Network Formation and Systemic Risk*

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Abstract

This paper introduces a model of endogenous network formation and systemic risk. In the model a link represents a trading opportunity that yields benefits only if the counterparty does not subsequently default. After links are formed, they are subjected to exogenous shocks that are either good or bad. Bad shocks reduce returns from links and incentivize default. Good shocks, the reverse. Defaults triggered by bad shocks might propagate via links. The model yields three insights. The first concerns the volatility paradox. A higher probability of good shocks generates a higher systemic risk because increased interconnectedness in the network offsets the effect of better fundamentals. Second, the networks formed in the model are utilitarian efficient. The former is a consequence of contagion being triggered too often, whereas the latter is a consequence of contagion spreading easily. Third, the network formed critically depends on the correlation between shocks to the links. As a consequence, an outside observer who misconceives the correlation structure of shocks, upon observing a highly interconnected network, will significantly underestimate the probability of system wide default.

JEL classification: D85, G01.

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1 Introduction

The awkward chain of events that so upset the bankers in 2008, began with the collapse of Lehmann Brothers. Panic spread, the dollar wavered and world markets fell. Interconnectedness of the financial system, it was suggested, allowed Lehmann's fall to threaten the stability of the entire system. This prompted scholars to characterize the network structures that allow shocks to one part of the financial network to be amplified and spread. Blume et al. (2013) as well as Vivier-Lirimonty (2006), for example, argue that dense interconnections pave the way to systemic failures. In contrast, Allen and Gale (2000) as well as Freixas et al. (2000), argue that a more interconnected architecture protects the system against contagion because the losses of a distressed institution can be divided among many creditors. With some exceptions, a common feature of these and other papers (Acemoglu et al. (2015), Eboli (2013), Elliott et al. (2014), Gai et al. (2011), Glasserman and Young (2015)) is an exogenously given network. A node (or subset of them) is subjected to a shock and its propagation studied as the size of the shock varies. Absent are reasons for the presence of links between agents.[1] This paper assumes that a link between two agents represents a potentially lucrative joint opportunity. However, every link increases the possibility of contagion. In the presence of such a trade-off we ask what kinds of networks would agents form? In particular, do they form networks that are susceptible to contagion?

In the model we use to answer these questions, agents first form links. The payoff to the counterparties that share a link is uncertain and depends upon the future realization of a random variable (which we call a shock) and actions taken subsequent to the shock. Specifically, there are three stages. In stage one, agents form links which can be interpreted as partnerships or joint ventures. In stage two, each link formed is subjected to a shock. In stage three, with full knowledge of the network and realized shocks, each agent decides whether to 'default' or not. The payoff to an agent depends on the action she takes in the third stage as well the actions of her counterparties (and their counterparties and so on) as well as the realized shocks. The default decision corresponds to exiting from every partnership formed in stage one. The event that the only Nash equilibrium of the game in stage three is that everyone defaults is called system wide failure. In our model, default is the result of a 'loss of confidence' rather than simple 'spillover' effects.[2] Our paper is also a contribution to the literature on contagion in networks. We build on the models of Morris (2000) as well as Goyal and Vega-Redondo (2005) which analyze contagion on a fixed network. We extend these by incorporating uncertain payoffs and endgogenizing the networks.

In the benchmark version of this model the network formed in stage one is utilitarian efficient. Efficiency is a consequence of the high risk of contagion which forces agents to form isolated clusters that serve as firebreaks. The main source of possible inefficiency, contagion spreading to distant parts of the network, is eliminated by the absence of links between clusters.[3] This outcome is not obvious because the high risk of contagion might cause agents to form inefficiently few links.

A second contribution is to examine how the probability of system wide failure as well as the expected number of defaults varies with a change in the distribution of shocks. When shocks are independent and binary (good or bad), the probability of system wide failure *increases* with an increase in the probability of a good shock, up to the point at which the formed network becomes a complete graph, i.e. every pair of agents

[1]Blume et al. (2013) and Farboodi (2015) are exceptions.

[2]Glasserman and Young (2015) argue that spillover effects have only a limited impact. They suggest that the "mere possibility (rather than the actuality) of a default can lead to a general and widespread decline in valuations....."

[3]Later, we consider variations in the strength of contagion and types of agents that produce other network structures.

is linked. After this point, the probability of system wide failure declines. Intuitively, as partnerships become less risky, agents are encouraged to form more partnerships increasing interconnectedness which increases the probability of system wide failure. This gives a network foundation for the volatility paradox described in Brunnermeier and Sannikov (2014). The expected number of defaults, in contrast, declines with an increase in the probability of a good shock. This illustrates that two plausible measures of systemic risk can move in different directions with a change in the fundamentals.

Our final contribution shows that the structure of the network formed in stage one depends critically on whether the shocks to the links are believed to be correlated or independent of each other. When shocks are perfectly correlated, the network formed in stage one is a complete graph. We think this finding relevant to the debate between two theories of financial destruction advanced to explain the 2008 financial crisis. The first, described above, is dubbed the 'domino theory'. The alternative, advocated most prominently by Edward Lazear, is dubbed 'popcorn'.[4] Lazear describes it thusly in a 2011 opinion piece in the Wall Street Journal:

> "The popcorn theory emphasizes a different mechanism. When popcorn is made (the old fashioned way), oil and corn kernels are placed in the bottom of a pan, heat is applied and the kernels pop. Were the first kernel to pop removed from the pan, there would be no noticeable difference. The other kernels would pop anyway because of the heat. The fundamental structural cause is the heat, not the fact that one kernel popped, triggering others to follow.
>
> Many who believe that bailouts will solve Europe's problems cite the Sept. 15, 2008 bankruptcy of Lehman Brothers as evidence of what allowing one domino to fall can do to an economy. This is a misreading of the historical record. Our financial crisis was mostly a popcorn phenomenon. At the risk of sounding defensive (I was in the government at the time), I believe that Lehman's downfall was more a result of the factors that weakened our economic structure than the cause of the crisis."

Our model suggests that underlying structural weaknesses (modeled by strong correlations between shocks) and greater interconnectedness can coexist. Therefore, it would be incorrect to highlight the interconnectedness of the system and suggest it *alone* as the cause of instability.

More importantly, it implies that a mistake in assessing the correlation structure of shocks can lead to disproportionately bigger mistakes in assessing the probability of systemwide failure. In the model, a complete network arises from perfectly correlated shocks, the popcorn world, no matter how likely the shocks are to be bad. However, a complete network arises from independent shocks, the dominoes world, only if the shocks are very likely to be good. Therefore, we suggest that Lazear's view might shed light on the possible causes for the underestimation of the likelihood of a financial crisis.

Our model differs from the prior literature in two important ways.

1. The networks we study are formed endogenously. Babus (2016) also has a model of network formation, but one in which agents share the goal of minimizing the probability of system wide default. In our model agents are concerned with their own expected payoffs and only indirectly with the possibility of system wide failure. Acemoglu et al. (2015) also discusses network formation but within a set of

[4]Chair of the US President's Council of Economic Advisers during the 2007-2008 financial crisis.

limited alternatives. Zawadowski (2013) models the decision of agents to purchase default insurance on their counter-parties. This can be interpreted as a model of network formation, but it is not a model of an agent choosing a particular counter-party because the counter-parties are fixed. Default insurance serves to change the terms of trade with an existing counter-party. The model in Farboodi (2015) includes network formation with the same solution concept we employ. Our model encompasses mutual cross-holdings whereas her model focuses on directional interbank lending. Furthermore, we are able to explicitly characterize all networks formed, and provide detailed comparative statics by determining the exact distribution of defaults. Blume et al. (2013) has networks that form endogenously. However, the risk of a node defaulting is *independent* of the network formed. In our model, the likelihood of a node defaulting depends on the structure of the network formed.

2. We examine the effects of a distribution that generates the shocks rather than the effects of fixed shocks applied to particular nodes. Glasserman and Young (2015) and Cabrales et al. (2017) are the only exceptions we are aware of, but the networks they consider are exogenously given. Endogenizing the network allows us to shed light on the volatility paradox as well as the popcorn vs. dominoes debate.

In section 2, we give a formal description of the model. Section 3.1 uses these results to characterize the structure of the realized networks and their efficiency. Section 3.2 investigates systemic risk and illustrates volatility paradox. Section 3.3 studies correlated shocks and discusses the popcorn vs. dominoes debate. Section 5 describes some extensions to the basic model. Future work is discussed in Section 6.

2 The Model

2.1 Environment

Denote by $N = \{1, 2, ..., n\}$ a finite set of agents. Each pair of agents in N can form a bilateral partnership. A potential edge $\{i, j\}$ represents the joint venture between nodes i and j. The payoffs to i and j from edge $\{i, j\}$ are contingent on the future realization of some shock, as well as the actions that the agents take.

The model has three stages. In stage one, network formation, agents, by mutual consent, decide which potential edges to pick. The edges picked are called realized. The set of realized edges is denoted E. The corresponding network, denoted (N, E), is called the realized network.

In stage two, for each realized edge $\{i, j\}$, two outcomes θ^{ij} and θ^{ji}, describing the payoffs to incident[5] nodes i and j, are chosen by nature identically and independently. θ^{ij} is the payoff to i from the edge, and θ^{ji} is the payoff to j. With probability α, the edge is good: $\theta^{ij} = \theta_0$, where $0 < \theta_0 < 1$. With probability $1 - \alpha$, the edge is bad:$\theta^{ij} = \theta_1$, where $\theta_1 < 0$. The same is true for θ^{ji}. Denote the realized network by $(N, E, \boldsymbol{\theta})$. In a sense, the network is directed since θ^{ij} and θ^{ji} can be different.

In stage three, with full knowledge of $(N, E, \boldsymbol{\theta})$ each agent i chooses one of two possible actions called B (business as usual) or D (default), denoted a_i. If agent i chooses to play D she receives an outside option 0. If i chooses to play C she receives the sum of θ^{ij}'s over all of his neighbors[6] j in (N, E). Furthermore, i

[5]A node v is incident to an edge e if $v \in e$.

[6]Two distinct nodes v and v' are neighbors if $\{v, v'\} \subset E$. In this case v and v' are also said to be adjacent.

incurs a cost 1 for each of its defaulting neighbors. That is,

$$u_i(C, a_{-i}|(N, E, \boldsymbol{\theta})) = \sum_{\substack{j\,:\,\{i,j\} \in E \\ \& \, a_j = C}} \theta^{ij} + \sum_{\substack{j\,:\,\{i,j\} \in E \\ \& \, a_j = D}} \left(\theta^{ij} - 1\right),$$

$$u_i(D, a_{-i}|(N, E, \boldsymbol{\theta})) = 0.$$

2.2 Discussion of the environment

In this section we make our interpretation of the links formed as joint ventures concrete. Each link is a joint project funded by deposits in stage one, with an interest rate $r > 1$ promised upon the realization of returns from project. The project requires a monetary investment from each counterparty and costly effort to be exerted at the end of stage two. The project succeeds if and only if both counterparties exert effort. In our model the payoff from the project is both uncertain and independent across the counterparts. This can be justified in at least two ways.

- The partners are responsible for different and independent parts of the project and the costs they bear are uncertain. This is not at all unusual in, for example, construction.
- The project may have (positive or negative) externalities on projects/products that do not belong to the scope of the joint venture (see Dessein (2005), Kumar (2010)).

While we assume that θ^{ij} and θ^{ji} are independent of each other, the results in the benchmark case (Section 3) carry over to the case when they are perfectly correlated (or even conditionally independent). This can be interpreted as a common cost or payoff shock to the counterparties.

Two features of the model deserve discussion. First, in contrast to prior literature, shocks apply to edges rather than nodes. While we extend our model (Section 5) to allow for shocks to nodes as well, we believe shocks to edges to be of independent interest. An agent's solvency depends on the outcomes of the many investments she has chosen to make. The interesting case is when these investments require coordination with at least one other agent, a joint venture if you will. It is the outcome of this joint venture that will determine whether the participants decide to continue or walk away.

Second, an agent must default on all partnerships or none. While demanding, it is not, we argue, unreasonable. Were an agent free to default on any subset of its partnerships, we could model this by splitting each node in (N, E) into as many copies of itself as its degree.[7] Each copy would be incident to exactly one of the edges that were previously incident to the original node. Thus, our model easily accommodates this possibility. However, this has the effect of treating a single entity as a collection of independent smaller entities which we think inaccurate. Institutions considering default face liquidity constraints, which restrict, at best, the number of parties they can repay. When a company fails to pay sufficiently many of its creditors, the creditors can force the company into bankruptcy. While some entities can selectively default, there is a knock-on effect. Agents that selectively default, have their credit ratings downgraded which raise their borrowing costs for the other activities they are engaged in. Thus, it is entirely reasonable to suppose that the default decisions

[7] The degree of a node in a graph is the number of edges incident to it.

associated with the edges a node is incident to must be linked. Ours is an extreme, but simple, version of such a linkage.

2.3 Solution concepts

Here we describe the solution concepts to be employed in stages one and three. We begin with stage three as the outcomes in this stage will determine the choices made by agents in stage one.

Agents enter stage three knowing $(N, E, \boldsymbol{\theta})$. With this knowledge, each agent simultaneously chooses action B or D. We do not allow actions chosen in stage three to be conditioned on what happens in earlier stages. The outcome in stage three is assumed to be a Nash equilibrium. Notice that 'everybody plays D' is a Nash equilibrium since $\theta_1 < \theta_0 < 1$, yet it need not be the only one.

For any given network and realization of shocks, the game in stage three is supermodular. Using Tarski's theorem, one can show that the set of Nash equilibria is a complete lattice, ordered via set inclusion on the set of nodes that play C. Therefore, -the strategy profile in which any node who plays C in at least one Nash equilibrium, plays C- is a Nash equilibrium. It is known that the maximum point of the lattice can be reached by applying iterated elimination of strictly dominated strategies, which is the notion of contagion here. Then, equivalently, the rationalizable strategy profile in which any agent who can rationalize C plays C is the "best Nash equilibrium". We work with this Nash equilibrium and call it the cooperating equilibrium in short. For more details on the contagion and some characteristics of cooperating equilibrium refer to Section 4.3.

A realized network along with shocks, $(N, E, \boldsymbol{\theta})$, exhibits system wide failure if in the cooperating equilibrium of the game all agents in N choose D.[8] In this case, agents can coordinate on nothing but action D. The probability of system wide failure of a realized network is called its systemic risk. We consider other measures as well.

In stage one, agents know the distribution by which nature assigns states and the equilibrium selection in stage three. Therefore, they are in a position to evaluate their expected payoff in each possible realized network. Using this knowledge they decide which edges to form. Next we describe how the realized network is formed.

Consider a candidate network (N, E) and a coalition of agents $V \subset N$. A feasible deviation by V allows agents in V

1. to add any absent edges within V, and
2. to delete any edges incident to at least one vertex in V.

A profitable deviation by V is a feasible deviation in which each member of V receives strictly higher expected payoff.[9]

[8] This is equivalent to saying that 'everybody plays D' is the only Nash equilibrium.

[9] The requirement that all agents in a profitable deviation receive strictly higher payoff prevents 'cycling'. To illustrate, consider three nodes $N = \{v_1, v_2, v_3\}$ and $E = \{\{v_1, v_2\}, \{v_2, v_3\}\}$. Suppose v_1 and v_3 deviating to $E' = \{\{v_1, v_3\}, \{v_2, v_3\}\}$, leaves v_1 indifferent and v_3 strictly better off. However, E' is just isomorphic to E and there is no good sense in which v_1 would bother deviating to E'. v_1 and v_2 could very well want to deviate back to E from E'. Precluding 'weak' deviations would be overly restrictive. In particular it almost trivially imposing very strong forms of symmetry on any candidate network to be formed.

A realized network (N, E) is called $k-$stable if there are no profitable deviations by any $V \subset N$ with $|V| \leq k$ (see Jackson (2010)). G is group stable[10] if there are no profitable deviations for any $V \subset N$. That is, group stability is ∞-stability. G is bilaterally stable if it is $2-$stable. We assume that the network formed in stage one is group stable. In the sequel we discuss how our main results change under weaker notions of stability. A widely used solution concept is pairwise stability, which is implied by bilateral stability. Pairwise stability allows for one pair i and j to add their missing edge $\{i, j\}$ if $\{i, j\} \notin E$ and to delete their existing edge $\{i, j\}$ if $\{i, j\} \in E$, just as bilateral stability does. However, pairwise stability does not allow i to delete $\{i, k\}$ or j to delete $\{j, k\}$ in a feasible deviation by the pair i and j, whereas bilateral stability allows for a node to delete any number of links of its own.

$$Group\, stable \implies k - stable\, (k \geq 2) \implies Bilaterally\, stable \implies Pairwise\, stable.$$

Group stability corresponds to the *strong* Nash equilibrium of an underlying non-cooperative network formation game played between the members of N. Each agent simultaneously proposes to a subset of agents to form an edge. The cost of each proposal is $c > 0$. If a proposal is reciprocated, the corresponding edge is formed. The owners of the edge are refunded c. If a proposal is not reciprocated, c is not refunded and the edge is not formed.

Notice that in any Nash equilibrium of this game, all proposals must be mutual. Consider a *strong* Nash equilibrium of the proposal game. A coalition V can make mutual proposals between themselves to form a missing edge, or undo a proposal by any member which would delete the corresponding edge. Therefore, *strong* Nash equilibria of this game correspond to group stable networks.

3 Benchmark case

Denote by b_i the number of edges incident to i that receive bad shocks, and let f_i be the number of neighbors of i that play D. Then,

$$u_i(C, a_{-i}|(N, E, \boldsymbol{\theta})) = d_i\theta_0 - b_i\,(\theta_0 - \theta_1) - f_i.$$

Therefore, i's best response is C if $b_i\,(\theta_0 - \theta_1) + f_i \leq d_i\theta_0$. In general, this specification makes it quite difficult to find the cooperating equilibrium for any network, and makes it even harder to find the network formed. In the benchmark case, we make two simplifying assumptions which we relax later on.

Assumption 1. ***(Strong contagion)*** $0 < (n-1)\theta_0 < 1$.

Assumption 2. ***(Strong seed)*** $(n-2)\theta_0 + \theta_1 < 0$.

The condition $(n-1)\theta_0 < (\theta_0 - \theta_1)$ ensures that anyone incident to at least one bad edge with θ_1 defaults as the unique best reply. The condition $(n-1)\theta_0 < 1$ ensures that anyone who has at least one defaulting neighbor also defaults as the unique best reply. Hence in any Nash equilibrium, all nodes in a maximally connected component[11] play D altogether or play C altogether. Under these two assumptions, in the coop-

[10] In Dutta and Mutuswami (1997) this is called strongly stable while Farboodi (2015) calls this solution concept group stable. We think the second more evocative. Also, Jackson and Van den Nouweland (2005) considers a stronger notion that rules out deviations that Pareto improve the deviating coalition.

[11] The maximally connected component of a node v is the set of all nodes connected to v via a path. Formally, a path between two nodes v_0 and v_{k+1} is a sequence of nodes $v_0, v_1, ..., v_k, v_{k+1}$ such that $\{v_i, v_{i+1}\} \subset E$ for all $i = 0, 1, ..., k$. Two nodes are connected nodes if there is path between them. A subset V of nodes is a connected set if any two elements of V are connected by a path that resides entirely in V. $V \subset N$ is maximally connected if V is connected and there is no strict superset of V that is connected.

erating equilibrium, all nodes of a maximally connected component play D if at least one of them received at least one bad shock. Otherwise they all play C.

Assumptions 1 and 2, which we relax later, are the extreme cases of how contagion starts and spreads. This simplifies contagion dynamics, buying great technical convenience. One can interpret these as a case where agents are very heavily exposed to each other which leads to a very significant risk of contagion. Under these assumptions, maximum possible sum of gains from trade scale linearly with n, and that if hypothetically the market were complete in stage three, the system as a whole could withstand bad shocks that make up a fraction of at most $1/n$ of all edges.

3.1 Network Formation

In this section we characterize the set of group stable networks under Assumptions 1 and 2. We show that a group stable network consists of a collection of node disjoint complete subgraphs[12]. This is illustrated in Figure 1. By forming into complete subgraphs agents increase the benefits they enjoy from partnerships. However, the complete subgraphs formed are limited in size and order, and are disjoint. In this way agents ensure that a default in one portion of the realized network does not spread to the entire network.[13] This extreme structure is a consequence of the spareness of our model. However, it suggests that more generally we should expect to see collections of densely connected clusters that are themselves sparsely connected to each other. Blume et al. (2013) have a similar finding.[14] We discuss this structure formally in Section 5.1 and show that richer network structures, in particular core-periphery, can emerge under various extensions without altering any qualitative insights.

We first need to determine an agent's expected payoff in various realized networks. Recall that nature determines shocks identically and independently across edges. Consider $i \in N$. Suppose that in the realized network i has degree d and the maximally connected component that contains it has e edges. The probability that no one in the relevant component defaults is α^{2e}. In this case, i gets $d\theta_0$. The probability that every node in the relevant component defaults is $1-\alpha^{2e}$. In this case, i gets 0. So i's expected payoff in stage two is $d\alpha^{2e}\theta_0$. For example, a node in a disjoint complete subgraph of order $d+1$ has expected payoff $U(d) := d\alpha^{d(d+1)}\theta_0$.

Lemma 1. *Any bilaterally stable network consists of disjoint complete subgraphs.*

Proof. Suppose, for a contradiction, a bilaterally stable network with two non-adjacent nodes v' and v'' in the same connected component. Take a path $v' = v_1, v_2, \ldots\ v_t = v''$ between v' and v''. Insert the edge $\{v', v''\}$ and delete $\{v', v_2\}$, as well as $\{v_{t-1}, v''\}$. The degrees of v' and v are unchanged but the number of edges in the component that contains them strictly decreases. Hence, this is a profitable pairwise deviation by v' and v'' which contradicts bilateral stability. Therefore, in any bilaterally stable network all nodes within the same connected component are adjacent, which completes the proof. □

Let $d^* = \arg\max_{d\in\mathbb{N}} U(d)$. For generic α, d^* is well defined. Note that $U(d)$ is strictly increasing in $d \in \mathbb{N}$ up to d^*, and strictly decreasing after d^*. Further, d^* is an increasing step function of α. A rough approximation is $d^* \approx \sqrt{-2log(\alpha)}^{-1}$. This makes the probability of failure in a disjoint complete subgraph of order d^*+1 roughly $1-\boldsymbol{e}^{-0.5}$.

[12]A graph (N', E') is a subgraph of (N, E) if $N' \subset N$ and $E' \subset E$.

[13]The size of a subgraph or a subset of edges is the number of edges in it. The order of a subgraph or a subset of nodes is the number of nodes in it.

[14]Cabrales et al. (2017) assume a similar structure. Our analysis complement theirs by showing that the this structure emerges endogenously.

Lemma 2. *A group stable network consists of a collection of disjoint complete subgraphs, all but one of order $d^* + 1$. The remaining complete subgraph is of order at most $d^* + 1$.*

Proof. By Lemma 1 a group stable network (if it exists) is composed of disjoint complete subgraphs. The payoff to an agent in a $(d+1)$-complete subgraph is $U(d) = d\alpha^{d(d+1)}\theta_0$. This is strictly increasing up to d^*. First, no complete subgraph can have order $d+1 > d^*+1$ in the realized network. Otherwise, d^*+1 members could deviate by forming a (d^*+1)-complete subgraph and cutting all other edges. This would be a strict improvement since d^* is the unique maximizer of $U(d)$.
Second, there cannot be two complete subgraphs of order $d+1 < d^*+1$. Suppose not. Let there be $d'+1$ nodes all together in these two complete subgraphs. Then $min\{d'+1, d^*+1\}$ nodes would have a profitable deviation by forming an isolated complete subgraph since $U(d)$ is increasing in d up to d^*. □

A realized network that is group stable necessarily consists of a collection of complete subgraphs of order d^*+1 and one 'left-over' complete subgraph with order different from d^*+1. To avoid dealing with the 'left-over' we make a parity assumption about n. For the remainder of the analysis we assume $n \equiv 0\,(mod\, d^*+1)$. In fact, without this assumption, the set of group stable networks may be empty. To see why, assume that the 'left-over' complete subgraph is of order 1. This single left-over node would like to have any number of edges rather than having none, and any other agent would be happy to form one edge since this extra edge does not carry excess risk of contagion.

Theorem 1. *For $n \equiv 0\,(mod\, d^*+1)$, there exists a unique (up to permutations) group stable network and it consists of disjoint (d^*+1)-complete subgraphs. For $n < d^*+1$, the unique group stable network is the n-complete subgraph.*

Proof. The parity assumption and Lemma 2 suffices to yield uniqueness of group stable networks once we have existence. It remains to show that a realized network $G = (N, E)$ consisting of disjoint complete subgraphs $C_1, C_2, ..., C_k$ all of order (d^*+1) (for k such that $N = k\,(d^*+1)$) is a group stable network.
For any profitable deviation by V' from G to G', define $\phi(V', G')$ to be the number of edges between V' and N/V' in G'. Let the minimum of ϕ be attained at (V^*, G^*).
Consider G^*. Take a node $v' \in V^*$ that is adjacent to a node in N/V^*. Suppose that there exists $v'' \in V^*$ such that v' is connected but not adjacent to. Cut one edge connecting v' to N/V^* and insert the absent edge between v' and v''. This new graph, say G'', is also a profitable deviation by V^* from G. This is because when we move from G^* to G'', the degrees of all nodes in V^* weakly increase, and their component sizes weakly decreases. However, $\phi(V^*, G'') < \phi(V^*, G^*)$, which is a contradiction. Therefore, any node in V^* that is connected to v' is adjacent to it. The same holds for any node that is adjacent to N/V^*.
Take a node in V^* with minimal degree, say v with degree d. Let $d' \geq 0$ be the number of v's neighbors in N/V^*. Suppose $d' \geq 1$. By the last paragraph, a node in V^* that is connected to a neighbor of v can only be a neighbor of v. Therefore, any neighbor of v in V^* has at most $d - d'$ neighbors in V^*, hence at least $d' \geq 1$ neighbors in N/V^*. So by the last paragraph, v and his $d - d'$ neighbors in V^* are all adjacent to each other, forming $(0.5)(d - d' + 1)(d - d')$ edges. Each of them have at least d' edges to N/V^*, so that makes $d'(d - d' + 1)$ edges. Finally, since nodes in N/V^* have not deviated from G and are connected to each other, they are all adjacent to each other, forming $(0.5)d'(d'-1)$ edges. Therefore, in v's maximally connected component, there are at least $(0.5)d(d+1)$ edges, so that his payoff is at most $U(d)$. Now suppose $d' = 0$. Then all v's d neighbors are in V^*, hence all have degree at least d. Then again, v's component has at least $d(d+1)/2$ edges, so that his payoff is at most $U(d)$. In both cases, v's payoff in G^* is at most $U(d) \leq U(d^*)$; contradiction with profitable deviation from G.

For the part $n < d^* + 1$, recall that $U(d)$ is increasing in d up to $d^* > n$. The remainder of the proof follows the previous steps hence we omit the details. □

This proof relies on the idea that one can construct profitable deviations using other profitable deviations up to a point that the worst-off deviator in the final deviation cannot achieve a strictly better payoff than the what the candidate network in the theorem yields. Figure 1 illustrates the structure of the group stable network. This structure is due to ex-ante homogeneity of agents, and certain forms of asymmetries can lead to more realistic network structure, such as the core-periphery network. See Section 5.1 for an extension of the model with some risk-free nodes. In this extension, nodes form a core-periphery network with risk-free nodes in the core as illustrated in Figure 11. The exact set of comparative statics that we are about the present go through under such heterogeneity. The topology of the formed network here with identical and risky nodes should be seen as a qualitative feature of the interaction between agents that are exposed to more risk than others, rather than a description of what one should expect in all possible scenarios. It is well known that most financial networks feature a core-periphery structure.
It is also noteworthy that we do not need the full strength of group stability in the proof. This is elaborated further in Section 5.3.

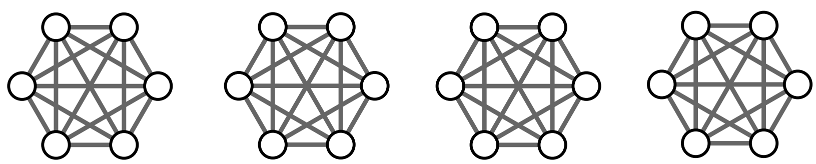


Figure 1: Structure of the group stable network

Call a realized network (N, E) (utilitarian) efficient if it maximizes the sum of expected payoffs of agents among all realized networks. Consider a connected subgraph with e edges. A node in the subgraph with degree d enjoys an expected payoff of $d\alpha^{2e}\theta_0$. Therefore, the sum of payoffs of nodes within the graph is $2e\alpha^{2e}\theta_0$. Here we use the well known fact that the sum of degrees is twice the number of edges. It follows then, that the problem of finding an efficient network devolves into two parts: how to partition nodes into maximally connected components, and how many edges to put into each component.

Theorem 2. *If $n \equiv 0 \,(mod\, d^* + 1)$, a network is efficient if and only if it is group stable.*

All pairwise stable networks other than the group stable network are, thus, inefficient.[15] We discuss the cause of efficiency in Section 4.
To see why group stability leads to efficiency in this case, consider an agent choosing the size clique they would like to join. If the cliques are the wrong size from an individual point of view, by symmetry, all agents would want to change clique sizes and all be better off. This would violate group stability. When an agent considers moving from a clique of size d, say, to $d + 1$: they get some benefit from increasing that clique size in terms of an added partnership, and that adds an extra chance of default. The extra benefit they get is $\frac{1}{d+1}$ of the total benefit from the larger clique where d is that size, and they also bear $\frac{1}{d+1}$ of the total cost

[15] Blume et al. (2013) find that their pairwise stable networks are not efficient. Their notion of efficient is a worst-case one, very different from the one employed here. Farboodi (2015) also finds that formed networks are inefficient, despite having group stability as the solution concept.

of default in expectation. Therefore, their incremental individual benefit is proportional to the incremental total benefit.

On the other hand, showing that the efficient structure is given by disjoint cliques is not trivial. Since contagion is strong, the number of edges in a maximally connected component is a sufficient statistic for the sum of payoffs of nodes within the component. A priori, the optimal number of edges need not be $d(d+1)/2$. We show in the proof of Theorem 2 that it does, so that the efficient structure is given by disjoint cliques, which then have to be sized appropriately.

3.2 Systemic risk: volatility paradox

Here we provide comparative statics concerning the endogenous network in α, the probability of good shocks. For a fixed d^*, as α increases, the economy is in a fundamentally better state and most measures of systemic risk decrease. For fixed α, as d^* increases, the economy is more interconnected and there is more room for contagion. This increases most measures of overall risk. In the endogenous network formed, as α increases, d^* increases, which is portrayed in Figure 2. Hence, it is not clear how overall risk changes in α.

Recall that systemic risk is defined as the probability that all nodes default. All nodes of a maximal connected component play D if at least one of the edges in the component is subject to a bad shock; otherwise they all choose action B. The probability that any node/all nodes in a maximal complete subgraph chooses D is $1-\alpha^{d^*(d^*+1)}$. Then systemic risk is

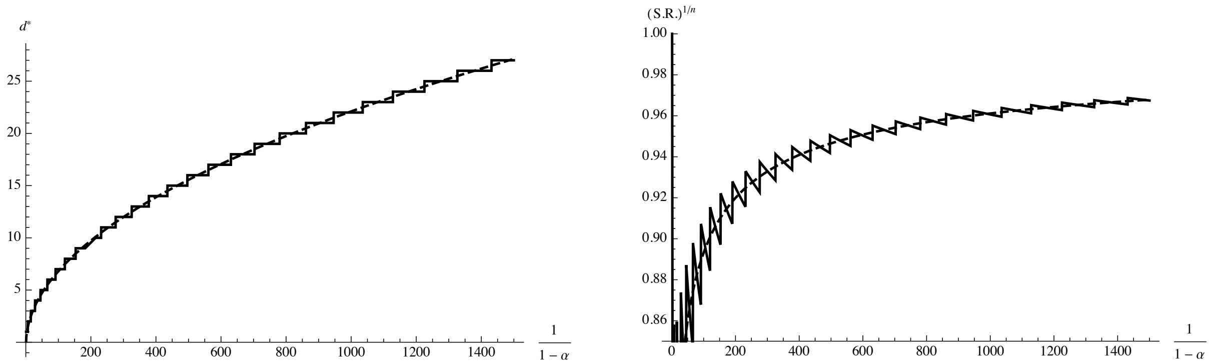
$$\left(1-\alpha^{d^*(d^*+1)}\right)^{\frac{n}{d^*+1}}. \tag{1}$$

For fixed α, expression (1) is increasing in $d^* < n$. An increase in d^* leads to fewer but larger complete subgraphs. Thus, for fixed α, higher interconnectedness translates into higher systemic risk. For fixed d^*, the expression decreases in α. However it is not apriori clear whether systemic risk increases or decreases with a change in α. Note that as α increases, the group stable network consists of fewer but larger clusters. As one can see in Figure 2, it turns out, systemic risk of the group stable/efficient network increases with α. d^* increases at such a rate that systemic risk of the group stable/efficient network also increases with α.[16] Recall that a rough estimate for d^* is $(-2log[\alpha])^{-0.5}$. Then systemic risk is roughly

$$\left(1-\boldsymbol{e}^{-0.5}\right)^{n\sqrt{-2log[\alpha]}}$$

which increases in α. This gives a network foundation for the volatility paradox.

[16] Since d^* is a step function of α, in intervals where d^* stays constant the probability decreases. However, this is an artifact of discreteness. When α^2 hits $\left(\frac{d-1}{d}\right)^{\frac{1}{d}}$, d^* jumps from $d-1$ to d. If one considers these jumping points of α, the probability is increasing. In order to clarify further, recall the definition of $d^* = argmax_{d\in\mathbb{N}} d\alpha^{d(d+1)}$. For a "smooth version" of d^* as a function of α, a real number $d^* = argmax_{d\in\mathbb{R}} d\alpha^{d(d+1)}$, the probability is strictly increasing.



Degree d^* Systemic risk

Figure 2: Degree d^* and systemic risk

As the economy gets fundamentally safer, agents form larger clusters. That is in their individual interest and the outcome is efficient. However, the tail risk from interconnectedness dominates the safety from α. Catastrophic events become more frequent. Once α becomes too large and hits $\left(\frac{n-1}{n}\right)^{\frac{1}{2n}}$, d^* becomes n and the clusters cannot get any larger. Hence, systemic risk cannot get any larger and it starts decreasing again. We can actually pin down the exact distribution of the number of nodes that default. Given α, the number of maximal complete subgraphs that fail is k with probability

$$\binom{\frac{n}{d^*+1}}{k}\left(1-\alpha^{d^*(d^*+1)}\right)^k\left(\alpha^{d^*(d^*+1)}\right)^{\frac{n}{d^*+1}-k}. \tag{2}$$

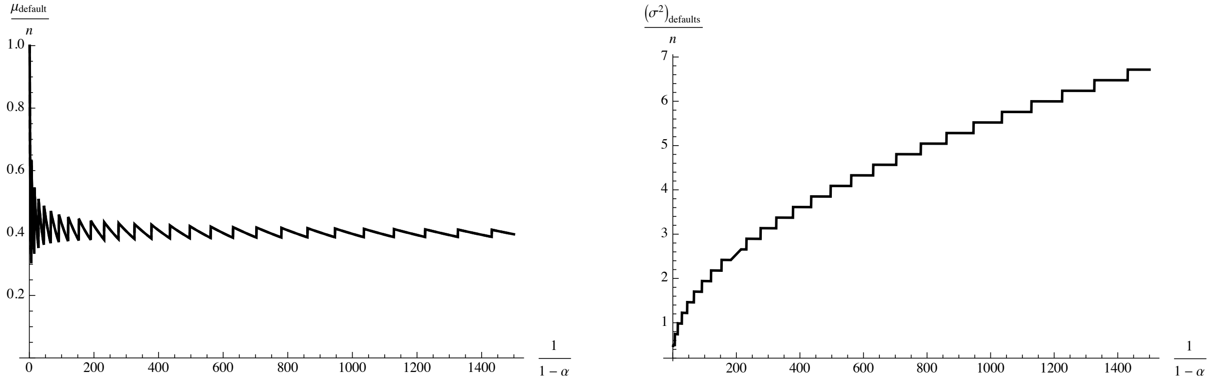
This is also the probability that $(d^*+1)k$ agents default and the rest do not. There is no first order stochastic dominance order among these distributions indexed by α. However, the distributions with larger α's second order stochastically dominate those with smaller α's. Approximately, $k \times (-2log[\alpha])^{-0.5}$ nodes default out of n, with probability $\mathbb{F}\left[k, \left\lfloor n\sqrt{-2log[\alpha]}\right\rfloor, 1-\boldsymbol{e}^{-0.5}\right]$, where $\mathbb{F}$ is the binomial P.D.F.
The mean and variance of the number of defaults are given by

$$\frac{\mu_{defaults}}{n} = \left(1-\alpha^{d^*(d^*+1)}\right) \approx \left(1-\boldsymbol{e}^{-0.5}\right),$$

$$\frac{\sigma^2_{defaults}}{n} = (d^*+1)\left(1-\alpha^{d^*(d^*+1)}\right)\left(\alpha^{d^*(d^*+1)}\right) \approx (-2log[\alpha])^{-0.5}\left(1-\boldsymbol{e}^{-0.5}\right)\boldsymbol{e}^{-0.5}.$$

The mean is roughly constant but the variance gets larger in α. It is remarkable that this is not a high mean and high variance case, but rather fixed mean, high variance for the number of defaults. Increasing variance is due to the fact that nodes in components have correlated risk and the components are getting larger in α.[17]
Figures 3 below show how the mean and variance vary with α.

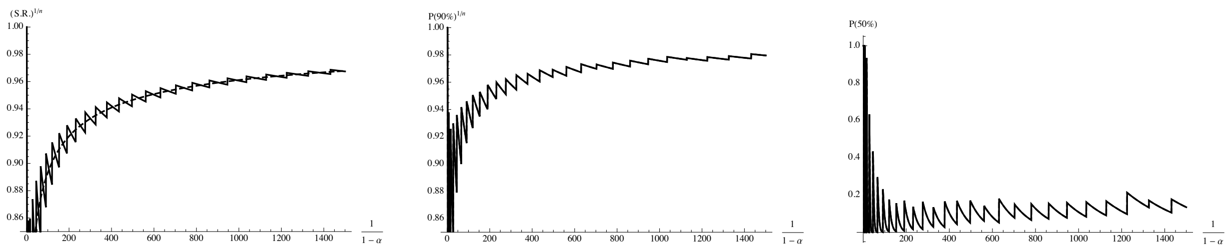
[17] Of course, the moment α hits the threshold that the network becomes complete, components can not get any larger, mean starts increasing and variance starts dropping.



Mean Variance

Figure 3: Mean and variance of the number of defaults

Figure 4 shows the probability that 100%, 90%, and 50% of agents default. It is interesting that even the probability that at least half of the agents default (slightly) increases in α.



100%, powered to $1/n$ 90%, powered to $1/n$ %50

Figure 4: Probability that at least a certain fraction of nodes default

Despite the increasing volatility in the number of defaults, welfare increases as a consequence of the increased number of edges. The sum off payoffs is $nd^{*}\alpha^{d^{*}(d^{*}+1)}$ which is clearly increasing in α by definition of d^{*} being the maximizer of $U(d)$.

The distribution of welfare can also be pinned down. Realized ex-post welfare is $(n-(d^{*}+1)k)\,d^{*}\theta_0$ with probability given in Equation 2. Hence welfare has mean and variance given by

$$\frac{\mu_{welfare}}{n} = d^{*}\alpha^{d^{*}(d^{*}+1)}\theta_0 \approx \boldsymbol{e}^{-0.5}\left(-2log[\alpha]\right)^{-1}\theta_0,$$

$$\frac{\sigma^2_{welfare}}{n} = (d^{*}+1)\,(d^{*})^2\,\theta_0^2\left(1-\alpha^{d^{*}(d^{*}+1)}\right)\left(\alpha^{d^{*}(d^{*}+1)}\right) \approx \left(-2log[\alpha]\right)^{-1.5}\left(1-\boldsymbol{e}^{-0.5}\right)\boldsymbol{e}^{-0.5}\theta_0^2.$$

Both are increasing in α. These are plotted in Figure 5. Welfare has higher mean and higher variance in fundamentally safer economies. Note that $\frac{\sigma^2}{\mu}$ is also increasing in α.

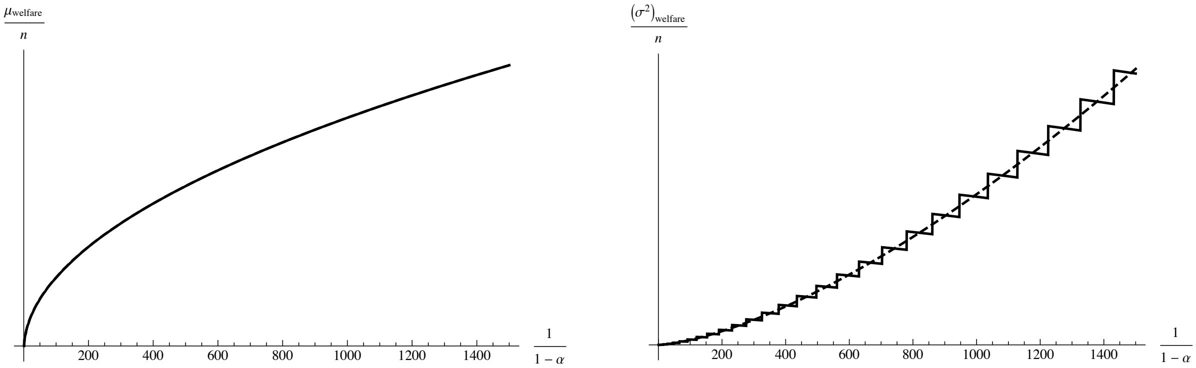


Figure 5: Welfare mean and variance

3.3 Correlation: popcorn vs. dominoes

We noted earlier a debate about whether interconnectedness of nodes is a significant contributor to systemic risk. An alternative theory is that the risk faced is via common exposures, i.e., popcorn.[18] Observed outcomes might be similar in both scenarios but the dynamics can be significantly different.

We model the popcorn story as perfect correlation in states of edges. With probability σ *all* edges have state θ_0, with probability $1-\sigma$ *all* edges are in state θ_1. There is no change in the analysis of stage three. As for stage one, now there is no risk of contagion.

Theorem 3. *Under 'popcorn', the unique group stable (and unique pairwise stable) network is the complete graph on n nodes, denoted K_n.*

Proof. In any given realized network, if all shocks are θ_0 then everybody plays B and if all shocks are θ_1 everybody play D. The payoff of an agent incident to d edges is $d\theta_0$ or 0 respectively. Thus, the expected payoff of each agent is $d\sigma\theta_0$. Then, it is clear that in a group stable (or pairwise stable) network there cannot be any missing edges because that would lead to a profitable pairwise deviation. The only candidate is K_N which is clearly group stable. □

When agents anticipate common exposures (popcorn) rather than contagion, they form highly interconnected networks in order to reap the benefits of trade. Under popcorn, all nodes are in the same ship and there is no risk of contagion. Hence, there is no need to refrain from forming as many edges as possible. If the underlying common shock is good, then they get higher returns by having more edges. If the underlying common shock is bad, then, there is no extra harm of having formed more edges.

In an independent shocks world, the probability that everybody defaults in K_n is $1-\alpha^n$, which is the highest systemic risk that any network can achieve. However, K_n is as safe as all the other possible realized networks in the popcorn world. This highlights the importance of identifying the shock structure before investigating a given network. A specific network and a particular shock structure might very well be incompatible.

Remark. An important feature of the popcorn argument is related to underestimating systemic risk. In the popcorn world, systemic risk is $1-\sigma$, which can take any arbitrarily large value. In the dominoes world, systemic risk is $1-\alpha^{n(n-1)}$. For the complete network to be formed, α must be large enough that

$$(n-1)\alpha^{n(n-1)} \geq (n-2)\alpha^{(n-1)(n-2)}.$$

[18] In other words, systematic risk.

Then systemic risk can be at most

$$1 - \left(\frac{n-2}{n-1}\right)^{\frac{n}{2}}$$

which is very close to 0.4 for $n \geq 6$, and asymptotes $1 - \frac{1}{\sqrt{e}} \approx 0.39$ for large n. Therefore, an outside observer who observes a complete network and believes it is the dominoes world would think that systemic risk is 0.4 or lower, whereas under the popcorn world, systemic risk can be arbitrarily close to 1.

Perfect correlation and complete independence are two extremes. Suppose that with some probability the economy operates as 'normal' and edges are subject to their own idiosyncratic shocks, while with complementary probability a common shock is realized. Formally, with probability σ_1 all edges are θ_1, while with probability σ_0 all states are θ_0. With probability $1 - \sigma_0 - \sigma_1$ all edges are i.i.d.: θ_0 with probability α and θ_1 with probability $1 - \alpha$.

In this setting, the expected payoff of an agent is $d\left(\sigma_0 + \alpha^{2e}\left(1 - \sigma_1 - \sigma_2\right)\right)\theta_0$. The proof of Theorem 1 can be replicated. For large enough σ_0, $d^* > n$ so that the unique group stable network is K_n. This illustrates that Theorem 3 is not an anomaly due to perfect correlation. In fact, it is a corollary of Theorem 1 the same result holds for sufficiently strong correlation not just perfect correlation.

4 Identifying the cause of the volatility paradox and efficiency

Now we relax assumptions 1 and 2 separately. Without assumption 1, we will not have not much to say about group stability. For consistency, the main solution concept we use here is pairwise stability. When possible we include results on group stability as well.

We focus on the particular structure that emerged in the benchmark case, the one that was the unique strongly stable under assumptions 1 and 2. Formally, for the general case, let $U(d)$ be the payoff of an agent in a disjoint complete subgraph with order $d+1$ and $d^* = argmax\, U(d)$. Call the network topology that consists of disjoint complete subgraphs of order d^*+1 the optimal clique structure. We first try to establish that a network in optimal clique structure is pairwise stable even if we relax assumptions 1 or 2. Therefore, the optimal clique structure are the only pairwise stable networks that are robust to stronger contagion/seed and to a stronger solution concept. We then look into which way our previous intuitions on volatility and efficiency go when we relax assumptions 1 and 2.

4.1 Weaker seed

Here we maintain assumption 1 but drop assumption 2. Now, it takes a non-trivial number of bad shocks to force a node into default, but the moment one node defaults, all nodes in the maximally connected component of the initial defaulting node also default. Let C_i denote the maximally connected component containing i. Let $\zeta = \frac{\theta_0}{\theta_0 - \theta_1}$, $F(d_j|\zeta) = \mathbb{P}\left[b_j \leq \zeta d_j\right]$, and $W(d_i) = F(d_i)\mathbb{E}\left[\zeta d - b_i | \zeta d \geq b_i\right]$.

Then, i's expected payoff is given (normalized by $\theta_0 - \theta_1$) by

$$W(d_i) \times \prod_{j \in C_i, j \neq i} F(d_j).$$

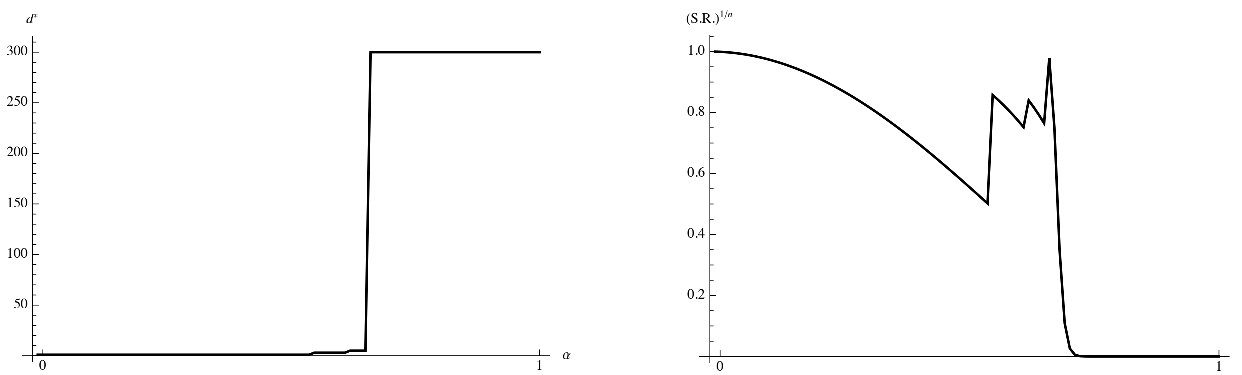
A nice feature of this payoff function is that the effect of i's neighbors and the effect of the rest of its component are completely separated. The only payoff relevant aspect of the network for i is the degree sequence of nodes in C_i. Accordingly,

$$U(d) = W(d)F(d)^d.$$

Stability and efficiency. Roughly speaking, if $1-\alpha < \zeta$ $(\alpha\theta_0 + (1-\alpha)\theta_1 > 0)$ then adding one more edge is beneficial because the probability of getting a bad shock from an edge is less than the resilience the extra edge adds to the node. So both W and F are increasing. For $1-\alpha > \zeta$, both W and F are decreasing. Such complementarity and the separable structure of payoffs simplifies the analysis.

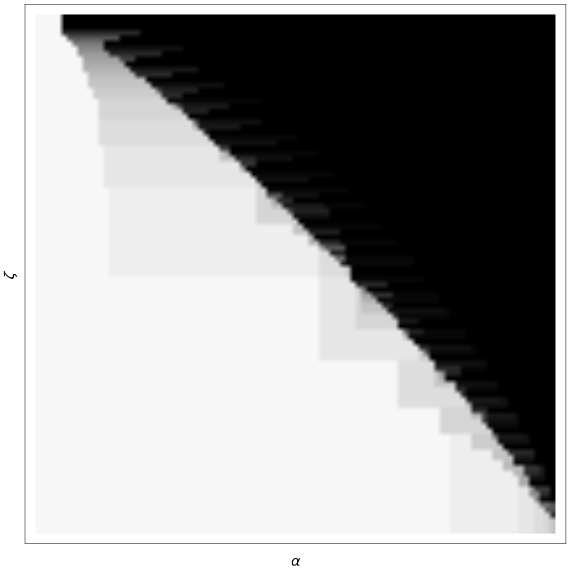
Proposition 1. *There exists ϕ_1, such that, if $1-\alpha > \zeta + \phi_1$, then $d^* = 1$. There exists ϕ_2 and $\bar{n}$ such that and if $1-\alpha < \zeta - \phi_2$ and $n > \bar{n}$, then $d^* = n-1$. In both cases, all pairwise stable networks consist of disjoint cliques, the optimal clique structure is the unique group stable network and the unique efficient network.*

Systemic risk. As α increases, the network starts from $d^* = 1$, then rapidly increases in size, and transitions to $d^* = n-1$. Systemic risk roughly decreases, with a sudden fall at the transition point. This is illustrated in Figure 6 for a fixed ζ and in Figure 7 as a heat map for all values of α and ζ. This goes against the volatility paradox intuition. Then it can be argued that the underlying cause of volatility is not the strength of contagion, but the strength of the seed. Next we analyze the case of weak contagion and strong seed.

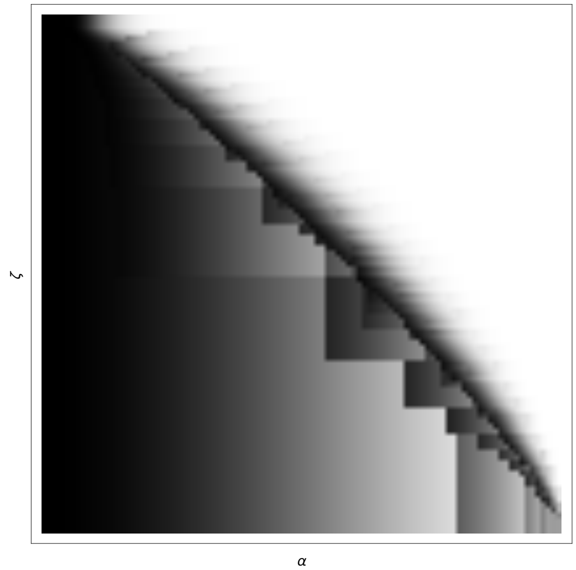


Optimal degree d^* in α Systemic risk in α

Figure 6: Optimal degree and systemic risk under weak seed and strong contagion.



Optimal degree



Systemic risk

Figure 7: Heat map of optimal degree and systemic risk under weak seed and strong contagion, in α and ζ. Darker values are larger.

4.2 Weaker contagion

Now we drop Assumption 1 but maintain Assumption 2. Any node that receives at least one bad shock defaults, but it takes a non-trivial number of defaulting neighbors to force a node into default. Under weaker contagion, it is harder to pin down the exact payoffs of nodes for any arbitrary network. $U(d)$, the payoff of a node in clique with $d+1$ nodes is

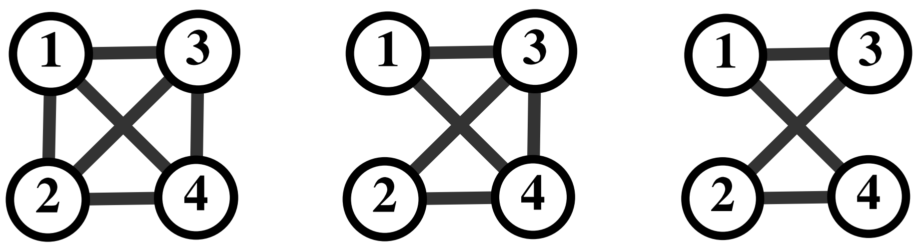
$$U(d) = \sum_b \mathbb{P}\left[x = b\right] \left(d\theta - b\right)^+$$

where x is a binomial random variable with d trials and $1 - \alpha^d$ success probability.
Stability. Among clique structures, clearly the only candidate for group stable networks is the optimal clique structure. Because, nodes can always jointly deviate to the optimal clique structure and improve their payoffs over a sub-optimal clique structure. Unfortunately, we do not know much more about group stable networks under weak contagion. Hence we examine pairwise stability. Figure 9 shows the values of (α, θ_0) at which the optimal clique structure is pairwise stable. For a large region of parameter values, the optimal cliques, under weaker contagion, do not have to be pairwise (let alone group stable). $\alpha^2 + \theta_0 < 1$ appears to be reasonably tight bound for the values at which the optimal clique structure is pairwise stable which is also shown on Figure 9. We can prove the following.

Proposition 2. *Suppose that $\alpha^2 + \theta_0 < 1$ and $d^* + 1$ divides n. A network that consists of disjoint optimal cliques is pairwise stable.*

Accordingly, the strong contagion assumption can be interpreted as a robustness check that verifies the optimal clique structure as the unique pairwise stable network that survives strong contagion. This justifies our focus on the optimal clique structure when studying weaker contagion.
Efficiency. We describe an example in which the optimal clique structure is pairwise stable but not efficient

under weak contagion. Consider $\alpha = 0.9$, $\theta_0 = 0.5$. This yields $d^* = 3$. Let n be divisible by 4. In this case, the optimal clique structure is pairwise stable. Now consider a clique of order 4, with nodes $1, 2, 3$, and 4, as illustrated in Figure 8(a). The subgraph in Figure 8(b) gives higher total payoff. It increases the payoff of 3 and 4 by making 1 and 2 individually more resilient. This comes at the expense of 1 and 2's payoff, but total payoff increases. Second, consider Figure 8(c), with edges $\{1, 2\}$ and $\{3, 4\}$ missing from the clique. In this subgraph, both pairs $\{1, 2\}$ and $\{3, 4\}$ give up some edges. Each pair, by cutting their link, give up some expected payoff from that link, but greatly reduces the risk of the other pair. When both pairs sever their links, both pairs improve.

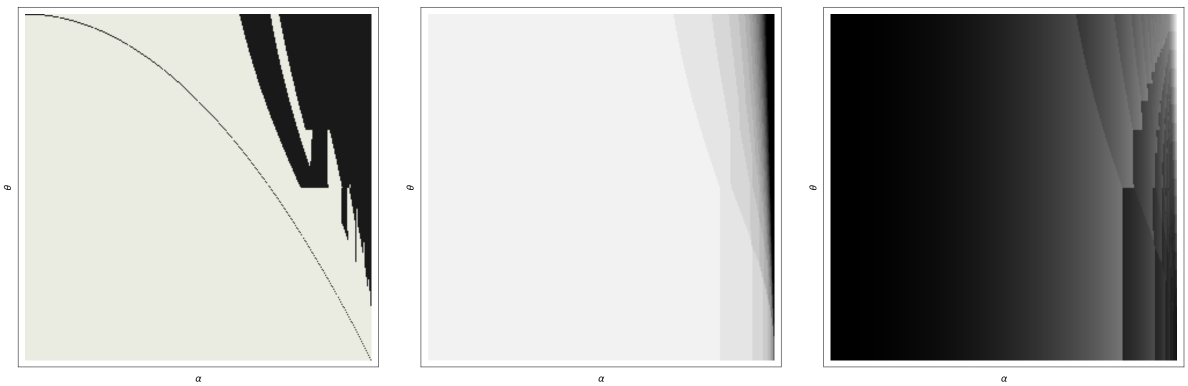


(a) (b) (c)

Figure 8: An example in which optimal structure clique is pairwise stable but not efficient under weak contagion

The second example also shows that there is no group stable network in clique structure for $\alpha = 0.9$ and $\theta_0 = 0.5$, because, the only candidate for a group stable network among clique structures is the optimal clique structure. However, the coalition of four nodes, by cutting two links as we have illustrated, improve their payoffs. So the only candidate for a group stable network is not group stable either.

Systemic risk. Notice in Figure 9 that for $\theta_0 > 0.5$, at the values of α that the optimal clique structure is pairwise stable, d^* is equal to 1 or 3. In this region, since the network is roughly constant, systemic risk appears to decline with α when the network is pairwise stable, but that is not what is really happening. It is an artifact of discreteness. The interesting interactions happen for $\theta_0 < 0.5$, where optimal degree is non-trivial. Notice that for smaller values of θ_0, d^* keeps growing and systemic risk indeed increases in α (modulo the drops between values of α that d^* increases, which are due to discreteness). In Figure 10 we present a plot of optimal degree and systemic risk for $\theta_0 = 0.2$ for some values of α that the optimal clique structure is pairwise stable.

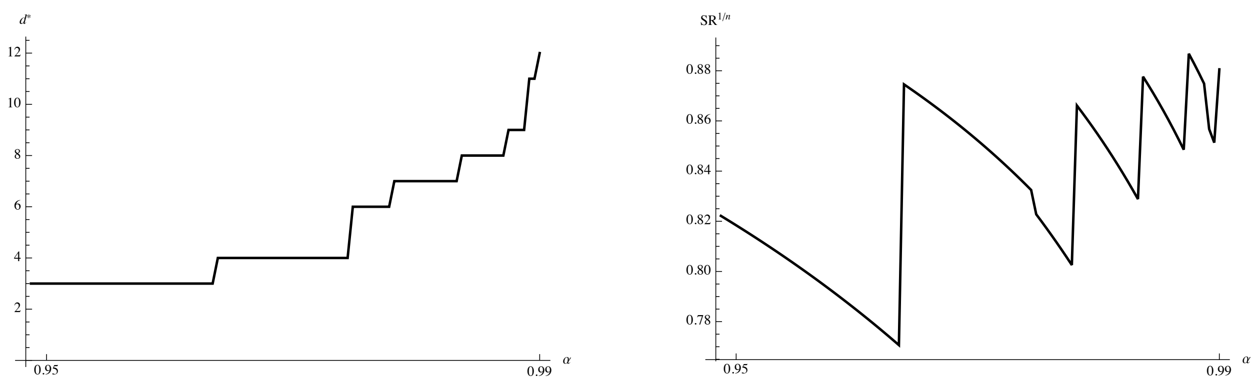


White region: optimal clique structure is pairwise stable. Thin line is $\alpha^2 + \theta_0 = 1$

Degree d^*. Darker values are larger.

Systemic risk. Darker values are larger.

Figure 9: The values at which optimal clique structure is pairwise stable, the degree at the optimal cliques, and the resulting systemic risk, in α and θ_0



Optimal degree d^* in α

Systemic risk in α

Figure 10: Optimal degree and systemic risk under weak contagion and strong seed.

As we have seen in the example of an inefficient pairwise stable optimal clique, the optimal clique structure can be pairwise stable for values of (α, θ_0) with $\alpha^2 + \theta_0 > 1$, but $\alpha^2 + \theta_0 < 1$ is reasonably close to a sufficient condition. As is the case in the example, computations show that if $\alpha^2 + \theta_0 > 1$, in a pairwise stable optimal clique structure, $d^*\theta_0$ can be larger than 1. However, for the case of $\alpha^2 + \theta_0 < 1$, it turns out, $d^*\theta_0 < 1$ even if $d^* > 1$. Despite the fact that contagion is arbitrarily weak (θ_0 can be arbitrarily close to 1), if α is sufficiently small ($\alpha < \sqrt{1 - \theta_0}$), the optimal clique structure is pairwise stable and the size optimal clique structure is such that it exhibits strong contagion. Nodes select a network that features strong contagion although they do not have to. Therefore, all comparative statics concerning systemic risk are valid under weak contagion, for the case of $\alpha^2 + \theta_0 < 1$. As for the case of $\alpha^2 + \theta_0 > 1$, Figure 9 shows that the regions in which the optimal clique structure is pairwise stable is vary hard to formulate in closed form, so we focus on an example. In Figure 10, where $\theta_0 = 0.2$, for large enough α, $d^*\theta_0 > 1$ meaning that realized contagion is not strong. Nonetheless, systemic risk still keeps increasing.

Remark. Apriori, the volatility paradox could be generated by two things. An increase in α could lead to a slight increase in individual risk undertaken, which then could be transmitted strongly throughout the system. Or an increase in α could lead to a great increase in individual risk taking, which then turns into systemic risk despite individual risks being transmitted weakly throughout the system. One might expect that contagion is a bigger systemic problem since individual risk taking is internalized, yet the externalities imposed by the network are not internalized. Hence strong contagion would be the environment with greatest extent of externalities imposed, greater systemic risk, and reduced efficiency.

Counterintuitively, the volatility paradox persists under weak contagion and strong seed, but not under strong contagion and weak seed. This is exactly because negative externalities are far greater under strong contagion. When contagion is strong, it becomes such a great risk that agents prioritize guarding against contagion, which makes the realized network as safe as possible. On the other hand, when contagion is weak, negative externalities are not too great, so agents can trade off between guarding against contagion and obtaining larger benefits from undertaking more ventures. When each agent takes on much more individual risk, the externalities, although weak, are triggered much more often, and systemic risk increases.

Efficiency, again counterintuitively, survives under strong contagion and weak seed, but not under weak contagion and strong seed. The reasoning is the same. Stronger contagion makes guarding against contagion a priority, whereas weak contagion allows for taking more individual risk which then makes the weak negative externalities kick in much more often.

In sum, strong seed is the root cause of the volatility paradox in the benchmark case while strong contagion is the root cause of efficiency in the benchmark case.

4.3 Weaker seed and weaker contagion

Now we drop both Assumption 1 and Assumption 2. We are unable to pin down the stable network. We conjecture that the networks formed would combine the intuition from both cases. As α grows, the cliques get larger in a way that systemic risk increases. This would be the volatility paradox region. Around a tipping point for α, the network would rapidly jump to a complete network featuring an abrupt fall in systemic risk. From this moment on, systemic risk declines to zero. In the remainder, we describe some features of the outcome of contagion using a more general payoff structure and relate these results to the previous literature on contagion.

Let $u_i\left(a_i, a_j; \theta^{ij}\right)$ denote the payoff i gets from his edge with j. It is given by 0 if $a_i = D$, θ^{ij} if $a_i = a_j = C$, and $\theta^{ij} - 1$ if $a_i = C, a_j = D$ in the model. For the following characterization of the cooperating equilibrium, u_i can be any supermodular function such that $u_i(D, D, \theta) > u_i(B, D, \theta)$.

Proposition 3. *A cooperating equilibrium is well-defined and unique.*

Proof. Fix $(N, E, \boldsymbol{\theta})$. The profile where all agents in N play D is a Nash equilibrium by since $u_i(D, D, \theta) > u_i(B, D, \theta)$. Hence, D is rationalizable for everyone. Let M be the set of agents who have the unique rationalizable action D. For agents in $N \setminus M$, both B and D are rationalizable.

Consider an agent $i \notin M$. B is rationalizable, i.e., B is a best response to some strategy profile, say a_{-i}, of agents $-i$ in which agents in M play D. Let $\Delta(s_{-i})$ be the difference in payoffs for i between playing B and D against strategy profile s_{-i} of $-i$. $\Delta(a_{-i}) \geq 0$ since B is a best reply to a_{-i}.

Now consider the strategy profile b_{-i} of agents $-i$ such that agents in M play D and the rest play B. We will prove that $\Delta(b_{-i}) \geq \Delta(a_{-i})$. In a_{-i}, players in $N \setminus M$ could be playing B or D. Let $K \subseteq N \setminus M$ be those agents who play D in a_{-i} and let N_i be the set of neighbors of i in the realized network (N, E). Then $\Delta(b_{-i}) - \Delta(a_{-i}) =$

$$\sum_{k \in K \cap N_i} \left(u_i\left(B, B; \theta^{ik}\right) - u_i\left(D, B; \theta^{ik}\right)\right) - \left(u_i\left(B, D; \theta^{ik}\right) - u_i\left(D, D; \theta^{ik}\right)\right)$$

which is positive by supermodularity.
As $\Delta(b_{-i}) \geq \Delta(a_{-i}) \geq 0$ it follows that B is a best reply by i to b_{-i}. This argument works for every agent in $N \setminus M$, not just i. Also, recall that D is the unique rationalizable action for agents in M so that D is the unique best reply to any strategy profile in which all agents in M play D. Therefore, a profile where all agents in M play D and all agents in $N \setminus M$ choose B is a Nash equilibrium.
Note that in any Nash equilibrium, everyone in M must play D since it is their unique rationalizable action. Therefore, "M plays D, M^c plays B" is the unique cooperating equilibrium. □

The proof suggests an *equivalent definition* of a cooperating equilibrium: the rationalizable strategy profile in which those who have the unique rationalizable action D play D, while the remainder play B. Recall that rationalizable actions are those which remain after the iterated elimination of strictly dominated actions. The iteration is as follows. Those agents who have a strictly dominant action D play D. Then, knowing that these agents play D, it becomes strictly dominant to play D for other agents to do so. This iteration stops in a finite number of steps as N is finite. The remaining action profiles are the rationalizable ones, and the cooperating equilibrium is given by the profile in which whoever is not reached in the iteration plays B.
There is a natural analogy between contagion of sequential defaults and rationalizable strategies.[19] First, agents whose incident edges have realized states that cause them to default in any best response, no matter what other players do, default. Then, some agents, knowing that some of their counter-parties will default in any best response, choose to default in any best response. Then some more agents and so on.
Structure of the Cooperating Equilibrium. For a given $(N, E, \boldsymbol{\theta})$ we characterize the structure of a cooperating equilibrium. In what follows, the following notation will be useful.
Let $\Theta = \{\theta_0, \theta_1, ..., \theta_k\}$ be the set of possible states. For each $v \in N$, let $\Delta_v(\theta) = u_v(B, D; \theta) - u_v(D, D; \theta)$, $\Delta'_v(\theta) = u_v(B, B; \theta) - u_v(D, B; \theta)$; be the gains to v from deviating to B from from D. Denote the vector of these gains by

$$\Delta_v = (\Delta_v(\theta_0), \Delta_v(\theta_1), ..., \Delta_v(\theta_k), \Delta'_v(\theta_0), \Delta'_v(\theta_1), ...\Delta'_v(\theta_k)) \in \mathbb{R}^{2k+2}.$$

Let $V^c = N \setminus V$ denote the complement of V in N for $V \subset N$. For a given $(N, E, \boldsymbol{\theta})$ let $d(v)$ be the degree of $v \in N$ and $d(v, V, \theta_s)$ be the number of edges in state θ_s which are incident to v and V. Let

$$\pi_s(V|v) = \frac{d(v, V, \theta_s)}{d(v)}$$

be the portion of v's edges that are incident to V and has state θ_s. Denote the vector of these ratios respectively for V^c and V by

$$\pi^v(V) = (\pi_0(V^c|v), \pi_1(V^c|v), ..., \pi_k(V^c|v), \pi_0(V|v), \pi_1(V|v), ..., \pi_k(V|v)) \in \mathbb{R}^{2k+2}.$$

Strictly speaking our notation should depend upon $(N, E, \boldsymbol{\theta})$. However, as these are all fixed in stage three we omit doing so.
Notice that $\Delta_v(\theta) < 0$ and $\Delta_v(\theta) \leq \Delta'_v(\theta)$ for all θ and v (by Assumptions 1 and 2). The following lemma characterizes an agent's best response to the actions of other agents.

Lemma 3. *Consider a $V \subset N$ and $v \in N$. Suppose that agents in $V \setminus \{v\}$ play B, and agents in $(N \setminus V) \setminus \{v\}$ play D. Then D (B) is the unique best reply of v if and only if $\Delta_v \cdot \pi^v(V) < 0$ ($\Delta_v \cdot \pi^v(V) > 0$).*

[19] See Milgrom and Roberts (1990) for more on this. Although not exactly the same, similar algorithms are used in Eisenberg and Noe (2001), Elliott et al. (2014), etc.

Call a $V \subset N$ strategically cohesive if for all $v \in V$

$$\Delta_v \cdot \pi^v(V) \geq 0.$$

Proposition 4. *In the cooperating equilibrium, an agent v plays B if and only if there exists a strategically cohesive set V with $v \in V$.*

Lemma 4. *If V and V' are both strategically cohesive, then $V \cup V'$ is also strategically cohesive.*

Call a set $V \subset N$ maximally cohesive if it is the largest strategically cohesive set. This is well-defined by Lemma 4.

Proposition 5. *In the cooperating equilibrium, all members of the maximally cohesive set play B, all the others play D.*

Resilience to system wide failure at stage three is determined by the existence of a strategically cohesive set.[20] Strategic cohesiveness is determined by both Δ_v and $\pi^v(V)$. The first captures the effect of payoffs, while the second captures the effect of the structure of the realized network with states. This suggests that the correct ex-post notion of fragility cannot rely on purely network centric measures. Even if one were to look for an appropriate network centric component of a good measure, it would not be measures like too-interconnected-to-fail (which is silent about the neighbors of the neighbors of the too-interconnected node), or degree sequences (which is silent about local structures), but rather cohesiveness which incorporates the idea of a group of nodes reinforcing each other and resisting contagion that began elsewhere.

To separate the effects of network and payoff structure we make some simplifying assumptions and examine their consequences below.

Separating network and payoff effects. Now let's go back to the original payoff function u_i of the model. $u_i\left(a_i, a_k; \theta^{ik}\right)$ is given by 0 if $a_i = D$, θ^{ik} if $a_i = a_k = C$, and $\theta^{ik} - 1$ if $a_i = C, a_k = D$. For each $V \subset N$ and $v \in N$ let $d(v, V)$ be the number of v's neighbors that are in V. Let

$$\pi(V|v) = \frac{d(v,V)}{d(v)}.$$

Let

$$\pi(v) = \left(\pi_0(N|v), \pi_1(N|v), ..., \pi_k(N|v)\right).$$

Given $(N, E, \boldsymbol{\theta})$, a set $V \subset N$ is said to be ex-post cohesive if $\pi(V|v) + \theta \cdot \pi(v) \geq 1$ for all $v \in V$. The term $\theta \cdot \pi(v)$ captures v's individual resilience from his payoffs, $\pi(V|v)$ captures the collective resilience of V as a function of network structure. If V is sufficiently resilient individually and collectively, then it is ex-post cohesive. Notice that under Assumption 3, strategic cohesiveness reduces to ex-post cohesiveness.

For a given $(N, E, \boldsymbol{\theta})$, vertices within an ex-post cohesive set all play B. Thus, they resist default through mutual 'support'. To illustrate, suppose a vertex v's incident edges all have states that are negative valued. Then, $1 - \theta \cdot \pi(v) > 1$ so that v cannot be part of any ex-post cohesive set. Thus, v defaults for sure. As another example, suppose all elements of Θ are positive. Then, the maximally cohesive set would be N itself for *any* possible case of $(N, E, \boldsymbol{\theta})$. Thus, in any realization, all agents play B. Similarly, if all states in Θ were negative, the maximally cohesive set would be the empty set. In every realization all agents would choose D, i.e., there would be certainty of system wide failure.

[20] One can think of strategic cohesive sets as 'firebreaks'.

p-cohesiveness. Ex-post cohesiveness is closely related to p-cohesiveness introduced in Morris (2000). The significance and relevance of p-cohesiveness is further illuminated in Glasserman and Young (2015). Given $p \in \mathbb{R}$, a set V is p-cohesive if for all $v \in V$, $\pi(V|v) \geq p$. p-cohesiveness imposes a uniform bound on the number of neighbors each vertex in V has in V. Ex-post cohesiveness imposes heterogeneous bounds on the same quantity that depend solely on the realized characteristics of v, particularly how v's edge states are distributed.[21] Notice that if Θ was a singleton, say $\{\theta_0\}$, ex-post cohesiveness would be equivalent to $(1-\theta_0)$-cohesiveness.

p-cohesiveness is an ex-ante concept relying only on the structure of (V, E). Ex-post cohesiveness, as its name suggests, is an ex-post concept that depends on $(N, E, \boldsymbol{\theta})$. To illustrate, consider a realized edge with a "bad state" $\theta < 0$ in which $\Delta_v(\theta)$ and $\Delta'_v(\theta)$ are very small. The presence of such an edge would help a set 'containing' the edge become "more" p-cohesive, however it makes it "less" ex-post cohesive. In this sense, lack of strategic cohesiveness is the appropriate ex-post notion of fragility taking into account the variety in states, while lack of p-cohesiveness is possibly an appropriate ex-ante notion of fragility when the states of edges are not yet realized.

5 Extensions of the benchmark case

5.1 Heterogeneity and core-periphery

Observed financial networks feature a core-periphery structure in which a small number of nodes are densely connected to each other (core) while the remainder are connected to the core and only sparsely connected to each other (see Afonso et al. (2013), Craig and Von Peter (2014), Cocco et al. (2009) among many others). This contrasts with the clique structure generated by our model. However, this is a function of the ex-ante symmetry of the agents. Introducing ex-ante heterogeneity in our model produces a core-periphery network while maintaining the essential intuition.

Suppose that there are m risk-free nodes, who receive $p > 0$ from edges in which the incident node continues, and $q \in [0, p)$ from edges in which the incident node defaults. One can think of these risk-free nodes as commercial banks that borrow from households, and lend out these funds to other institutions or individuals at fixed interest rates, with no skin-in-the-game. p is a proxy for the gross interest rate and q can be thought of as the liquidation value of the project that the lending bank can recover in case of default.

Clearly, all nodes are going to connect with all risk-free nodes. As for the links between risky nodes, the same analysis goes through. Set

$$d^* = argmax\,(m+d)\alpha^{(d+k)(d+1)}.$$

Proposition 6. *Suppose that* $n \equiv m\,(mod\,d^*+1)$. *Then, the unique group stable network has a core-periphery structure. Risk-free nodes are the core and are adjacent to all nodes. No-risk free nodes form cliques of order* d^*+1 *among each other besides their edges with the core. This is the unique efficient network.*

[21] Ex-post cohesiveness can trivially be applied to situation in which edges have heterogeneous volumes.

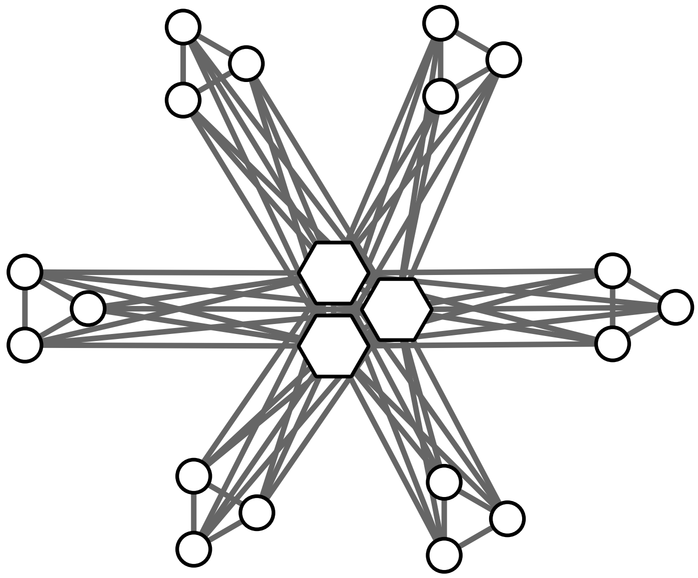


Figure 11: Structure of the group stable network

Figure 11 is a visualization of the network formed. The proof of proposition is identical to previous proofs, and all the intuition goes through. Risk is not transmitted through the core, but cliques, among each other, are subject to the exact same risks we have discussed.

5.2 Node shocks

Suppose that nodes are subject to shocks of their own, in addition to shocks to the edges. There are two ways to interpret such shocks. The first is an idiosyncratic shock that affects an institution without any direct effect to any other institution, such as liquidity shocks. The second is one in which the financial sector has ties with the real sector and these ties are subject to shocks as well. In the model, each node (financial institution) is incident to an (imaginary) edge outside of the network. The shock to this edge is effectively an idiosyncratic shock to the node itself.

Formally, after stage two has ended and before we move on to stage three, each 'imaginary' edge independently defaults with probability $1-\beta$ or proceed as normal with probability β. In stage three, similar to the case with only edge shocks, a maximally connected component with all edges good and all nodes normal, all play B in the cooperating equilibrium. Otherwise the entire component plays D. Then in stage two, the expected payoff of a node with degree d in a maximally connected component which with e total number of edges and f total number nodes (clearly $f \geq d+1$) nodes has payoff, $d\alpha^e\beta^f\theta_0$.

As for stage one, the earlier results apply. A group stable network will consist of disjoint complete subgraphs. Let $d^* := \arg\max_{d\in\mathbb{N}} d\alpha^{d(d+1)}\beta^{d+1}$. Theorems and comparative statics concerning the group stable networks apply. Note that d^* increases in β. When agents are exposed to external individual risks, they form less interconnected networks, but network topology remains unaltered.

5.3 $k-$stability

Bilaterally stable networks. Group stability assumes the ability of any coalition to get together and 'block'. Networks that survive weaker notions of blocking are also of interest. Two natural candidates are Nash networks and bilaterally stable networks. The first preclude deviations by single nodes only, while the second by pairs only. All group stable networks are pairwise stable, and all pairwise stable networks are Nash networks.

Robustness to unilateral deviations is too permissive. Most (permutation classes of) graphs with degree less than k^* are Nash networks.[22] This is because no node can add an edge in a feasible Nash deviation. As for deleting edges, for graphs that are sufficiently well connected a unilateral deletion will not reduce the cluster size very much. Hence, agents are not going to delete edges since they already have less than k^* edges.

Recall Lemma 1. Any bilaterally stable network consists of disjoint complete subgraphs. The orders of these complete subgraphs are not arbitrary. Let $h^* \geq d^*$ be the largest integer h such that $U(1) \leq U(h)$. Let $h^{**} \leq d^*$ be the largest integer such that $\frac{1}{\alpha^2} \leq \frac{h+1}{h}\alpha^{h(h+1)} = \frac{U(h+1)}{h\alpha^{2h}}$.

Proposition 7. *Any network that consists of disjoint complete subgraphs, each with order between $h^{**}+1$ and h^*+1, is bilaterally stable. Call these uniform stable networks.*[23]

Let uniform stable networks whose maximal complete subgraphs all have order larger than or equal to d^*+1 be called upper-uniform stable networks, and those with all maximal complete subgraphs having order smaller than d^*+1 be called lower-uniform stable networks.

Proposition 8. *Let $n \equiv 0\,(mod\, d^*+1)$. Upper-uniform (lower-uniform) stable networks have higher (lower) systemic risk than the group stable/efficient network.*

Proposition 9. *For any $k \geq d^*+1$, the unique k-stable network is group stable.*

Keeping in mind that we typically think of d^*+1 as being relatively small with respect to n, Proposition 9 shows us that the results in the paper don't need the full power of group stability that precludes profitable deviations by any coalition. A restriction on relatively small sized coalitions is sufficient. The next theorem concerns $k \leq d^*$.

Proposition 10. *Take any $k \leq d^*$. Let $h^*(k) \geq d^*$ be the largest integer such that $U(k) \leq U(h^*(k))$. Any network that consists of disjoint complete subgraphs, each with order between d^*+1 and $h^*(k)+1$, is k-stable. Call these upper-uniform t-stable networks.*

Notice that as $k \leq d^*$ gets smaller, upper-uniform k-stable networks become similar to upper-uniform stable networks. As $k \leq d^*$ gets larger, $h^*(k)$ approaches d^*+1, so that upper-uniform k-stable networks become closer to group stable networks. After d^*, for $k \geq d^*+1$ the only k-stable network is the group stable network itself (the upper-uniform (d^*+1)-stable network). These results bridge the gap between the group stability and pairwise stability.

As k gets larger, k-stable-complete networks become more efficient in a sense. Networks are subjected to further constraints by precluding deviations by larger coalitions, and the remaining set of networks get closer to the efficient/group stable networks, increasing the efficiency. Similarly, systemic risk of upper-uniform k-stable networks decline with larger values of k.

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[22] Let $k^* = \arg\max_{y \in \mathbb{N}} y\alpha^{2y}$. For generic α this is well defined. Note that $y\alpha^{2y}$ is strictly increasing in $y \in \mathbb{N}$ up to k^* and strictly decreasing after k^*. Note also that when maximizing $y\alpha^{2y}$ over the non-negative reals, the maximum occurs at a number $y^* = -\frac{1}{2log(\alpha)}$. Note that y^* lies in the interval $\left(\frac{\alpha^2}{1-\alpha^2}, \frac{1}{1-\alpha^2}\right)$.

[23] This is close to a complete characterization of all bilaterally stable networks in the following sense. Any complete subgraph in any bilaterally stable network has to be of order at most h^*+1. Moreover, there can be at most one complete subgraph with order less than $h^{**}+1$. The bound on the smallest order depends on what the second smallest order is, and is more involved to characterize.

6 Future Work

The model we introduce is tractable and rich. While we have examined some extensions, many more important extensions are possible. We list some of them here.

A major extension is allowing for government intervention in the contagion and/or network formation stages. Would the anticipation of government intervention be harmful due to moral hazard costs, or would the ex-post gains from intervention outweigh moral hazard costs? Should there be caps on the ability of a government to intervene ? What are the welfare implications of specific policies? Furthermore, government reputation can be considered when the model is cast into a dynamic framework.

As we illustrated in the asymmetry section, borrowing and lending can be incorporated into the model and endogenous prices can be tractably determined.

An important but difficult extension is introducing asymmetric information. For example in stage three, nodes could be modeled to know the states of their incident edges but not the rest. It is important to see the what happens in that case, yet it is significantly harder to solve for technical reasons.

In the network formation stage, we have introduced a proposal game to micro-found the solution concepts. The agents could have started off with an existing status-quo network, and build extra edges on top of the the existing ones. It would be interesting to see how this will alter the resulting network. Furthermore, one can think of a dynamic proposal game to see whether first-movers tend to become too central.

Other extensions can include allowing for more than two actions; allowing for moderate strength of contagion; allowing for heterogeneous volumes of edges; allowing for bilateral transfers between neighbors and allowing for different forms of correlations of shocks. A very important one among these is to build on the general contagion results and find which networks would be formed.

7 Conclusion

In our model, rational agents who anticipate the possibility of system wide failure during network formation, guard against it by segregating themselves into densely connected clusters that are sparsely connected to each other. As the economy gets fundamentally safer, they organize into larger clusters which results in an increase in systemic risk.

Whether the networks formed efficiently trade-off the benefits of surplus generation against systemic risk depends on two factors. First is the ability of agents to coordinate among themselves during network formation. If the networks formed are robust to bilateral deviations only, they are inefficient. If robust to deviations by relatively larger subsets, they are fully efficient. Second, is the infectiousness of counter-party risk, which serves as a natural mechanism for agents to internalize externalities. With strong contagion, agents recognize they are in the same boat during network formation.

Our model highlights that assessing the vulnerability of a network to system wide failure cannot be done in ignorance of the beliefs of agents who formed that network. Efficient markets generate structures that are safe under the correct specification of shocks, which will appear fragile under the wrong specification of the shock structure. Thus, mistakes in policy can arise from a misspecification in the correlation of risks.

Asymmetries between firms can lead to the emergence of 'central' institutions. However, it does not follow that they are 'too-big' or 'too-interconnected' if the networks formed are multilaterally stable. If the networks are robust to bilateral deviations only, then, there can be excess interconnectedness around these central institutions which can generate an excessive risk of contagion. However, in a large enough economy, these central groups become marginal and isolated.

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8 Appendix

Proof. **(Theorem 2)** Recall that $U(x) = x\alpha^{(0.5)x(x+1)}$. Let $\mathcal{U} = \{u \in \mathbb{R} \,|\, u = U(x) \text{ for some } x \in \mathbb{N}\}$. The maximum of $\mathcal{U}$ is achieved, uniquely, at $x = d^*$. Let $\bar{u} = U(d^*)$, this is the average payoff at the group stable network. We will prove that the average is strictly smaller in any other network.
Consider an efficient network G and suppose it to be made up of a collection of disjoint connected components: $C^1, C^2, C^3, \ldots$. Consider component C^i and suppose it has q_i edges. The total payoff of C^i scales with $2q_i\alpha^{q_i}$. If $q_i \neq k^*$ we can improve total payoff by deleting or adding (if not complete) edges to C^i. Therefore, we can assume that $q_i = k^*$, or that C^i is complete.
Let r_i be the largest integer such that $r_i(r_i-1)/2 < q_i \leq r_i(r_i+1)/2$. Let w_i be such that $q_i = r_i(r_i-1)/2+w_i$, where $1 \leq w_i \leq r_i$. Note that there must be at least $r_i + 1$ nodes in C^i.
Case 1: $1 \leq w_i \leq \frac{r_i-1}{2}$.
The average degree of nodes in C^i is at most $\frac{2k^*}{r_i+1} = \frac{r_i(r_i-1)+2w_i}{r_i+1} \leq r_i-1$. Note that $k^* = q_i \geq (r_i-1)r_i/2+1$. Hence the average payoff per node is at most $(r_i-1)\alpha^{k^*} < (r_i-1)\alpha^{\frac{(r_i-1)r_i}{2}} \leq \bar{u}$. So the average payoff is strictly less than $\bar{u}$.

Case 2: $r_i - 1 \geq w_i \geq \frac{r_i}{2}$.
Since $w_i < r_i$, $k^* = q_i \leq r_i(r_i+1)/2 - 1$. The average degree of nodes in C^i is at most $\frac{2k^*}{r_i+1} \leq \frac{r_i(r_i+1)-2}{r_i+1} \leq r_i - \frac{2}{r_i+1}$. Note that $k^* = q_i = (r_i-1)r_i/2 + w_i \geq r_i^2/2$. Hence the average payoff per node is at most $\left(r_i - \frac{2}{r_i+1}\right)\alpha^{r_i^2/2}$. Now we show that this is strictly less than $(r_i-1)\alpha^{(r_i^2-r_i)/2} = U(r_i-1)$. That is equivalent to showing that $\alpha < \left(\frac{r_i+1}{r_i+2}\right)^{2/r_i}$. Recall that k^* is the unique integer between $\alpha/(1-\alpha)$ and $1/(1-\alpha)$. Therefore, $\alpha \leq 1 - \frac{1}{k^*+1} \leq 1 - \frac{2}{r_i(r_i+1)}$. Hence, it suffices to verify that

$$1 - \frac{2}{r_i(r_i+1)} < \left(\frac{r_i+1}{r_i+2}\right)^{\frac{2}{r_i}} \iff \left(\frac{r_i+1}{r_i+2}\right)^{\frac{2}{r_i}} > \frac{(r_i+2)(r_i-1)}{(r_i)(r_i+1)}$$

$$\iff (r_i+2)\log\left(1 - \frac{1}{r_i+2}\right) > r_i \log\left(1 - \frac{1}{r_i}\right)$$

which is true since the function $f(x) = x\log(1-\frac{1}{x})$ is strictly increasing. Therefore, the average payoff is strictly less than $U(r_i-1) \leq U(d^*) = \bar{u}$.

Case 3: $w_i = r_i$. (This covers the case in which C^i is complete as well.)
Then, the average payoff per node is less than $U(r_i) \leq \bar{u}$, and the inequality is strict unless C^i is a $(d^*+1)-$complete graph. □

Proof. **(Proposition 1)** Chernoff bounds can be used to find ϕ_2 such that for $1-\alpha < \zeta - \phi_2$, W and F are increasing, W grows unboundedly in d, and furthermore, $F(d)^d > 1 - d \times \omega^{-d}$ for some $\omega > 1$. Then, one can also show that for large d, $W(d+1)F(d+1)^{d+1} > W(d) > W(d)F(d)^d$. Then, for large n, $d^* = n-1$. Now take any pairwise stable network. If there is a missing edge in a component, the incident nodes would clearly add the edge. This would improve each other's W and F. Hence all maximally connected components must be cliques in any pairwise stable network. Finally, take the complete network (N, E). Suppose that (N, E') is a profitable deviation by V. Note that all nodes in N/V are adjacent to each other. Take any $i \in V$ and let C_i by the component of i in E'. If i is not connected to N/V, then the deviation in which all nodes in C_i formed a clique must also be profitable deviation for them. But that gives less than the complete

network. If i is connected to N/V, then, the deviation in which C_i did not cut any edges with N/V, and formed a clique among themselves, and cut all edges with $N/\{V \cup C_i\}$ is also a profitable deviation. But that also gives a clique payoff, which is not more than the complete network payoff. Hence there is no profitable deviation by any coalition from the complete network. As for efficiency, notice that adding edges one by one in a maximally connected component improves the total payoff in the component at every step. Hence any efficient network must be disjoint cliques. But since $W(d)F(d)^d$ is increasing unboundedly after $\bar{n}$, the largest clique K_n gives the highest average payoff among cliques, making K_n the unique efficient network.
Again, Chernoff bounds can be used to find ϕ_1 such that for $1-\alpha > \zeta + \phi_1$, both $W(d)$ and $F(d)$ are decreasing for $d \geq 1$. Clearly, any node i would always want to cut all links but one. This strictly improves $W(d_i)$, weakly improves $F(d_j)$ for any j that i is still connected to, and weakly decreases the number of nodes that i is connected to. Hence all pairwise stable networks consist of disjoint edges. By the same token, the optimal clique features $d^* = 1$. This is clearly the group stable network too. In any other network, there will be a node with degree at least 2, which has smaller payoff. □

Proof. **(Proposition 2)** Let F and G denote the binomial PDF and CDF.
$V(d) = \sum_{t=0}^{t=d} F[t, d, 1-\alpha^d]\,(d\theta - t)^+$ is the payoff of a node in a clique with $d+1$ nodes. $d^* = argmaxV(d)$.
If two nodes in a clique of order d^*+1 cut an edge, their payoff becomes

$$\sum_{t=0}^{t=d-1} F\left[t, d-1, 1-\alpha^d\right](d\theta - \theta - t) < V(d^*-1) < V(d^*).$$

This can not be profitable. If two nodes in separate cliques of order $d+1$ add their missing link, their payoff becomes

$$W(d) = \sum_{t=0}^{t=d} F\left[t, d, 1-\alpha^d\right]\Big(q(d)\,(d\theta + \theta - t)^+ + (1 - q(d)\,(d\theta + \theta - t - 1)^+\Big)$$

where

$$q(d) = \begin{cases} \alpha^{d+1} G\left[d\theta_0, d, 1-\alpha^d\right] & if\ (d+1)\theta_0 < d \\ \alpha^{d+1} & if\ (d+1)\theta_0 \geq d \end{cases}$$

We need to show that $V(d^*) \geq W(d^*)$.
For the ease of exposition in the proof, we use d for d^*, G_t for $G\left[t, d, 1-\alpha^d\right]$, F_t for $F\left[t, d, 1-\alpha^d\right]$, q for $q(d)$, m for $\lfloor d\theta \rfloor$, s for $\lfloor d\theta + \theta \rfloor$.
Case 1: $m = s$.

$$\frac{W(d)}{\alpha V(d)} = \frac{\sum_{t=0}^{t=d} F_t\Big(q\,(d\theta + \theta - t)^+ + (1-q)\,(d\theta + \theta - t - 1)^+\Big)}{\sum_{t=0}^{t=d} F_t\,(d\theta - t)^+}$$

$$= \frac{\sum_{t=0}^{t=m-1} F_t\left[(d\theta + \theta - t) - (1-q)\right] + F_m q(d\theta + \theta - m)}{\sum_{t=0}^{t=m} F_t\,(d\theta - t)}$$

$$= \frac{\sum_{t=0}^{t=m} F_t\,(d\theta - t) - F_m(d\theta - m) + G_{m-1}(\theta + q - 1) + F_m q(d\theta + \theta - m)}{\sum_{t=0}^{t=m} F_t\,(d\theta - t)}$$

$$= 1 + \frac{G_{m-1}(\theta + q - 1) + F_m\left(q\theta - (d\theta - m)(1-q)\right)}{\sum_{t=0}^{t=m} F_t\,(d\theta - t)}.$$

If the numerator is negative, we are done. If it is positive:

$$\frac{W(d)}{\alpha V(d)} \leq 1 + \frac{G_{m-1}(\theta+q-1) + F_m\left(q\theta - (d\theta - m)(1-q)\right)}{G_{m-1}}$$

$$= 1 + \theta + q - 1 + \frac{F_m\left(q\theta - (d\theta - m)(1-q)\right)}{G_{m-1}}$$

$$\leq \theta + q + \frac{F_m\left(q\theta - (d\theta - m)(1-q)\right)}{F_{m-1}}$$

$$= \theta + q + \frac{d+1-m}{m} \times \frac{1-\alpha^d}{\alpha^d} \times \left(q\theta - (d\theta - m)(1-q)\right).$$

Define $\epsilon = d\theta - m$.

$$\frac{W(d)}{\alpha V(d)} \leq \theta + q + \left(\frac{d+1}{d\theta - \epsilon} - 1\right) \times \frac{1-\alpha^d}{\alpha^d} \times \left(q\theta - \epsilon(1-q)\right).$$

Consider the function Φ of ϵ keeping all else fixed:

$$\Phi(\epsilon) = \left(\frac{d+1}{d\theta - \epsilon} - 1\right)\left(q\theta - \epsilon(1-q)\right).$$

$$\Phi'(\epsilon) = (d+1)q\theta\frac{-1}{(d\theta-\epsilon)^2} - (d+1)(1-q)\frac{d\theta}{(d\theta-\epsilon)^2} + (1-q)$$

$$< (d+1)q\theta\frac{-1}{(d\theta-\epsilon)^2} - (1-q) + (1-q) < 0.$$

So $\Phi(\epsilon)$ is decreasing and maxed at $\epsilon = 0$. That is,

$$\frac{d+1-m}{m} \times \left(q\theta - (d\theta - m)(1-q)\right) < \frac{d+1}{d\theta} \times \left(q\theta - \epsilon(1-q)\right)$$

and

$$\frac{W(d)}{\alpha V(d)} \leq \theta + q + \left(\frac{d+1}{d\theta} - 1\right) \times \frac{1-\alpha^d}{\alpha^d} \times q\theta = \theta + q + \left(1 - \theta + \frac{1}{d}\right) \times \left(\frac{1-\alpha^d}{\alpha^d}\right) \times q.$$

Case 1.1: $d\theta \geq 1$. In this case, $1 - \theta + \frac{1}{d} \leq 1$ so that

$$\frac{W(d)}{\alpha V(d)} \leq \theta + q + \left(\frac{1}{\alpha^d} - 1\right) \times q = \theta + \frac{q}{\alpha^d} \leq \theta + \alpha \leq \frac{1}{\alpha}.$$

Case 1.2: $d\theta < 1$. Then $m = s = 0$. Since $s = 0$, $(d+1)\theta < 1$. Optimal d can't be zero, so $(d+1)\theta < 1 \leq d$, hence $q(d) = \alpha^{d+1}G_m = \alpha^{d+1}G_0 = \alpha^{d+1}F_0 = \alpha^{d+1+d^2}$. Then $W(d) = \alpha^{d+1}F_0 \times q(d\theta+\theta) = \alpha^{2d+2+2d^2}(d\theta+\theta)$. On the other hand $V(d+1) = \alpha^{d+1+(d+1)^2}(d\theta+\theta)$. Since $d \geq 1$, we have $2d+2+2d^2 \geq d+1+(d+1)^2$ so that $V(d+1) \geq W(d)$, which implies $V(d) \geq W(d)$.

Case 2: $m = s - 1$.

Case 2.1: $(d+1)\theta < d$.

$$\frac{V(d)}{\alpha^d} - \frac{W(d)}{\alpha^{d+1}} = \sum_{t=0}^{t=d} F_t\left(d\theta - t\right)^+ - \left[\sum_{t=0}^{t=d} F_t\left(q\left(d\theta + \theta - t\right)^+ + (1-q)\left(d\theta + \theta - t - 1\right)^+\right)\right]$$

$$= \sum_{t=0}^{t=m} F_t\left(d\theta - t\right) - \left[\sum_{t=0}^{t=s-1=m} F_t\left(d\theta + \theta - t - 1 + q\right) + F_s q(d\theta + \theta - s)\right]$$

$$= G_m(1 - \theta - q) - F_{m+1} q(d\theta + \theta - s).$$

$d\theta < m + 1 = s$ so $d\theta + \theta - s < \theta$. Also $(d+1)\theta < d$, so that $q(d) = \alpha^{d+1} G_m$. Also $F_{m+1} < 1 - G_m$. Plug all these in:

$$\frac{V(d)}{\alpha^d} - \frac{W(d)}{\alpha^{d+1}} > G_m\left(1 - \theta - \alpha^{d+1} G_m - (1 - G_m)\alpha^{d+1}\theta\right).$$

$$> G_m\left(1 - \theta - \alpha^{d+1} G_m(1-\theta) - \alpha^{d+1}\theta\right) > G_m\left(1 - \theta - \alpha^{d+1}(1-\theta) - \alpha^{d+1}\theta\right)$$

$$= G_m\left(1 - \theta - \alpha^{d+1}\right) \geq 0.$$

That is $V(d) > \frac{1}{\alpha} W(d) > W(d)$.

Case 2.2: $(d+1)\theta \geq d$. Then $s = d$, so that $m = d - 1$. Note that $(d+1)\theta \geq \theta$ implies $(d+1)\alpha^2 \leq 1$.

Case 2.2.1: If $d \geq 3$, then

$$W(d) < \alpha^{d+1}(d\theta + \theta) \leq \alpha^4 (d+1)\theta \leq \alpha^2\theta = V(1) \leq V(d).$$

Case 2.2.2: If $d = 2$, then $s = 2$ so $\theta > \frac{2}{3} > \frac{1}{2}$. Then

$$\frac{V(2)}{\alpha^2} = \alpha^4 2\theta + 2\alpha^2(1-\alpha^2)(2\theta - 1)$$

$$= 2\theta\left[2\alpha^2 - \alpha^4\right] - 2\alpha^2\left(1 - \alpha^2\right) < 2\theta\left[2\alpha^2 - \alpha^4\right] - 2\alpha^2\theta$$

$$= 2\theta\left[\alpha^2 - \alpha^4\right] < \frac{\theta}{2} < \theta = \frac{V(1)}{\alpha^2}$$

which is a contradiction. $d = 2$ is not possible in this case.

Case 2.2.3: If $d = 1$, then $s = 1$ and $\theta > \frac{1}{2}$.

$$\frac{W(1)}{\alpha^2} = \alpha\left(\alpha^2 2\theta + (1-\alpha^2)(2\theta - 1)\right) + (1-\alpha)\left(\alpha^2(2\theta - 1)\right)$$

$$= 2\theta\left[\alpha + \alpha^2 - \alpha^3\right] - \left[\alpha + \alpha^2 - 2\alpha^3\right].$$

$$\frac{V(1) - W(1)}{\alpha^2} = 2\theta\left[\frac{1}{2} - \alpha - \alpha^2 + \alpha^3\right] + \left[\alpha + \alpha^2 - 2\alpha^3\right].$$

If the term in the first bracket is positive, we are done. If it is negative, then replace insert $\theta < 1 - \alpha^2$.

$$\frac{V(1) - W(1)}{\alpha^2} > 2\left(1 - \alpha^2\right)\left[\frac{1}{2} - \alpha - \alpha^2 + \alpha^3\right] + \left[\alpha + \alpha^2 - 2\alpha^3\right]$$

$$= (1 - \alpha)\left(1 - \alpha^2\right) \geq 0.$$

□

Proof. **(Proposition 4)** (If part) By the 3, $\Delta_v \cdot \pi^v(V) \geq 0$ implies that B is a best reply by v if all players in V play B and others play D. D is rationalizable for every player, therefore, B can never be eliminated by players in V. For all players in V playing B is rationalizable. Hence, in the cooperating equilibrium, all of

V, in particular v, play B.
(Only if part) Suppose not. Then N is not strategically cohesive (since $v \in N$) and there exists $v_1 \in N$ such that $\Delta_{v_1} \cdot \pi^{v_1}(N) < 0$. Notice that $\pi^{v'}(V') = \pi^{v'}(V'/\{v'\}) = \pi^{v'}(V' \cup \{v'\})$ for any V' and v' since nodes are not adjacent to themselves. Then, $\Delta_{v_1} \cdot \pi^{v_1}(N/\{v_1\}) < 0$. By Lemma 3, v_1's best response to $N/\{v_1\}$ playing B is D. By Assumption 2, v_1's best response to any strategy profile, in particular any strategy profile not eliminated, is D. Thus, v_1 plays D in a cooperating equilibrium. Hence $v_1 \neq v$. Let $N_1 = N/\{v_1\}$. $v \in N_1$. Therefore, by supposition, N_1 is not strategically cohesive. Hence, there exists $v_2 \in N$ such that $\Delta_{v_2} \cdot \pi^{v_2}(N_1) < 0$. Similarly, by Lemma 3 and Assumption 2, v_2's best response to any profile in which $N_1/\{v_2\}$ plays B, in particular any strategy profile not eliminated, is D. Thus v_2 plays D in the cooperating equilibrium, and $v_2 \neq v$. Let $N_2 = N_1/\{v_2\}$. Since N is finite and v plays D in the cooperating equilibrium, we reach a contradiction in a finite number of steps. □

Proof. **(Lemma 4)** Consider a $v \in V$. We show that $\Delta_v \cdot [\pi^v(V \cup V') - \pi^v(V)] \geq 0$. In this summation the t'th component is $\Delta_v(\theta_t) \times \left[\pi_t((V \cup V')^c \,|v) - \pi_t(V^c|v)\right]$ and $k+t$'th component is $\Delta'_v(\theta_t) \times [\pi_t((V \cup V')\,|v) - \pi_t(V|v)]$. The terms in the brackets add up to 0. Hence the sum of these two terms is equal to $[\Delta'_v(\theta_t) - \Delta_v(\theta_t)] \times [\pi_t((V \cup V')\,|v) - \pi_t(V|v)] \geq 0$ by Assumption 2.
Therefore, $\Delta_v \cdot \pi^v(V \cup V') = \Delta_v \cdot \pi^v(V) + \Delta_v \cdot [\pi^v(V \cup V') - \pi^v(V)] \geq \Delta_v \cdot \pi^v(V) \geq 0$. □

Proof. **(Proposition 7)** Consider a uniform-stable network and suppose that there is a profitable bilateral deviation by two nodes. Take one of them, let her have degree d, and let her have $e = d(d+1)/2$ edges in her complete subgraph. Suppose that in the bilateral profitable deviation she deletes x of her incident edges in her complete subgraph, and adds $t \in \{0, 1\}$ new edges.
If $x = d$, her payoff is at most $\alpha = U(1) \leq U(d)$ (since $1 \leq d \leq h^*$) which cannot be a profitable deviation. So $x < d$, which means she is still incident to $e - x$ edges in her old component. Then her payoff is at most $(d - x + t)\alpha^{2(e-x+t)}$. If $t - x \leq 0$, this is less than $d\alpha^{2e}$ since $y\alpha^{2y}$ is strictly increasing up to k^* in $y \in \mathbb{N}$ and $h \leq k^*$. Then $t - x > 0$, which is possible only when $t = 1$ and $x = 0$. This is true for the other deviator as well. Therefore, these two deviators keep all their previous edges and connect to each other with a new edge. Let the other deviator have degree d'. Without loss of generality, let $d \leq d'$. Then, the deviator with the smaller degree has her payoff moved from $d\alpha^{d(d+1)}$ to $(d+1)\alpha^{2+d(d+1)+d'(d'+1)}$ which is less than or equal to $(d+1)\alpha^{2+d(d+1)}$. This being a profitable deviation immediately implies $d < h^{**}$, which is a contradiction. □

Proof. **(Proposition 8)** Recall that $\left(1 - \alpha^{x(x+1)}\right)^{1/x}$ is increasing in x. Take any complete subgraph with order $d + 1 \geq d^* + 1$.

$$1 - \alpha^{d(d+1)} = \left(1 - \alpha^{d(d+1)}\right)^{(d+1)/(d+1)} \geq \left(1 - \alpha^{(d^*+1)}\right)^{(d+1)/(d^*+1)}.$$

Let $d_t + 1$'s be the orders of maximally complete subgraphs of a upper-uniform stable network. Then

$$\prod_t \left(1 - \alpha^{(d_t+1)}\right) \geq \left(1 - \alpha^{d^*(d^*+1)}\right)^{\frac{1}{d^*+1}\sum d_t+1} = \left(1 - \alpha^{d^*(d^*+1)}\right)^{\frac{n}{d^*+1}}.$$

The case for lower-uniform stable networks have the similar proof. □