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# Intermediary Leverage Cycles and Financial Stability \*

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## Abstract

We develop a theory of financial intermediary leverage cycles in the context of a dynamic model of the macroeconomy. The interaction between a production sector, a financial intermediation sector, and a household sector gives rise to amplification of fundamental shocks that affect real economic activity. The model features two state variables that represent the dynamics of the economy: the net worth and the leverage of financial intermediaries. The leverage of the intermediaries is procyclical owing to risk-sensitive funding constraints. Relative to an economy with constant leverage, financial intermediaries generate higher output and consumption growth and lower consumption volatility in normal times, but at the cost of systemic solvency and liquidity risks. We show that tightening intermediaries' risk constraints affects the systemic risk-return trade-off, by lowering the likelihood of systemic crises at the cost of higher pricing of risk. Our model thus represents a conceptual framework for cyclical macroprudential policies within a dynamic stochastic general equilibrium model.

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\*The views expressed here are the authors' and are not representative of the views of the Federal Reserve Bank of New York or the Federal Reserve System.

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# 1 Introduction

The financial crisis of 2007-09 demonstrated the important role of financial intermediaries in the amplification of fundamental shocks, spurring a renewed interest in policies toward financial stability. In this paper, we develop a dynamic stochastic general equilibrium model that captures the leverage cycle of financial intermediaries and the relation between asset returns and intermediary leverage in an empirically relevant way. Moreover, the model features endogenous systemic solvency and liquidity risk, allowing us to study the impact of macroprudential policies on the systemic risk-return trade-off. We thus provide a conceptual framework capturing the systemic risk of the financial sector within a dynamic macroeconomic model.

While our paper shares many features with the recently emerging literature<sup>1</sup> on financially intermediated macroeconomies, we assume risk-sensitive intermediary funding constraints. This assumption allows us to match empirical regularities about the intermediation and pricing of credit. In particular, risk-based funding constraints give rise to the procyclical leverage behavior emphasized by [Adrian and Shin \(2010a\)](#). In our setup, households can invest in intermediary debt, or directly in the capital stock of the productive sector, which generates the procyclical substitution between intermediated and directly granted credit documented by [Adrian et al. \(2011a\)](#).

In our theory, there are two sources of uncertainty: productivity and liquidity shocks. As there are two sources of uncertainty, equilibrium dynamics are represented by two state variables. These state variables are the (relative) net worth of the financial sector and the leverage of financial intermediaries. The role of the net worth variable goes back to the seminal contributions of [Bernanke and Gertler \(1989\)](#) and [Kiyotaki and Moore \(1997\)](#), who emphasize the easing of financial conditions when net worth is high. One contribution of our paper is to complement the net worth variable with a second state variable, the leverage of financial intermediaries. Movements in the leverage state variable are closely tied to the liquidity shocks that households experience, as these leverage shocks represent funding

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<sup>1</sup>[Brunnermeier and Sannikov \(2011, 2012\)](#), [He and Krishnamurthy \(2012a,b\)](#), [Gertler and Kiyotaki \(2012\)](#), and [Gertler et al. \(2011\)](#) all have recently proposed equilibrium theories with a financial sector.

liquidity shocks to intermediaries.<sup>2</sup> In fact, [Adrian et al. \(2011b\)](#) and [Adrian et al. \(2010\)](#) show that fluctuations in intermediary leverage are tightly linked to the time series and cross section of asset risk premia. [Adrian et al. \(2010\)](#) document a negative relationship between intermediary leverage growth and risk premia which we can generate in the model. In addition, we can express the equilibrium pricing kernel as a function of shocks to leverage, which as been demonstrated to be a good empirical asset pricing approach by [Adrian et al. \(2011b\)](#).

Our model features systemic liquidity and solvency risk. Systemic liquidity risk occurs when the intermediary issues debt with negative expected excess returns, while systemic solvency risk occurs when intermediary’s net worth falls below a threshold. Because our setup features a representative intermediary, its solvency and liquidity risks are systemic by nature. We show that both notions of systemic risk exhibit the “volatility paradox” of [Adrian and Brunnermeier \(2010\)](#): Times of low volatility tend to be associated with a buildup of leverage, which in turn increases forward-looking systemic risk. We also study the systemic risk-return trade-off: Low prices of risk today tend to be associated with larger forward-looking systemic risk measures, suggesting that measures of asset price valuations are useful indicators for systemic risk assessments.

Since our theory captures important empirical regularities about the dynamic interactions between the financial sector and the macroeconomy, it provides a conceptual framework for financial stability policies. In this paper, we focus primarily on one form of prudential policy, which concerns the tightness of intermediaries’ funding constraint. Changes in the tightness of the risk based capital constraint can be viewed as changes to the capital requirement faced by institutions. Our paper is among the few that consider the role of (macro)prudential policies in dynamic equilibrium models explicitly (see [Goodhart et al. \(2012\)](#), [Angelini et al. \(2011\)](#), and [Bianchi and Mendoza \(2011\)](#) for alternative settings). Our main findings are intuitive. We demonstrate that tighter prudential requirements are associated with more expensive credit intermediation and lower forward-looking systemic default risk. While these implications are often discussed by policymakers, they are the

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<sup>2</sup>Notice that, unlike in [Rampini and Viswanathan \(2012\)](#), the second variable is leverage, and not household wealth, which mirrors recent empirical work.

result of complex interactions between the different parts of the economy that can, under some circumstances, lead to results that run counter to simple intuitions. The interactions between the households, the financial intermediaries, and the productive sector lead to a highly nonlinear system. We consider the nonlinearity a desirable feature, as the model is able to capture strong amplification effects. Our theory features both endogenous risk amplification (where fundamental volatility is amplified as in [Danielsson et al. \(2011\)](#)), as well as the creation of endogenous systemic risk. The amplification effects are best understood in comparison to a benchmark setup. As a benchmark, we adopt the same economic structure, but restrict the capital constraint to be constant. <sup>3</sup>

The benchmark model features constant investment, a constant price of capital, and a constant risk-free rate. Fluctuations in output of the benchmark economy are entirely due to productivity shocks, and output is fully insulated from liquidity shocks. In contrast, in our model with a risk based funding constraint, liquidity shocks spill over to real activity, and productivity shocks are amplified. While the existence of the financial sector generates more fluctuations, intermediaries provide consumption smoothing services to households during normal times. As a result, overall growth rates in the economy with the intermediary leverage cycles are higher, but systemic risk states with high volatility and low growth sometimes occur. There is therefore a risk-return trade-off in comparing economies with various funding constraints.

This paper is related to several strands of the literature. [Geanakoplos \(2003\)](#) and [Fostel and Geanakoplos \(2008\)](#) show that leverage cycles can cause contagion and issuance rationing in a general equilibrium model with heterogeneous agents, incomplete markets, and endogenous collateral. [Brunnermeier and Pedersen \(2009\)](#) further show that market liquidity and traders' access to funding are co-dependent, leading to liquidity spirals. Our model differs from that of [Fostel and Geanakoplos \(2008\)](#) as our asset markets are dynamically complete and debt contracts are not collateralized. The leverage cycle in our model comes from the risk-based leverage constraint of the financial intermediaries and is intimately related to the funding liquidity of [Brunnermeier and Pedersen \(2009\)](#). Unlike in their model, however, the

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<sup>3</sup>[Brunnermeier and Sannikov \(2011, 2012\)](#), [He and Krishnamurthy \(2012a,b\)](#), [Gertler and Kiyotaki \(2012\)](#), and [Gertler et al. \(2011\)](#) exhibit various alternative assumptions about intermediaries' leverage constraints.

funding liquidity that matters in our setup is that of the financial intermediaries, not that of speculative traders.

This paper is also related to studies of amplification in models of the macroeconomy. The seminal paper in this literature is [Bernanke and Gertler \(1989\)](#), which shows that the condition of borrowers' balance sheets is a source of output dynamics. Net worth increases during economic upturns, increasing investment and amplifying the upturn, while the opposite dynamics hold in a downturn. [Kiyotaki and Moore \(1997\)](#) show that small shocks can be amplified by credit restrictions, giving rise to large output fluctuations. Instead of focusing on financial frictions in the demand for credit as Bernanke–Gertler and Kiyotaki–Moore do, our theory focuses on frictions in the supply of credit. Another important distinction is that the intermediaries in our economy face leverage constraints that depend on current volatility, which give rise to procyclical leverage. In contrast, the leverage constraints of Bernanke–Gertler and Kiyotaki–Moore are state independent.

[Gertler and Kiyotaki \(2012\)](#) and [Gertler et al. \(2011\)](#) extend the accelerator mechanism of [Bernanke and Gertler \(1989\)](#) and [Kiyotaki and Moore \(1997\)](#) to financial intermediaries. [Gertler et al. \(2011\)](#) consider a model in which financial intermediaries can issue outside equity and short-term debt, making intermediary risk exposure an endogenous choice. [Gertler and Kiyotaki \(2012\)](#) further extend the model to allow for household liquidity shocks as in [Diamond and Dybvig \(1983\)](#). While these models are similar in spirit to that presented in this paper, our model is more parsimonious in nature and allows for endogenous defaultable debt. We can thus investigate the creation of both systemic default and liquidity risk, and the effectiveness of macroprudential policy in mitigating these risks.

Our theory is closely related to the work of [He and Krishnamurthy \(2012a,b\)](#) and [Brunnermeier and Sannikov \(2011, 2012\)](#), who explicitly introduce a financial sector into dynamic models of the macroeconomy. While our setup shares many conceptual and technical features of this work, our points of departure are empirically motivated. We allow households to invest via financial intermediaries as well as directly in the capital stock, a feature strongly supported by the data, which gives rise to important substitution effects between directly granted and intermediated credit. In the setup of He–Krishnamurthy, investment is always intermediated. Furthermore, our model features procyclical intermediary leverage, while

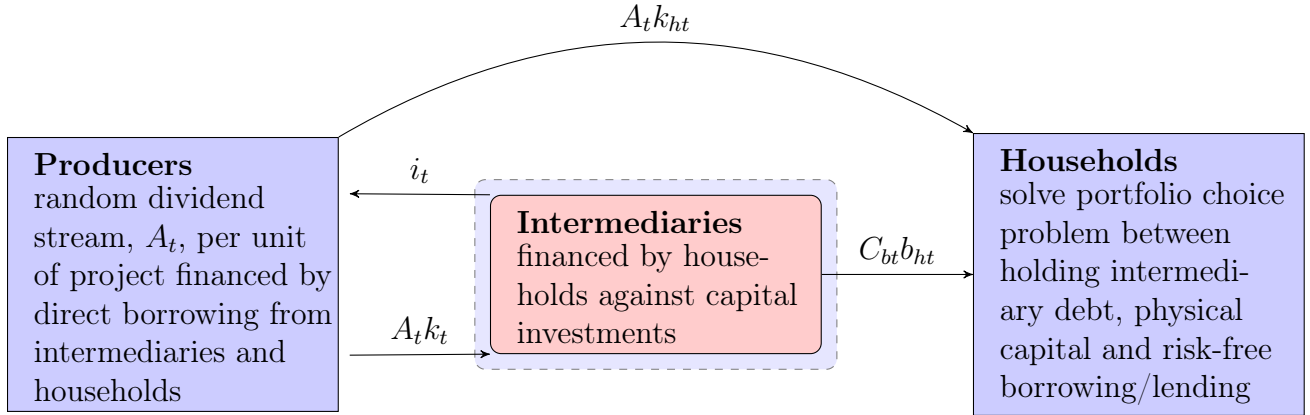


Figure 1: Economy structure

theirs is countercyclical. Finally, systemic risk of the intermediary sector is at the heart of our analysis, while He–Krishnamurthy and Brunnermeier–Sannikov focus primarily on the amplification of shocks.

The rest of the paper is organized as follows. We describe the model in Section 2. The equilibrium interactions and outcomes are outlined in Section 3. We investigate the creation of systemic risk in Section 4 and the amplifications and distortions due to the existence of the financial intermediation system in Section 5. Section 6 concludes. Technical details are relegated to the appendix.

## 2 A Model

We begin with a single consumption good economy, where the unique good at time  $t > 0$  is used as the numeraire. There are three types of agents in the economy: producers, financial intermediaries, and households. In the basic formulation, we abstract from modeling the decisions of the producers and focus instead on the interaction between the intermediary sector and the households. The basic structure of the economy is represented in Figure 1.

### 2.1 Production

We consider an economy with two active types of agents: financially sophisticated intermediaries and unsophisticated households. While both types of agents can own capital, only

financial intermediaries can create new capital through investment. We denote by  $K_t$  the aggregate amount of capital in the economy at time  $t \geq 0$  and assume that each unit of capital produces  $A_t$  units of the consumption good. The total output in the economy at time  $t$  is given by

$$Y_t = A_t K_t,$$

where the stochastic productivity of capital  $\{A_t = e^{a_t}\}_{t \geq 0}$  follows a geometric diffusion process of the form

$$da_t = \bar{a}dt + \sigma_a dZ_{at},$$

where  $(Z_{at})_{0 \leq t < +\infty}$  is a standard Brownian motion defined on the filtered probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ . Each unit of capital in the economy depreciates at a rate  $\lambda_k$ , so that the capital stock in the economy evolves as

$$dK_t = (I_t - \lambda_k) K_t dt,$$

where  $I_t$  is the reinvestment rate per unit of capital in place. Thus, output in the economy evolves according to

$$dY_t = \left( I_t - \lambda_k + \bar{a} + \frac{\sigma_a^2}{2} \right) Y_t dt + \sigma_a Y_t dZ_{at}.$$

Notice that the quantity  $A_t K_t$  corresponds to the “efficiency” capital of [Brunnermeier and Sannikov \(2012\)](#), with a constant productivity rate of 1.

There is a fully liquid market for physical capital in the economy, in which both the financial intermediaries and the households are allowed to participate. To keep the economy scale-invariant, we denote by  $p_{kt} A_t$  the price of one unit of capital at time  $t$  in terms of the consumption good.

## 2.2 Household sector

There is a unit mass of risk-averse, infinitely lived households in the economy. We assume that the households in the economy are identical, such that the equilibrium outcomes are determined by the decisions of the representative household. The households, however, are exposed to a preference shock, modeled as a change-of-measure variable in the household's utility function. This reduced-form approach allows us to remain agnostic as to the exact source of this second shock: With this specification, it can arise either from time-variation in the households' risk aversion or from time-variation in households' beliefs. In particular, we assume that the representative household evaluates different consumption paths  $\{c_t\}_{t \geq 0}$  according to

$$\mathbb{E} \left[ \int_0^{+\infty} e^{-(\xi_t + \rho_h t)} \log c_t dt \right],$$

where  $\rho_h$  is the subjective time discount of the representative household, and  $c_t$  is the consumption at time  $t$ . Here,  $\exp(-\xi_t)$  is the Radon-Nikodym derivative of the measure induced by households' time-varying preferences or beliefs with respect to the physical measure. For simplicity, we assume that  $\{\xi_t\}_{t \geq 0}$  evolves as a Brownian motion, correlated with the productivity shock,  $Z_{at}$ :

$$d\xi_t = \sigma_\xi \rho_{\xi,a} dZ_{at} + \sigma_\xi \sqrt{1 - \rho_{\xi,a}^2} dZ_{\xi t},$$

where  $\{Z_{\xi t}\}$  is a standard Brownian motion of  $(\Omega, \mathcal{F}_t, \mathbb{P})$ , independent of  $Z_{at}$ . In the current setting, with households constrained in their portfolio allocation,  $\exp(-\xi_t)$  can be interpreted as a time-varying liquidity preference shock, as in [Allen and Gale \(1994\)](#) and [Diamond and Dybvig \(1983\)](#), or as a time-varying shock to the preference for early resolution of uncertainty, as in [Bhamra et al. \(2010a,b\)](#). In particular, when the households receive a positive  $d\xi_t$  shock, their effective discount rate increases, leading to a higher demand for liquidity.

The households finance their consumption through holdings of physical capital, holdings of risky intermediary debt, and short-term risk-free borrowing and lending. Unlike the intermediary sector, the households do not have access to the investment technology. Thus,

the physical capital  $k_{ht}$  held by households evolves according to

$$dk_{ht} = -\lambda_k k_{ht} dt.$$

When a household buys  $k_{ht}$  units of capital at price  $p_{kt}A_t$ , by Itô's lemma, the value of capital evolves according to

$$\frac{d(k_{ht}p_{kt}A_t)}{k_{ht}p_{kt}A_t} = \frac{dA_t}{A_t} + \frac{dp_{kt}}{p_{kt}} + \frac{dk_{ht}}{k_{ht}} + \left\langle \frac{dp_{kt}}{p_{kt}}, \frac{dA_t}{A_t} \right\rangle.$$

Each unit of capital owned by the household also produces  $A_t$  units of output, so the total return to one unit of household wealth invested in capital is

$$dR_{kt} = \underbrace{\frac{A_t k_{ht}}{k_{ht} p_{kt} A_t} dt}_{\text{dividend-price ratio}} + \underbrace{\frac{d(k_{ht} p_{kt} A_t)}{k_{ht} p_{kt} A_t}}_{\text{capital gains}} \equiv \mu_{Rk,t} dt + \sigma_{ka,t} dZ_{at} + \sigma_{k\xi,t} dZ_{\xi t}.$$

For future use, notice that, with this notation, we have

$$\frac{dp_{kt}}{p_{kt}} = \left( \mu_{Rk,t} - \frac{1}{p_{kt}} + \lambda_k - \bar{a} - \frac{\sigma_a^2}{2} - \sigma_a (\sigma_{ka,t} - \sigma_a) \right) dt + (\sigma_{ka,t} - \sigma_a) dZ_{at} + \sigma_{k\xi,t} dZ_{\xi t}.$$

In addition to direct capital investment, the households can invest in risky intermediary debt. To keep the balance sheet structure of the financial institutions time-invariant, we assume that the bonds mature at a constant rate  $\lambda_b$ , so that the time  $t$  probability of a bond maturing before time  $t + dt$  is  $\lambda_b dt$ . Notice that this corresponds to an infinite-horizon version of the “stationary balance sheet” assumption of [Leland and Toft \(1996\)](#). Thus, the risky debt holdings  $b_{ht}$  of households follow

$$db_{ht} = (\beta_t - \lambda_b) b_{ht} dt,$$

where  $\beta_t$  is the issuance rate of new debt. The bonds pay a floating coupon  $C_{bt}$  until maturity, with the coupon payment determined in equilibrium to clear the risky bond market. Similarly to capital, risky bonds are liquidly traded, with the price of a unit of intermediary debt at time  $t$  in terms of the consumption good given by  $p_{bt}A_t$ . Hence, the total return from one

unit of household wealth invested in risky debt is

$$dR_{bt} = \underbrace{\frac{(C_{bt} + \lambda_b) A_t b_{ht}}{b_{ht} p_{bt} A_t} dt}_{\text{dividend-price ratio}} + \underbrace{\frac{d(b_{ht} p_{bt} A_t)}{b_{ht} p_{bt} A_t}}_{\text{capital gains}} \equiv \mu_{Rb,t} dt + \sigma_{ba,t} dZ_{at} + \sigma_{b\xi,t} dZ_{\xi t}.$$

When a household with total wealth  $w_{ht}$  buys  $k_{ht}$  units of capital and  $b_{ht}$  units of risky intermediary debt, it invests the remaining  $w_{ht} - p_{kt} k_{ht} - p_{bt} b_{ht}$  at the risk-free rate  $r_{ft}$ , so that the wealth of the household evolves as

$$dw_{ht} = r_{ft} w_{ht} + p_{kt} A_t k_{ht} (dR_{kt} - r_{ft} dt) + p_{bt} A_t b_{ht} (dR_{bt} - r_{ft} dt) - c_t dt. \quad (2.1)$$

We assume that the household faces no-shorting constraints, such that

$$k_{ht} \geq 0$$

$$b_{ht} \geq 0.$$

Thus, the household solves

$$\max_{\{c_t, k_{ht}, b_{ht}\}} \mathbb{E} \left[ \int_0^{+\infty} e^{-(\xi_t + \rho_{ht} t)} \log c_t dt \right], \quad (2.2)$$

subject to the household wealth evolution (2.1) and the no-shorting constraints.

### 2.3 Financial intermediary sector

There is a unit mass of risk-neutral, infinitely lived financial intermediaries in the economy. In the basic formulation, we assume that all the financial intermediaries are identical and thus the equilibrium outcomes are determined by the behavior of the representative intermediary. We abstract from modeling the dividend payment decision (“consumption”) of the intermediary sector and assume that an intermediary invests maximally if the opportunity arises. In particular, financial intermediaries create new capital through capital investment. Denote by  $k_t$  the physical capital held by the representative intermediary at time  $t$  and by  $i_t A_t$  the investment rate per unit of capital. Then the stock of capital held by the representative

intermediary evolves according to

$$dk_t = (\Phi(i_t) - \lambda_k) k_t dt.$$

Here,  $\Phi(\cdot)$  reflects the costs of (dis)investment. We assume that  $\Phi(0) = 0$ , so in the absence of new investment, capital depreciates at the economy-wide rate  $\lambda_k$ . Notice that the above formulation implies that costs of adjusting capital are higher in economies with a higher level of capital productivity, corresponding to the intuition that more developed economies are more specialized. We follow [Brunnermeier and Sannikov \(2012\)](#) in assuming that investment carries quadratic adjustment costs, so that  $\Phi$  has the form

$$\Phi(i_t) = \phi_0 \left( \sqrt{1 + \phi_1 i_t} - 1 \right),$$

for positive constants  $\phi_0$  and  $\phi_1$ .

Each unit of capital owned by the intermediary produces  $A_t(1 - i_t)$  units of output net of investment. As a result, the total return from one unit of intermediary capital invested in physical capital is given by

$$dr_{kt} = \underbrace{\frac{(1 - i_t) A_t k_t}{k_t p_{kt} A_t} dt}_{\text{dividend-price ratio}} + \underbrace{\frac{d(k_t p_{kt} A_t)}{k_t p_{kt} A_t}}_{\text{capital gains}} = dR_{kt} + \left( \Phi(i_t) - \frac{i_t}{p_{kt}} \right) dt.$$

Compared to the households, the financial intermediary earns an extra return to holding firm capital to compensate it for the cost of investment. This extra return is partially passed on to the households as coupon payments on the intermediaries' debt.

It should be noted that financial intermediaries serve two functions in our economy. First, they generate new investment. Second, they provide capital that provides risk-bearing capacity to the households. Compare this with the notion of intermediation of [He and Krishnamurthy \(2012a,b,c\)](#). In their model, intermediaries allow households to access the risky investment technology: Without the intermediary sector, the households can only invest in the risk-free rate. Instead, the households enter into a profit-sharing agreement with the intermediary, with the profits distributed according to the initial wealth contributions.

The intermediaries finance their investment in new capital projects by issuing risky floating coupon bonds to the households. Denoting by  $\beta_t$  the issuance rate of bonds at time  $t$ , the stock of bonds  $b_t$  on a representative intermediary's balance sheet evolves as

$$db_t = (\beta_t - \lambda_b) b_t dt.$$

Each unit of debt issued by the intermediary pays  $C_{bt}$  units of output until maturity and one unit of output at maturity. The total net cost of one unit of intermediary debt is therefore given by

$$dr_{bt} = \underbrace{\frac{(C_{bt} + \lambda_b - \beta_t p_{bt}) A_t b_t}{b_t p_{bt} A_t} dt}_{\text{dividend-price ratio}} + \underbrace{\frac{d(b_t p_{bt} A_t)}{b_t p_{bt} A_t}}_{\text{capital gains}} = dR_{bt}.$$

Thus, the cost of debt to the intermediary equals the return on holding bank debt for the households.

Consider now the budget constraint of an intermediary in this economy. An intermediary in this economy holds capital investment projects ( $k_t$ ) on the assets side of its balance sheet and has bonds ( $b_t$ ) on the liability side. In mathematical terms, we can express the corresponding budget constraint as

$$p_{kt} A_t k_t = p_{bt} A_t b_t + w_t, \tag{2.3}$$

where  $w_t$  is the implicit value of equity in the intermediary. Thus, in terms of flows, the intermediary's equity value evolves according to

$$dw_t = k_t p_{kt} A_t dr_{kt} - b_t p_{bt} A_t dr_{bt}. \tag{2.4}$$

In this paper, we assume that intermediary borrowing is restricted by a risk-based capital constraint, similar to the value at risk ( $VaR$ ) constraint of [Danielsson et al. \(2011\)](#). In

particular, we assume that

$$\alpha \sqrt{\frac{1}{dt} \langle k_t d(p_{kt} A_t) \rangle^2} = w_t, \quad (2.5)$$

where  $\langle \cdot \rangle^2$  is the quadratic variation operator. That is, we assume that the intermediaries are restricted to retain enough equity to cover a certain fraction of losses on their assets. Unlike a traditional *VaR* constraint, this does not keep the volatility of intermediary equity constant, leaving the intermediary sector exposed to solvency risk. The risk-based capital constraint implies a time-varying leverage constraint  $\theta_t$ , defined by

$$\theta_t = \frac{p_{kt} A_t k_t}{w_t} = \frac{1}{\alpha \sqrt{\frac{1}{dt} \langle \frac{d(p_{kt} A_t)}{p_{kt} A_t} \rangle^2}}.$$

Thus, the ratio of the total *VaR* of asset to total assets is negatively related to intermediary leverage, a feature documented by [Adrian and Shin \(2010a\)](#).

Finally, the representative intermediary maximizes equity holder value to solve

$$\max_{\{k_t, \beta_t, i_t\}} \mathbb{E} \left[ \int_0^{\tau_D} e^{-\rho t} w_t dt \right], \quad (2.6)$$

subject to the dynamic intermediary budget constraint (2.4) and the risk-based capital constraint (2.5). Here,  $\rho$  is the subjective discount rate of the intermediary and  $\tau_D$  is the default time of the representative intermediary. We assume that the intermediary defaults when its equity falls below zero, so that

$$\tau_D = \inf_{t \geq 0} \{w_t \leq 0\}.$$

When the intermediary defaults, a fraction  $1 - \bar{\kappa}$  of intermediary capital is lost in the liquidation process. A new intermediary is created to operate the remaining remaining capital,  $\bar{\kappa} k_{\tau_D}^-$ . We assume that the new intermediary is initially all-equity-financed, so that  $w_{\tau_D} = \bar{\kappa} k_{\tau_D}^- p_{k\tau_D} A_{\tau_D}$ .

In addition to solvency risk, the intermediary sector is also exposed to liquidity risk. We

define liquidity to be the state of the economy when the excess return on intermediary debt falls below zero. That is, if we denote by  $\tau_I$  the random time at which the intermediary becomes liquidity-constrained,  $\tau_I$  is the first hitting time of the set  $\mathcal{R}_D = \{(w_t, \theta_t) \in \mathbb{R}^2 : \mu_{Rb,t} - r_{ft} \leq 0\}$

$$\tau_I = \inf_{t \geq 0} \{\mu_{Rb,t} - r_{ft} \leq 0\}.$$

This notion of liquidity risk captures the intuition that intermediaries are liquidity-constrained when they have difficulty rolling over their debt obligations. Since the intermediaries in our economy issue floating coupon debt, this corresponds to intermediaries promising negative excess returns on their debt.

We also introduce the term structure of systemic solvency and liquidity risks to be, respectively,

$$\delta_t(T) = \mathbb{P}(\tau_D \leq T | (w_t, \theta_t))$$

$$\delta_{It}(T) = \mathbb{P}(\tau_I \leq T | (w_t, \theta_t)).$$

Here,  $\delta_t(T)$  is the time  $t$  probability of default occurring before time  $T$ , while  $\delta_{It}(T)$  is the time  $t$  probability of the intermediary becoming liquidity constrained before time  $T$ . Notice that, since the fundamental shocks in the economy are Brownian, and all the agents in the economy have perfect information, the local solvency and liquidity risk is zero.

## 2.4 Equilibrium

**Definition 2.1.** *An equilibrium in this economy is a set of price processes  $\{p_{kt}, p_{bt}, C_{bt}\}_{t \geq 0}$ , a set of household decisions  $\{k_{ht}, b_{ht}, c_t\}_{t \geq 0}$ , and a set of intermediary decisions  $\{k_t, \beta_t, i_t, \theta_t\}_{t \geq 0}$  such that the following apply:*

1. *Taking the price processes and the intermediary decisions as given, the household's choices solve the household's optimization problem (2.2), subject to the household budget constraint (2.1).*

2. Taking the price processes and the household decisions as given, the intermediary's choices solve the intermediary optimization problem (2.6), subject to the intermediary wealth evolution (2.3) and the risk-based capital constraint (2.5).

3. The capital market clears:

$$K_t = k_t + k_{ht}.$$

4. The risky bond market clears:

$$b_t = b_{ht}.$$

5. The risk-free debt market clears:

$$w_{ht} = p_{kt}A_t k_{ht} + p_{bt}A_t b_{ht}.$$

6. The goods market clears:

$$c_t = A_t (K_t - i_t k_t).$$

Notice that the bond markets' clearing conditions imply

$$p_{kt}A_t K_t = w_{ht} + w_t.$$

Notice also that the aggregate capital in the economy evolves as

$$dK_t = -\lambda_k K_t dt + \Phi(i_t) k_t dt.$$

### 3 Solution

To solve for the equilibrium, we introduce two additional state variables. In particular, we define the fraction of the total wealth in the economy held by the financial intermediaries as

$$\omega_t = \frac{w_t}{w_t + w_{ht}} = \frac{w_t}{p_{kt}A_tK_t}.$$

With this definition, the share of total wealth in the economy held by the households is  $(1 - \omega_t)$ . The second state variable we use is  $\theta_t$ , the leverage of the intermediary sector. The vector of state variables in the economy is then

$$(\theta_t, \omega_t).$$

Notice that, by construction, the household belief shocks are expectation-neutral, and thus their level is not a state variable in the economy. Similarly, we have defined prices in the economy to scale with the level of productivity,  $A_t$ , so productivity itself is not a state variable in the scaled version of the economy. We will characterize the equilibrium outcomes in terms of these variables, with the equilibrium conditions determining the time series evolution of  $\theta_t$  and  $\omega_t$  in terms of the primitive shocks in the economy,  $(Z_{at}, Z_{\xi,t})$ . In particular, we will make use of the following representations

$$\begin{aligned} \frac{d\omega_t}{\omega_t} &= \mu_{\omega t}dt + \sigma_{\omega a,t}dZ_{at} + \sigma_{\omega \xi,t}dZ_{\xi t} \\ \frac{d\theta_t}{\theta_t} &= \mu_{\theta t}dt + \sigma_{\theta a,t}dZ_{at} + \sigma_{\theta \xi,t}dZ_{\xi t}. \end{aligned}$$

Notice that, by observing the evolution of  $A_t$ , as well as the two state variables in the economy, we can isolate the time series evolution of the shocks to household beliefs,  $(Z_{\xi t})_{t \geq 0}$ . Notice finally that the *VaR* constraint implies

$$\alpha^{-2}\theta_t^{-2} = \sigma_{ka,t}^2 + \sigma_{k\xi,t}^2.$$

Thus, the riskiness of the return to holding capital increases as intermediary leverage de-

creases. This relation forms the crux of the volatility paradox discussed in detail below: Periods of low volatility of the return to holding capital coincide with high intermediary leverage, which leads to high systemic solvency and liquidity risk. Thus, the riskiness of the return to holding capital increases as intermediary leverage decreases. This relation forms the crux of the volatility paradox discussed in detail below: Periods of low volatility of the return to holding capital coincide with high intermediary leverage, which leads to high systemic solvency and liquidity risk. We plot the theoretical and the empirical trade-off between leverage growth and volatility in Figure 2. Clearly, higher levels of the VIX tend to precede declines in broker-dealer leverage (right panel). In the model, this translates into a negative relationship between contemporaneous volatility and expected leverage growth (left panel). In the left panel of Figure 2, we show that this relationship survives both as intermediary wealth is decreased (going from solid to dashed lines) and as the funding constraint is tightened (going from blue to red). The negative relationship between broker-dealer leverage and the VIX is further investigated in [Adrian and Shin \(2010b,a\)](#).<sup>4</sup>

### 3.1 Capital evolution

Recall from the intermediary’s leverage constraint that

$$\theta_t = \frac{p_{kt}A_t k_t}{w_t}.$$

Using our definition of  $\omega_t$ , we can thus express the amount of capital held by the financial institutions as

$$k_t = \frac{\theta_t w_t}{p_{kt}A_t} = \theta_t \omega_t K_t.$$

---

<sup>4</sup>While [Adrian and Shin \(2010b\)](#) show that fluctuations in primary dealer repo—which is a good proxy for fluctuations in broker-dealer leverage — tend to forecast movements in the VIX, Figure 2 shows that higher levels of the VIX precede declines in broker-dealer leverage. We use the lagged VIX as the VIX is implied volatility and hence a forward-looking measure (though the negative relationship also holds for contemporaneous VIX). [Adrian and Shin \(2010a\)](#) use the *VaR* data of major securities broker-dealers to show a negative association between broker-dealer leverage growth and the *VaRs* of the broker dealers. All of these additional empirical results are fully consistent with our setup.

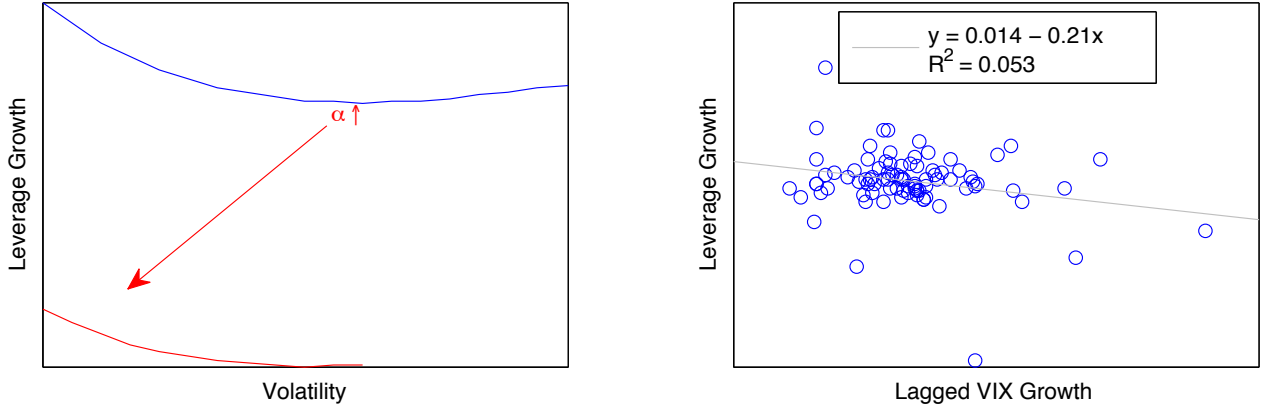


Figure 2: The trade-off between the growth rate of leverage of financial institutions and local volatility. The left panel investigates the shift in the trade-off as intermediary wealth share is decreased (going from the solid line to the dashed line) and the risk-based capital constraint is tightened, increasing  $\alpha$  (going from the blue to the red lines). Data on broker-dealer leverage comes from Flow of Funds Table L.129.

Applying Itô's lemma, we obtain

$$dk_t = \omega_t K_t d\theta_t + \theta_t K_t d\omega_t + \theta_t \omega_t dK_t + K_t \langle d\theta_t, d\omega_t \rangle.$$

Recall, on the other hand, that the intermediary's capital evolves as

$$dk_t = (\Phi(i_t) - \lambda_k) k_t dt.$$

Equating coefficients, we obtain

$$\sigma_{\theta a, t} = -\sigma_{\omega a, t}$$

$$\sigma_{\theta \xi, t} = -\sigma_{\omega \xi, t}$$

$$\mu_{\theta t} = \underbrace{\Phi(i_t)(1 - \theta_t \omega_t)}_{\text{asset growth rate}} - \mu_{\omega t} + \underbrace{\sigma_{\theta a, t}^2 + \sigma_{\theta \xi, t}^2}_{\text{risk adjustment}}.$$

Thus, intermediary leverage is perfectly negatively correlated with the share of wealth held by the financial intermediaries. This reflects the fact that capital stock is not immediately adjustable, so changes in the value of intermediary assets translate one-for-one into changes

in intermediary leverage. Notice further that the intermediary faces a trade-off in the growth rate of its leverage,  $\mu_{\theta t}$ , and the growth rate of its wealth share in the economy,  $\mu_{\omega t}$ .

Figure 3 plots the growth of the share of intermediated credit as a function of total credit growth, showing the strong positive relationship in the model and the data. This positive relationship has been previously documented in Adrian et al. (2011a) and shows the procyclical nature of intermediated finance. The middle panel of Figure 3 shows the procyclical nature of the leverage of financial intermediaries. Leverage tends to expand when balance sheets grow, a fact that has been documented by Adrian and Shin (2010b) for the broker-dealer sector and by Adrian et al. (2011a) for the commercial banking sector. The lower panel shows that the procyclical leverage translates into countercyclical equity growth, both in the data and in the model. We should note that the procyclical leverage of financial intermediaries is closely tied to the risk-based capital constraint. In fact, previous literature has found it a challenge to generate this feature (see Brunnermeier and Sannikov (2011, 2012), He and Krishnamurthy (2012a,b), Bernanke and Gertler (1989), Kiyotaki and Moore (1997), Gertler and Kiyotaki (2012), and Gertler et al. (2011)).

In all three panels of Figure 3, the movement from the solid to the dashed lines represents a decrease in intermediary wealth share. In the lower two panels, such a decline in intermediary wealth corresponds to a steepening of the corresponding relationship. Notice also that tightening of the funding constraint (so that  $\alpha$  is increased going from the blue lines to the red lines) does not change the shape of the relation but, rather, shifts the relations to lower ranges of intermediary leverage growth.

### 3.2 Household's problem

Consider now solving for the optimal consumption and investment rules of the household, given asset prices. Recall that the household solves

$$V_t = \max_{\{c_t, k_{ht}, b_{ht}\}} \mathbb{E} \left[ \int_0^{+\infty} e^{-(\xi_t + \rho_h t)} \log c_t dt \right],$$

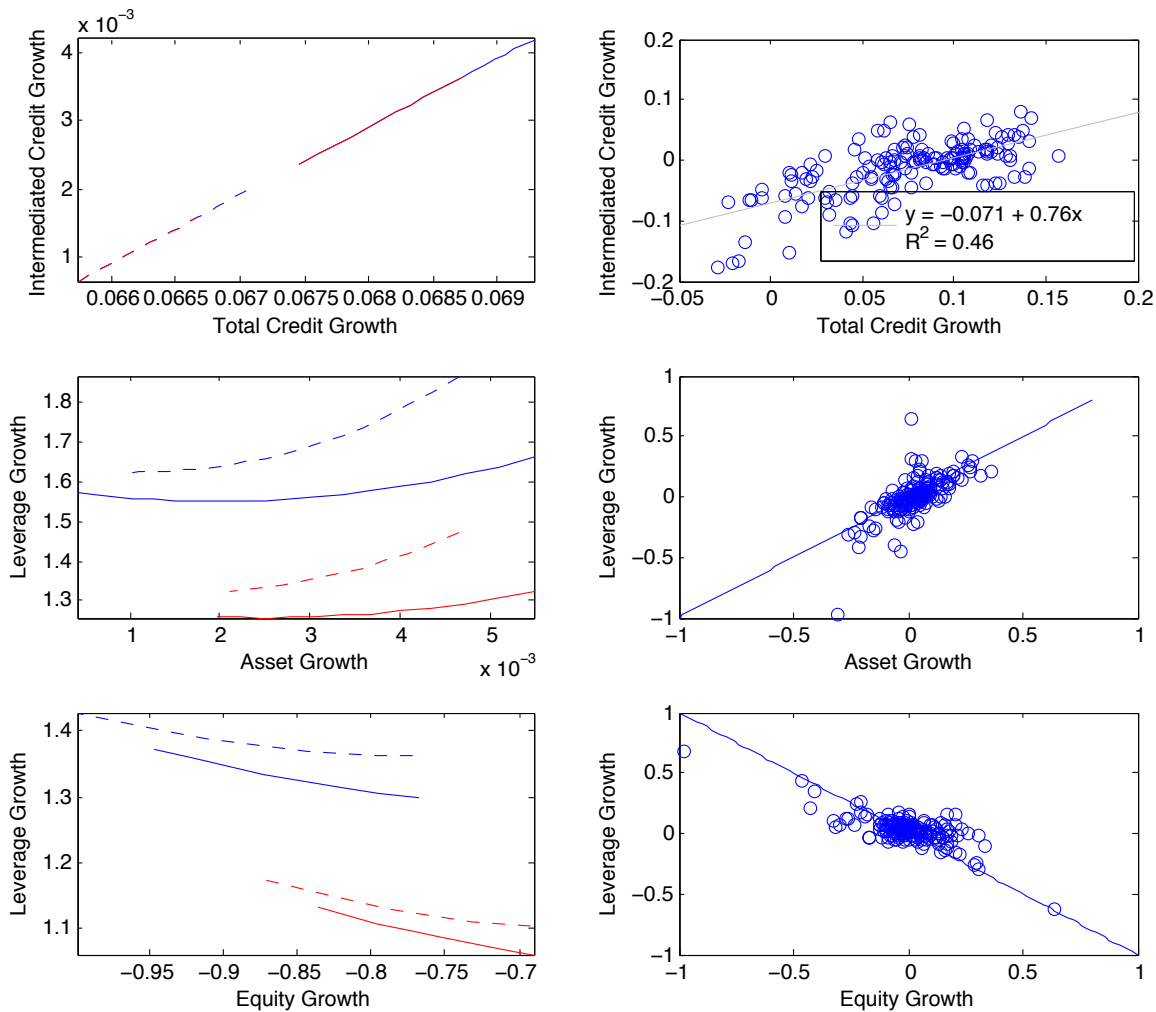


Figure 3: Top panels: The share of intermediated credit increases as a function of total credit extended to nonfinancials, showing the procyclicality of financial intermediation. Middle panels: the trade-off between the growth rate of intermediary leverage and intermediary asset growth predicted by the model (left panel) and observed empirically (right panel), showing the procyclical leverage of financial intermediaries. Lower panels: the trade-off between the growth rate of intermediary leverage and intermediary equity growth predicted by the model (left panel) and observed empirically (right panel). The figure investigates the shift in the trade-off as intermediary wealth share is decreased (going from the solid line to the dashed line) and the risk-based capital constraint is tightened, increasing  $\alpha$  (going from the blue to the red lines). Data on total credit to the nonfinancial corporate sector and the share of intermediated finance come from Flow of Funds Table L.102. Data on broker-dealer leverage, equity, and assets come from Flow of Funds Table L.129.

subject to the household wealth evolution

$$dw_{ht} = r_{ft}w_{ht}dt + p_{kt}A_t k_{ht} (dR_{kt} - r_{ft}dt) + p_{bt}A_t b_{ht} (dR_{bt} - r_{ft}dt) - c_t dt$$

and no-shorting constraints

$$k_{ht} \geq 0$$

$$b_{ht} \geq 0.$$

Denote by  $\pi_{kt} = (p_{kt}A_t k_{ht})/w_{ht}$  the fraction of household wealth invested in the physical capital and by  $\pi_{bt} = (p_{bt}A_t b_{ht})/w_{ht}$  the fraction of household wealth invested in the risky intermediary debt at time  $t$ . Accordingly, we can express the household's budget constraint as

$$dw_{ht} = r_{ft}w_{ht}dt + w_{ht}\pi_{kt} (dR_{kt} - r_{ft}dt) + w_{ht}\pi_{bt} (dR_{bt} - r_{ft}dt) - c_t dt.$$

We have the following result.

**Lemma 3.1.** *The household's optimal consumption choice satisfies*

$$c_t = \left( \rho_h - \frac{\sigma_\xi^2}{2} \right) w_{ht}.$$

*In the unconstrained region, the household's optimal portfolio choice is given by*

$$\begin{aligned} \begin{bmatrix} \pi_{kt} \\ \pi_{bt} \end{bmatrix} &= \left( \begin{bmatrix} \sigma_{ka,t} & \sigma_{k\xi,t} \\ \sigma_{ba,t} & \sigma_{b\xi,t} \end{bmatrix} \begin{bmatrix} \sigma_{ka,t} & \sigma_{ba,t} \\ \sigma_{k\xi,t} & \sigma_{b\xi,t} \end{bmatrix} \right)^{-1} \begin{bmatrix} \mu_{Rk,t} - r_{ft} \\ \mu_{Rb,t} - r_{ft} \end{bmatrix} \\ &\quad - \sigma_\xi \begin{bmatrix} \sigma_{ka,t} & \sigma_{ba,t} \\ \sigma_{k\xi,t} & \sigma_{b\xi,t} \end{bmatrix}^{-1} \begin{bmatrix} \rho_{\xi,a} \\ \sqrt{1 - \rho_{\xi,a}^2} \end{bmatrix}. \end{aligned}$$

*Proof.* See Appendix A.1. □

Thus, the household with the time-varying beliefs chooses consumption as a myopic investor but with a lower rate of discount. The optimal portfolio choice of the household, on the other hand, also includes a hedging component for variations in the Radon-Nikodym derivative,

$\exp(-\xi_t)$ . Since intermediary debt is locally risk-less, however, households do not self-insure against intermediary default.

Notice that, with the notation introduced above, we also have

$$\begin{aligned}\pi_{kt} &= \frac{p_{kt}A_t k_{ht}}{w_{ht}} = \frac{p_{kt}A_t(K_t - k_t)}{(1 - \omega_t)p_{kt}A_t K_t} = \frac{1 - \theta_t \omega_t}{1 - \omega_t} \\ \pi_{bt} &= 1 - \pi_{kt} = \frac{\omega_t(\theta_t - 1)}{1 - \omega_t}.\end{aligned}$$

Thus, the household holds a non-zero amount of intermediary debt while intermediary leverage exceeds one, and a non-zero amount of capital while the unlevered value of intermediary capital share is less than one. Using this result, we can express the excess return to holding capital as

$$\begin{aligned}\mu_{Rk,t} - r_{ft} &= \underbrace{(\sigma_{ka,t}^2 + \sigma_{k\xi,t}^2) \frac{1 - \theta_t \omega_t}{1 - \omega_t}}_{\text{compensation for own risk}} + \underbrace{(\sigma_{ka,t}\sigma_{ba,t} + \sigma_{k\xi,t}\sigma_{b\xi,t}) \frac{\omega_t(\theta_t - 1)}{1 - \omega_t}}_{\text{compensation for risk of correlated asset}} \\ &\quad + \underbrace{\sigma_\xi \left( \sigma_{ka,t}\rho_{\xi,a} + \sigma_{k\xi,t}\sqrt{1 - \rho_{\xi,a}^2} \right)}_{\text{compensation for beliefs risk}}.\end{aligned}$$

Thus, the excess return on holding capital directly has three components. The first compensates households for the direct risk of holding a claim to the volatile output stream, while the second compensates households for the riskiness of holding the correlated asset (risky intermediary debt). The remaining component is the hedging motive for holding capital and compensates households for the risk associated with fluctuations in the Radon-Nikodym derivative,  $\exp(-\xi_t)$ .

Similarly, the excess return to holding risky intermediary debt is given by

$$\begin{aligned}\mu_{Rb,t} - r_{ft} &= \underbrace{(\sigma_{ba,t}^2 + \sigma_{b\xi,t}^2) \frac{\omega_t(\theta_t - 1)}{1 - \omega_t}}_{\text{compensation for own risk}} + \underbrace{(\sigma_{ka,t}\sigma_{ba,t} + \sigma_{k\xi,t}\sigma_{b\xi,t}) \frac{1 - \theta_t \omega_t}{1 - \omega_t}}_{\text{compensation for risk of correlated asset}} \\ &\quad + \underbrace{\sigma_\xi \left( \sigma_{ba,t}\rho_{\xi,a} + \sigma_{b\xi,t}\sqrt{1 - \rho_{\xi,a}^2} \right)}_{\text{compensation for beliefs risk}}.\end{aligned}$$

As with the excess return to direct capital investment, the excess return on risky intermediary

debt has three components. The first compensates households for the direct risk of holding a claim to the volatile coupons, while the second compensates households for the riskiness of holding the correlated asset (direct capital investment). As with capital, the remaining component is the hedging motive for holding capital and compensates households for the risk associated with fluctuations in the Radon-Nikodym derivative,  $\exp(-\xi_t)$ .

In Figure 4 and 5, we plot the equilibrium risk-return trade-off for capital and intermediary debt, and investigate how the equilibrium outcome changes as intermediary wealth share in the economy decreases (top panels), leverage increases (middle panels), and the risk-based capital constraint becomes tighter (bottom panels). Consider first the impact of a decrease in the intermediary's wealth share in the economy. In the capital market, this leads to lower expected excess returns  $\mu_{Rk,t} - r_{ft}$ , lower loading on the shock to productivity,  $\sigma_{ka,t}$ , and higher loading on the shock to household beliefs,  $\sigma_{k\xi,t}$ . In the debt market, on the other hand, a decrease in the intermediary's wealth share in the economy leads to an increase in expected excess return  $\mu_{Rb,t} - r_{ft}$ , but also to a lower loading on the shock to productivity,  $\sigma_{ba,t}$ , and higher loading on the shock to household beliefs,  $\sigma_{b\xi,t}$ . Intuitively, as the intermediary owns a smaller share of wealth in the economy, household beliefs become more prevalent, increasing the sensitivity of excess returns to beliefs shocks. A decrease in intermediary wealth is accompanied by a decrease in investment, making capital less valuable and decreasing the expected excess return to holding capital. As financial intermediaries decrease the rate of investment, they reduce their dependence on outside financing, making intermediary debt more scarce in the economy and increasing the expected excess return to holding intermediary debt.

When the intermediary increases its leverage, the equilibrium in the capital market is almost unchanged. In the debt market, however, this leads to lower expected excess returns  $\mu_{Rb,t} - r_{ft}$ , lower loading on the shock to productivity,  $\sigma_{ba,t}$ , and higher loading on the shock to household beliefs,  $\sigma_{b\xi,t}$ . Intuitively, higher leverage increases the probability of default and, thus, decreases the expected excess return to holding intermediary debt. The trade-off between higher leverage (growth) and the excess return to intermediary debt is illustrated further in the left panels of Figure 6. In particular, we see that the excess return to intermediary debt and capital increase as the growth rate of intermediary leverage decreases.

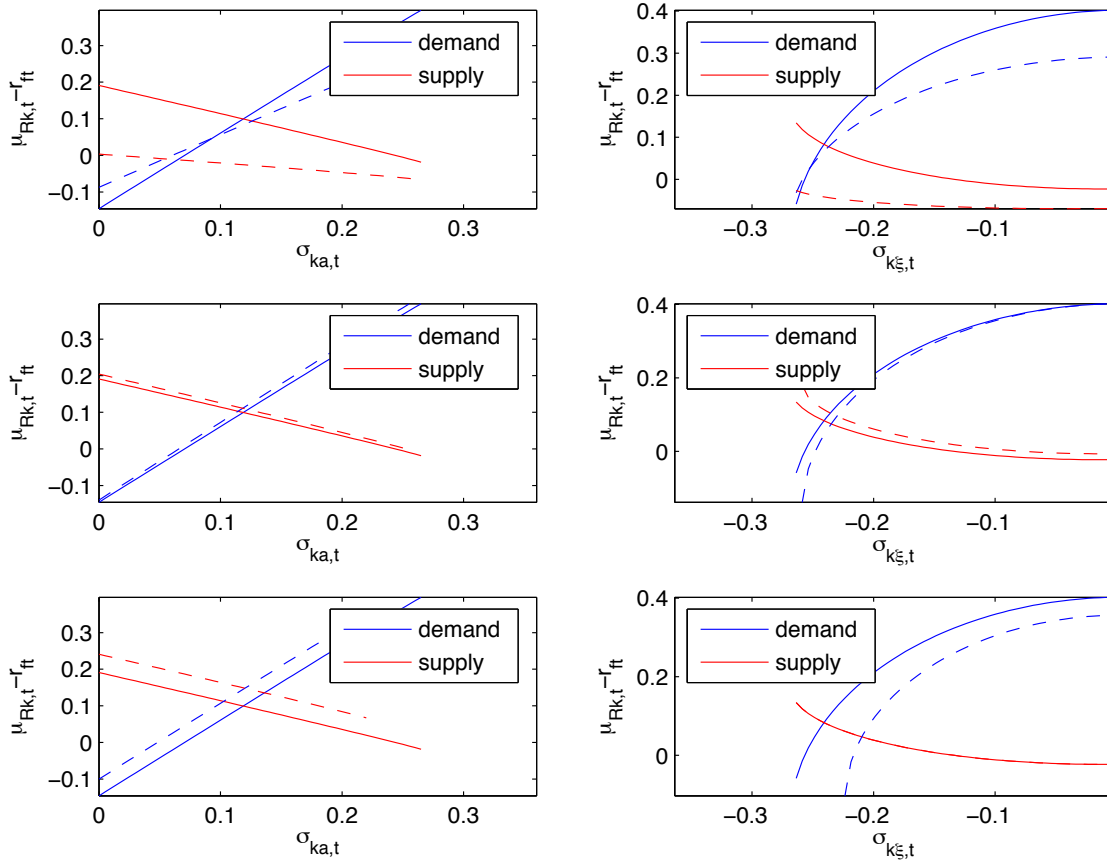


Figure 4: Capital market clearing. “Demand” refers to the capital demand by the households; “supply” refers to the capital demand by the financial institutions. The upper panels investigate the shift in equilibrium as intermediary wealth share is decreased (going from the solid lines to the dashed lines). The middle panels investigate the shift in equilibrium as intermediary leverage is increased (going from the solid lines to the dashed lines). The lower panels investigate the shift in equilibrium as the risk-based capital constraint is tightened, increasing  $\alpha$  (going from the solid lines to the dashed lines).

Furthermore, a decrease in the intermediary’s wealth share in the economy,  $\omega_t$ , makes the relationship between the growth rate of leverage and the excess return to intermediary debt steeper. As a result, the excess return declines faster with an increased rate of leverage growth, while a tightening of the risk-based capital constraint does not mitigate the severity of the trade-off. The right panels of Figure 6 show that this negative relationship between returns and lagged broker-dealer leverage growth holds empirically. In fact, [Adrian et al. \(2010\)](#) document that broker-dealer leverage growth is a good empirical proxy for the time variation of expected returns for a variety of stock and bond portfolios.

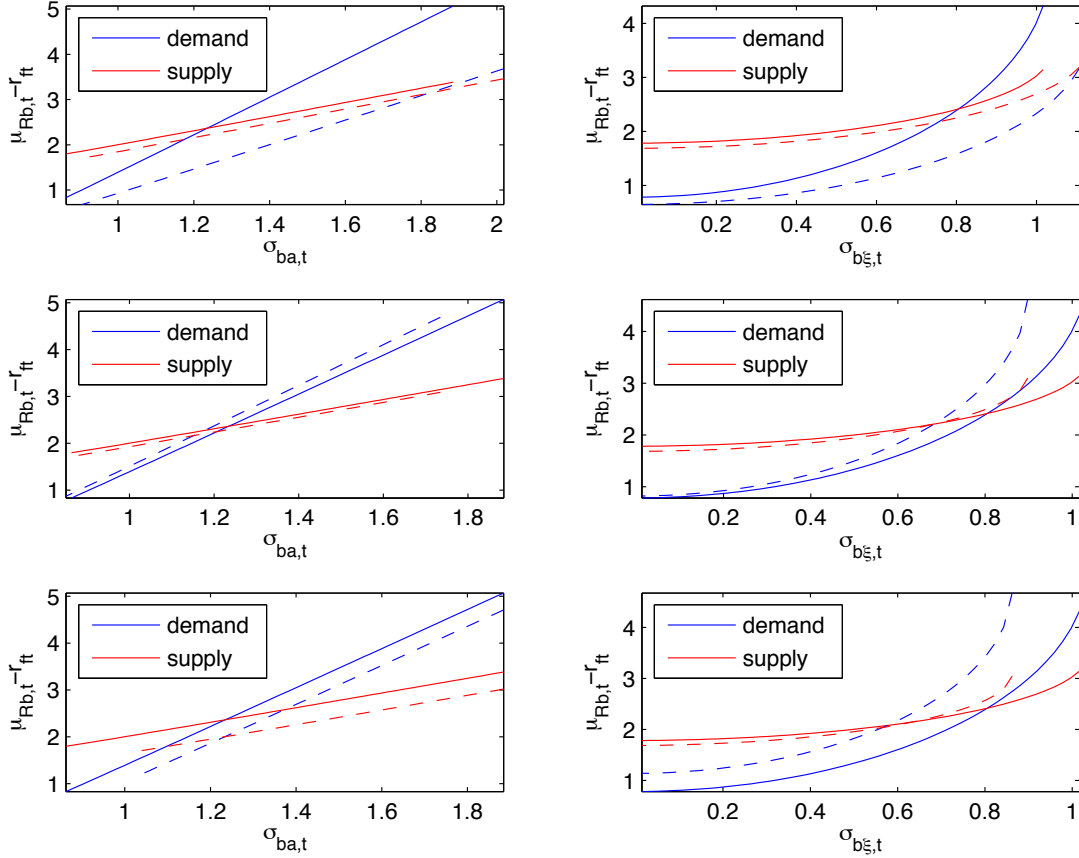


Figure 5: Debt market clearing. “Demand” refers to the debt demand by the households; “supply” refers to the debt supply by the financial institutions. The upper panels investigate the shift in equilibrium as intermediary wealth share is decreased (going from the solid lines to the dashed lines). The middle panels investigate the shift in equilibrium as intermediary leverage is increased (going from the solid lines to the dashed lines). The lower panels investigate the shift in equilibrium as the risk-based capital constraint is tightened, increasing  $\alpha$  (going from the solid lines to the dashed lines).

The left panels of Figure 6 show that a tightening of the risk-based capital constraint does impact the sensitivity of the returns to holding capital and intermediary debt to the fundamental shocks in the economy. In the capital market, an increase in  $\alpha$  leads to an increase in expected excess return and a decrease in sensitivity to the liquidity shock. In the debt market, an increase in  $\alpha$  leads to a decrease in expected excess return, as well as a decrease in the sensitivity to productivity shocks and household liquidity shocks. Intuitively, as the risk-based capital constraint becomes tighter, the financial intermediaries cannot take on as much leverage, making the system more resilient to the transmission of shock to household

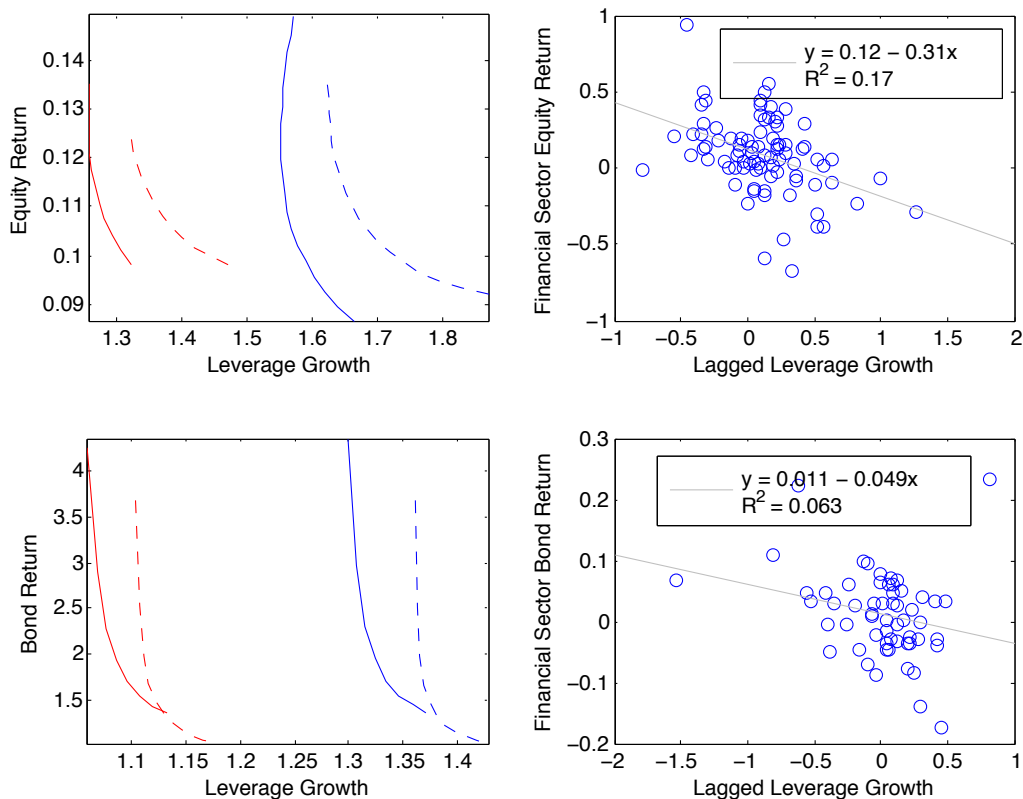


Figure 6: Upper-left panel: the trade-off between the excess return to holding capital and the growth rate of intermediary leverage. Lower-left panel: the trade-off between the excess return to intermediary debt and the growth rate of intermediary leverage. Upper-right panel: the quarterly return to holding the S&P Financial Index as a function of lagged annual broker-dealer leverage growth. Lower-right panel: the quarterly return to holding the Barclays Bond Financial Index as a function of lagged annual broker-dealer leverage growth. The figures in the left panels investigate the shift in equilibrium as intermediary wealth share is decreased (going from the solid line to the dashed line) and as the funding constraint is tightened (going from the blue to the red line). Data on broker-dealer leverage come from Flow of Funds Table L.129 and that on the return to the S&P Financial Index from Haver Analytics and Barclays.

liquidity preference. This makes intermediary debt less risky, reducing the compensation for the risk associated with holding intermediary debt. Since households are less able to insure against liquidity shocks, capital becomes more risky from their viewpoint, increasing the expected excess return to holding capital.

As an aside, notice that, while the economy is unconstrained, the household's pricing kernel

is given by

$$\begin{aligned} \frac{d\Lambda_t}{\Lambda_t} &= -r_{ft}dt - \left( \frac{1 - \theta_t \omega_t}{1 - \omega_t} \sigma_{ka,t} + \frac{\omega_t (\theta_t - 1)}{1 - \omega_t} \sigma_{ba,t} + \sigma_\xi \rho_{\xi,a} \right) dZ_{at} \\ &\quad - \left( \frac{1 - \theta_t \omega_t}{1 - \omega_t} \sigma_{k\xi,t} + \frac{\omega_t (\theta_t - 1)}{1 - \omega_t} \sigma_{b\xi,t} + \sigma_\xi \sqrt{1 - \rho_{\xi,a}^2} \right) dZ_{\xi t}. \end{aligned}$$

While it is natural to express the pricing kernel as a function of the fundamental shocks  $\xi$  and  $a$ , these are not readily observable. Instead, we follow the empirical literature and express the pricing kernel in terms of shocks to output and leverage. Define the innovation to (log) output as

$$d\hat{y}_t = \sigma_a^{-1} (d \log Y_t - \mathbb{E}_t [d \log Y_t]) = dZ_{at}$$

and the innovation to the growth rate of leverage of the intermediaries as

$$d\hat{\theta}_t = (\sigma_{\theta a,t}^2 + \sigma_{\theta \xi,t}^2)^{-\frac{1}{2}} \left( \frac{d\theta_t}{\theta_t} - \mathbb{E}_t \left[ \frac{d\theta_t}{\theta_t} \right] \right) = \frac{\sigma_{\theta a,t}}{\sqrt{\sigma_{\theta a,t}^2 + \sigma_{\theta \xi,t}^2}} dZ_{at} + \frac{\sigma_{\theta \xi,t}}{\sqrt{\sigma_{\theta a,t}^2 + \sigma_{\theta \xi,t}^2}} dZ_{\xi t}.$$

Thus, we can express the pricing kernel as

$$\begin{aligned} \frac{d\Lambda_t}{\Lambda_t} &= -r_{ft}dt - \sqrt{1 + \frac{\sigma_{\theta a,t}^2}{\sigma_{\theta \xi,t}^2}} \left( \frac{1 - \theta_t \omega_t}{1 - \omega_t} \sigma_{k\xi,t} + \frac{\omega_t (\theta_t - 1)}{1 - \omega_t} \sigma_{b\xi,t} + \sigma_\xi \sqrt{1 - \rho_{\xi,a}^2} \right) d\hat{\theta}_t \\ &\quad - \left( \frac{1 - \theta_t \omega_t}{1 - \omega_t} \sigma_{ka,t} + \frac{\omega_t (\theta_t - 1)}{1 - \omega_t} \sigma_{ba,t} + \sigma_\xi \rho_{\xi,a} \right) d\hat{y}_t \\ &\quad + \frac{\sigma_{\theta a,t}}{\sigma_{\theta \xi,t}} \left( \frac{1 - \theta_t \omega_t}{1 - \omega_t} \sigma_{k\xi,t} + \frac{\omega_t (\theta_t - 1)}{1 - \omega_t} \sigma_{b\xi,t} + \sigma_\xi \sqrt{1 - \rho_{\xi,a}^2} \right) d\hat{y}_t. \end{aligned}$$

Hence, the price of risk associated with shocks to the growth rate of intermediary leverage is

$$\eta_{\theta t} = \sqrt{1 + \frac{\sigma_{\theta a,t}^2}{\sigma_{\theta \xi,t}^2}} \left( \frac{1 - \theta_t \omega_t}{1 - \omega_t} \sigma_{k\xi,t} + \frac{\omega_t (\theta_t - 1)}{1 - \omega_t} \sigma_{b\xi,t} + \sigma_\xi \sqrt{1 - \rho_{\xi,a}^2} \right),$$

and the price of risk associated with shocks to output is

$$\begin{aligned} \eta_{yt} = & \frac{1 - \theta_t \omega_t}{1 - \omega_t} \left( \sigma_{ka,t} - \frac{\sigma_{\theta a,t}}{\sigma_{\theta \xi,t}} \sigma_{k\xi,t} \right) + \frac{\omega_t (\theta_t - 1)}{1 - \omega_t} \left( \sigma_{ba,t} - \frac{\sigma_{\theta a,t}}{\sigma_{\theta \xi,t}} \sigma_{b\xi,t} \right) \\ & + \sigma_{\xi} \left( \rho_{\xi,a} - \frac{\sigma_{\theta a,t}}{\sigma_{\theta \xi,t}} \sqrt{1 - \rho_{\xi,a}^2} \right). \end{aligned}$$

Thus, while the households are unconstrained, a two-factor [Merton \(1973\)](#) ICAPM holds, with shocks to intermediary leverage driving the uncertainty about future investment opportunities.

Finally, notice that

$$\frac{dw_{ht}}{w_{ht}} = \frac{d((1 - \omega_t) p_{kt} A_t K_t)}{(1 - \omega_t) p_{kt} A_t K_t}.$$

Thus, the expected rate of change in the financial intermediaries' wealth share in the economy is given by

$$\begin{aligned} \mu_{\omega t} = & \underbrace{(\theta_t - 1) (\mu_{Rkt} - \mu_{Rb,t})}_{\text{expected portfolio return}} - \underbrace{(\sigma_{ka,t} \sigma_{\omega a,t} + \sigma_{k\xi,t} \sigma_{\omega \xi,t})}_{\text{compensation for portfolio risk}} \\ & + \underbrace{\frac{1 - \omega_t}{\omega_t} \left[ \left( \rho_h - \frac{\sigma_{\xi}^2}{2} \right) - \frac{1}{p_{kt}} + \Phi(i_t) \theta_t \omega_t \right]}_{\text{consumption provision to households}}, \end{aligned}$$

and the loadings of the financial intermediaries' wealth share in the economy on the two sources of fundamental risk are given by

$$\begin{aligned} \sigma_{\omega a,t} &= (\theta_t - 1) (\sigma_{ka,t} - \sigma_{ba,t}) \\ \sigma_{\omega \xi,t} &= (\theta_t - 1) (\sigma_{k\xi,t} - \sigma_{b\xi,t}). \end{aligned}$$

That is, the risk loadings of the financial intermediaries' relative wealth reflect the ability of the financial intermediaries to absorb shocks to their balance sheets. The negative sign on the volatility of bond returns reflects the fact that losses in the value of the bonds benefit the intermediaries by reducing the liabilities side of their balance sheets.

### 3.3 Goods market clearing and price of capital

Recall that goods market clearing implies the households consume all output, except that used for investment

$$c_t = A_t (K_t - i_t k_t).$$

Recall further that the only real choice the intermediary has to make (since financing is restricted by the risk-based capital constraint) is in its optimal investment, given by

$$\frac{1}{p_{kt}} = \Phi'(i_t),$$

such that the equilibrium rate of investment is given by

$$i_t = \frac{1}{\phi_1} \left( \frac{\phi_0^2 \phi_1^2}{4} p_{kt}^2 - 1 \right).$$

As the price of capital increases, the book value of intermediary assets increases and the intermediaries are able to invest at a higher rate. Thus, in equilibrium, we must have

$$\underbrace{\left( \rho_h - \frac{\sigma_\xi^2}{2} \right) p_{kt} (1 - \omega_t)}_{\text{household demand}} = \underbrace{1 - \frac{\theta_t \omega_t}{\phi_1} \left( \frac{\phi_0^2 \phi_1^2}{4} p_{kt}^2 - 1 \right)}_{\text{total supply}}.$$

The households' demand for the consumption good is driven by the households' wealth share in the economy,  $1 - \omega_t$ , and the capital price  $p_{kt}$ . The supply of the consumption good, on the other hand, is determined by the financial intermediaries' wealth share in the economy,  $\omega_t$ , financial intermediaries' leverage,  $\theta_t$ , and the capital price. Denoting

$$\beta = \left( \frac{4}{\phi_0^2 \phi_1} \left( \rho_h - \frac{\sigma_\xi^2}{2} \right) \right),$$

the price of capital solves

$$0 = p_{kt}^2 \theta_t \omega_t + \beta p_{kt} (1 - \omega_t) - \frac{4}{\phi_0^2 \phi_1} - \frac{4 \theta_t \omega_t}{\phi_0^2 \phi_1^2},$$

or

$$p_{kt} = \frac{-\beta(1-\omega_t) + \sqrt{\beta^2(1-\omega_t)^2 + \frac{16}{\phi_0^2\phi_1^2}\theta_t\omega_t(\phi_1 + \theta_t\omega_t)}}{2\theta_t\omega_t}. \quad (3.1)$$

As an aside, notice that, for the intermediary to disinvest, we must have

$$(1-\omega_t) \geq \frac{\phi_0\phi_1}{2\left(\rho_h - \frac{\sigma_\xi^2}{2}\right)}.$$

Thus, the intermediary disinvests when the household is a large fraction of the economy—that is, when the intermediary has a relatively low value of equity. Notice that the time  $t$  probability of the intermediary becoming illiquid before time  $T$  is given by

$$\delta_{It}(T) = \mathbb{P}\left(\inf_{t \leq s \leq T} \omega_s \leq 1 - \frac{2}{\beta\phi_0} \middle| \omega_t\right).$$

Applying Itô's lemma and equating coefficients, we obtain

$$\begin{aligned} [dZ_{at}] : \quad & \beta\omega_t\sigma_{\omega a,t} = (2\theta_t\omega_t p_{kt} + \beta(1-\omega_t))(\sigma_{ka,t} - \sigma_a) \\ [dZ_{\xi t}] : \quad & \beta\omega_t\sigma_{\omega \xi,t} = (2\theta_t\omega_t p_{kt} + \beta(1-\omega_t))\sigma_{k\xi,t} \\ [dt] : \quad & 0 = \theta_t\omega_t p_{kt}^2 \left( 2 \left( \mu_{Rk,t} - \frac{1}{p_{kt}} - \bar{a} - \frac{\sigma_a^2}{2} + \lambda_k - \sigma_a(\sigma_{ka,t} - \sigma_a) \right) + \Phi(i_t)(1 - \theta_t\omega_t) \right) \\ & + \beta(1-\omega_t)p_{kt} \left( \mu_{Rk,t} - \frac{1}{p_{kt}} - \bar{a} - \frac{\sigma_a^2}{2} + \lambda_k - \sigma_a(\sigma_{ka,t} - \sigma_a) \right) \\ & - \beta p_{kt}\omega_t\mu_{\omega t} - \frac{4}{\phi_0^2\phi_1^2}\Phi(i_t)(1-\theta_t\omega_t)\theta_t\omega_t + \theta_t\omega_t p_{kt}^2 ((\sigma_{ka,t} - \sigma_a)^2 + \sigma_{k\xi,t}^2) \\ & - \beta p_{kt}\omega_t ((\sigma_{ka,t} - \sigma_a)\sigma_{\omega a,t} + \sigma_{k\xi,t}\sigma_{\omega \xi,t}). \end{aligned}$$

Thus, in equilibrium, the financial intermediaries' wealth ratio in the economy reacts to shocks in the households' beliefs in the same direction as the return to capital.

### 3.4 Equilibrium

We summarize the resulting equilibrium outcomes in the following lemma.

**Lemma 3.2.** *In equilibrium, the expected excess return on capital and risky intermediary debt, as well as the expected return on intermediary equity, the risk-free rate, and the volatility of intermediary equity and intermediary debt, depends linearly on the volatility of the return to holding capital. In particular, we can express the endogenous variables as*

$$\begin{aligned}
\mu_{Rk,t} &= \mathcal{K}_0(\omega_t, \theta_t) + \mathcal{K}_a(\omega_t, \theta_t) \sigma_{ka,t} + \sigma_\xi \sqrt{1 - \rho_{\xi,a}^2} \sigma_{k\xi,t} \\
\mu_{Rb,t} &= \mathcal{B}_0(\omega_t, \theta_t) + \mathcal{B}_a(\omega_t, \theta_t) \sigma_{ka,t} + \mathcal{B}_\xi(\omega_t, \theta_t) \sigma_{k\xi,t} \\
\mu_{\omega t} &= \mathcal{O}_0(\omega_t, \theta_t) + \mathcal{O}_a(\omega_t, \theta_t) \sigma_{ka,t} + \mathcal{O}_\xi(\omega_t, \theta_t) \sigma_{k\xi,t} \\
\mu_{\theta t} &= \mathcal{S}_0(\omega_t, \theta_t) + \mathcal{S}_a(\omega_t, \theta_t) \sigma_{ka,t} - \mathcal{O}_\xi(\omega_t, \theta_t) \sigma_{k\xi,t} \\
r_{ft} &= \mathcal{R}_0(\omega_t, \theta_t) + \mathcal{R}_a(\omega_t, \theta_t) \sigma_{ka,t} \\
\sigma_{ba,t} &= \frac{2\theta_t \omega_t p_{kt} + \beta(1 - \omega_t)}{\beta \omega_t (\theta_t - 1)} \sigma_a - \frac{2\theta_t \omega_t p_{kt} + \beta(1 - \theta_t \omega_t)}{\beta \omega_t (\theta_t - 1)} \sigma_{ka,t} \\
\sigma_{b\xi,t} &= -\frac{2\theta_t \omega_t p_{kt} + \beta(1 - \theta_t \omega_t)}{\beta \omega_t (\theta_t - 1)} \sigma_{k\xi,t} \\
\sigma_{\theta a,t} &= -\frac{2\theta_t \omega_t p_{kt} + \beta(1 - \omega_t)}{\beta \omega_t} (\sigma_{ka,t} - \sigma_a) \\
\sigma_{\theta \xi,t} &= -\frac{2\theta_t \omega_t p_{kt} + \beta(1 - \omega_t)}{\beta \omega_t} \sigma_{k\xi,t},
\end{aligned}$$

where the coefficients  $(\mathcal{K}_0, \mathcal{K}_a, \mathcal{B}_0, \mathcal{B}_a, \mathcal{B}_\xi, \mathcal{O}_0, \mathcal{O}_a, \mathcal{O}_\xi, \mathcal{S}_0, \mathcal{S}_a, \mathcal{R}_0, \mathcal{R}_a)$  are non linear functions of the state variables  $(\omega_t, \theta_t)$ , given by (A.1)-(A.14). The loadings of the return to holding capital on the shock to household beliefs,  $\sigma_{k\xi,t}$ , and on the shock to productivity,  $\sigma_{ka,t}$ , are given, respectively, by

$$\begin{aligned}
\sigma_{k\xi,t} &= -\sqrt{\frac{\theta_t^{-2}}{\alpha^2} - \sigma_{ka,t}^2} \\
\sigma_{ka,t} &= \frac{\theta_t^{-2}}{\alpha^2} + \sigma_a^2 \left( 1 + \frac{1 - \omega_t}{\omega_t (2\theta_t \omega_t p_{kt} + \beta(1 - \omega_t))} \right).
\end{aligned}$$

*Proof.* See Appendix A.2. □

Notice that we pick the negative root in determining the exposure of capital to the household liquidity shocks,  $\sigma_{k\xi,t}$ . Intuitively, when the household experiences a negative liquidity shock, such that  $dZ_{\xi t} < 0$ , the household discount rate is increased, making households more

impatient and decreasing the return to holding capital.

Recall that the price of the risk associated with shocks to intermediary leverage is given by

$$\eta_{\theta t} = \sqrt{1 + \frac{\sigma_{\theta a,t}^2}{\sigma_{\theta \xi,t}^2}} \left( \frac{1 - \theta_t \omega_t}{1 - \omega_t} \sigma_{k\xi,t} + \frac{\omega_t (\theta_t - 1)}{1 - \omega_t} \sigma_{b\xi,t} + \sigma_{\xi} \sqrt{1 - \rho_{\xi,a}^2} \right).$$

Substituting the equilibrium expressions for  $\sigma_{\theta a,t}$ ,  $\sigma_{\theta \xi,t}$  and  $\sigma_{b\xi,t}$ , we obtain

$$\eta_{\theta t} = \sqrt{1 + \frac{(\sigma_{ka,t} - \sigma_a)^2}{\sigma_{k\xi,t}^2}} \left( -\frac{2\theta_t \omega_t p_{kt}}{\beta (1 - \omega_t)} \sigma_{k\xi,t} + \sigma_{\xi} \sqrt{1 - \rho_{\xi,a}^2} \right).$$

Since capital has a negative exposure to the households' preference shocks, the price of risk associated with shocks to intermediary leverage is positive, so leverage risk commands a positive risk premium. While the sign of the risk premium is always positive, the dependence of the price of leverage risk on the leverage growth rate is nonmonotonic. The empirical literature strongly favors the positive price of leverage risk for stock and bond returns (see [Adrian et al. \(2011b\)](#)) and a negative relationship between the price of risk and the growth rate of leverage (see [Adrian et al. \(2010\)](#)).

Similarly, the price of risk associated with shocks to output is given, in equilibrium, by

$$\begin{aligned} \eta_{yt} &= \frac{1 - \theta_t \omega_t}{1 - \omega_t} \left( \sigma_{ka,t} - \frac{\sigma_{\theta a,t}}{\sigma_{\theta \xi,t}} \sigma_{k\xi,t} \right) + \frac{\omega_t (\theta_t - 1)}{1 - \omega_t} \left( \sigma_{ba,t} - \frac{\sigma_{\theta a,t}}{\sigma_{\theta \xi,t}} \sigma_{b\xi,t} \right) \\ &\quad + \sigma_{\xi} \left( \rho_{\xi,a} - \frac{\sigma_{\theta a,t}}{\sigma_{\theta \xi,t}} \sqrt{1 - \rho_{\xi,a}^2} \right) \\ &= \sigma_a + \sigma_{\xi} \left( \rho_{\xi,a} - \frac{\sigma_{ka,t} - \sigma_a}{\sigma_{k\xi,t}} \sqrt{1 - \rho_{\xi,a}^2} \right). \end{aligned}$$

Unlike the price of leverage risk, the price of risk associated with shocks to output changes signs, depending on whether the equilibrium sensitivity of the return to holding capital to output shocks is lower or higher than the fundamental volatility. The time-varying nature of the direction of the risk premium for output shocks makes it difficult to detect in observed returns, suggesting an explanation for the poor performance of the production CAPM in the data.<sup>5</sup> Figure 7 shows the risk prices as a function of leverage growth, and Figure 8 plots the

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<sup>5</sup>For example, [Jermann \(1998\)](#) finds that, in a general equilibrium setting, both capital adjustment costs and habit preferences on the part of consumers are necessary to generate reasonable levels of the equity risk

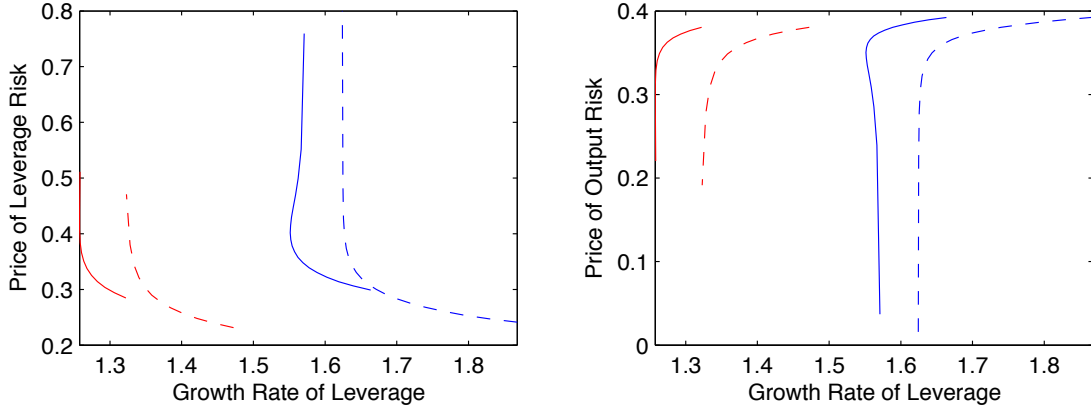


Figure 7: Upper panel: the price of risk associated with shocks to leverage ( $\eta_{\theta t}$ ) as a function of leverage growth. Lower panel: the price of risk associated with shocks to output ( $\eta_{y t}$ ) as a function of leverage growth. The figure investigates the change in equilibrium as intermediary wealth share is reduced (going from the solid line to the dashed line) and as the risk-based capital constraint is tightened (going from the blue to the red line).

risk-return trade-off between the prices of leverage and output risks and the probabilities of insolvency and illiquidity.

## 4 Solvency and Liquidity

We turn now to characterizing the term structure of the probabilities of default,  $\delta_t(T)$ , and of liquidity,  $\delta_{I t}(T)$ . Recall that we have defined intermediary insolvency as the states of the economy in which the expected excess return on holding intermediary debt falls below 0, such that  $\mu_{R b, t} - r_{f t} < 0$ , while liquidity is defined as the states of the economy in which the intermediary disinvests from the capital projects, such that  $i_t < 0$ . Although the corresponding probabilities,  $\delta_t(T)$  and  $\delta_{I t}(T)$ , do not have closed-form solutions, we can easily compute them using Monte Carlo simulations. In this section, we consider the term structure of the probabilities of liquidity and of default, the distribution of outcomes leading to liquidity and default, and the relation between local volatility and the probabilities of liquidity and of default.

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premium.

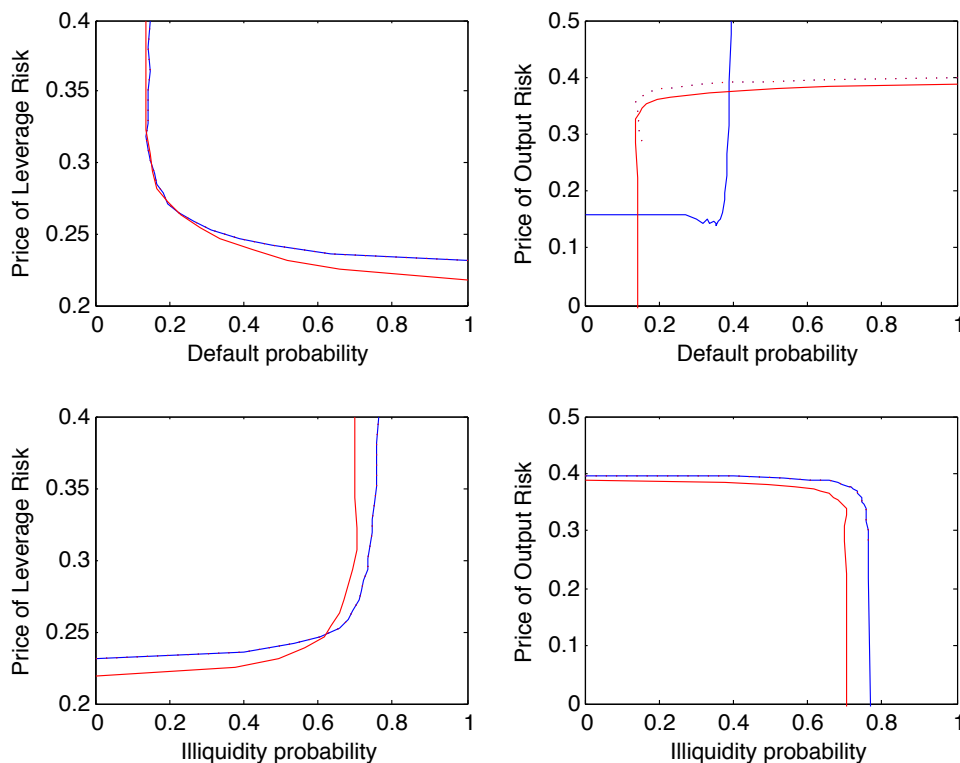


Figure 8: Left panels: the trade-off between the price of risk associated with shocks to leverage ( $\eta_{\theta t}$ ) and the probability of default (upper panel) and the probability of illiquidity (lower panel). Right panels: the trade-off between the price of risk associated with shocks to output ( $\eta_{y t}$ ) and the probability of default (upper panel) and the probability of illiquidity (lower panel). For the red lines, the risk-based capital constraint is tightened, increasing  $\alpha$ .

## 4.1 Term structure of default and liquidity

We begin by considering the term structure of the probabilities of liquidity (left panels) and of default (right panels), plotted in Figure 9. An increase in the intermediary's wealth share in the economy (upper panels) decreases both the probability of liquidity and the probability of default. Intuitively, an increase in the intermediary's wealth share in the economy implies that the intermediary is further from the liquidity boundary. From the previous section, we know that, while an increase in the intermediary's wealth share in the economy reduces the expected excess return to holding intermediary debt, it also reduces the volatility of the excess return, making it less likely that the excess return to intermediary debt will become negative. An increase in intermediary leverage (lower panels) has the opposite effect, increasing both

probabilities. Once again, this acts through an increase in the volatility of the growth rate of the intermediary's wealth share in the economy (making liquidity more likely) and an increase in the volatility of the excess return to intermediary debt (making default more likely). The more interesting effect, however, is that of a tightening of the risk-based capital constraint. While a higher  $\alpha$  does make default less likely, it increases the probability of liquidity. From the previous section, we know that a tightening of the risk-based capital constraint reduces the expected excess return to intermediary debt, reducing equilibrium leverage and, thus, the amount of funds financial intermediaries have to allocate to investment problems. While this has a stabilizing effect in terms of default, it does increase the probability of the intermediary disinvesting from capital projects.

## 4.2 The volatility paradox

We turn finally to the trade-off between the instantaneous riskiness of capital investment, as measured by the local volatility of the return to holding capital, and the long-run fragilities in the economy, as measured by the probabilities of liquidity and of default. Figure 10 plots the trade-off between local volatility and the probability of the intermediaries becoming liquidity constrained, and the trade-off between local volatility and the probability of default. As local volatility increases, the probability of liquidity decreases. This relationship persists even as the initial share of the intermediary's wealth in the economy increases and even as the risk-based capital constraint is tightened. Notice that, while an increase in the initial share of the intermediary's wealth in the economy has a negligible effect on the trade-off for lower values of  $\alpha$ , for a tighter risk-based capital constraint, an increase in the initial share of the intermediary's wealth in the economy leads to a steepening of the trade-off.

While the probability of default exhibits the same behavior for low levels of local volatility, the relationship reverses for more volatile returns to holding capital, with the probability of default increasing as the local volatility increases. Recall that an increase in local volatility of the return to holding capital is associated with lower leverage. The relation between leverage and the volatility of the return to intermediary debt, on the other hand, is not monotone: *Ceteris paribus*, the volatility of the return to intermediary debt increases for only very low levels of leverage. Since the probability of default is primarily determined by the volatility

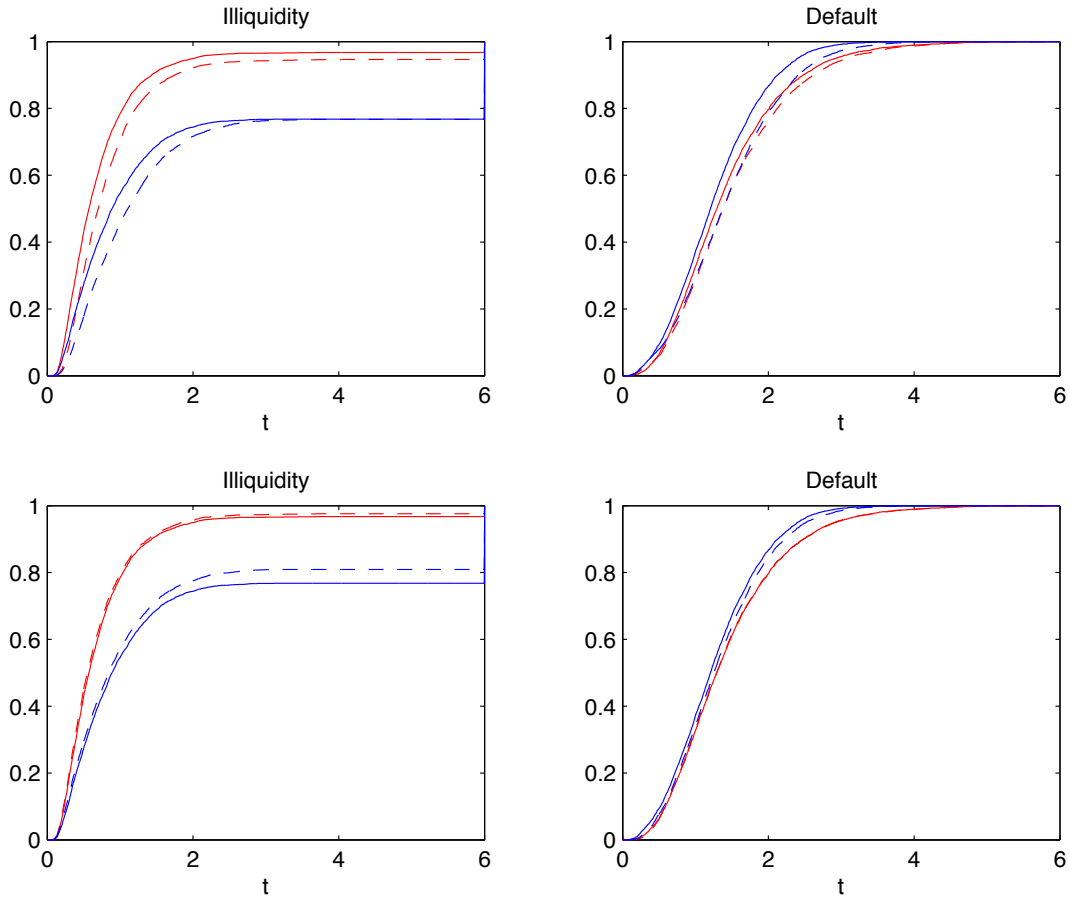


Figure 9: Term structure of the probabilities of liquidity (left panels) and of default (right panels). The upper panels investigate the shift in equilibrium as intermediary wealth share is reduced (going from the solid lines to the dashed lines). The lower panels investigate the shift in equilibrium as the intermediary leverage is increased (going from the solid lines to the dashed lines). For the red lines, the risk-based capital constraint is tightened, increasing  $\alpha$ .

of the excess return to intermediary debt, this results in a nonmonotone relation between the probability of default and the local volatility of the return to holding capital.

### 4.3 Stress tests

By introducing preferences for the financial intermediary, we can extend our model to study the impact of the use of stress tests as a macroprudential tool. By further introducing preferences for the prudential regulator, the model also provides implications for the optimal

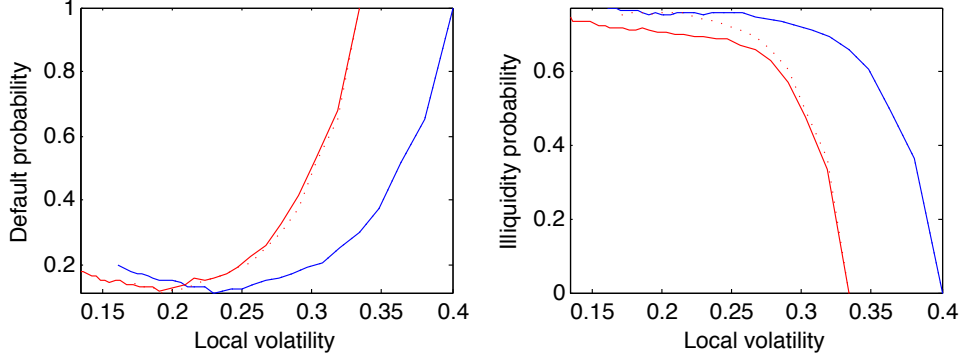


Figure 10: Left panel: the trade-off between local volatility ( $x$ -axis) and the probability of default ( $y$ -axis). Right panel: the trade-off between local volatility ( $x$ -axis) and the probability of intermediaries becoming liquidity constrained ( $y$ -axis). Red lines correspond to a tighter risk-based capital constraint, while dashed lines correspond to a higher initial level of the intermediary's wealth share in the economy,  $\omega_t$ .

design of stress tests. We leave the formal treatment of these extensions for future work and provide here a sketch of how stress tests can be incorporated in the current setting.

Recall that, in our model, intermediary debt is subject to the risk-based capital constraint, which is a constraint on the local volatility of the asset side of the intermediary balance sheet

$$\theta_t^{-1} \geq \alpha \sqrt{\sigma_{ka,t}^2 + \sigma_{k\xi,t}^2}.$$

Stress tests, on the other hand, can be interpreted as a constraint on the total volatility of the asset side of the balance sheet over a fixed time interval

$$\theta_t^{-1} \geq \vartheta \sqrt{\mathbb{E}_t \left[ \int_t^T (\sigma_{ka,s}^2 + \sigma_{k\xi,s}^2) ds \right]}.$$

Thus, in effect, stress tests can be thought of as a Stackelberg game between the policymaker and the financial intermediary, with the policymaker moving first to choose the maximal allowable level of volatility over a time interval, and the intermediary moving second to allocate the volatility allowance between different periods. Under the assumption that the prudential regulator designs stress tests to minimize total volatility, while the intermediary maximizes the expected discounted value of equity, the optimization problem for the intermediary re-

sembles the optimal robust control problem under model misspecification studied by Hansen et al. (2006); Hansen and Sargent (2001); Hansen et al. (1999); Hansen and Sargent (2007), among others

$$V_t(\vartheta) = \max_{\{i,\beta,k\}} \min_{q \in \mathcal{Q}(\vartheta)} \int \int_t^{\tau_D} e^{-\rho(s-t)} w_t(i, \beta, k) ds dq$$

subject to

$$\theta_t^{-1} \geq \vartheta \sqrt{\int_t^T \int (\sigma_{ka,s}^2 + \sigma_{k\xi,s}^2) dq_s ds}.$$

Notice that, in the limit at  $T \rightarrow t + dt$ , this reduces to the risk-based capital constraint described above. In the language of Hansen et al. (2006), this is a *nonsequential* problem since the constraint is over a non-infinitesimal time horizon. The density function  $q$  is a density over the future realizations of the fundamental shocks ( $dZ_{at}, dZ_{\xi t}$ ) in the economy, and  $\mathcal{Q}$  is the set of densities that satisfies the stress-test constraint. Hansen et al. (2006) show how to move from the nonsequential robust controls problems to sequential problems. In particular, for the constraint formulation, they augment the state-space to include the continuation value of entropy and solve for the optimal value function that also depends on this continuation entropy.

In our setting, we can reformulate the optimization problem of the financial intermediary as

$$V_t(\vartheta) = \max_{\{i,\beta,k,\alpha_s\}} \mathbb{E}_t \left[ \int_t^{\tau_D} e^{-\rho(s-t)} w_t(i, \beta, k) ds \right]$$

subject to

$$\begin{aligned} \frac{\theta_s^{-1}}{\alpha_s} &\geq \sqrt{\sigma_{ka,s}^2 + \sigma_{k\xi,s}^2} \\ \theta_t^{-1} &\geq \vartheta \sqrt{\mathbb{E}_t \left[ \int_t^T \frac{\theta_s^{-2}}{\alpha_s^2} ds \right]}. \end{aligned}$$

That is, the intermediary chooses an optimal capital plan at the time of the stress test to maximize the discounted present value of equity subject to satisfying the intertemporal

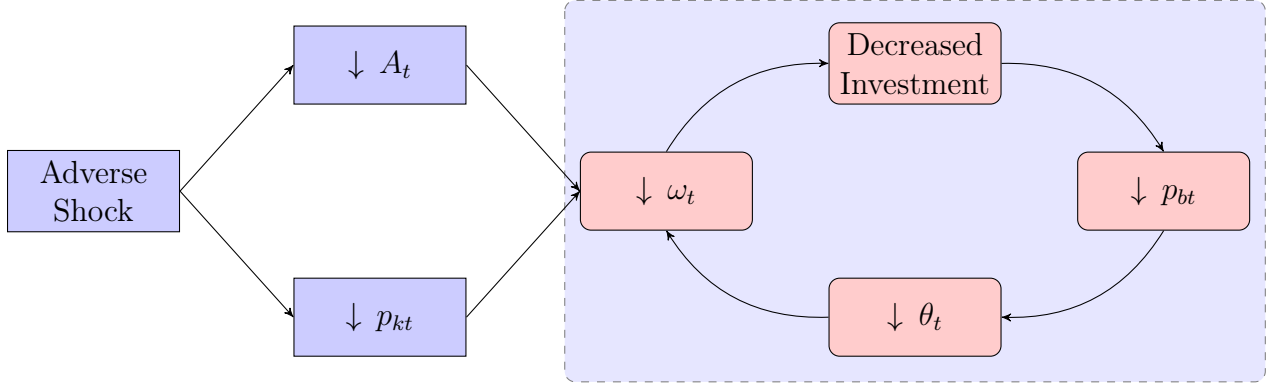


Figure 11: Shock amplification

volatility constraint imposed by the stress test. Locally, the portfolio allocation decision of the intermediary satisfies a risk-based capital constraint, albeit with a time-varying  $\alpha$ . However, along a given capital plan, the optimal decisions of both the households and the intermediary are as described above. Stress tests are hence a natural but technically challenging extension of the current setup and are left for future exploration.

## 5 Distortions and Amplifications

In this section, we investigate how financial frictions amplify the transmission of fundamental shocks in the economy and distort equilibrium outcomes. The basic amplification mechanism, illustrated in Figure 11, acts through the risk-based capital constraint of the representative intermediary: A shock to the relative wealth of the intermediaries reduces the equilibrium level of investment, reducing the price of capital, which makes the risk-based capital constraint bind more, reducing further the financial intermediaries' relative wealth. The amplification mechanism acts through the time-varying leverage constraint that is induced by the risk-sensitive capital constraint. To understand the mechanism better, we describe the equilibrium outcomes in an economy with constant leverage, and contrast the resulting dynamics with those in the full model.

## 5.1 Constant leverage benchmark

In this benchmark, we consider a more conventional economy, in which the financial intermediaries face a constant leverage constraint. In particular, instead of facing the risk-based capital constraint, the intermediaries face a constant leverage constraint, such that

$$\frac{p_{kt}A_tk_t}{w_t} = \bar{\theta},$$

where  $\bar{\theta}$  is a constant set by the prudential regulator. The equilibrium asset prices are summarized in the following lemma.

**Lemma 5.1.** *In the economy with constant leverage and no shocks to households' liquidity preferences, the wealth share of the intermediary in the economy is locally riskless, such that*

$$\begin{aligned}\sigma_{\omega a,t} &= 0 \\ \sigma_{\omega \xi,t} &= 0,\end{aligned}$$

with the growth rate of the intermediary's share given by

$$\mu_{\omega t} = \Phi(i_t) (1 - \bar{\theta}\omega_t) dt.$$

The expected returns to holding capital directly and to holding intermediary debt are given, respectively, by

$$\begin{aligned}\mu_{Rk,t} &= \frac{1}{p_{kt}} + \bar{a} + \frac{\sigma_a^2}{2} - \lambda_k - \frac{\Phi(i_t)(1 - \bar{\theta}\omega_t)}{p_{kt}(2\bar{\theta}\omega_t p_{kt} + \beta(1 - \omega_t))} \left( \frac{4\bar{\theta}}{\phi_0^2\phi_1^2} - \frac{\omega_t}{1 - \omega_t} \beta p_{kt} + \bar{\theta}\omega_t p_{kt}^2 \right) \\ \mu_{Rb,t} &= \mu_{Rk,t} + \left( \rho_h - \frac{\sigma_\xi^2}{2} - \frac{1}{p_{kt}} \right) \frac{1 - \omega_t}{\omega_t(\bar{\theta} - 1)} + \Phi(i_t),\end{aligned}$$

with the riskiness of the returns equal to the riskiness of the productivity growth

$$\begin{aligned}\sigma_{ka,t} &= \sigma_{ba,t} = \sigma_a \\ \sigma_{k\xi,t} &= \sigma_{b\xi,t} = 0.\end{aligned}$$

*Proof.* See Appendix [A.3](#). □

Thus, when the financial intermediaries face a constant leverage constraint, the intermediary sector does not amplify the fundamental shocks in the economy. Notice further that the dynamics for the intermediary's wealth share imply that the economy converges to a constant intermediary wealth share, equal to the inverse of the leverage constraint  $\bar{\theta}$ . In that steady state, the intermediary sector owns all the capital in the economy, with the expected excess return to holding capital given by

$$\mu_{Rk,t} - r_{ft} = \frac{1}{p_k} + \sigma_a^2 - \left( \rho_h - \frac{\sigma_\xi^2}{2} \right) - \Phi(i_t),$$

and the expected excess return to holding bank debt given by:

$$\mu_{Rb,t} - r_{ft} = \sigma_a^2.$$

Hence, in the economy with constant leverage, there is no risk of either default or liquidity crises. Notice, however, that the excess return to holding capital compensates investors for the cost of capital adjustment. Thus, the financial system provides a channel through which market participants can share the cost of capital investment.

The benefit of having a financial system with a flexible leverage constraint is, then, increased output growth and more valuable capital, albeit at the cost of global stability. Figure [12](#) plots the evolution of the growth rate of output and that of the price of capital in the full model of Section [2](#) as well as those in the benchmark case discussed here. For the mean simulated path, both the output growth and the price of capital in the full model are well above those in the benchmark model during normal times. During periods when the intermediary is liquidity-constrained, output growth in the full model is slower than in the benchmark as the intermediary disinvests. Similarly, the disinvestment that accompanies liquidity episodes causes the capital price in the economy to dip below that of the benchmark model.

Since the rate of investment and the capital price are constant in this benchmark, the volatility of consumption growth equals the volatility of productivity growth, and the expected consumption growth rate equals the expected productivity growth rate. In our model, the

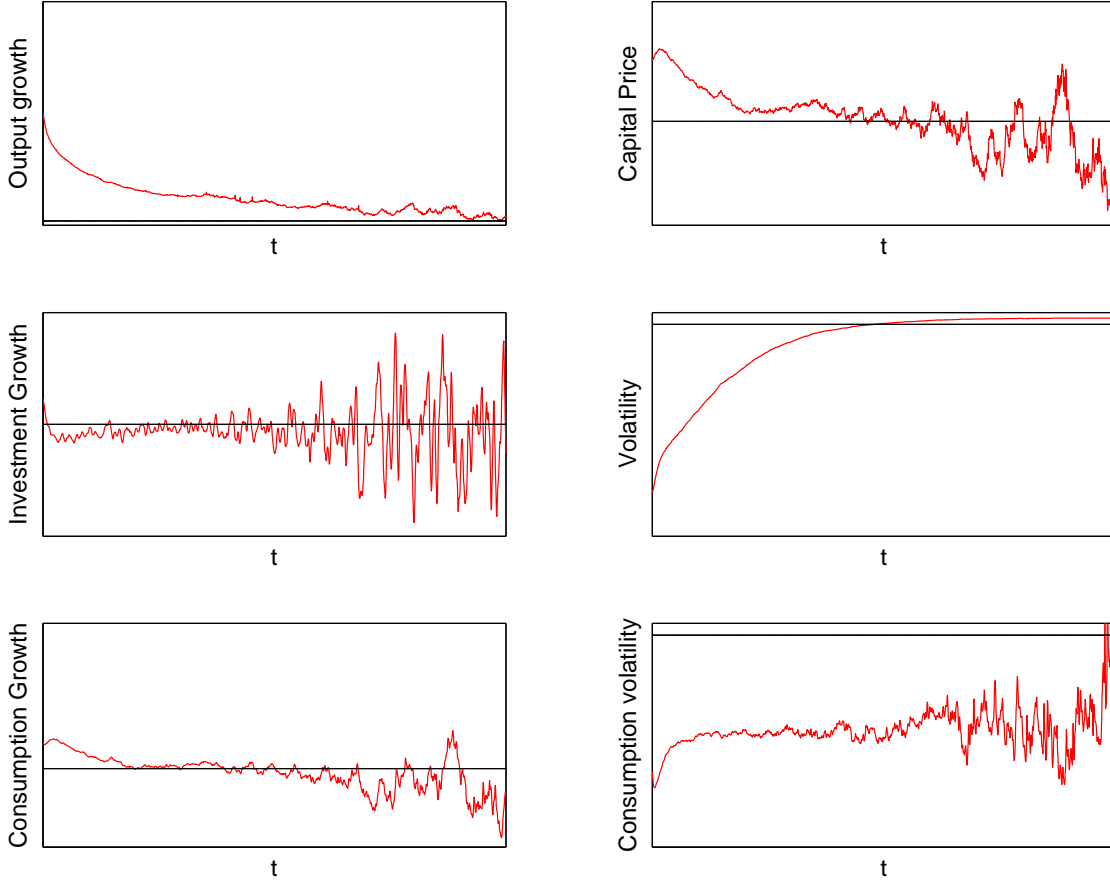


Figure 12: Mean simulated paths of output growth rate (upper-left panel), investment growth rate (middle-left panel), consumption growth rate (lower-left panel), capital price (upper-right panel), volatility of the return to capital (middle-right panel), and the volatility of consumption growth under the full model in red and the corresponding benchmark quantities in black. The mean is taken across 5,000 simulations, with each simulation terminating the first time the financial system defaults along the path.

financial intermediary sector allows households to smooth consumption, reducing the instantaneous volatility of consumption during good times, but at the cost of higher consumption growth volatility during times of financial distress. In particular, notice that, in the full model, volatility of consumption growth is given by

$$\left\langle \frac{dc_t}{c_t} \right\rangle^2 = \left( -\frac{2\theta_t\omega_t}{\beta(1-\omega_t)}p_{kt}(\sigma_{ka,t} - \sigma_a) + \sigma_a \right)^2 + \left( \frac{2\theta_t\omega_t}{\beta(1-\omega_t)}p_{kt}\sigma_{k\xi,t} \right)^2,$$

which is lower than the fundamental volatility  $\sigma_a^2$  when  $\sigma_{ka,t}$  is bigger than  $\sigma_a$ . In the

lower panels of Figure 12, we plot the mean simulated path of the consumption growth and consumption volatility together with their counterparts from the economy without the financial sector. The figure shows that, while the financial system is unconstrained, the volatility of consumption growth in the economy with the financial sector is lower than in the pure production economy. At the same time, the expected growth rate of consumption is higher.

## 6 Conclusion

We present a dynamic, general equilibrium theory of financial intermediaries' leverage cycle as a conceptual basis for policies geared toward financial stability. In this setup, any change in prudential policies has general equilibrium effects that impact the pricing of financial and nonfinancial credit, the equilibrium volatilities of financial and real assets, and the allocation of consumption and investment goods. From a normative point of view, such effects are important to understand, as they ultimately determine the effectiveness of prudential policies.

The assumptions of our model are empirically motivated, and our theory captures many important stylized facts about financial intermediary dynamics that have been documented in the literature. There is both direct and intermediated credit by households, giving rise to a substitution from intermediated credit to directly granted credit in times of tighter intermediary constraints. The risk-based funding constraint leads to procyclical intermediary leverage, matching empirical observations. Our theory gives rise to the volatility paradox, where times of low contemporaneous volatility give rise to high intermediary leverage and increases in forward-looking systemic risk. Finally, high leverage growth is associated with low expected asset returns, another feature that is strongly borne out in the data.

The most important contribution of the paper is to directly study the impact of prudential policies on the likelihood of systemic liquidity and solvency risks. We uncover a systemic risk-return trade-off: Tighter intermediary capital requirements tend to shift the term structure of systemic downward, at the cost of increased risk pricing today. The impacts on contemporaneous volatility and contemporaneous leverage are ambiguous. Our setup further allows

us to characterize the impact of policy changes on the dynamic evolution of endogenous real and financial decisions.

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# A Appendix: Proofs

## A.1 Household's optimization (Proof of Lemma 3.1)

Recall that the household solves the portfolio optimization problem:

$$\max_{\{c_t, \pi_{kt}, \pi_{bt}\}} \mathbb{E} \left[ \int_0^{+\infty} e^{-\xi t - \rho_h t} \log c_t dt \right],$$

subject to the wealth evolution equation:

$$\begin{aligned} dw_{ht} = & r_{ft} w_{ht} dt + w_{ht} \pi_{kt} \{ (\mu_{Rk,t} - r_{ft}) dt + \sigma_{ka,t} dZ_{at} + \sigma_{k\xi,t} dZ_{\xi,t} \} \\ & + w_{ht} \pi_{bt} \{ (\mu_{Rb,t} - r_{ft}) dt + \sigma_{ba,t} dZ_{at} + \sigma_{b\xi,t} dZ_{\xi,t} \} - c_t dt, \end{aligned}$$

and the no-shorting constraints:

$$\pi_{kt}, \pi_{bt} \geq 0.$$

Instead of solving the dynamic optimization problem, we follow Cvitanić and Karatzas (1992) and rewrite the household problem in terms of a static optimization. Cvitanić and Karatzas (1992) extend the Cox and Huang (1989) martingale method approach to constrained optimization problems, such as the one that the households face in our economy.

Define  $K = \mathbb{R}_+^2$  to be the convex set of admissible portfolio strategies and introduce the support function of the set  $-K$  to be

$$\begin{aligned} \delta(x) = \delta(x|K) &\equiv \sup_{\bar{\pi} \in K} (-\bar{\pi}'x) \\ &= \begin{cases} 0, & x \in K \\ +\infty, & x \notin K \end{cases}. \end{aligned}$$

We can then define an auxiliary unconstrained optimization problem for the household, with the returns in the auxiliary asset market defined as

$$\begin{aligned} r_{ft}^v &= r_{ft} + \delta(\vec{v}_t) \\ dR_{kt}^v &= (\mu_{Rk,t} + v_{1t} + \delta(\vec{v}_t)) dt + \sigma_{ka,t} dZ_{at} + \sigma_{k\xi,t} dZ_{\xi,t} \\ dR_{bt}^v &= (\mu_{Rb,t} + v_{2t} + \delta(\vec{v}_t)) dt + \sigma_{ba,t} dZ_{at} + \sigma_{b\xi,t} dZ_{\xi,t}, \end{aligned}$$

for each  $\vec{v}_t = [v_{1t} \ v_{2t}]'$  in the space  $V(K)$  of square-integrable, progressively measurable processes taking values in  $K$ . Corresponding to the auxiliary returns processes is an auxiliary state-price density

$$\frac{d\eta_t^v}{\eta_t^v} = - (r_{ft} + \delta(\vec{v}_t)) dt - (\vec{\mu}_{Rt} - r_{ft} + \vec{v}_t)' (\sigma'_{Rt})^{-1} d\vec{Z}_t,$$

where

$$\vec{\mu}_{Rt} = \begin{bmatrix} \mu_{Rk,t} \\ \mu_{Rb,t} \end{bmatrix}; \quad \sigma_{Rt} = \begin{bmatrix} \sigma_{ka,t} & \sigma_{k\xi,t} \\ \sigma_{ba,t} & \sigma_{b\xi,t} \end{bmatrix}; \quad \vec{Z}_t = \begin{bmatrix} Z_{at} \\ Z_{\xi t} \end{bmatrix}.$$

The auxiliary unconstrained problem of the representative household then becomes

$$\max_{c_t} \mathbb{E} \left[ \int_0^{+\infty} e^{-\xi_t - \rho_h t} \log c_t dt \right]$$

subject to the static budget constraint:

$$w_{h0} = \mathbb{E} \left[ \int_0^{+\infty} \eta_t^v c_t dt \right].$$

The solution to the original constrained problem is then given by the solution to the unconstrained problem for the  $v$  that solves the dual problem

$$\min_{v \in V(K)} \mathbb{E} \left[ \int_0^{+\infty} e^{-\xi_t - \rho_h t} \tilde{u}(\lambda \eta_t^v) dt \right],$$

where  $\tilde{u}(x)$  is the convex conjugate of  $-u(-x)$

$$\tilde{u}(x) \equiv \sup_{z > 0} [\log(zx) - zx] = -(1 + \log x)$$

and  $\lambda$  is the Lagrange multiplier of the static budget constraint. [Cvitanic and Karatzas \(1992\)](#) show that, for the case of logarithmic utility, the optimal choice of  $v$  satisfies

$$\begin{aligned} v_t^* &= \arg \min_{x \in K} \left\{ 2\delta(x) + \left\| (\vec{\mu}_{Rt} - r_{ft} + x)' \sigma_{Rt}^{-1} \right\|^2 \right\} \\ &= \arg \min_{x \in K} \left\| (\vec{\mu}_{Rt} - r_{ft} + x)' \sigma_{Rt}^{-1} \right\|^2. \end{aligned}$$

Thus,

$$\begin{aligned} v_{1t} &= \begin{cases} 0, & \mu_{Rk,t} - r_{ft} \geq 0 \\ r_{ft} - \mu_{Rk,t}, & \mu_{Rk,t} - r_{ft} < 0 \end{cases} \\ v_{2t} &= \begin{cases} 0, & \mu_{Rb,t} - r_{ft} \geq 0 \\ r_{ft} - \mu_{Rb,t}, & \mu_{Rb,t} - r_{ft} < 0 \end{cases}. \end{aligned}$$

Consider now solving the auxiliary unconstrained problem. Taking the first order condition, we obtain

$$[c_t]: \quad 0 = \frac{e^{-\xi_t - \rho_h t}}{c_t} - \lambda \eta_t^v,$$

or

$$c_t = \frac{e^{-\xi_t - \rho_h t}}{\lambda \eta_t^v}.$$

Substituting into the static budget constraint, we obtain

$$\eta_t^v w_{ht} = \mathbb{E}_t \left[ \int_t^{+\infty} \eta_s^v c_s ds \right] = \mathbb{E}_t \left[ \int_t^{+\infty} \frac{e^{-\xi_s - \rho_h s}}{\lambda} ds \right] = \frac{e^{-\xi_t - \rho_h t}}{\lambda (\rho_h - \sigma_\xi^2/2)}.$$

Thus

$$c_t = \left( \rho_h - \frac{\sigma_\xi^2}{2} \right) w_{ht}.$$

To solve for the household's optimal portfolio allocation, notice that:

$$\begin{aligned} \frac{d(\eta_t^v w_{ht})}{\eta_t^v w_{ht}} &= -\rho_h dt - d\xi_t + \frac{1}{2} d\xi_t^2 \\ &= \left( -\rho_h + \frac{1}{2} \sigma_\xi^2 \right) dt - \sigma_\xi \rho_{\xi,a} dZ_{at} - \sigma_\xi \sqrt{1 - \rho_{\xi,a}^2} dZ_{\xi t}. \end{aligned}$$

On the other hand, applying Itô's lemma, we obtain

$$\frac{d(\eta_t^v w_{ht})}{\eta_t^v w_{ht}} = \frac{d\eta_t^v}{\eta_t^v} + \frac{dw_{ht}}{w_{ht}} + \frac{dw_{ht}}{w_{ht}} \frac{d\eta_t^v}{\eta_t^v}.$$

Equating the coefficients on the stochastic terms, we obtain

$$\vec{\pi}'_t = (\vec{\mu}_{Rt} - r_{ft} + \vec{v}_t)' (\sigma'_{Rt} \sigma_{Rt})^{-1} - \sigma_\xi \begin{bmatrix} \rho_{\xi a} & \sqrt{1 - \rho_{\xi a}^2} \end{bmatrix} \sigma_{Rt}^{-1}.$$

## A.2 Equilibrium outcomes (Proof of Lemma 3.2)

To summarize, in equilibrium, we must have

$$\begin{aligned}
\mu_{\theta t} &= \Phi(i_t) (1 - \theta_t \omega_t) - \mu_{\omega t} + \sigma_{\theta a, t}^2 + \sigma_{\theta \xi, t}^2 \\
\mu_{Rk, t} - r_{ft} &= (\sigma_{ka, t}^2 + \sigma_{k\xi, t}^2) \frac{1 - \theta_t \omega_t}{1 - \omega_t} + (\sigma_{ka, t} \sigma_{ba, t} + \sigma_{k\xi, t} \sigma_{b\xi, t}) \frac{\theta_t \omega_t - \omega_t}{1 - \omega_t} \\
&\quad + \sigma_{\xi} \left( \sigma_{ka, t} \rho_{\xi, a} + \sigma_{k\xi, t} \sqrt{1 - \rho_{\xi, a}^2} \right) \\
\mu_{Rb, t} - r_{ft} &= (\sigma_{ba, t}^2 + \sigma_{b\xi, t}^2) \frac{\theta_t \omega_t - \omega_t}{1 - \omega_t} + (\sigma_{ka, t} \sigma_{ba, t} + \sigma_{k\xi, t} \sigma_{b\xi, t}) \frac{1 - \theta_t \omega_t}{1 - \omega_t} \\
&\quad + \sigma_{\xi} \left( \sigma_{ba, t} \rho_{\xi, a} + \sigma_{b\xi, t} \sqrt{1 - \rho_{\xi, a}^2} \right) \\
\mu_{\omega t} &= (\theta_t - 1) (\mu_{Rkt} - \mu_{Rbt}) + (\sigma_{ka, t} \sigma_{\theta a, t} + \sigma_{k\xi, t} \sigma_{\theta \xi, t}) \\
&\quad + \frac{1 - \omega_t}{\omega_t} \left[ \left( \rho_h - \frac{\sigma_{\xi}^2}{2} \right) - \frac{1}{p_{kt}} + \Phi(i_t) \theta_t \omega_t \right] \\
\sigma_{\theta a, t} &= -(\theta_t - 1) (\sigma_{ka, t} - \sigma_{ba, t}) \\
\sigma_{\theta \xi, t} &= -(\theta_t - 1) (\sigma_{k\xi, t} - \sigma_{b\xi, t}) \\
\beta (\theta_t \omega_t - \omega_t) \sigma_{ba, t} &= -(\beta (1 - \theta_t \omega_t) + 2\theta_t \omega_t p_{kt}) \sigma_{ka, t} + (2\theta_t \omega_t p_{kt} + \beta (1 - \omega_t)) \sigma_a \\
\beta (\theta_t \omega_t - \omega_t) \sigma_{b\xi, t} &= -(\beta (1 - \theta_t \omega_t) + 2\theta_t \omega_t p_{kt}) \sigma_{k\xi, t} \\
\alpha^{-2} \theta_t^{-2} &= \sigma_{ka, t}^2 + \sigma_{k\xi, t}^2 \\
0 &= \theta_t \omega_t (1 - \theta_t \omega_t) \Phi(i_t) \left( p_{kt}^2 - \frac{4}{\phi_0^2 \phi_1^2} \right) \\
&\quad + p_{kt} (2\theta_t \omega_t p_{kt} + \beta (1 - \omega_t)) \left( \mu_{Rk, t} - \frac{1}{p_{kt}} - \bar{a} - \frac{\sigma_a^2}{2} + \lambda_k - \sigma_a (\sigma_{ka, t} - \sigma_a) \right) \\
&\quad - \beta p_{kt} \omega_t \mu_{\omega t} - p_{kt} (\theta_t \omega_t p_{kt} + \beta (1 - \omega_t)) \left( (\sigma_{ka, t} - \sigma_a)^2 + \sigma_{k\xi, t}^2 \right).
\end{aligned}$$

Notice that the first eight equations describe the evolutions of  $\theta_t \omega_t$ ,  $\omega_t$ , the return of risky intermediary debt  $R_{bt}$ , and the expected excess return to direct capital holding in terms of the two state variables,  $(\theta_t \omega_t, \omega_t)$  and the loadings,  $\sigma_{ka, t}$  and  $\sigma_{k\xi, t}$ , of the return to direct capital holding on the two fundamental shocks in the economy.<sup>6</sup> The final two equations, then, express  $\sigma_{ka, t}$  and  $\sigma_{k\xi, t}$  in terms of the state variables.

Before solving the final two equations, we simplify the equilibrium conditions. Notice first that

$$(\sigma_{ka, t} \sigma_{\theta a, t} + \sigma_{k\xi, t} \sigma_{\theta \xi, t}) = -(\theta_t - 1) (\sigma_{ka, t} (\sigma_{ka, t} - \sigma_{ba, t}) + \sigma_{k\xi, t} (\sigma_{k\xi, t} - \sigma_{b\xi, t})),$$

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<sup>6</sup>Recall from Equation (3.1) that we have also expressed the price of capital in terms of the state variables.

and

$$\begin{aligned}\mu_{Rkt} - \mu_{Rb,t} &= \left( \sigma_{ka,t}^2 + \sigma_{k\xi,t}^2 - \sigma_{ka,t}\sigma_{ba,t} - \sigma_{k\xi,t}\sigma_{b\xi,t} \right) \frac{1 - \theta_t\omega_t}{1 - \omega_t} \\ &\quad - \left( \sigma_{ba,t}^2 + \sigma_{b\xi,t}^2 - \sigma_{ka,t}\sigma_{ba,t} - \sigma_{k\xi,t}\sigma_{b\xi,t} \right) \frac{\theta_t\omega_t - \omega_t}{1 - \omega_t} \\ &\quad + \sigma_\xi \left( (\sigma_{ka,t} - \sigma_{ba,t}) \rho_{\xi,a} + (\sigma_{k\xi,t} - \sigma_{b\xi,t}) \sqrt{1 - \rho_{\xi,a}^2} \right).\end{aligned}$$

Thus,

$$\begin{aligned}(\mu_{Rkt} - \mu_{Rb,t}) + \frac{1}{\theta_t - 1} (\sigma_{ka,t}\sigma_{\theta a,t} + \sigma_{k\xi,t}\sigma_{\theta\xi,t}) &= -\frac{\theta_t\omega_t - \omega_t}{1 - \omega_t} \left( (\sigma_{ka,t} - \sigma_{ba,t})^2 + (\sigma_{k\xi,t} - \sigma_{b\xi,t})^2 \right) \\ &\quad + \sigma_\xi \left( (\sigma_{ka,t} - \sigma_{ba,t}) \rho_{\xi,a} + (\sigma_{k\xi,t} - \sigma_{b\xi,t}) \sqrt{1 - \rho_{\xi,a}^2} \right).\end{aligned}$$

Using

$$\begin{aligned}\beta (\theta_t\omega_t - \omega_t) (\sigma_{ka,t} - \sigma_{ba,t}) &= (2\theta_t\omega_t p_{kt} + \beta (1 - \omega_t)) (\sigma_{ka,t} - \sigma_a) \\ \beta (\theta_t\omega_t - \omega_t) (\sigma_{k\xi,t} - \sigma_{b\xi,t}) &= (2\theta_t\omega_t p_{kt} + \beta (1 - \omega_t)) \sigma_{k\xi,t}\end{aligned}$$

we can thus express the drift of  $\omega_t$  as

$$\begin{aligned}\mu_{\omega t} &= -\frac{1}{\beta^2\omega_t} (2\theta_t\omega_t p_{kt} + \beta (1 - \omega_t))^2 \left[ (\sigma_{ka,t} - \sigma_a)^2 + \sigma_{k\xi,t}^2 \right] \\ &\quad + \frac{\sigma_\xi}{\beta\omega_t} (2\theta_t\omega_t p_{kt} + \beta (1 - \omega_t)) \left( (\sigma_{ka,t} - \sigma_a) \rho_{\xi,a} + \sigma_{k\xi,t} \sqrt{1 - \rho_{\xi,a}^2} \right) \\ &\quad + \frac{1 - \omega_t}{\omega_t} \left[ \left( \rho_h - \frac{\sigma_\xi^2}{2} \right) - \frac{1}{p_{kt}} + \Phi(i_t) \theta_t\omega_t \right].\end{aligned}$$

Substituting the risk-based capital constraint, this becomes

$$\begin{aligned}\mu_{\omega t} &= -\frac{1}{\beta^2\omega_t} (2\theta_t\omega_t p_{kt} + \beta (1 - \omega_t))^2 \left[ \sigma_a^2 - 2\sigma_a\sigma_{ka,t} + \frac{\theta_t^{-2}}{\alpha^2} \right] \\ &\quad + \frac{\sigma_\xi}{\beta\omega_t} (2\theta_t\omega_t p_{kt} + \beta (1 - \omega_t)) \left( (\sigma_{ka,t} - \sigma_a) \rho_{\xi,a} + \sigma_{k\xi,t} \sqrt{1 - \rho_{\xi,a}^2} \right) \\ &\quad + \frac{1 - \omega_t}{\omega_t} \left[ \left( \rho_h - \frac{\sigma_\xi^2}{2} \right) - \frac{1}{p_{kt}} + \Phi(i_t) \theta_t\omega_t \right] \\ &\equiv \mathcal{O}_0(\omega_t, \theta_t) + \mathcal{O}_a(\omega_t, \theta_t) \sigma_{ka,t} + \mathcal{O}_\xi(\omega_t, \theta_t) \sigma_{k\xi,t},\end{aligned}$$

where

$$\mathcal{O}_0(\omega_t, \theta_t) = -\frac{1}{\beta^2 \omega_t} (2\theta_t \omega_t p_{kt} + \beta(1 - \omega_t))^2 \left[ \sigma_a^2 + \frac{\theta_t^{-2}}{\alpha^2} \right] - \frac{\sigma_\xi \sigma_a \rho_{\xi,a}}{\beta \omega_t} (2\theta_t \omega_t p_{kt} + \beta(1 - \omega_t)) \quad (\text{A.1})$$

$$+ \frac{1 - \omega_t}{\omega_t} \left[ \left( \rho_h - \frac{\sigma_\xi^2}{2} \right) - \frac{1}{p_{kt}} + \Phi(i_t) \theta_t \omega_t \right]$$

$$\mathcal{O}_a(\omega_t, \theta_t) = \frac{2\sigma_a}{\beta^2 \omega_t} (2\theta_t \omega_t p_{kt} + \beta(1 - \omega_t))^2 + \frac{\sigma_\xi \rho_{\xi,a}}{\beta \omega_t} (2\theta_t \omega_t p_{kt} + \beta(1 - \omega_t)) \quad (\text{A.2})$$

$$\mathcal{O}_\xi(\omega_t, \theta_t) = \frac{\sigma_\xi \sqrt{1 - \rho_{\xi,a}^2}}{\beta \omega_t} (2\theta_t \omega_t p_{kt} + \beta(1 - \omega_t)). \quad (\text{A.3})$$

Substituting into the drift rate of intermediary leverage

$$\begin{aligned} \mu_{\theta t} &= \Phi(i_t) (1 - \theta_t \omega_t) - \mu_{\omega t} + \sigma_{\theta a,t}^2 + \sigma_{\theta \xi,t}^2 \\ &= \mathcal{S}_0(\omega_t, \theta_t) + \mathcal{S}_a(\omega_t, \theta_t) \sigma_{ka,t} + \mathcal{S}_\xi(\omega_t, \kappa_t) \sigma_{k\xi,t}, \end{aligned}$$

where

$$\mathcal{S}_0(\omega_t, \theta_t) = \Phi(i_t) (1 - \theta_t \omega_t) - \mathcal{O}_0(\omega_t, \theta_t) + \left( \frac{2\theta_t \omega_t p_{kt} + \beta(1 - \omega_t)}{\beta \omega_t} \right)^2 \frac{\theta_t^{-2}}{\alpha^2} \quad (\text{A.4})$$

$$\mathcal{S}_a(\omega_t, \theta_t) = -2\sigma_a \left( \frac{2\theta_t \omega_t p_{kt} + \beta(1 - \omega_t)}{\beta \omega_t} \right) - \mathcal{O}_a(\omega_t, \theta_t) \quad (\text{A.5})$$

$$\mathcal{S}_\xi(\omega_t, \theta_t) = -\mathcal{O}_\xi(\omega_t, \theta_t). \quad (\text{A.6})$$

Similarly, the excess return on capital is given by

$$\begin{aligned} \mu_{Rk,t} - r_{ft} &= (\sigma_{ka,t}^2 + \sigma_{k\xi,t}^2) \frac{1 - \theta_t \omega_t}{1 - \omega_t} + (\sigma_{ka,t} \sigma_{ba,t} + \sigma_{k\xi,t} \sigma_{b\xi,t}) \frac{\theta_t \omega_t - \omega_t}{1 - \omega_t} \\ &\quad + \sigma_\xi \left( \sigma_{ka,t} \rho_{\xi,a} + \sigma_{k\xi,t} \sqrt{1 - \rho_{\xi,a}^2} \right) \\ &= \frac{\theta_t^{-2}}{\alpha^2} \frac{1 - \theta_t \omega_t}{1 - \omega_t} - \frac{\beta(1 - \theta_t \omega_t) + 2\theta_t \omega_t p_{kt}}{\beta(1 - \omega_t)} \frac{\theta_t^{-2}}{\alpha^2} + \frac{\beta(1 - \omega_t) + 2\theta_t \omega_t p_{kt}}{\beta(1 - \omega_t)} \sigma_a \sigma_{ka,t} \\ &\quad + \sigma_\xi \left( \sigma_{ka,t} \rho_{\xi,a} + \sigma_{k\xi,t} \sqrt{1 - \rho_{\xi,a}^2} \right) \\ &= \frac{-2\omega_t \theta_t p_{kt}}{\beta(1 - \omega_t)} \frac{\theta_t^{-2}}{\alpha^2} + \frac{\beta(1 - \omega_t) + 2\theta_t \omega_t p_{kt}}{\beta(1 - \omega_t)} \sigma_a \sigma_{ka,t} + \sigma_\xi \left( \sigma_{ka,t} \rho_{\xi,a} + \sigma_{k\xi,t} \sqrt{1 - \rho_{\xi,a}^2} \right). \end{aligned}$$

The excess return on intermediary debt is given by

$$\begin{aligned}
\mu_{Rb,t} - r_{ft} &= (\sigma_{ba,t}^2 + \sigma_{b\xi,t}^2) \frac{\theta_t \omega_t - \omega_t}{1 - \omega_t} + (\sigma_{ka,t} \sigma_{ba,t} + \sigma_{k\xi,t} \sigma_{b\xi,t}) \frac{1 - \theta_t \omega_t}{1 - \omega_t} \\
&\quad + \sigma_\xi \left( \sigma_{ba,t} \rho_{\xi,a} + \sigma_{b\xi,t} \sqrt{1 - \rho_{\xi,a}^2} \right) \\
&= \left( \frac{\theta_t \omega_t - \omega_t}{1 - \omega_t} \right) \left( \frac{\beta (1 - \theta_t \omega_t) + 2\theta_t \omega_t p_{kt}}{\beta (\theta_t \omega_t - \omega_t)} \right)^2 \frac{\theta_t^{-2}}{\alpha^2} \\
&\quad + \left( \frac{\theta_t \omega_t - \omega_t}{1 - \omega_t} \right) \left( \frac{\beta (1 - \omega_t) + 2\theta_t \omega_t p_{kt}}{\beta (\theta_t \omega_t - \omega_t)} \right)^2 \sigma_a^2 \\
&\quad - 2 \left( \frac{\theta_t \omega_t - \omega_t}{1 - \omega_t} \right) \left( \frac{\beta (1 - \theta_t \omega_t) + 2\theta_t \omega_t p_{kt}}{\beta (\theta_t \omega_t - \omega_t)} \right) \left( \frac{\beta (1 - \omega_t) + 2\theta_t \omega_t p_{kt}}{\beta (\theta_t \omega_t - \omega_t)} \right) \sigma_{ka,t} \sigma_a \\
&\quad - \left( \frac{1 - \theta_t \omega_t}{\theta_t \omega_t - \omega_t} \right) \frac{\beta (1 - \theta_t \omega_t) + 2\theta_t \omega_t p_{kt}}{\beta (1 - \omega_t)} \frac{\theta_t^{-2}}{\alpha^2} + \left( \frac{1 - \theta_t \omega_t}{1 - \omega_t} \right) \frac{\beta (1 - \omega_t) + 2\theta_t \omega_t p_{kt}}{\beta (\theta_t \omega_t - \omega_t)} \sigma_a \sigma_{ka,t} \\
&\quad + \sigma_\xi \rho_{\xi,a} \left[ - \frac{\beta (1 - \theta_t \omega_t) + 2\theta_t \omega_t p_{kt}}{\beta (\theta_t \omega_t - \omega_t)} \sigma_{ka,t} + \frac{\beta (1 - \omega_t) + 2\theta_t \omega_t p_{kt}}{\beta (\theta_t \omega_t - \omega_t)} \sigma_a \right] \\
&\quad - \sigma_\xi \sqrt{1 - \rho_{\xi,a}^2} \frac{\beta (1 - \theta_t \omega_t) + 2\theta_t \omega_t p_{kt}}{\beta (\theta_t \omega_t - \omega_t)} \sigma_{k\xi,t}.
\end{aligned}$$

Notice also that we can now derive the risk-free rate. Recall that, in the unconstrained region, the risk-free rate satisfies the household Euler equation

$$r_{ft} = \left( \rho_h - \frac{\sigma_\xi^2}{2} \right) + \frac{1}{dt} \mathbb{E} \left[ \frac{dc_t}{c_t} \right] - \frac{1}{dt} \mathbb{E} \left[ \frac{\langle dc_t \rangle^2}{c_t^2} + \frac{\langle dc_t, d\xi_t \rangle^2}{c_t} \right].$$

Applying Itô's lemma to the goods clearing condition, we obtain

$$\begin{aligned}
dc_t &= d(K_t A_t - i_t k_t A_t) \\
&= A_t dK_t + (K_t - i_t k_t) dA_t - A_t k_t di_t - A_t i_t dk_t - k_t \langle di_t, dA_t \rangle.
\end{aligned}$$

From the financial intermediaries' optimal investment choice, we have

$$\begin{aligned}
di_t &= \frac{\phi_0^2 \phi_1 p_{kt}^2}{4} \left( 2 \left( \mu_{Rk,t} - \frac{1}{p_{kt}} - \bar{a} - \frac{\sigma_a^2}{2} + \lambda_k - \sigma_a (\sigma_{ka,t} - \sigma_a) \right) + (\sigma_{ka,t} - \sigma_a)^2 + \sigma_{k\xi,t}^2 \right) dt \\
&\quad + \frac{\phi_0^2 \phi_1 p_{kt}^2}{2} (\sigma_{ka,t} - \sigma_a) dZ_{at} + \frac{\phi_0^2 \phi_1 p_{kt}^2}{2} \sigma_{k\xi,t} dZ_{\xi t}.
\end{aligned}$$

Thus

$$\begin{aligned}
\frac{1}{dt} \mathbb{E} \left[ \frac{dc_t}{c_t} \right] &= \bar{a} + \frac{\sigma_a^2}{2} - \lambda_k + \frac{\theta_t \omega_t}{1 - i_t \theta_t \omega_t} \Phi(i_t) (1 - i_t) \\
&\quad - \left( \frac{\theta_t \omega_t}{1 - i_t \theta_t \omega_t} \right) \frac{\phi_0^2 \phi_1 p_{kt}^2}{4} \left( 2 \left( \mu_{Rk,t} - \frac{1}{p_{kt}} - \left( \bar{a} + \frac{\sigma_a^2}{2} - \lambda_k \right) \right) + (\sigma_{ka,t} - \sigma_a)^2 + \sigma_{k\xi,t}^2 \right) \\
\frac{1}{dt} \mathbb{E} \left[ \frac{\langle dc_t \rangle^2}{c_t^2} \right] &= \left( \sigma_a - \left( \frac{\theta_t \omega_t}{1 - i_t \theta_t \omega_t} \right) \frac{\phi_0^2 \phi_1 p_{kt}^2}{2} (\sigma_{ka,t} - \sigma_a) \right)^2 + \left( \left( \frac{\theta_t \omega_t}{1 - i_t \theta_t \omega_t} \right) \frac{\phi_0^2 \phi_1 p_{kt}^2}{2} \sigma_{k\xi,t} \right)^2 \\
\frac{1}{dt} \mathbb{E} \left[ \frac{\langle dc_t d\xi_t \rangle^2}{c_t} \right] &= \sigma_\xi \left( \sigma_a - \left( \frac{\theta_t \omega_t}{1 - i_t \theta_t \omega_t} \right) \frac{\phi_0^2 \phi_1 p_{kt}^2}{2} \sigma_{ka,t} \right) \rho_{\xi,a} \\
&\quad - \sigma_\xi \left( \left( \frac{\theta_t \omega_t}{1 - i_t \theta_t \omega_t} \right) \frac{\phi_0^2 \phi_1 p_{kt}^2}{2} \sigma_{k\xi,t} \right) \sqrt{1 - \rho_{\xi,a}^2}.
\end{aligned}$$

Figure 13 plots the three components of the risk-free rate as a function of intermediary leverage (upper left panel), the wealth share of the intermediary (upper right panel), and the volatility of the excess return to capital (lower panels). We see that the precautionary savings motive dominates the consumption growth component, and even more so as leverage increases.

Recall that, in equilibrium, we have

$$1 - i_t \theta_t \omega_t = \left( \rho_h - \frac{\sigma_\xi^2}{2} \right) p_{kt} (1 - \omega_t),$$

so that

$$\frac{1}{1 - i_t \theta_t \omega_t} \frac{\phi_0^2 \phi_1}{4} p_{kt}^2 = \left( \left( \rho_h - \frac{\sigma_\xi^2}{2} \right) p_{kt} (1 - \omega_t) \right)^{-1} \frac{\phi_0^2 \phi_1}{4} p_{kt}^2 = \frac{p_{kt}}{\beta (1 - \omega_t)}$$

and

$$1 + \frac{\theta_t \omega_t}{1 - i_t \theta_t \omega_t} \frac{\phi_0^2 \phi_1}{4} p_{kt}^2 = \frac{\beta (1 - \omega_t) + 2 \theta_t \omega_t p_{kt}}{\beta (1 - \omega_t)}.$$

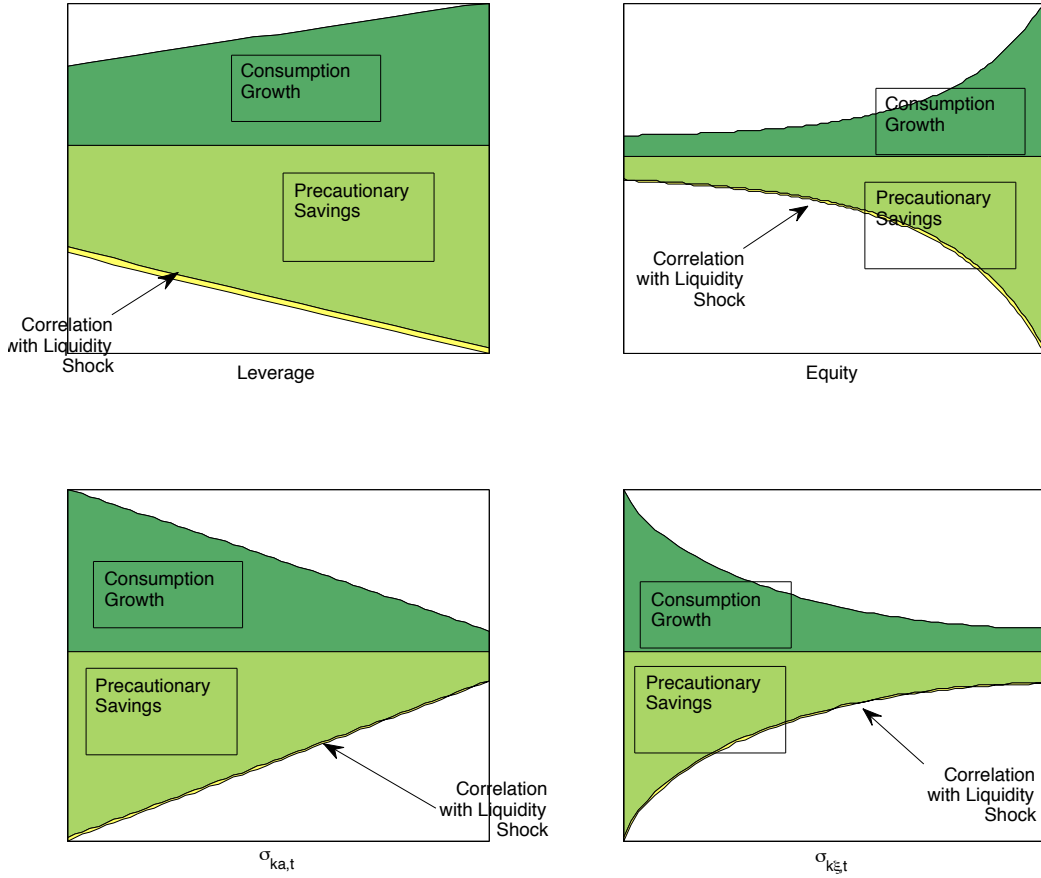


Figure 13: Equilibrium risk-free rate as a function of intermediary leverage (upper left panel), the wealth share of the intermediary (upper right panel), and the volatility of the excess return to capital (lower panels). “Consumption growth” refers to the component of the risk-free rate that’s due to the expected consumption growth rate. “Precautionary saving” is measured as the contribution to the risk-free of the variance of the consumption growth rate. “Correlation with Liquidity Shock” is measured as the contribution to the risk-free of the correlation between the consumption growth rate and the liquidity shock  $d\xi_t$ . The equilibrium risk-free rate is then given as the lower boundary of the shaded region in each graph.

Substituting into the expression for the risk-free rate, we obtain

$$\begin{aligned}
r_{ft} = & \left( \rho_h - \frac{\sigma_\xi^2}{2} \right) + \left( \bar{a} + \frac{\sigma_a^2}{2} - \lambda_k \right) \frac{\beta(1-\omega_t) + 2\theta_t\omega_t p_{kt}}{\beta(1-\omega_t)} + \frac{\theta_t\omega_t}{1-i_t\theta_t\omega_t} \Phi(i_t)(1-i_t) \\
& - \frac{2\theta_t\omega_t p_{kt}}{\beta(1-\omega_t)} \left( \mu_{Rk,t} - \frac{1}{p_{kt}} \right) - \frac{\theta_t\omega_t p_{kt}}{\beta(1-\omega_t)} \left( \sigma_a^2 + \frac{\theta_t^{-2}}{\alpha^2} - 2\sigma_{ka,t}\sigma_a \right) \\
& - \sigma_a^2 \left( \frac{\beta(1-\omega_t) + 2\theta_t\omega_t p_{kt}}{\beta(1-\omega_t)} \right)^2 + \frac{4p_{kt}\theta_t\omega_t}{\beta(1-\omega_t)} \left( \frac{\beta(1-\omega_t) + 2\theta_t\omega_t p_{kt}}{\beta(1-\omega_t)} \right) \sigma_{ka,t}\sigma_a \\
& - \sigma_a\sigma_\xi\rho_{\xi,a} \left( \frac{\beta(1-\omega_t) + 2\theta_t\omega_t p_{kt}}{\beta(1-\omega_t)} \right) - \left( \frac{2p_{kt}\theta_t\omega_t}{\beta(1-\omega_t)} \right)^2 \frac{\theta_t^{-2}}{\alpha^2} \\
& + \frac{2p_{kt}\theta_t\omega_t}{\beta(1-\omega_t)} \sigma_\xi\rho_{\xi,a}\sigma_{ka,t} + \frac{2p_{kt}\theta_t\omega_t}{\beta(1-\omega_t)} \sigma_\xi \sqrt{1 - \rho_{\xi,a}^2} \sigma_{k\xi,t}.
\end{aligned}$$

We can now solve for the return on capital. In particular, we have

$$\begin{aligned}
\mu_{Rk,t} &= r_{ft} - \frac{2\omega_t\theta_t p_{kt}}{\beta(1-\omega_t)} \frac{\theta_t^{-2}}{\alpha^2} + \frac{\beta(1-\omega_t) + 2\theta_t\omega_t p_{kt}}{\beta(1-\omega_t)} \sigma_a \sigma_{ka,t} + \sigma_\xi \left( \sigma_{ka,t} \rho_{\xi,a} + \sigma_{k\xi,t} \sqrt{1-\rho_{\xi,a}^2} \right) \\
&= \left( \rho_h - \frac{\sigma_\xi^2}{2} \right) + \left( \bar{a} + \frac{\sigma_a^2}{2} - \lambda_k \right) \frac{\beta(1-\omega_t) + 2\theta_t\omega_t p_{kt}}{\beta(1-\omega_t)} + \frac{\theta_t\omega_t}{1-i_t\theta_t\omega_t} \Phi(i_t)(1-i_t) \\
&\quad - \frac{2\theta_t\omega_t p_{kt}}{\beta(1-\omega_t)} \left( \mu_{Rk,t} - \frac{1}{p_{kt}} \right) - \frac{3\theta_t\omega_t p_{kt}}{\beta(1-\omega_t)} \frac{\theta_t^{-2}}{\alpha^2} + \frac{\beta(1-\omega_t) + 4\theta_t\omega_t p_{kt}}{\beta(1-\omega_t)} \sigma_a \sigma_{ka,t} \\
&\quad - \sigma_a^2 \left( \frac{\theta_t\omega_t p_{kt}}{\beta(1-\omega_t)} + \left( \frac{\beta(1-\omega_t) + 2\theta_t\omega_t p_{kt}}{\beta(1-\omega_t)} \right)^2 \right) + \frac{4p_{kt}\theta_t\omega_t}{\beta(1-\omega_t)} \left( \frac{\beta(1-\omega_t) + 2\theta_t\omega_t p_{kt}}{\beta(1-\omega_t)} \right) \sigma_{ka,t} \sigma_a \\
&\quad - \sigma_a \sigma_\xi \rho_{\xi,a} \left( \frac{\beta(1-\omega_t) + 2\theta_t\omega_t p_{kt}}{\beta(1-\omega_t)} \right) - \left( \frac{2p_{kt}\theta_t\omega_t}{\beta(1-\omega_t)} \right)^2 \frac{\theta_t^{-2}}{\alpha^2} \\
&\quad + \frac{\beta(1-\omega_t) + 2\theta_t\omega_t p_{kt}}{\beta(1-\omega_t)} \sigma_\xi \rho_{\xi,a} \sigma_{ka,t} + \frac{\beta(1-\omega_t) + 2\theta_t\omega_t p_{kt}}{\beta(1-\omega_t)} \sigma_\xi \sqrt{1-\rho_{\xi,a}^2} \sigma_{k\xi,t}.
\end{aligned}$$

Solving for  $\mu_{Rk,t}$ , we obtain

$$\mu_{Rk,t} = \mathcal{K}_0(\omega_t, \theta_t) + \mathcal{K}_a(\omega_t, \theta_t) \sigma_{ka,t} + \mathcal{K}_\xi(\omega_t, \theta_t) \sigma_{k\xi,t},$$

where

$$\mathcal{K}_0(\omega_t, \theta_t) = \frac{\beta(1-\omega_t)}{\beta(1-\omega_t) + 2\theta_t\omega_t p_{kt}} \left( \rho_h - \frac{\sigma_\xi^2}{2} + \frac{\theta_t\omega_t}{1-i_t\theta_t\omega_t} \Phi(i_t)(1-i_t) + \frac{2\theta_t\omega_t}{\beta(1-\omega_t)} \right) \quad (\text{A.7})$$

$$\begin{aligned}
&+ \bar{a} + \frac{\sigma_a^2}{2} - \lambda_k - \frac{\theta_t\omega_t p_{kt}}{\beta(1-\omega_t) + 2\theta_t\omega_t p_{kt}} \left( 3 + \frac{4p_{kt}\theta_t\omega_t}{\beta(1-\omega_t)} \right) \frac{\theta_t^{-2}}{\alpha^2} \\
&- \sigma_a^2 \left( \frac{\theta_t\omega_t p_{kt}}{\beta(1-\omega_t) + 2\theta_t\omega_t p_{kt}} + \left( \frac{\beta(1-\omega_t) + 2\theta_t\omega_t p_{kt}}{\beta(1-\omega_t)} \right) \right) - \sigma_a \sigma_\xi \rho_{\xi,a}
\end{aligned}$$

$$\mathcal{K}_a(\omega_t, \theta_t) = \sigma_\xi \rho_{\xi,a} + \frac{\beta(1-\omega_t) + 4\theta_t\omega_t p_{kt}}{\beta(1-\omega_t) + 2\theta_t\omega_t p_{kt}} \sigma_a + \frac{4p_{kt}\theta_t\omega_t}{\beta(1-\omega_t)} \sigma_a \quad (\text{A.8})$$

$$\mathcal{K}_\xi(\omega_t, \theta_t) = \sigma_\xi \sqrt{1-\rho_{\xi,a}^2}. \quad (\text{A.9})$$

We can now express the risk-free rate in the economy as

$$\begin{aligned}
r_{ft} &= \mu_{Rk,t} + \frac{2\omega_t\theta_t p_{kt}}{\beta(1-\omega_t)} \frac{\theta_t^{-2}}{\alpha^2} - \frac{\beta(1-\omega_t) + 2\theta_t\omega_t p_{kt}}{\beta(1-\omega_t)} \sigma_a \sigma_{ka,t} - \sigma_\xi \left( \sigma_{ka,t} \rho_{\xi,a} + \sigma_{k\xi,t} \sqrt{1-\rho_{\xi,a}^2} \right) \\
&\equiv \mathcal{R}_0(\omega_t, \theta_t) + \mathcal{R}_a(\omega_t, \theta_t) \sigma_{ka,t},
\end{aligned}$$

where

$$\mathcal{R}_0(\omega_t, \theta_t) = \mathcal{K}_0(\omega_t, \theta_t) + \frac{2\omega_t\theta_t p_{kt}}{\beta(1-\omega_t)} \frac{\theta_t^{-2}}{\alpha^2} \quad (\text{A.10})$$

$$\mathcal{R}_a(\omega_t, \theta_t) = \mathcal{K}_a(\omega_t, \theta_t) - \frac{\beta(1-\omega_t) + 2\theta_t\omega_t p_{kt}}{\beta(1-\omega_t)} \sigma_a - \sigma_\xi \rho_{\xi,a}. \quad (\text{A.11})$$

Substituting into the excess return on holding intermediary debt, we obtain

$$\mu_{Rb,t} = \mathcal{B}_0(\omega_t, \theta_t) + \mathcal{B}_a(\omega_t, \theta_t) \sigma_{ka,t} + \mathcal{B}_\xi(\omega_t, \theta_t) \sigma_{k\xi,t},$$

where

$$\begin{aligned} \mathcal{B}_0(\omega_t, \theta_t) &= \mathcal{R}_0(\omega_t, \theta_t) + \left( \frac{\theta_t\omega_t - \omega_t}{1-\omega_t} \right) \left( \frac{\beta(1-\theta_t\omega_t) + 2\theta_t\omega_t p_{kt}}{\beta(\theta_t\omega_t - \omega_t)} \right)^2 \frac{\theta_t^{-2}}{\alpha^2} \\ &+ \left( \frac{\theta_t\omega_t - \omega_t}{1-\omega_t} \right) \left( \frac{\beta(1-\omega_t) + 2\theta_t\omega_t p_{kt}}{\beta(\theta_t\omega_t - \omega_t)} \right)^2 \sigma_a^2 + \sigma_\xi \rho_{\xi,a} \sigma_a \frac{\beta(1-\omega_t) + 2\theta_t\omega_t p_{kt}}{\beta(\theta_t\omega_t - \omega_t)} \\ &- \left( \frac{1-\theta_t\omega_t}{\theta_t\omega_t - \omega_t} \right) \frac{\beta(1-\theta_t\omega_t) + 2\theta_t\omega_t p_{kt}}{\beta(1-\omega_t)} \frac{\theta_t^{-2}}{\alpha^2} \end{aligned} \quad (\text{A.12})$$

$$\begin{aligned} \mathcal{B}_a(\omega_t, \theta_t) &= \mathcal{R}_a(\omega_t, \theta_t) + \left( \frac{1-\theta_t\omega_t}{1-\omega_t} \right) \frac{\beta(1-\omega_t) + 2\theta_t\omega_t p_{kt}}{\beta(\theta_t\omega_t - \omega_t)} \sigma_a \\ &- 2 \left( \frac{\theta_t\omega_t - \omega_t}{1-\omega_t} \right) \left( \frac{\beta(1-\theta_t\omega_t) + 2\theta_t\omega_t p_{kt}}{\beta(\theta_t\omega_t - \omega_t)} \right) \left( \frac{\beta(1-\omega_t) + 2\theta_t\omega_t p_{kt}}{\beta(\theta_t\omega_t - \omega_t)} \right) \sigma_a \\ &- \sigma_\xi \rho_{\xi,a} \left( \frac{\beta(1-\theta_t\omega_t) + 2\theta_t\omega_t p_{kt}}{\beta(\theta_t\omega_t - \omega_t)} \right) \end{aligned} \quad (\text{A.13})$$

$$\mathcal{B}_\xi(\omega_t, \theta_t) = \mathcal{R}_\xi(\omega_t, \theta_t) - \sigma_\xi \sqrt{1-\rho_{\xi,a}^2} \frac{\beta(1-\theta_t\omega_t) + 2\theta_t\omega_t p_{kt}}{\beta(\theta_t\omega_t - \omega_t)}. \quad (\text{A.14})$$

Notice that

$$p_{kt} (2\theta_t\omega_t p_{kt} + \beta(1-\omega_t)) \mathcal{K}_\xi(\omega_t, \theta_t) - \beta p_{kt} \omega_t \mathcal{O}_\xi(\omega_t, \theta_t) = 0.$$

Using these results and the risk-based capital constraint, we can rewrite

$$\begin{aligned} 0 &= \theta_t\omega_t (1-\theta_t\omega_t) \Phi(i_t) \left( p_{kt}^2 - \frac{4}{\phi_0^2 \phi_1^2} \right) \\ &+ p_{kt} (2\theta_t\omega_t p_{kt} + \beta(1-\omega_t)) \left( \mu_{Rk,t} - \frac{1}{p_{kt}} - \bar{a} - \frac{\sigma_a^2}{2} + \lambda_k - \sigma_a (\sigma_{ka,t} - \sigma_a) \right) \\ &- \beta p_{kt} \omega_t \mu_{\omega_t} - p_{kt} (\theta_t\omega_t p_{kt} + \beta(1-\omega_t)) ((\sigma_{ka,t} - \sigma_a)^2 + \sigma_{k\xi,t}^2) \end{aligned}$$

as

$$\begin{aligned}
0 &= \theta_t \omega_t (1 - \theta_t \omega_t) \Phi(i_t) \left( p_{kt}^2 - \frac{4}{\phi_0^2 \phi_1^2} \right) \\
&+ p_{kt} (2\theta_t \omega_t p_{kt} + \beta (1 - \omega_t)) \left( \mu_{Rk,t} - \frac{1}{p_{kt}} - \bar{a} - \frac{\sigma_a^2}{2} + \lambda_k - \sigma_a (\sigma_{ka,t} - \sigma_a) \right) \\
&- \beta p_{kt} \omega_t \mu_{\omega_t} - p_{kt} (\theta_t \omega_t p_{kt} + \beta (1 - \omega_t)) \left( \frac{\theta_t^{-2}}{\alpha^2} + \sigma_a^2 - 2\sigma_{ka,t} \sigma_a \right) \\
&\equiv \mathcal{C}_0(\omega_t, \theta_t) + \mathcal{C}_a(\omega_t, \theta_t) \sigma_{ka,t},
\end{aligned}$$

where

$$\mathcal{C}_0(\omega_t, \theta_t) = \theta_t \omega_t (1 - \theta_t \omega_t) \Phi(i_t) \left( p_{kt}^2 - \frac{4}{\phi_0^2 \phi_1^2} \right) - p_{kt} (\theta_t \omega_t p_{kt} + \beta (1 - \omega_t)) \left( \frac{\theta_t^{-2}}{\alpha^2} + \sigma_a^2 \right) \tag{A.15}$$

$$+ p_{kt} (2\theta_t \omega_t p_{kt} + \beta (1 - \omega_t)) \left( \mathcal{K}_0(\omega_t, \theta_t) - \frac{1}{p_{kt}} - \bar{a} + \frac{\sigma_a^2}{2} + \lambda_k \right) - \beta p_{kt} \omega_t \mathcal{O}_0(\omega_t, \theta_t)$$

$$\mathcal{C}_a(\omega_t, \theta_t) = p_{kt} (2\theta_t \omega_t p_{kt} + \beta (1 - \omega_t)) \mathcal{K}_a(\omega_t, \theta_t) - \beta p_{kt} \omega_t \mathcal{O}_a(\omega_t, \theta_t) + p_{kt} \beta (1 - \omega_t) \sigma_a. \tag{A.16}$$

Solving for  $\sigma_{ka,t}$ , we obtain

$$\sigma_{ka,t} = -\frac{\mathcal{C}_0(\omega_t, \theta_t)}{\mathcal{C}_a(\omega_t, \theta_t)}.$$

Substituting into the risk-based capital constraint, we obtain

$$\frac{\theta_t^{-2}}{\alpha^2} = \sigma_{k\xi,t}^2 + \left( \frac{\mathcal{C}_0(\omega_t, \theta_t)}{\mathcal{C}_a(\omega_t, \theta_t)} \right)^2,$$

so that

$$\sigma_{k\xi,t} = \sqrt{\frac{\theta_t^{-2}}{\alpha^2} - \left( \frac{\mathcal{C}_0(\omega_t, \theta_t)}{\mathcal{C}_a(\omega_t, \theta_t)} \right)^2}.$$

We can further simplify the above expressions by substituting for  $\mathcal{O}_0$ ,  $\mathcal{O}_a$ ,  $\mathcal{K}_0$ , and  $\mathcal{K}_a$ . Notice

first that

$$\begin{aligned}
\mathcal{C}_a(\omega_t, \theta_t) &= p_{kt} (2\theta_t \omega_t p_{kt} + \beta(1 - \omega_t)) \mathcal{K}_a(\omega_t, \theta_t) - \beta p_{kt} \omega_t \mathcal{O}_a(\omega_t, \theta_t) + p_{kt} \beta (1 - \omega_t) \sigma_a \\
&= p_{kt} (2\theta_t \omega_t p_{kt} + \beta(1 - \omega_t)) (\mathcal{K}_a(\omega_t, \theta_t) - \sigma_\xi \rho_{\xi, a}) \\
&\quad + p_{kt} \left\{ -\frac{2\sigma_a}{\beta} (2\theta_t \omega_t p_{kt} + \beta(1 - \omega_t))^2 + \beta(1 - \omega_t) \sigma_a \right\} \\
&= p_{kt} \left( \beta(1 - \omega_t) + 4\theta_t \omega_t p_{kt} + \frac{4p_{kt} \theta_t \omega_t}{\beta(1 - \omega_t)} (2\theta_t \omega_t p_{kt} + \beta(1 - \omega_t)) \right) \sigma_a \\
&\quad + p_{kt} \left\{ -\frac{2\sigma_a}{\beta} (2\theta_t \omega_t p_{kt} + \beta(1 - \omega_t))^2 + \beta(1 - \omega_t) \sigma_a \right\} \\
&= \frac{2\sigma_a p_{kt}}{\beta} \left( \frac{\omega_t}{1 - \omega_t} \right) (2\theta_t \omega_t p_{kt} + \beta(1 - \omega_t))^2.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\mathcal{C}_0(\omega_t, \theta_t) &= \theta_t \omega_t (1 - \theta_t \omega_t) \Phi(i_t) \left( p_{kt}^2 - \frac{4}{\phi_0^2 \phi_1^2} \right) - p_{kt} (\theta_t \omega_t p_{kt} + \beta(1 - \omega_t)) \left( \frac{\theta_t^{-2}}{\alpha^2} + \sigma_a^2 \right) \\
&\quad + p_{kt} (2\theta_t \omega_t p_{kt} + \beta(1 - \omega_t)) \left( \mathcal{K}_0(\omega_t, \theta_t) - \frac{1}{p_{kt}} - \bar{a} + \frac{\sigma_a^2}{2} + \lambda_k \right) - \beta p_{kt} \omega_t \mathcal{O}_0(\omega_t, \theta_t) \\
&= \theta_t \omega_t (1 - \theta_t \omega_t) \Phi(i_t) \left( p_{kt}^2 - \frac{4}{\phi_0^2 \phi_1^2} \right) - p_{kt} (\theta_t \omega_t p_{kt} + \beta(1 - \omega_t)) \left( \frac{\theta_t^{-2}}{\alpha^2} + \sigma_a^2 \right) \\
&\quad + \beta p_{kt} (1 - \omega_t) \frac{\theta_t \omega_t}{1 - i_t \theta_t \omega_t} \Phi(i_t) (1 - i_t) \\
&\quad - \theta_t \omega_t p_{kt}^2 \left( 3 + \frac{4p_{kt} \theta_t \omega_t}{\beta(1 - \omega_t)} \right) \frac{\theta_t^{-2}}{\alpha^2} - \sigma_a^2 p_{kt} \left( \theta_t \omega_t p_{kt} + \frac{(\beta(1 - \omega_t) + 2\theta_t \omega_t p_{kt})^2}{\beta(1 - \omega_t)} \right) \\
&\quad + \frac{p_{kt}}{\beta} (2\theta_t \omega_t p_{kt} + \beta(1 - \omega_t))^2 \left( \sigma_a^2 + \frac{\theta_t^{-2}}{\alpha^2} \right) - \beta p_{kt} (1 - \omega_t) \Phi(i_t) \theta_t \omega_t
\end{aligned}$$

Collecting like terms, we obtain

$$\begin{aligned}
\mathcal{C}_0(\omega_t, \theta_t) &= \theta_t \omega_t (1 - \theta_t \omega_t) \Phi(i_t) \left( p_{kt}^2 - \frac{4}{\phi_0^2 \phi_1^2} \right) - \beta p_{kt} (1 - \omega_t) \theta_t \omega_t \Phi(i_t) \left( 1 - \frac{1 - i_t}{1 - i_t \theta_t \omega_t} \right) \\
&\quad + \frac{\theta_t^{-2}}{\alpha^2} \frac{p_{kt}}{\beta} \left\{ (2\theta_t \omega_t p_{kt} + \beta(1 - \omega_t))^2 - \beta(\theta_t \omega_t p_{kt} + \beta(1 - \omega_t)) - \theta_t \omega_t p_{kt} \left( 3\beta + \frac{4p_{kt} \theta_t \omega_t}{1 - \omega_t} \right) \right\} \\
&\quad + \sigma_a^2 \frac{p_{kt}}{\beta} \left\{ (2\theta_t \omega_t p_{kt} + \beta(1 - \omega_t))^2 - \beta(2\theta_t \omega_t p_{kt} + \beta(1 - \omega_t)) - \frac{(\beta(1 - \omega_t) + 2\theta_t \omega_t p_{kt})^2}{(1 - \omega_t)} \right\} \\
&= -\frac{\theta_t^{-2}}{\alpha^2} \frac{p_{kt}}{\beta} \frac{\omega_t}{1 - \omega_t} (2\theta_t \omega_t p_{kt} + \beta(1 - \omega_t))^2 \\
&\quad - \sigma_a^2 \frac{p_{kt}}{\beta} (2\theta_t \omega_t p_{kt} + \beta(1 - \omega_t)) \frac{\omega_t (2\theta_t \omega_t p_{kt} + \beta(1 - \omega_t)) + 1 - \omega_t}{1 - \omega_t}
\end{aligned}$$

Thus

$$\begin{aligned}\sigma_{ka,t} &= -\frac{C_0(\omega_t, \theta_t)}{C_a(\omega_t, \theta_t)} \\ &= \frac{\theta_t^{-2}}{\alpha^2} + \sigma_a^2 \left( 1 + \frac{1 - \omega_t}{\omega_t (2\theta_t \omega_t p_{kt} + \beta (1 - \omega_t))} \right).\end{aligned}$$

Figure 14 plots the equilibrium sensitivities of the excess return to capital (top panels), the excess return to intermediary debt (middle panels), and the growth rate of the intermediary wealth (lower panels), as a function of intermediary's wealth share in the economy  $\omega_t$  and intermediary leverage  $\theta_t$ . The sensitivity of the return to holding capital to both the productivity and the household beliefs shock decreases in the leverage of the intermediary sector, while an increase in the intermediary's wealth share in the economy has little impact. Intuitively, since total volatility of the return to holding capital is inversely proportional to intermediary leverage, an increase in leverage has to lead to a decrease in volatility. The sensitivity of the return to intermediary debt, on the other hand, has an inverse relationship to the intermediary's wealth share in the economy, with the total volatility of the excess return to intermediary debt increasing as the intermediary owns a smaller fraction of wealth in the economy. Intuitively, as the intermediary's wealth share in the economy decreases, shocks impact intermediary equity more, increasing the volatility of the return to holding intermediary debt.

### A.3 Constant leverage benchmark (Proof of Lemma 5.1)

We begin by solving for the equilibrium dynamics of the intermediaries' wealth share in the economy. Recall that the capital held by the intermediaries is given by

$$k_t = \theta \omega_t K_t.$$

Applying Itô's lemma, we obtain

$$\begin{aligned}\frac{dk_t}{k_t} &= \frac{d\omega_t}{\omega_t} + \frac{dK_t}{K_t} \\ &= (\mu_{\omega t} + \Phi(i_t) \theta \omega_t - \lambda_k) dt + \sigma_{\omega a, t} dZ_{at} + \sigma_{\omega \xi, t} dZ_{\xi, t}.\end{aligned}$$

Recall, on the other hand, that intermediary capital evolves as

$$\frac{dk_t}{k_t} = (\Phi(i_t) - \lambda_k) dt.$$

Thus, equating coefficients, we obtain

$$\begin{aligned}\sigma_{\omega a, t} &= 0 \\ \sigma_{\omega \xi, t} &= 0 \\ \mu_{\omega t} &= \Phi(i_t) (1 - \theta \omega_t).\end{aligned}$$

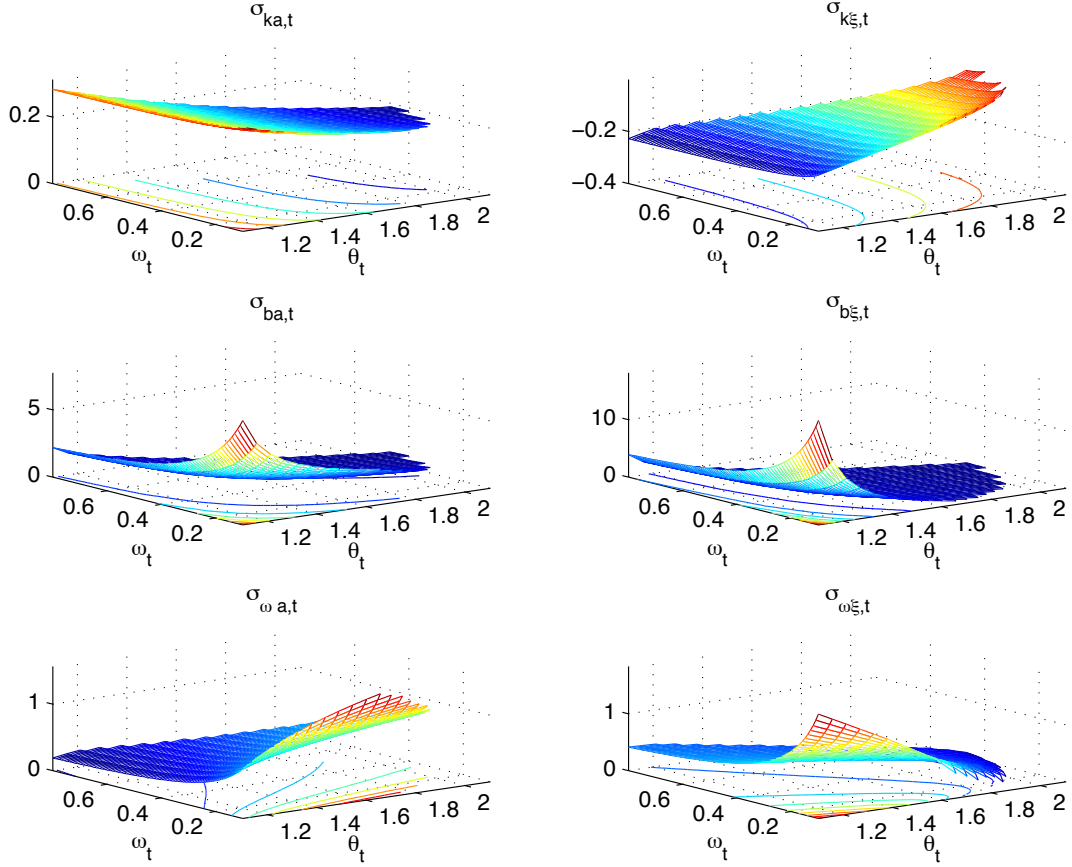


Figure 14: Loadings on the shock to productivity (left panels) and the shock to beliefs (right panels) as a function of intermediary leverage  $\theta_t$  ( $x$ -axis) and financial intermediaries' wealth share in the economy  $\omega_t$  ( $y$ -axis). Upper panels: loadings ( $\sigma_{ka,t}, \sigma_{k\xi,t}$ ) of the return to capital; middle panels: loadings ( $\sigma_{ba,t}, \sigma_{b\xi,t}$ ) of the return to intermediary debt; lower panels: loadings ( $\sigma_{\omega a,t}, \sigma_{\omega\xi,t}$ ) of the growth rate of intermediary equity.

Consider now the wealth evolution of the representative household. From the households' budget constraint, we have

$$\frac{dw_{ht}}{w_{ht}} = \left( r_{ft} - \rho_h + \frac{\sigma_\xi^2}{2} \right) dt + \frac{1 - \bar{\theta}\omega_t}{1 - \omega_t} (dR_{kt} - r_{ft}dt) + \frac{\omega_t (\bar{\theta} - 1)}{1 - \omega_t} (dR_{bt} - r_{ft}dt).$$

On the other hand, from the definition of  $\omega_t$ , we obtain

$$\begin{aligned} \frac{dw_{ht}}{w_{ht}} &= \frac{d((1 - \omega_t) p_{kt} A_t K_t)}{(1 - \omega_t) p_{kt} A_t K_t} \\ &= \frac{dp_{kt}}{p_{kt}} + \frac{dA_t}{A_t} + \frac{dK_t}{K_t} - \frac{\omega_t}{1 - \omega_t} \frac{d\omega_t}{\omega_t} + \left\langle \frac{dp_{kt}}{p_{kt}}, \frac{dA_t}{A_t} \right\rangle. \end{aligned}$$

Equating coefficients once again and simplifying, we obtain

$$\begin{aligned}\sigma_{ba,t} &= \sigma_{ka,t} \\ \sigma_{b\xi,t} &= \sigma_{k\xi,t} \\ \mu_{Rb,t} &= \mu_{Rk,t} + \frac{1 - \omega_t}{\omega_t (\bar{\theta} - 1)} \left( \rho_h - \frac{\sigma_\xi^2}{2} - \frac{1}{p_{kt}} \right) + \Phi(i_t).\end{aligned}$$

We now turn to solving for the equilibrium price of capital. The goods clearing condition in this economy reduces to

$$\left( \rho_h - \frac{\sigma_\xi^2}{2} \right) (1 - \omega_t) p_{kt} = 1 - i_t \bar{\theta} \omega_t.$$

Substituting the optimal level of investment

$$i_t = \frac{1}{\phi_1} \left( \frac{\phi_0^2 \phi_1^2}{4} p_{kt}^2 - 1 \right),$$

we obtain that the price of capital satisfies

$$\left( \rho_h - \frac{\sigma_\xi^2}{2} \right) (1 - \omega_t) p_{kt} = 1 - \frac{\bar{\theta} \omega_t}{\phi_1} \left( \frac{\phi_0^2 \phi_1^2}{4} p_{kt}^2 - 1 \right).$$

Then the price of capital satisfies

$$0 = \bar{\theta} \omega_t p_{kt}^2 + \beta (1 - \omega_t) p_{kt} - \frac{4}{\phi_0^2 \phi_1^2} (\bar{\theta} \omega_t + \phi_1),$$

or:

$$p_{kt} = \frac{-\beta (1 - \omega_t) + \sqrt{\beta^2 (1 - \omega_t)^2 + \frac{16 \bar{\theta} \omega_t}{\phi_0^2 \phi_1^2} (\bar{\theta} \omega_t + \phi_1)}}{2 \bar{\theta} \omega_t}.$$

Applying Itô's lemma, we obtain

$$0 = \bar{\theta} \omega_t p_{kt}^2 \left( 2 \frac{dp_{kt}}{p_{kt}} + \left\langle \frac{dp_{kt}}{p_{kt}} \right\rangle^2 + \frac{d\omega_t}{\omega_t} \right) + \beta (1 - \omega_t) p_{kt} \frac{dp_{kt}}{p_{kt}} - \frac{\omega_t}{1 - \omega_t} \beta p_{kt} \frac{d\omega_t}{\omega_t} + \frac{4}{\phi_0^2 \phi_1^2} \bar{\theta} \frac{d\omega_t}{\omega_t}.$$

Equating coefficients and simplifying, we obtain

$$\begin{aligned}\sigma_{ka,t} &= \sigma_a \\ \sigma_{k\xi,t} &= 0 \\ \mu_{Rk,t} &= \frac{1}{p_{kt}} + \bar{a} + \frac{\sigma_a^2}{2} - \lambda_k - \frac{\Phi(i_t) (1 - \bar{\theta} \omega_t)}{p_{kt} (2 \bar{\theta} \omega_t p_{kt} + \beta (1 - \omega_t))} \left( \frac{4 \bar{\theta}}{\phi_0^2 \phi_1^2} - \frac{\omega_t}{1 - \omega_t} \beta p_{kt} + \bar{\theta} \omega_t p_{kt}^2 \right).\end{aligned}$$

Finally, consider the equilibrium risk-free rate. Notice that

$$\begin{aligned}\frac{dc_t}{c_t} &= \frac{d((1 - i_t \bar{\theta} \omega_t) A_t K_t)}{(1 - i_t \bar{\theta} \omega_t) A_t K_t} \\ &= \frac{dA_t}{A_t} + \frac{dK_t}{K_t} - \frac{\bar{\theta} \omega_t}{1 - i_t \bar{\theta} \omega_t} di_t - \frac{i_t \bar{\theta} \omega_t}{1 - i_t \bar{\theta} \omega_t} \frac{d\omega_t}{\omega_t} - \frac{\bar{\theta} \omega_t}{1 - i_t \bar{\theta} \omega_t} \left\langle di_t, \frac{dA_t}{A_t} \right\rangle,\end{aligned}$$

and:

$$di_t = d \left( \frac{\left( \rho_h - \frac{\sigma_\xi^2}{2} \right)}{\beta} p_{kt}^2 - \frac{1}{\phi_1} \right) = \frac{\left( \rho_h - \frac{\sigma_\xi^2}{2} \right)}{\beta} p_{kt}^2 \left( 2 \frac{dp_{kt}}{p_{kt}} + \left\langle \frac{dp_{kt}}{p_{kt}} \right\rangle^2 \right).$$

Using:

$$1 - i_t \bar{\theta} \omega_t = \left( \rho_h - \frac{\sigma_\xi^2}{2} \right) (1 - \omega_t) p_{kt},$$

the risk-free rate is thus given by

$$\begin{aligned}r_{ft} &= \left( \rho_h - \frac{\sigma_\xi^2}{2} \right) + \frac{1}{dt} \mathbb{E}_t \left[ \frac{dc_t}{c_t} \right] - \frac{1}{dt} \mathbb{E}_t \left[ \left\langle \frac{dc_t}{c_t} \right\rangle \right] \\ &= \left( \rho_h - \frac{\sigma_\xi^2}{2} \right) + \bar{a} - \frac{\sigma_a^2}{2} + \Phi(i_t) \bar{\theta} \omega_t - \lambda_k - \frac{2\bar{\theta} \omega_t p_{kt}}{\beta (1 - \omega_t)} \left( \mu_{Rk,t} - \frac{1}{p_{kt}} + \lambda_k - \bar{a} - \frac{\sigma_a^2}{2} \right) \\ &\quad - \frac{i_t \bar{\theta} \omega_t}{1 - i_t \bar{\theta} \omega_t} \mu_{\omega t}.\end{aligned}$$