be equivalently expressed as

$$\frac{W_t^*(\ell)}{P_t} = \frac{\mathbb{E}_t \sum_{s=0}^{\infty} \zeta_p^s \frac{\bar{\beta}^s \xi_{t+s}}{\xi_t \lambda^w(\mathrm{n}_{t,t+s}(\ell))} n_{t+s}(\ell)[1+\lambda^w(\mathrm{n}_{t+s}(\ell))](1+\tau_{t+s}^c) n_{t+s}^{\nu} [C_{t+s}^{RA} - hC_{t+s-1}^{RA}]}{\mathbb{E}_t \sum_{s=0}^{\infty} \zeta_p^s \frac{\bar{\beta}^s \xi_{t+s}}{\xi_t \lambda^w(\mathrm{n}_{t,t+s}(\ell))} n_{t+s}(\ell)(1-\tau_{t+s}^n) \frac{\chi_{t,t+s}^w(\ell)}{P_{t+s}/P_t}} \tag{9.40}$$

Aggregate wages evolve as

$$W_t = (1-\zeta_w) W_t^* H'^{-1}\left(\frac{W_t^* n_t}{\Xi_t^n}\right) + \zeta_w \pi_{t-1}^{\iota_w} \bar{\pi}^{1-\iota_w} W_{t-1} H'^{-1}\left(\frac{\pi_{t-1}^{\iota_w} \bar{\pi}^{1-\iota_w} W_{t-1} n_t}{\Xi_t^n}\right), \tag{9.41}$$

Along the deterministic balanced growth path, the detrended desired real wage is given by a constant mark-up over the marginal rate of substitution. Given constant inflation, the symmetric deterministic growth path also implies, from equation (9.41), that the desired real wage equals the actual real wage:

$$\bar{w} = \bar{w}^* = (1+\bar{\lambda}_w)\bar{w}^h = (1+\bar{\lambda}_w)\frac{1+\bar{\tau}^c}{1-\bar{\tau}^n}\bar{n}^\nu \bar{c}^{RA}[1-h/\mu], \tag{9.42}$$

where the second equality uses (9.31).

9.3 Government

The government sets nominal interest R_t according to an interest rate rule, purchases goods and services for government consumption G_t, pays transfers S_t to households, and provides public capital for the production of intermediate goods, K_t^g. It finances its expenditures by levying taxes on capital and labor income, a tax on consumption expenditure, and one period nominal bond issues. We consider a setup in which monetary policy is active in the neighborhood of the balanced growth path.

9.3.1 Fiscal policy

In modelling the government sector, we take as given the tax structure along the balanced growth path as in **?**, who used NIPA data to compute the capital and labor income and comsumption expenditure tax rates for the US. Off the balanced growth path, we follow **?** in assuming that labor tax rates adjust gradually to balance the budget in the long-run, whereas in the short-run much of any additional government expenditure is tax financed.

The government flow budget constraint is given by:

$$G_t + X_t^g + S_t + \frac{B_{t-1}}{P_t} \leq \frac{B_t}{R_t^{gov} P_t} + \tau_t^c C_t + \tau_t^n n_t \frac{W_t}{P_t} + \tau_t^k \left[u_t \frac{R_t^k}{P_t} - a(u_t) - \delta \right] K_{t-1}^p. \tag{9.43}$$

Detrended, the government budget constraint is given by:

$$\bar{y} g_t + x_t^g + s_t + \frac{b_{t-1}}{\mu \pi_t} \leq \frac{b_t}{R_t^{gov}} + \tau_t^c c_t + \tau_t^n n_t w_t + \tau_t^k k_t^s r_t^k - \tau_t^k [a(u_t) + \delta] \frac{k_{t-1}^p}{\mu}. \tag{9.44}$$

Government consumption $g_t = \frac{G_t}{\bar{y}\mu^t}$ is given exogenously and is stochastic, driven by genuine spending shocks as well as by technology shocks.

By introducing a wedge between the federal funds rate and government bonds, we capture both short-term liquidity premia as well as changes in the term structure of government debt. Since the latter is absent with only one period bonds, in the estimation the bond premium may also reflect differences in the borrowing cost due to a more complex maturity structure. [6]

Labor tax rates have both a stochastic and a deterministic component. They adjust deterministically to ensure long-run budget balance at a speed governed by the parameter $\psi_\tau \in [\underline{\psi}_\tau, 1]$, where $\underline{\psi}_\tau$ is some positive number large enough to guarantee stability. To simplify notation denote the remaining detrended deficit prior to new debt and changes in labor tax rates as d_t:

$$d_t \equiv \bar{y} g_t + x_t^g + \bar{s} + s_t^{exo} + \frac{b_{t-1}}{\mu \pi_t} - \bar{\tau}^c c_t - \bar{\tau}^n w_t n_t - \bar{\tau}^k k_t^s r_t^k + \bar{\tau}^k \delta \frac{k_{t-1}^p}{\mu}.$$

In the baseline case, labor tax rates are adjusted according to the following rule:

$$(\tau_t^n - \bar{\tau}^n) w_t n_t + \epsilon_t^\tau = \psi_\tau (d_t - \bar{d}), \tag{9.45}$$

[6] Historical data by the Federal Reserve implies a maturity between 10 and 22 quarters with an average between 16 and 20 quarters (The Federal Reserve Board Bulletin, 1999, Figure 4).

where ϵ_t^τ is an exogenous shock to the tax rate.

In general:

$$\psi_\tau(d_t - \bar{d}) - \epsilon_t^\tau = \begin{cases} (\tau_t^n - \bar{\tau}^n) w_t n_t & \text{Baseline}, \tau_t^c = \tau_t^k = s_t^{endo} = 0, \\ (\tau_t^c - \bar{\tau}^c) c_t & \text{Alternative 1}, \tau_t^n = \tau_t^k = s_t^{endo} = 0, \\ (\tau_t^k - \bar{\tau}^k) k_t^s (r_t^k - \delta) & \text{Alternative 2}, \tau_t^n = \tau_t^c = s_t^{endo} = 0, \\ -(s_t^{endo} - \bar{s}) & \text{Alternative 3}, \tau_t^n = \tau_t^c = \tau_t^k = 0. \end{cases} \tag{9.46}$$

Debt issues are then given by the budget constraint or equivalently as the residual from (9.45): $\frac{b_t}{R_t^{gov}} = (1 - \psi_\tau)(d_t - \bar{d}) + \epsilon_t^\tau$.

Government investment is chosen optimally for a given tax structure. Given the congestion effect of production on public infrastructure, a tax on production would be optimal (Barro and Sala-i Martin, 1992). Similarly, we neglect the potential cost of financing of productive government expenditure via distortionary taxes. To motivate this assumption note that along the balanced growth path, government capital can be completely debt-financed or privatized and financed through government bond issues, whereas other government expenditures such as transfers which are not backed by real assets have to backed by the government's power to levy taxes.

Formally, the government chooses investment and capital stock to maximize the present discounted value of output net of investment expenditure along the balanced growth path:

$$\max_{\{K_{t+s}^g, X_{t+s}^g\}_{s=0}^\infty} \mathbb{E}_t \sum_{s=0}^\infty \beta^s \frac{\Xi_{t+s}}{\Xi_t} [Y_{t+s} - X_{t+s}^g],$$

given K_{t-1}^g and subject to the aggregate production function (9.8) and to the capital accumulation equation

$$K_{t+s}^g = (1-\delta) K_{t+s-1}^g + q_{t+s}^{x,g} \left[1 - S_g \left(\frac{[X_{t+s}^g + \tilde{u}_{t+s}^{x,g}]}{[X_{t+s-1}^g + \tilde{u}_{t+s-1}^{x,g}]} \right) \right] (X_{t+s}^g + \tilde{u}_{t+s}^{x,g}). \tag{9.47}$$

The government is subject to similar adjustment costs as the private sector

$S_g(\mu) = S_g'(\mu) = 0, S_g'' > 0$ and investment is subject to shocks to its relative efficiency $q_{t+s}^{x,g}$. We assume that government capital depreciates at the same rate as private physical capital. $\tilde{u}^{x,g}$ represents exogenous shocks to government investment spending – such as stimulus spending.

Denote the Lagrange multiplier on (9.47) at time $t+s$ as $\beta^s \frac{\Xi_{t+s}^g}{\Xi_t}$. Then the first order conditions are:

$$\begin{aligned}
[X_t^g] \quad 1 &= \frac{\Xi_t^g}{\Xi_t} q_t^x \left(1 - S_g\Big(\frac{[\tilde{u}_t^{x,g} + X_t^g]}{[\tilde{\epsilon}_{t-1}^{x,g} + X_{t-1}^g]}\Big) - S_g'\Big(\frac{[\tilde{\epsilon}_t^{x,g} + X_t^g]}{[\tilde{\epsilon}_{t-1}^{x} + X_{t-1}^g]}\Big)\Big(\frac{[\tilde{u}_t^{x,g} + X_t^g]}{[\tilde{\epsilon}_{t-1}^{x,g} + X_{t-1}^g]}\Big)\right) \\
&\quad + \beta \mathbb{E}_t \left(\frac{\Xi_{t+1}^g}{\Xi_t} q_{t+1}^x S_g'\Big(\frac{[\tilde{\epsilon}_{t+1}^{x,g} + X_{t+1}^g]}{[\tilde{u}_t^{x,g} + X_t^g]}\Big)\Big(\frac{[\tilde{\epsilon}_{t+1}^{x,g} + X_{t+1}^g]}{[\tilde{u}_t^{x,g} + X_t^g]}\Big)^2\right) \\
[K_t^g] \quad \frac{\Xi_t^g}{\Xi_t} &= \beta \mathbb{E}_t \left(\frac{\Xi_{t+1}}{\Xi_t} \zeta \frac{Y_t + \mu^t \Phi}{K_{t-1}^g} + (1-\delta) \frac{\Xi_{t+1}^g}{\Xi_t}\right)
\end{aligned}$$

Defining the shadow price of government capital as $Q_t^g \equiv \frac{\Xi_t^g}{\Xi_t}$ and detrending, the first order conditions can be equivalently written as:

$$\begin{aligned}
1 = Q_t^g q_t^x &\left(1 - S_g\Big(\frac{[\epsilon_t^{x,g} + {}_t^g]\mu}{[\epsilon_{t-1}^{x,g} + x_{t-1}^g]}\Big) - S_g'\Big(\frac{[\epsilon_t^{x,g} + x_t^g]\mu}{[\tilde{\epsilon}_{t-1}^{x} + x_{t-1}^g]}\Big)\Big(\frac{[\epsilon_t^{x,g} + x_t^g]\mu}{[\epsilon_{t-1}^{x,g} + x_{t-1}^g]}\Big)\right) \\
&+ \bar{\beta} \mathbb{E}_t \left(Q_{t+1}^g \frac{\xi_{t+1}}{\xi_t} q_{t+1}^x S_g'\Big(\frac{[\epsilon_{t+1}^{x,g} + x_{t+1}^g]\mu}{[\epsilon_t^{x,g} + x_t^g]}\Big)\Big(\frac{[\tilde{\epsilon}_{t+1}^{x,g} + x_{t+1}^g]\mu}{[\epsilon_t^{x,g} + x_t^g]}\Big)^2\right)
\end{aligned} \tag{9.48a}$$

$$Q_t^g = \bar{\beta} \mathbb{E}_t \left(\frac{\xi_{t+1}}{\xi_t} \zeta \frac{y_t + \Phi}{k_{t-1}^g/\mu} + \frac{\xi_{t+1}}{\xi_t}(1-\delta) Q_{t+1}^g\right), \tag{9.48b}$$

where $\epsilon_t^{x,g} \equiv \frac{1}{\mu}\tilde{\epsilon}_t^{x,g}$ denotes the detrended investment spending shock.

Along the balanced growth path, $S_g(\mu) = S_g'(\mu) = 0, \bar{q}^{x,g} = 1, \bar{\epsilon}^{x,g} = 0$ ensure that the shadow price of capital equals unity. Introduce r_t^g as a shorthand for the implied rental rate on government capital:

$$r_t^g = \zeta \frac{y_t + \Phi}{k_t^g/\mu}. \tag{9.49}$$

In the steady state, from (9.48b):

$$\bar{r}^g = \bar{\beta}^{-1} - (1-\delta) \tag{9.50}$$

Equation (9.48b) determines the optimal ratio of government capital to gross output. Importantly, the law of motion for government capital (9.47) and (9.48b) evaluated at the balanced growth path allow to back out the share of government capital in the aggregate production function, for any given government investment to net output ratio $\frac{\bar{x}^g}{\bar{y}}$. From the law of motion along the balanced growth path:

$$\bar{x}^g = \left(1 - \frac{1-\delta}{\mu}\right)\bar{k}^g \quad \Leftrightarrow \quad \frac{\bar{x}^g}{\bar{y}} = [\mu - (1-\delta) frac \bar{k}^g \mu \bar{y}$$

From the equation for r_t^g we have that $\frac{\bar{k}^g}{\mu \bar{y}} = \zeta \frac{\bar{y}+\Phi}{\bar{y}} \frac{1}{\bar{r}^g}$. Combined with the previous equation this allows to solve for the government capital share ζ:

$$\zeta = \frac{\bar{y}}{\bar{y}+\Phi} \frac{\bar{r}^g}{1-(1-\delta)} \frac{\bar{x}}{\bar{y}} \tag{9.51}$$

9.3.2 Monetary policy

The specification of the interest rate rule follows Smets and Wouters (2007). The Federal Reserve sets interest rates according to the following rule:

$$\frac{R_t^{FFR}}{\bar{R}} = \left(\frac{R_{t-1}^{FFR}}{\bar{R}}\right)^{\rho_R} \left[\left(\frac{\pi_t}{\bar{\pi}}\right)^{\psi_1} \left(\frac{Y_t}{Y_t^f}\right)^{\psi_2}\right]^{1-\rho_R} \left(\frac{Y_t/Y_{t-1}}{Y_t^f/Y_{t-1}^f}\right)^{\psi_3} \epsilon_t^r, \tag{9.52}$$

where ρ_R determines the degree of interest rate smoothing and Y_t^f denotes the level of output that would prevail in the economy in the absence of nominal frictions and with constant markups, i.e. the flexible output level. $\psi_1 > 1$ determines the reaction to inflation to deviations of inflation from its long-run average and $\psi_2, \psi_3 > 0$ determine the reaction to the deviation of actual output from the flexible economy output and to the change in the gap between actual and flexible output.

Due to financial market frictions, the return on government bonds differs from the federal funds rate:

$$R_t^{gov} = R_t^{FFR}(1 + \omega_t^b)$$

The flexible economy is the limit point of the economy characterized above with $\zeta_p = \zeta_w = 0$ and no markup shocks: $\epsilon_t^{\lambda,p} = \epsilon_t^{\lambda,w} = 0$. From the pricing and wages setting rules this limiting solution implies:

$$\frac{P_t^f(i)}{P_t^f} = [1 + \lambda_p(\mathrm{y}_t^f(i))] mc_t^f(i), \tag{9.53}$$

$$\frac{W_t^f(\ell)}{P_t^f} = [1 + \lambda_w(\mathrm{n}_t^f(\ell))] \frac{1+\tau_t^c}{1-\tau_t^{n,f}} {n_t^f}^{\nu} [C_t^f - hC_{t-1}^f], \tag{9.54}$$

where the superscript f denotes variables in the flexible economy. Given that final goods are the numeraire and given that firms are symmetric and can freely set their prices:

$$1 = P_t^f = P_t^f(i) = [1 + \lambda_p(1)] mc_t^f(i) \quad \forall t, \tag{9.55}$$

implying that marginal costs are constant for all firms.

Similarly, since all unions face a symmetric problem and can freely reset wages we have that, using that the numeraire equals unity and diving be trend growth:

$$\frac{W_t^f(\ell)}{\mu} = \frac{W_t^f}{\mu} = w_t^f = [1 + \lambda_w(1)] \frac{1+\tau_t^c}{1-\tau_t^{n,f}} {n_t^f}^{\nu} [c_t^f - (h/\mu) c_{t-1}^f]. \tag{9.56}$$

Money does not enter explicitly in the economy: the Federal Reserve supplies the amount of money demanded at interest rate R_t.

9.4 Exogenous processes

The exogenous processes are assumed to be log-normally distributed and, with the exception of government spending shocks, to be independent. Government spending shocks are correlated with technology shocks. Shocks to the two mark-up processes follow an ARMA(1,1) process, whereas the other shocks are AR(1) processes.

$$\log \epsilon_t^a = \rho_a \log \epsilon_{t-1}^a + u_t^a, \qquad u_t^a \overset{iid}{\sim} \mathcal{N}(0, \sigma_a^2) \tag{9.57a}$$

$$\log \epsilon_t^r = \rho_r \log \epsilon_{t-1}^r + u_t^r, \qquad u_t^r \overset{iid}{\sim} \mathcal{N}(0, \sigma_r^2) \tag{9.57b}$$

$$\log g_t = \log g_t^a + \tilde{u}_t^g, \tag{9.57c}$$

$$\log g_t^a = (1-\rho_g) \log \bar{g} + \rho_g \log g_{t-1}^a + \sigma_{ga} u_t^a + u_t^g, \qquad u_t^a \overset{iid}{\sim} \mathcal{N}(0, \sigma_a^2) \tag{9.57d}$$

$$\log s_t^{exo} = \tilde{u}_t^s, \tag{9.57e}$$

$$\log \epsilon_t^\tau = \rho_\tau \log \epsilon_{t-1}^\tau + u_t^\tau, \qquad u_t^\tau \overset{iid}{\sim} \mathcal{N}(0, \sigma_\tau^2) \tag{9.57f}$$

$$\log \tilde{\epsilon}_t^{\lambda,p} = \rho_{\lambda,p} \log \tilde{\epsilon}_{t-1}^{\lambda,p} + u_t^{\lambda,p} - \theta_{\lambda,p} u_{t-1}^{\lambda,p}, \qquad u_t^{\lambda,p} \overset{iid}{\sim} \mathcal{N}(0, \sigma_{\lambda,p}^2) \tag{9.57g}$$

$$\log \tilde{\epsilon}_t^{\lambda,w} = \rho_{\lambda,w} \log \tilde{\epsilon}_{t-1}^{\lambda,w} + u_t^{\lambda,w} - \theta_{\lambda,w} u_{t-1}^{\lambda,w}, \qquad u_t^{\lambda,w} \overset{iid}{\sim} \mathcal{N}(0, \sigma_{\lambda,w}^2) \tag{9.57h}$$

$$\log(1+\omega_t^b) \equiv \log q_t^b = \rho_b \log q_{t-1}^b + u_t^b, \qquad u_t^b \overset{iid}{\sim} \mathcal{N}(0, \sigma_b^2) \tag{9.57i}$$

$$\log(1-\omega_t^k) \equiv \log q_t^k = \rho_k \log q_{t-1}^k + u_t^k, \qquad u_t^k \overset{iid}{\sim} \mathcal{N}(0, \sigma_k^2) \tag{9.57j}$$

$$\log q_t^x = \rho_x \log q_{t-1}^x + u_t^x, \qquad u_t^x \overset{iid}{\sim} \mathcal{N}(0, \sigma_x^2) \tag{9.57k}$$

$$\log q_t^{x,g} = \rho_{x,g} \log q_{t-1}^{x,g} + u_t^{x,g}, \qquad u_t^{x,g} \overset{iid}{\sim} \mathcal{N}(0, \sigma_{x,g}^2) \tag{9.57l}$$

Three shocks are deterministic and used for policy counterfactuals only:

$$\tilde{u}_t^s, \tilde{u}_t^g, \tilde{u}_t^{x,g}.$$

9.5 Equilibrium conditions

9.5.1 Aggregation

From the final goods producers' problem (9.1) and using the zero profit condition in the competitive market, net output in nominal and real terms is given by

$$P_t Y_t = \int_0^1 P_t(i) Y_t(i) di \quad \Leftrightarrow \quad Y_t = \int_0^1 \frac{P_t(i)}{P_t} Y_t(i) di.$$

Outside the flexible economy, relative prices differ from unity, so that output is not simply the average production of intermediates. However, to a first order price dispersion is irrelevant because $y_t(i) \approx y_t - \eta_p(1) y_t \left(\frac{P_t(i)}{P_t} - 1 \right)$, so that the dispersion term averages out in the aggregate $\int_0^1 y_t(i) di \approx y_t$.

In the presence of heterogeneous labor, the measurement of labor supply faces similar issues because

$$n_t = \int_0^1 \frac{W_t(\ell)}{W_t} n_t(\ell) d\ell,$$

which, by analogy to the above argument for output, generally differs from average hours. However, to a first order:

$$\int_0^1 n_t(\ell) d\ell \approx n_t \tag{9.58}$$

Non-credit constrained households are indexed by $j \in [0, 1-\phi]$ and there is measure $1-\phi$ of these households in the economy. Each non-credit constraint household supplies $K_t(j) = K_t^{RA}$ units of capital services, so that total holdings of capital capital and government bonds per intertemporally optimizing household are given by $\frac{1}{1-\phi}$ times the aggregate quantity. Similarly, household investment is a multiple of aggregate investment. To see this, note that aggregate quantities of bond holdings B_t, investment X_t, physical capital K_t^p, and capital services K_t are computed as:

$$K_t = \int_0^{1-\phi} K_t(j)\Lambda(dj) = K_t(1-\phi)^{-1}\Lambda([0,1-\phi]) = K_t.$$

Aggregate consumption is given by:

$$C_t = \int_0^1 C_t(j)\Lambda(dj) = \int_0^{1-\phi} C_t^{RA}\Lambda(dj) + \int_{1-\phi}^1 C_t^{RoT}\Lambda(dj) = (1-\phi)C^R A_t + \phi C_t^{RoT}. \tag{9.59}$$

Given the consumption of rule-of-thumb agents (9.36), that of intertemporally optimizing agents is given by:

$$\bar{c}^{RA} = \frac{\bar{c} - \phi\bar{c}^{RoT}}{1-\phi}. \tag{9.60}$$

Similarly, aggregate transfers are given by

$$S_t = (1-\phi)S_t^{RA} + \phi S_t^{RoT}, \tag{9.61}$$

where equation (9.37) implies that:

$$\bar{s} = \bar{s}^{RA} + \bar{s}^{RoT}.$$

Aggregate labor supply coincides with individual labor supply of either type of household.

9.5.2 Market Clearing

Labor market clearing requires that labor demanded by intermediaries equals labor supplied by labor packers:

$$\int_0^1 n_t(i)di = n_t = n_t \int_0^1 \frac{W_t(\ell)}{W_t} n_t(\ell)d\ell,$$

where $n_t(\ell)$ is measured in units of the differentiated labor supplies and n_t is measured in units which differs from those supplied by households.

Adding the government and the budget constraints of the two types of households, integrated over $[0, 1-\phi]$ and $(1-\phi, 1]$, respectively, and substituting $\int_0^1 n_t(j) W_t^h (1+\lambda_{t,w}) dj = W_t n_t$, which results from combining the labor packers' zero profit condition with the union problem into the household budget constraint, yields the following equation:

$$C_{t+s} + X_{t+s}(j) + G_t + X_{t+s}^g = n_t \frac{W_{t+s}}{P_{t+s}} + \left[\frac{R_{t+s}^k u_{t+s}}{P_{t+s}} - a(u_{t+s})\right] K_{t+s-1}^p + \frac{\Pi_{t+s}^p \mu^{t+s}}{P_{t+s}},$$

Detrending and substituting in for real profits from (9.17a), using that $w_t \int_0^1 n_t(i) di = w_t n_t$:

$$c_{t+s} + x_{t+s} + \bar{y} g_{t+s} + x_{t+s}^g = y_{t+s} - a(u_{t+s}) \mu k_{t+s-1}^p, \tag{9.62}$$

which is the goods market clearing condition: Production is used for government and private consumption, government and private investment, as well as variations in capacity utilization.

9.6 Linearized equilibrium conditions

9.6.1 Firms

Log-linearizing the production function around the symmetric balanced growth path:

$$\hat{y}_t = \frac{\bar{y} + \Phi}{\bar{y}} \left(\hat{\epsilon}_t^a + \zeta \hat{k}_{t-1}^g + \alpha(1-\zeta)\hat{k}_t + (1-\alpha)(1-\zeta)\hat{n}_t\right). \tag{9.63}$$

The capital-labor ratio is approximated by (9.9):

$$\hat{k}_t = \hat{n}_t + \hat{w}_t - \hat{r}_t^k, \tag{9.64}$$

where symmetry around the balanced growth path was used.

Marginal costs in (9.65) are approximated by

$$\widehat{mc}_t = (1-\alpha)\hat{w}_t + \alpha\hat{r}_t^k - \frac{1}{1-\zeta}\left(\zeta\hat{k}_t^g - \zeta\frac{\bar{y}}{\bar{y}+\Phi}\hat{y}_t + \hat{\epsilon}_t^a\right)\left(\frac{k_t^g}{y_t+\Phi}\right)^{\frac{\zeta}{1-\zeta}}\tilde{\epsilon}_t^a, \tag{9.65}$$

and in the flexible economy from (9.55):

$$\widehat{mc}_t^f = 0 \tag{9.66}$$

To-log linearize the pricing FOC (9.14), note that to a first order the common terms in numerator and denominator, i.e. $\frac{\xi_{t+s}y_{t,t+s}(i)}{\lambda_p(\mathrm{y}_{t+s}(i))\xi_t}$, cancel out, using equation (9.16). As a preliminary step notice that in the absence of mark-up shocks:

$$\begin{aligned}
\overline{mc}d\left(\frac{\eta_p(\mathrm{y}_{t+s}(i))}{1-\eta_p(\mathrm{y}_{t+s}(i))}\right)\Big|_{\mathrm{y}_{t+s}(i)=1} &= \overline{mc}\frac{\bar{\eta}_p}{1-\bar{\eta}_p}\frac{-1}{1-\bar{\eta}_p}\frac{d\eta_p(\mathrm{y}_{t+s}(i))|_{\mathrm{y}_{t+s}(i)=1}}{\bar{\eta}_p} \\
&= -\bar{\lambda}_p\hat{\eta}_p(1)d\left(\frac{P_t^*(i)}{P_{t+s}}\right)\Big|_{\frac{P_{t+s}(i)}{P_{t+s}}=1}, \\
d\left(\frac{P_{t+s}(i)}{P_{t+s}}\right)\Big|_{\frac{P_t^*(i)}{P_{t+s}}=1} &= d\left(\frac{\chi_{t,t+s}}{\prod_{l=1}^s \pi_{t+l}}\right) + d\left(\frac{P_t^*(i)}{P_t}\right).
\end{aligned}$$

Notice that from (9.22):

$$1 + \bar{\lambda}_p\hat{\eta}_p = \frac{1}{\bar{A}_p}$$

To simplify notation and to address mark-up shocks use $\bar{\epsilon}^{\lambda,p} = 1$ define

$$\begin{aligned}
p_t^*(i) &\equiv \frac{P_t^*(i)}{P_t}, \\
\hat{\epsilon}_{t+s}^{\lambda,p} &\equiv \frac{\partial}{\partial\epsilon_{t+s}^{\lambda,p}}\left(\frac{\eta_p(\mathrm{y}_{t+s}(i))}{1-\eta_p(\mathrm{y}_{t+s}(i))}\right)\Big|_{\mathrm{y}_{t+s}(i)=1}\hat{\tilde{\epsilon}}_{t+s}^{\lambda,p} = \frac{\eta_p(1)}{[1-\eta_p(1)]^2}\left(\frac{G'_\epsilon(1)}{G'(1)} - \frac{G''_\epsilon(1)}{G''(1)}\right).
\end{aligned}$$

Now, taking a first-order approximation of (9.14) and using symmetry yields

$$0 = \mathbb{E}_t\sum_{s=0}^{\infty}(\mu\bar{\beta}\zeta_p)^s\left[\hat{p}_t^*(i) + \sum_{l=1}^{s}[\iota_p\hat{\pi}_{t+l-1} - \hat{\pi}_{t+l}]\right](1+\bar{\lambda}_p\hat{\eta}(1)) - [\widehat{mc}_{t+s} + \hat{\epsilon}_{t+s}^{\lambda,p}]$$

$$\Leftrightarrow \frac{1}{1-\bar{\beta}\zeta_p\mu}\frac{1}{\bar{A}_p}\hat{p}_t^* = \mathbb{E}_t\sum_{s=0}^{\infty}(\mu\bar{\beta}\zeta_p)^s[\widehat{mc}_{t+s}+\hat{\epsilon}_{t+s}^{\lambda,p}] - \sum_{l=1}^{s}[\iota_p\hat{\pi}_{t+l-1}-\hat{\pi}_{t+l}]\frac{1}{\bar{A}_p}$$

$$= \widehat{mc}_t + \hat{\epsilon}_t^{\lambda,p} - \frac{\bar{\beta}\mu\zeta_p}{1-\bar{\beta}\mu\zeta_p}\frac{1}{\bar{A}_p}[\iota_p\hat{\pi}_t - \mathbb{E}_t\hat{\pi}_{t+1}]$$

$$+ \mu\bar{\beta}\zeta_p\mathbb{E}_t\mathbb{E}_{t+s}\sum_{s=0}^{\infty}(\mu\bar{\beta}\zeta_p)^s[\widehat{mc}_{t+1+s}+\hat{\epsilon}_{t+1+s}^{\lambda,p}] - \sum_{l=1}^{s}[\iota_p\hat{\pi}_{t+l}-\hat{\pi}_{t+1+l}]\frac{1}{\bar{A}_p}$$

$$= \widehat{mc}_t + \hat{\epsilon}_t^{\lambda,p} - \frac{\bar{\beta}\mu\zeta_p}{1-\bar{\beta}\mu\zeta_p}\frac{1}{\bar{A}_p}[\iota_p\hat{\pi}_t - \mathbb{E}_t\hat{\pi}_{t+1}] + \mu\bar{\beta}\zeta_p\mathbb{E}_t\hat{p}_{t+1}^*.$$

Now, linearizing the evolution of the price index (9.15):

$$\hat{p}_t^* = \frac{\zeta_p}{1-\zeta_p}[\hat{\pi}_t - \iota_p\hat{\pi}_{t-1}] \quad \Leftrightarrow \quad \hat{\pi}_t = \frac{1-\zeta_p}{\zeta_p}\hat{p}_t^* + \iota_p\hat{\pi}_{t-1}.$$

Forwarding the equation once and substituting in and solving for $\hat{\pi}_t$ yields:

$$\hat{\pi}_t = \frac{\iota_p}{1+\iota_p\bar{\beta}\mu}\hat{\pi}_{t-1} + \frac{1-\zeta_p\bar{\beta}\mu}{1+\iota_p\bar{\beta}\mu}\frac{1-\zeta_p}{\zeta_p}\bar{A}_p(\widehat{mc}_t+\hat{\epsilon}_t^{\lambda,p}) + \frac{\bar{\beta}\mu}{1+\iota_p\bar{\beta}\mu}\mathbb{E}_t\hat{\pi}_{t+1} \quad (9.67)$$

9.6.2 Households

The law of motion for capital (9.26) and the fact that individual capital holdings are proportional to aggregate capital holdings implies:

$$\hat{k}_t^p = \left(1-\frac{\bar{x}}{\bar{k}^p}\right)\hat{k}_{t-1}^p + \frac{\bar{x}}{\bar{k}^p}(\hat{x}_t + \hat{q}_{t+s}^x). \quad (9.68)$$

From (9.27), capital services evolve as:

$$\hat{k}_t = \hat{u}_t + \hat{k}_{t-1}^p \quad (9.69)$$

From the static optimality condition (9.31)

$$\hat{w}_t^h = \nu\hat{n}_t + \frac{\hat{c}_t^{RA}-(h/\mu)\hat{c}_{t-1}^{RA}}{1-h/\mu} + \frac{d\tau_t^n}{1-\bar{\tau}^n} + \frac{d\tau_t^c}{1+\tau^c}. \quad (9.70)$$

In the flexible economy, given the absence of mark-up shocks equation (9.56) implies:

$$\hat{w}_t^f = \nu \hat{n}_t^f + \frac{\hat{c}_t^{RA,f} - (h/\mu)\hat{c}_{t-1}^{RA,f}}{1-h/\mu} + \frac{d\tau_t^{n,f}}{1-\bar{\tau}^n} + \frac{d\tau_t^{c,f}}{1+\bar{\tau}^c}. \tag{9.71}$$

In the presence of rigidities, the dynamic wage setting equation (9.40) can be linearized as in the derivation of (9.67), recognizing that the analogue to marginal costs is given by (9.70):[7]

$$\begin{aligned}\hat{w}_t &= \frac{\hat{w}_{t-1}}{1+\bar{\beta}\mu} + \frac{\bar{\beta}\mu \mathbb{E}_t[\hat{w}_{t+1}]}{1+\bar{\beta}\mu} \\ &+ \frac{(1-\zeta_w\bar{\beta}\mu)(1-\zeta_w)}{(1+\bar{\beta}\mu)\zeta_w}\bar{A}_w\left[\frac{1}{1-h/\mu}[\hat{c}_t - (h/\mu)\hat{c}_{t-1}] + \nu\hat{n}_t - \hat{w}_t + \frac{d\tau_t^n}{1-\tau_n} + \frac{d\tau_t^c}{1+\tau_c}]\right] \\ &- \frac{1+\bar{\beta}\mu\iota_w}{1+\bar{\beta}\mu}\hat{\pi}_t + \frac{\iota_w}{1+\bar{\beta}\mu}\hat{\pi}_{t-1} + \frac{\bar{\beta}\mu}{1+\bar{\beta}\mu}\mathbb{E}_t[\hat{\pi}_{t+1}] + \frac{\hat{\epsilon}_t^{\lambda,w}}{1+\bar{\beta}\mu}. \end{aligned} \tag{9.72}$$

From the consumption Euler equation (9.32):

$$\begin{aligned}\mathbb{E}_t[\hat{\xi}_{t+1} - \hat{\xi}_t] &= \mathbb{E}_t\left((\sigma-1)\bar{n}^{1+\nu}[\hat{n}_{t+1} - \hat{n}_t] - \frac{\sigma}{1-h/\mu}\left[\hat{c}_{t+1}^{RA} - \left(1+\frac{h}{\mu}\right)c_t^{RA} + \frac{h}{\mu}\hat{c}_{t+1}^{RA}\right]\right) \\ &= \frac{1}{1-h/\mu}\mathbb{E}_t\big((\sigma-1)\frac{\bar{n}^{1+\nu}[\bar{c}^{RA} - h/\mu\bar{c}^{RA}]}{\bar{c}^{RA}}[\hat{n}_{t+1} - \hat{n}_t]\end{aligned}$$

---

[7] Here, the analogy with marginal costs holds only to a first order. Noting that common terms drop out the first order condition (9.39) and using (9.42) as well as $A_w \equiv [1 + \bar{\lambda}_w\hat{\eta}_w(1)]^{-1}$ linearizes as follows:

$$\begin{aligned} 0 &= \mathbb{E}_t\left(\sum_{s=0}^{\infty}(\zeta_w\mu\bar{\beta})^s\frac{\bar{n}}{\bar{\lambda}_w}\bar{w}^*\left(\big[\hat{w}_t^* + \sum_{l=1}^{s}(\iota_w\hat{\pi}_{t+l-1} - \hat{\pi}_{t+l})\big](1+\bar{\lambda}\hat{\eta}_w(1)) - \bar{\lambda}_w\hat{\eta}_w(1)\hat{w}_{t+s} + \hat{w}_{t+s}^h + \hat{\bar{\epsilon}}_{t+s}^{\lambda,w}\right)\right) \\ &\propto \frac{1}{1-\zeta_w\mu\bar{\beta}}A_w^{-1}[\hat{w}_t^* + \iota_w\hat{\pi}_t - \mathbb{E}_t(\hat{\pi}_{t+1})] \\ &+ \mathbb{E}_t\left(\sum_{s=0}^{\infty}(\zeta_w\mu\bar{\beta})^s\left(\big[A_w^{-1}\sum_{l=1}^{s-1}(\iota_w\hat{\pi}_{t+l} - \hat{\pi}_{t+l+1})\big](1+\bar{\lambda}\hat{\eta}_w(1)) - [A_w^{-1} - 1]\hat{w}_{t+s} - \hat{w}_{t+s}^h - \hat{\bar{\epsilon}}_{t+s}^{\lambda,w}\right)\right) \\ &\propto \frac{1}{1-\zeta_w\mu\bar{\beta}}A_w^{-1}[\hat{w}_t^* + \iota_w\hat{\pi}_t - \mathbb{E}_t(\hat{\pi}_{t+1}) - \zeta_w\mu\bar{\beta}\mathbb{E}_t(w_{t+1}^*)] - \hat{w}_t^h - \hat{\bar{\epsilon}}_t^{\lambda,w} - (1-A_w^{-1})\hat{w}_t \end{aligned}$$

Log-linearizing the law of motion for aggregate wages (9.41) around the symmetric balanced growth path yields:

$$\hat{w}_t^* = \frac{1}{1-\zeta_w}[\hat{w}_t - \zeta_w\hat{w}_{t-1} - \zeta\iota_w\hat{\pi}_{t-1} + \zeta_w\hat{\pi}_t.$$

Substituting this equation into the above for $\hat{w}_t^*, \hat{w}_{t+1}^*$ and re-arranging yields (9.72).

$$
\begin{aligned}
&\quad - \sigma\left[\hat{c}^{RA}_{t+1} - \left(1+\frac{h}{\mu}\right)c^{RA}_t + \frac{h}{\mu}\hat{c}^{RA}_{t+1}\right]\Bigg) \\
&= \frac{1}{1-h/\mu}\mathbb{E}_t\Big((\sigma-1)\frac{1}{1+\bar{\lambda}_w}\frac{1-\bar{\tau}^n}{1+\tau^c}\frac{\bar{w}\bar{n}}{\bar{c}^{RA}}[\hat{n}_{t+1}-\hat{n}_t] \\
&\quad - \sigma\left[\hat{c}^{RA}_{t+1} - \left(1+\frac{h}{\mu}\right)c^{RA}_t + \frac{h}{\mu}\hat{c}^{RA}_{t+1}\right]\Bigg),
\end{aligned}
$$

where the last equality uses (9.42). Solving for current consumption growth:

$$
\begin{aligned}
\hat{c}^{RA}_t &= \frac{1}{1+h/\mu}\mathbb{E}_t[\hat{c}^{RA}_{t+1}] + \frac{h/\mu}{1+h/\mu}\hat{c}^{RA}_{t-1} + \frac{1-h/\mu}{\sigma[1+h/\mu]}\mathbb{E}_t[\hat{\xi}_{t+1}-\hat{\xi}_t] \\
&- \frac{[\sigma-1][\bar{w}\bar{n}/\bar{c}]}{\sigma[1+h/\mu]}\frac{1}{1+\lambda_w}\frac{1-\tau^n}{1+\tau^c}(\mathbb{E}_t[\hat{n}_{t+1}]-\hat{n}_t). \qquad (9.73)
\end{aligned}
$$

The remaining households' FOC linearize as:

$$
\mathbb{E}_t[\hat{\xi}_{t+1}-\hat{\xi}_t] = -\hat{q}^b_t - \hat{R}_t + \mathbb{E}_t[\hat{\pi}_{t+1}], \qquad (9.74a)
$$

$$
\begin{aligned}
\hat{Q}_t &= -\hat{q}^b_t - (\hat{R}_t - \mathbb{E}_t[\pi_{t+1}]) + \frac{1}{\bar{r}^k(1-\tau^k)+\delta\tau^k+1-\delta}\times \\
&\times\left[(\bar{r}^k(1-\tau^k)+\delta\tau^k)\hat{q}^k_t - (\bar{r}^k-\delta)d\tau^k_{t+1} + \bar{r}^k(1-\tau^k)\mathbb{E}_t(\hat{r}^k_{t+1}) + (1-\delta)\mathbb{E}_t(\hat{Q}_{t+1})\right],
\end{aligned}
$$

(9.74b)

$$
\hat{x}_t = \frac{1}{1+\bar{\beta}\mu}\left[\hat{x}_{t-1} + \bar{\beta}\mu\mathbb{E}_t(\hat{x}_{t+1}) + \frac{1}{\mu^2 S''(\mu)}[\hat{Q}_t + \hat{q}^x_t]\right], \qquad (9.74c)
$$

$$
\hat{u}_t = \frac{a'(1)}{a''(1)}\hat{r}^k_t \equiv \frac{1-\psi_u}{\psi_u}\hat{r}^k_t. \qquad (9.74d)
$$

For the credit constrained households, (9.35) implies the following linear consumption process: consumption evolves as

$$
\hat{c}^{RoT}_t = \frac{1}{1+\tau^c}\left(\frac{\bar{s}^{RoT}}{\bar{c}^{RoT}}\hat{s}_t + \frac{\bar{w}\bar{n}}{\bar{c}^{RoT}}[(1-\tau^n)(\hat{w}_t+\hat{n}_t) - d\tau^n_t] - d\tau^c_t + \frac{\bar{y}}{\bar{c}^{RoT}}\frac{d\Pi^p_t}{\bar{y}}\right), \qquad (9.75)
$$

where the change in profits is given by:

$$
\frac{d\Pi^p_t}{\bar{y}} = \frac{1}{1+\lambda_p}\hat{y}_t - \widehat{mc}_t.
$$

9.6.3 Government

The financing need evolves as:

$$\frac{dd_t}{\bar{y}} = \frac{1}{\mu}\Bigg[\mu[\hat{g}^a_t + \hat{g}^s] + \mu\frac{\bar{s}}{\bar{y}}\hat{s}^{exog}_t + \frac{\bar{b}}{\bar{y}}\frac{\hat{b}_{t-1} - \hat{\pi}_t}{\bar{\pi}} - \mu\tau^n\frac{\bar{w}\bar{n}}{\bar{c}}\frac{\bar{c}}{\bar{y}}\Big(\hat{w}_t + \hat{n}_t\Big) - \mu\tau_c\frac{\bar{c}}{\bar{y}}\hat{c}_t - \tau^k[\bar{r}^k r^k_t + (r^k_t - \delta)\hat{k}^p_{t-1}]\mu\frac{\bar{k}}{\bar{y}}\Bigg]. \tag{9.76}$$

In the benchmark case of distortionary labor taxes, Labor tax rates evolve according to (9.45), which is linearized as:

$$\begin{aligned}\bar{\tau}^n\frac{\bar{w}\bar{n}}{\bar{c}}\frac{\bar{c}}{\bar{y}}\left[\frac{d\tau^n_t}{\tau_n}\right] + \hat{\epsilon}^\tau_t &= \psi_\tau\frac{dd_t}{\bar{y}} \\ &= \frac{\psi_\tau}{\mu}\Bigg[\mu[\hat{g}^a_t + \hat{g}^s] + \mu\frac{\bar{s}}{\bar{y}}\hat{s}^{exog}_t + \frac{\bar{b}}{\bar{y}}\frac{\hat{b}_{t-1} - \hat{\pi}_t}{\bar{\pi}} - \mu\tau^n\frac{\bar{w}\bar{n}}{\bar{c}}\frac{\bar{c}}{\bar{y}}\Big(\hat{w}_t + \hat{n}_t\Big) \\ &\quad - \mu\tau_c\frac{\bar{c}}{\bar{y}}\hat{c}_t - \tau^k[\bar{r}^k r^k_t + (r^k_t - \delta)\hat{k}^p_{t-1}]\mu\frac{\bar{k}}{\bar{y}}\Bigg].\end{aligned} \tag{9.77}$$

In general, tax rates, or endogenous transfers satisfy from (9.46):

$$\bar{\tau}^n\frac{\bar{w}\bar{n}}{\bar{c}}\frac{\bar{c}}{\bar{y}}\left[\frac{d\tau^n_t}{\tau_n}\right] + \tau^c\frac{\bar{c}}{\bar{y}}\frac{d\tau^c_t}{\tau^c} + \tau^k\frac{[\bar{r}^k - \delta]\bar{k}}{\bar{y}}\frac{d\tau^k_t}{\tau^k} - \frac{\bar{s}}{\bar{y}}\hat{s}^{endog}_t + \hat{\epsilon}^\tau_t = \psi_\tau\frac{dd_t}{\bar{y}} \tag{9.78}$$

Note how the bond shock is treated here! Might want to change it for estimation etc. purposes. Check!!! Debt holdings are determined from the budget constraint (9.44):

$$\frac{1}{\bar{R}}\frac{\bar{b}}{\bar{y}}[\hat{b}_t - \hat{R}_t - \hat{q}^b_t] = (1-\psi_\tau)\frac{dd_t}{\bar{y}} - \bar{\tau}^n\frac{\bar{w}\bar{n}}{\bar{c}}\frac{\bar{c}}{\bar{y}}\left[\frac{d\tau^n_t}{\tau_n}\right] - \tau^c\frac{\bar{c}}{\bar{y}}\frac{d\tau^c_t}{\tau^c} - \tau^k\frac{[\bar{r}^k - \delta]\bar{k}}{\bar{y}}\frac{d\tau^k_t}{\tau^k} + \frac{\bar{s}}{\bar{y}}\hat{s}^{endog}_t - \hat{\epsilon}^\tau_t \tag{9.79}$$

The linearized counterpart to the law of motion for government capital (9.47) is given by:

$$\hat{k}^g = \left(1 - \frac{\bar{x}^g}{\bar{k}^g}\right)\hat{k}^g_{t-1} + \frac{\bar{x}^g}{\bar{k}^g}\hat{q}^{x,g}_t + \frac{\bar{x}^g}{\bar{k}^g}[\hat{x}^g_t + \hat{\epsilon}^{xg}_t], \tag{9.80}$$

where $u_t^{x,g} \equiv \frac{\tilde{u}_t^{x,g}}{\bar{x}^g}$.

The marginal product of government capital (9.49) is approximated by

$$\hat{r}_t^g = \frac{\bar{y}}{\bar{y}+\Phi}\hat{y}_t - \hat{k}_{t-1}^g \tag{9.81}$$

The shadow price of government capital (9.48b) has the following linear approximation:

$$\hat{Q}_t^g = -(\hat{R}_t + \hat{q}_t^b - \mathbb{E}_t[\pi_{t+1}]) + \frac{1}{\bar{r}^g + 1 - \delta}[\bar{r}^g \mathbb{E}_t(\hat{r}_{t+1}^g) + (1-\delta)\mathbb{E}_t(\hat{Q}_{t+1}^g)], \tag{9.82}$$

The Euler equation for government investment (9.48a) is approximated as:

$$\hat{x}_t^g = \frac{1}{1+\bar{\beta}\mu}\left[\hat{x}_{t-1} + u_{t-1}^{xg} + \bar{\beta}\mu\mathbb{E}_t([\hat{x}_{t+1}^g + u_{t+1}^{xg}]) + \frac{1}{\mu^2 S_g''(\mu)}[\hat{Q}_t^g + \hat{q}_t^{x,g}]\right] - u_t^{xg} \tag{9.83}$$

The monetary policy rule (9.52) is approximated by:

$$\hat{R}_t = \rho_R \hat{R}_{t-1} + (1-\rho_R)[\psi_1 \hat{\pi}_t + \psi_2(\hat{y}_t - \hat{y}_t^f)] + \psi_3 \Delta(\hat{y}_t - \hat{y}_t^f) + \hat{\epsilon}_t^r \tag{9.84}$$

9.6.4 Exogenous processes

The shock processes (9.57) are linearized as

$$\hat{\epsilon}_t^a = \rho_a \hat{\epsilon}_{t-1}^a + u_t^a, \tag{9.85a}$$

$$\hat{\epsilon}_t^r = \rho_r \hat{\epsilon}_{t-1}^r + u_t^r, \tag{9.85b}$$

$$\hat{g}_t = \hat{g}_t^a + \tilde{u}_t^g, \tag{9.85c}$$

$$\hat{g}_t^a = \rho_g \hat{g}_{t-1}^a + \sigma_{ga} u_t^a + u_t^g, \tag{9.85d}$$

$$\hat{s}_t = \tilde{u}_t^s, \tag{9.85e}$$

$$\hat{\epsilon}_t^\tau = \rho_\tau \hat{\epsilon}_{t-1}^\tau + u_t^\tau, \tag{9.85f}$$

$$\hat{\tilde{\epsilon}}_t^{\lambda,p} = \rho_{\lambda,p} \hat{\tilde{\epsilon}}_{t-1}^{\lambda,p} + u_t^{\lambda,p} - \theta_{\lambda,p} u_{t-1}^{\lambda,p}, \tag{9.85g}$$

$$\hat{\tilde{\epsilon}}_t^{\lambda,w} = \rho_{\lambda,w} \hat{\tilde{\epsilon}}_{t-1}^{\lambda,w} + u_t^{\lambda,w} - \theta_{\lambda,w} u_{t-1}^{\lambda,w}, \tag{9.85h}$$

$$\hat{q}_t^b = \rho_b \hat{q}_{t-1}^b + u_t^b, \tag{9.85i}$$

$$\hat{q}_t^k = \rho_k \hat{q}_{t-1}^k + u_t^k, \tag{9.85j}$$

$$\hat{q}_t^x = \rho_x \hat{q}_{t-1}^x + u_t^x, \tag{9.85k}$$

$$\hat{q}_t^{x,g} = \rho_{x,g} \hat{q}_{t-1}^{x,g} + u_t^{x,g}. \tag{9.85l}$$

9.6.5 Aggregation

Aggregate consumption (9.59) and transfers (9.61) are linearized as

$$\hat{c}_t = (1-\phi)\frac{\bar{c}^{RA}}{\bar{c}}\hat{c}_t^{RA} + \phi\frac{\bar{c}^{RoT}}{\bar{c}}\hat{c}_t^{RoT}, \tag{9.86}$$

$$\hat{s}_t = (1-\phi)\frac{\bar{s}^{RA}}{\bar{s}}\hat{s}_t^{RA} + \phi\frac{\bar{s}^{RoT}}{\bar{s}}\hat{s}_t^{RoT}. \tag{9.87}$$

9.6.6 Market Clearing

Goods market clearing:

$$\hat{y}_t = \frac{\bar{c}}{\bar{y}}\hat{c}_t + \frac{\bar{x}}{\bar{y}}\hat{x}_t + \frac{\bar{x}^g}{\bar{y}}\hat{x}_t^g + \hat{g}_t + \frac{\bar{r}^k\bar{k}}{\bar{y}}\hat{u}_t. \tag{9.88}$$

9.6.7 Solution

In addition to the exogenous processes in (9.85), the economy with frictions is reduced to 21 variables, whereas the flexible economy is characterized by 19 variables only, given perfectly flexible prices and wages. Table 13 on the following page lists the remaining variables and the corresponding equations. For the flexible economy, all variables other than those with an "n/a" entry have an ^{f} superscript. The markup shock processes affect only the economy with frictions. Table 14 on page 83 lists the steady state relationships which enter the linearized equations.

Variable	Economy with frictions	Economy without frictions
$\hat{c}$	(9.86)	(9.86)
$\hat{c}^{RA}$	(9.73)	(9.73)
$\hat{c}^{RoT}$	(9.75)	(9.75)
$\hat{x}$	(9.74a) in (9.74c)	(9.74c), (9.74a)
$\hat{k}^{p}$	(9.68)	(9.68)
$\hat{k}$	(9.69)	(9.69)
$\hat{u}$	(9.74d)	(9.74d)
$\hat{Q}$	(9.74a) in (9.74b)	(9.74b), (9.74a)
$\hat{r}^{k}$	(9.64)	(9.64)
$\hat{x}^{g}$	(9.74a) in (9.83)	(9.83), (9.74a)
$\hat{k}^{g}$	(9.80)	(9.80)
$\hat{Q}^{g}$	(9.74a) in (9.82)	(9.82), (9.74a)
$\hat{r}^{g}$	(9.81)	(9.81)
$d\tau^{n}, d\tau^{c}, d\tau^{k}, \hat{s}^{endo}$	one variable according to (9.78) with (9.76) other three variables = 0	(9.78) with (9.76) other three variables = 0
$\hat{b}$	(9.79)	(9.79)
$\hat{R}$	(9.84)	indirectly via (9.66)
$\hat{\pi}$	(9.67)	=0
$\widehat{mc}$	(9.65)	=0
$\hat{w}$	(9.72)	(9.71)
$\hat{y}$	(9.88)	(9.88)
$\hat{n}$	(9.63)	(9.63)

Table 13: Unknowns and equations

Constant	Equation	Expression
$\frac{\bar{c}}{\bar{y}}$	(9.62)	$1 - \frac{\bar{x}}{\bar{y}} - \frac{\bar{x}^g}{\bar{y}} - g$
$\frac{\bar{c}^{RA}}{\bar{y}}$	(9.60)	$\frac{\bar{c} - \phi \bar{c}^{RoT}}{\bar{y}(1-\phi)}$
$\frac{\bar{c}^{RoT}}{\bar{y}}$	(9.36)	$\frac{\bar{s}^{RoT} + (1-\tau^n)\bar{w}\bar{n}}{\bar{y}(1+\tau^c)}$
$\frac{\bar{x}}{\bar{k}^p}$	(9.30)	$1 - \frac{1-\delta}{\mu}$
$\frac{\bar{x}}{\bar{k}}$	(9.30)	$\mu - (1-\delta)$
$\frac{\bar{k}}{\bar{y}}$	(9.8)	$\left(\frac{\bar{y}+\Phi}{\bar{y}}\right)^{\frac{1}{1-\zeta}} \left(\frac{\bar{k}^g}{\bar{y}}\right)^{\frac{-\zeta}{1-\zeta}} \left(\frac{\bar{k}}{\bar{n}}\right)^{1-\alpha}$
$\bar{u}$	normalization	$a'^{-1}(\bar{r}^k)$
$\bar{\beta}$	definition	$\beta\mu^{-1}$
$\bar{r}^k$	(9.33c)	$\frac{\bar{\beta}^{-1} - \delta\tau^k - (1-\delta)}{1-\tau^k}$
$\frac{\bar{k}^g}{\bar{y}}$	(9.47)	$\left(1 - \frac{1-\delta}{\mu}\right)^{-1} \frac{\bar{x}^g}{\bar{y}}$
ζ	(9.51)	$\frac{\bar{y}}{\bar{y}+\Phi} \frac{\bar{r}^g}{1-(1-\delta)/\mu} \frac{\bar{x}}{\bar{y}}$
$\bar{r}^g$	(9.50)	$\bar{\beta}^{-1} - (1-\delta)$
$\bar{R}$	(9.33b)	$\bar{\beta}^{-1}\bar{\pi}$
$\overline{mc}$	(9.16)	$(1+\bar{\lambda}_p)^{-1}$
$\bar{\lambda}_p$	(9.18)	$\frac{\Phi}{\bar{y}}$
$\bar{w}$	(9.11)	$\frac{\alpha^{\frac{\alpha}{1-\alpha}}(1-\alpha)}{(1+\lambda_w)^{\frac{1}{(1-\zeta)(1-\alpha)}}} \frac{\left(\frac{\bar{k}^g}{\bar{y}}\right)^{\frac{\zeta}{(1-\zeta)(1-\alpha)}}}{\bar{r^k}^{\frac{\alpha}{1-\alpha}}}$
$\frac{\bar{w}\bar{n}}{\bar{y}}$	$[n_t(i)], [K_t(i)]$,(9.16),(9.18)	$1 - \bar{r}^k \frac{\bar{k}}{\bar{y}}$
$\frac{\bar{k}}{\bar{n}}$	(9.9)	$\frac{\alpha}{1-\alpha} \frac{\bar{w}}{\bar{r}^k}$

Table 14: Steady state relationships

9.7 Measurement equations

For the estimation of the model, the following measurement equations are appended to the model:

$$\Delta Y_t = 100(\hat{y}_t - \hat{y}_{t-1}) + 100(\mu - 1), \tag{9.89a}$$

$$\Delta C_t = 100(\hat{c}_t - \hat{c}_{t-1}) + 100(\mu - 1), \tag{9.89b}$$

$$\Delta X_t = 100(\hat{x}_t - \hat{x}_{t-1}) + 100(\mu - 1), \tag{9.89c}$$

$$\Delta X_t^g = 100(\hat{x}_t^g - \hat{x}_{t-1}^g) + 100(\mu - 1), \tag{9.89d}$$

$$\Delta \frac{W_t}{P_t} = 100(\hat{w}_t - \hat{w}_{t-1}) + 100(\mu - 1), \tag{9.89e}$$

$$\hat{\pi}_t^{obs} = 100\hat{\pi}_t + 100(\bar{\pi} - 1), \tag{9.89f}$$

$$\hat{R}_t^{obs} = 100\hat{R}_t + 100(\bar{R} - 1), \tag{9.89g}$$

$$\hat{q}_t^{k,obs} = 100\hat{q}_t^k + \bar{\hat{q}}^{k,obs}, \tag{9.89h}$$

$$\hat{n}_t^{obs} = 100\hat{n}_t + \bar{\hat{n}}^{obs}, \tag{9.89i}$$

$$\hat{b}_t^{obs} = 100\hat{b}_t + \bar{\hat{b}}^{obs}. \tag{9.89j}$$

The constants give the inflation rate $\bar{\pi}$ along the balanced growth path and the trend growth rates. $100(\mu - 1)$ represents the deterministic net trend growth imposed on the data,. Note that apart from the trend growth rate and the constant nominal interest rate, the discount factor can be backed out of the constants:

$$\beta = \frac{\bar{\pi}}{\bar{R}}\mu^{\sigma}.$$

The constant terms in the measurement equation are necessary even if the data is demeaned for the particular observation sample because the allocation in the flexible economy cannot be attained in the economy with frictions. Given a non-zero output gap, also other variables will deviate from zero. To see why notice that for the allocations to be the same in both the economy with frictions and the its frictionless counterpart required that the Calvo constraints on price and wage setting were slack – otherwise the equilibrium allocations would differ from that in the flexible economy. Slack

Calvo constraints in turn required that aggregate prices and wages were constant, which implied a constant real wage. Finally, a constant real wage would be inconsistent with the allocation in the flexible economy.

9.8 Welfare implications

To evaluate welfare implications, we approximate the compensating variation in terms of quarterly consumption of each type of agent separately as well as the population weighted average.

Independent of whether a household is constrained or not, equation (9.24) gives the preferences of the household. Using the log-linearized model solution around the deterministic balanced growth path, the lifetime utility of any time-path of consumption and hours worked can be computed as:

$$\begin{aligned}
U_t(\{\hat{c}_{t+s}, \hat{n}_{t+s}\}) &= \sum_{s=0}^{\infty} \beta^s \left[\frac{(\mu^{1-\sigma})^{t+s}}{1-\sigma} \left(\bar{c} \exp[\hat{c}_{t+s}] - \frac{h}{\mu} \bar{c} \exp[\hat{c}_{t+s-1}] \right)^{1-\sigma} \right] \\
&\qquad \times \exp \left[\frac{\sigma - 1}{1+\nu} (\bar{n} \exp[\hat{n}_{t+s}])^{1+\nu} \right] \\
&= (\mu^{1-\sigma})^t \sum_{s=0}^{\infty} [\beta \mu^{1-\sigma}]^s \left[\frac{\bar{c}^{1-\sigma}}{1-\sigma} \left(\exp[\hat{c}_{t+s}] - \frac{h}{\mu} \exp[\hat{c}_{t+s-1}] \right)^{1-\sigma} \right] \\
&\qquad \times \exp \left[- \frac{\bar{n}^{1+\nu}}{1+\nu} \exp[(1+\nu)\hat{n}_{t+s}] \right]^{1-\sigma} \\
&= (\mu^{1-\sigma})^t \frac{\bar{c}^{1-\sigma}}{1-\sigma} \sum_{s=0}^{\infty} [\beta \mu^{1-\sigma}]^s \left[\left(e^{\hat{c}_{t+s}} - \frac{h}{\mu} e^{\hat{c}_{t+s-1}} \right) \exp \left[- \frac{\bar{n}^{1+\nu}}{1+\nu} \exp[(1+\nu)\hat{n}_{t+s}] \right] \right]^{1-\sigma} .
\end{aligned}$$

Now we can compute the compensating variation between to paths of consumption and leisure, with and without the fiscal stimulus as:

$$\Gamma = \left[\frac{\sum_{s=0}^{\infty} [\beta \mu^{1-\sigma}]^s \left(e^{\hat{c}_{t+s}^{ARRA}} - \frac{h}{\mu} e^{\hat{c}_{t+s-1}^{ARRA}} \right) \exp \left[- \frac{\bar{n}^{1+\nu}}{1+\nu} \left(\exp[(1+\nu)\hat{n}_{t+s}^{ARRA}] - 1 \right) \right] \Big)^{1-\sigma}}{\sum_{s=0}^{\infty} [\beta \mu^{1-\sigma}]^s \left(e^{\hat{c}_{t+s}^{wo}} - \frac{h}{\mu} e^{\hat{c}_{t+s-1}^{wo}} \right) \exp \left[- \frac{\bar{n}^{1+\nu}}{1+\nu} \left(\exp[(1+\nu)\hat{n}_{t+s}^{wo}] - 1 \right) \right] \Big)^{1-\sigma}} \right]^{\frac{1}{1-\sigma}} - 1. \tag{9.90}$$

An individual with discount factor β would be willing to give up a fraction Γ of consumption in each period to live in an otherwise identical work with the fiscal stimulus in place.

For large s the deviations from the balanced growth path are numerically indistinguishable from zero. However, since $\beta\mu^{1-\sigma}$ is in practice close to unity, even for $s = 1,000$, the infinite sum has not converged. We therefore approximate:

$$\begin{aligned}
&\sum_{s=0}^{\infty}[\beta\mu^{1-\sigma}]^s \left[\left(e^{\hat{c}_{t+s}} - \frac{h}{\mu}e^{\hat{c}_{t+s-1}}\right)\exp\Big[-\frac{\bar{n}^{1+\nu}}{1+\nu}\big(\exp[(1+\nu)\hat{n}_{t+s}] - 1\big)\Big]\right]^{1-\sigma} \\
&\qquad \approx \sum_{s=0}^{T}[\beta\mu^{1-\sigma}]^s \left[\left(e^{\hat{c}_{t+s}} - \frac{h}{\mu}e^{\hat{c}_{t+s-1}}\right)\exp\Big[-\frac{\bar{n}^{1+\nu}}{1+\nu}\big(\exp[(1+\nu)\hat{n}_{t+s}] - 1\big)\Big]\right]^{1-\sigma} \\
&\qquad\quad + \frac{[\beta\mu^{1-\sigma}]^{T+1}}{1-\beta\mu^{1-\sigma}]^s}(1-h/\mu)^{1-\sigma},
\end{aligned}$$

for some large T. In practice, we use $T = 1000$ but checked the results for $T = 5,000$.

To obtain $\bar{n}^{1+\nu}$, multiply equation (9.42) by $\bar{n}$ and divide by $\bar{y}$. This shows that $\bar{n}^{1+\nu} = \frac{\bar{w}\bar{n}}{\bar{y}}\frac{1}{(1+\bar{\lambda}^w)}\frac{1}{\bar{c}^{RA}/\bar{y}}\frac{1}{1-\frac{h}{\mu}}\frac{1-\bar{\tau}^n}{1+\tau^c}$, which is in terms of the constants in table 14.