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DYNAMIC VALUATION DECOMPOSITION WITHIN STOCHASTIC ECONOMIES

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I explore the equilibrium value implications of economic models that incorporate responses to a stochastic environment with growth. I propose dynamic valuation decompositions (DVD's) designed to distinguish components of an underlying economic model that influence values over long investment horizons from components that impact only the short run. A DVD represents the values of stochastically growing claims to consumption payoffs or cash flows using a stochastic discount process that both discounts the future and adjusts for risk. It is enabled by constructing operators indexed by the elapsed time between the trading date and the date of the future realization of the payoff. Thus formulated, methods from applied mathematics permit me to characterize valuation behavior and the term structure of risk prices in a revealing manner. I apply this approach to investigate how investor beliefs and the associated uncertainty are reflected in current-period values and risk-price elasticities.

KEYWORDS: Valuation, macroeconomic uncertainty, intertemporal risk prices, stochastic growth, Perron–Frobenius theory, recursive utility, dynamic value decomposition.

The manner in which risk operates upon time preference will differ, among other things, according to the particular periods in the future to which the risk applies. Irving Fisher (*Theory of Interest* (1930))

1. INTRODUCTION

IN THIS PAPER, I propose to augment the toolkit for modeling economic dynamics with methods that reveal important economic components of valuation in economies with stochastic growth. These tools enable informative decompositions of a model's dynamic implications for valuation. They are the outgrowth of my observation of and participation in an empirical literature that seeks to understand better the links between financial market indicators and macroeconomic aggregates at alternative time horizons.

A more direct source of motivation is the burgeoning quantitative literature in macroeconomics and finance that features contributions of risk and uncertainty to valuation over alternative investment horizons, including, for

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instance, the work by Parker and Julliard (2003), Campbell and Vuolteenaho (2004), Lettau and Wachter (2007), Hansen, Heaton, and Li (2008), Bansal, Dittmar, and Kiku (2009), and van Binsbergen, Brandt, and Koijen (2011). This literature is a direct outgrowth of Rubinstein's (1976) fundamental paper on valuing flows that grow stochastically over time. I will not attempt to survey this literature, but instead I will suggest new ways to understand the valuation implications over alternative investment horizons.

Current dynamic models that relate macroeconomics and asset pricing are constructed from an amalgam of assumptions about preferences (such as risk aversion or habit persistence), technology (productivity of capital or adjustment costs to investment), markets, and exposure to unforeseen shocks. Some of these components have more transitory effects, while others have a lasting impact. In part, my aim is to illuminate the roles of these model ingredients by presenting a structure that uses long-term implications for value as a reference point. By *value* I mean either market or shadow prices of physical, financial, or even hypothetical assets. These methods provide a sharp contrast to more typically short-term characterizations of risk–return trade-offs.

These methods are designed to address three questions:

- What are the value implications of nonlinear economic models with stochastic growth and volatility?
- How do risk exposures and prices change as we alter the investment horizon?
- To which components of the *uncertainty* are long-term valuations most sensitive?

Recent models in both asset-pricing and macroeconomic literature incorporate volatility specifications that evolve stochastically over time, and the resulting state dependence is reflected in risk pricing. These literatures motivate my interest in the first question. Although insightful characterizations of intertemporal asset pricing have been obtained using log-linear models and log-linear approximations around a growth trajectory, the methods I describe offer a different vantage point. These methods are designed for the study of valuation in the presence of stochastic inputs that have long-run consequences, including stochastic volatility. In this paper, I will develop these methods and in so doing, I will draw upon some diverse results from stochastic process theory and time series analysis applied in novel ways.

This leads me to the second question. Many researchers study valuation under uncertainty by risk prices and exposures, and through them, the equilibrium risk–return trade-off. In equilibrium, expected returns change in response to shifts in the exposure to various components of macroeconomic risk. The trade-off is typically depicted over a single period in a discrete-time model or over an instant of time in a continuous-time model. I will extend the log-linear analysis in Hansen, Heaton, and Li (2008) and Bansal, Dittmar, and Kiku (2009) by deriving counterparts to this familiar exercise for alternative investment horizons. Specifically, I will perform a sensitivity analysis that recovers price elasticities for exposure to the components of risk that endure for alternative investment horizons.

I consider the third question because when we, as researchers, build dynamic economic models, we typically specify transitional dynamics over a unit of time for discrete-time models or an instant of time for continuous-time models. Long-term implications are encoded in such specifications, but they can be hard to decipher, particularly in nonlinear stochastic models in which stochastic growth or stochastic discounting is compounded over time. For convenience I explore methods that use a characterization of long-term limiting behavior as a frame of reference. I see three reasons why this is important. First, some economic inputs are more credible when they target low frequency behavior. Second, these inputs may be essential for meaningful long-run extrapolation of value. Nonparametric statistical alternatives suffer because of limited empirical evidence on the long-term behavior of macroeconomic aggregates and financial cash flows. More meaningful extrapolation of value implications may be obtained within a framework with interpretable limits over long investment horizons. Finally, such an approach allows me to investigate how investor beliefs about the distant future, including uncertainty, influence current-period values and risk prices. While it is hard to project the future accurately, the approach I describe shows formally the circumstances under which investors' concern about the future can have important consequences for current-period behavior and valuation.

My analysis is supported by two mathematical representations: (i) an additive decomposition of what is called an additive functional and (ii) a multiplicative factorization of what is called a multiplicative functional. An additive functional Y has increments represented as time invariant functions of an underlying Markov process. A multiplicative functional $M = \exp(Y)$ is the exponential of an additive functional. I will use the process M to model stochastic growth, stochastic discounting, or the product of the two. Thus the process Y will be the logarithmic counterpart, which is of interest because statistical models of economic time series data typically apply to logarithms. In this paper, I consider

$$
\begin{aligned}
(1) \qquad & Y_t = \nu t + \widetilde{Y}_t + [g(X_0) - g(X_t)] \quad \text{additive,} \\
& M_t = \exp(\rho t) \times \widetilde{M}_t \times \left[\frac{e(X_0)}{e(X_t)}\right] \quad \text{multiplicative,}
\end{aligned}
$$

where $\widetilde{Y}$ is an additive martingale and $\widetilde{M}$ is a multiplicative martingale. The first relation in (1) is an additive decomposition that is familiar from time series analysis (see Section 3). While it has a variety of uses, I use this decomposition as a way to identify a permanent shock represented as the increment to the martingale $\widetilde{Y}$. The second relation in (1) is a multiplicative factorization. It is novel in the time series literature, but it is germane for the study of valuation (see Sections 4, 5, and 6) and it has close ties to an applied mathematics literature that gives probabilistic characterizations of the behavior of large deviations. The idea that the decomposition and the factorization are related can be anticipated from the work of Alvarez and Jermann (2005) and Hansen, Heaton, and Li (2008). As I will argue, the decomposition and the factorization differ in important ways, but they are also related and can be used as complementary tools in the study of growth and valuation (see Section 7). I illustrate the usefulness of these tools in Section 8 in a study of a model with recursive utility investors. In the next section, I describe the underlying mathematical framework.

2. PROBABILISTIC SPECIFICATION

While there are variety of ways to introduce nonlinearity into time series models, for tractability I concentrate on introducing this nonlinearity within a Markov specification. I use a Markov process that summarizes the state of the economy as a building block in models of economic variables that can grow stochastically over time. I feature continuous-time models with sharp distinctions between two kinds of shocks: small shocks modeled as Brownian increments and large shocks modeled as Poisson jumps.[2]

2.1. *Underlying Markov Process*

Let X be a Markov process defined on a state space $\mathcal{E}$. Throughout my discussion, I will use the notation x to be a hypothetical realized value of the X_t for any calendar date t. Suppose that the process X can be decomposed into two components: $X^c + X^d$, where X^c has a continuous sample path and X^d records the jumps. The process X is right continuous with left limits. With this in mind, I define

$$X_{t-} = \lim_{u \downarrow 0} X_{t-u}.$$

I depict local evolution of X^c as

$$dX_t^c = \mu(X_{t-})\,dt + \sigma(X_{t-})\,dW_t,$$

where W is a possibly multivariate standard Brownian motion. The process X^d is a jump process. This process is modeled using a finite conditional measure $\eta(dx^*|x)$, where $\int \eta(dx^*|X_{t-})$ is the jump intensity; that is, $\varepsilon \int \eta(dx^*|X_{t-})$ is the approximate probability that there will be a jump for small time interval ε. The conditional measure $\eta(dx^*|x)$, scaled by the jump intensity, is the probability distribution for the jump conditioned on a jump occurring. Thus the entire Markov process is parameterized by (μ, σ, η).

[2]Some of what follows can be extended to a more general class of Levy processes.

I will often think of the process X as stationary, but strictly speaking this is not necessary for some of my results. As I next show, nonstationary processes can be constructed from X.

2.2. *Convenient Functions of the Markov Process*

To obtain flexible characterizations, I model processes in a flexible and tractable way that is typically derived from a more complete specification of an economic model. Motivated in part by work like that of Rubinstein's (1976) study of equilibrium valuation of stochastically growing cash flows, I consider two processes and how they interact: one process captures stochastic discounting and the other captures stochastic growth. Both processes are modeled as functionals built from an underlying Markov process. Specifically, the stochastic discount functional captures the compounding of discount rates and adjustments for risk for these alternative investment horizons. The representation of asset prices in this manner is well known from the theory of frictionless markets. The stochastic growth functional captures the compounding of growth rates over alternative time horizons. The growth functional could be aggregate consumption, a technology shock process, or some other economic aggregate that is used as a reference point for macroeconomic growth. Alternatively, it could be a cash flow whose value is to be ascertained. While stochastic discount factor functionals decay over the investment horizon, stochastic growth functionals increase. I model both conveniently as multiplicative functionals of the underlying Markov process or their additive counterparts formed by taking logarithms. Time series econometric models typically apply to logarithms, and thus I describe additive functionals of a Markov process.

Formally, an additive functional Y is constructed from the underlying Markov process such that $Y_{t+\tau} - Y_t = \phi_\tau(X_u \text{ for } t < u \leq t + \tau)$ for any $t \geq 0$ and any $\tau \geq 0$. For convenience, it is initialized at $Y_0 = 0$. Notice that what I call an additive functional is actually a stochastic process defined for all $t \geq 0$. Even when the underlying Markov process X is stationary, an additive functional will typically not be. Instead it will have increments that are stationary and hence the Y process can display arithmetic growth (or decay) even when the underlying process X does not. An additive functional can be normally distributed, but I will also be interested in other specifications. For instance, I will feature some specifications with stochastic volatility. The implied state dependence in the volatility specification mixes the normal distributions as the process evolves over time. Conveniently, the linear combination of two additive functionals is additive.

I consider a family of such functionals parameterized by (β, ξ, χ), where the following statements hold:

(i) $\beta : \mathcal{E} \to \mathbb{R}$ and $\int_0^t |\beta(X_u)|\, du < \infty$ for every positive t.

(ii) $\xi : \mathcal{E} \to \mathbb{R}^m$ and $\int_0^t |\xi(X_u)|^2\, du < \infty$ for every positive t.

(iii) $\chi : \mathcal{E} \times \mathcal{E} \to \mathbb{R}$, $\chi(x, x) = 0$, and

$$Y_t = \int_0^t \beta(X_u)\, du + \int_0^t \xi(X_{u-}) \cdot dW_u + \sum_{0<u\le t} \chi(X_u, X_{u-}). \tag{2}$$

The additive functional Y in (2) has three components, each of which accumulates linearly over time. The first component is a simple integral, $\int_0^t \beta(X_u)\, du$, and as a consequence it is locally predictable. The second component is a stochastic integral, $\int_0^t \xi(X_u) \cdot dW_u$, and it reflects how "small shocks" alter the functional Y. These small shocks are modeled as Brownian increments. This component is a so-called local martingale (defined using stopping times), but I will feature cases in which it is a global martingale. Recall that the best forecast of the future value of a martingale is the current value of the martingale. The third component shows how jumps in the underlying process X induce jumps in the additive functional. If X jumps at date t, Y also jumps at date t by the amount $\chi(X_t, X_{t-})$. The term $\sum_{0<u\le t} \chi(X_u, X_{u-})$ thus reflects the impact of "large shocks." This component is not necessarily a martingale because the jumps may have a predictable component. The integral

$$\tilde{\beta}(x) = \int_{\mathcal{E}} \chi(x^*, x)\eta(dx^*|x) \tag{3}$$

captures this predictability locally. Integrating $\tilde{\beta}$ over time and subtracting it from the jump component of Y gives an additive local martingale:

$$\sum_{0<u\le t} \chi(X_u, X_{u-}) - \int_0^t \tilde{\beta}(X_u)\, du.$$

I will be primarily interested in specifications of χ for which this constructed process is a martingale.

In summary, an additive functional grows or decays stochastically in a linear way. Its dynamic evolution can reflect the impact of small shocks represented as a state-dependent weighting of a Brownian increment and the impact of large shocks represented by a possibly nonlinear response to jumps in the underlying process X. As I mentioned previously, the logarithms of economic aggregates can be conveniently represented as additive functionals as can the logarithms of stochastic discount factors used to represent economic values.[3] I next consider the level counterparts to such functionals.

While a multiplicative functional can be defined more generally, I will consider ones that are constructed as exponentials of additive functionals:

[3]For economic aggregates, it is necessary to subtract the date zero logarithms so that $Y_0 = 0$. The restriction that $Y_0 = 0$ is essentially a convenient normalization.

$M = \exp(Y)$. Thus the ratio $M_{t+\tau}/M_t$ is constructed as a function of X_u for $t < u \leq t + \tau$.[4] Multiplicative functionals are necessarily initialized at 1.

Even when X is stationary, a multiplicative process can grow (or decay) stochastically in an exponential fashion. Although its logarithm will have stationary increments, these increments are not restricted to have a zero mean.

3. ADDITIVE DECOMPOSITION

In this section, I review and extend existing methods to identify permanent shocks. I develop these methods to allow me to investigate how such permanent shocks affect valuation over different investment horizons.

Applied macroeconomic researchers often characterize steady-state relations by inferring a scaling process or processes that capture growth components common to several time series. The steady-state implications apply after appropriately scaling the economic time series. Similarly, econometricians feature the co-integration of multiple times series that grow together because their logarithms depend linearly on a small number of permanent shocks; see Engle and Granger (1987).[5] Thus by applying economic reasoning, we expect certain time series to move together. In related literature, Beveridge and Nelson (1981), Blanchard and Quah (1989), and many others used long-run implications to identify shocks; they asserted that supply or technology shocks broadly conceived are the only shocks that influence output in the long run. These methods aim to measure the potency of shocks in the long run while permitting short-run dynamics. I explore these methods for identifying permanent shocks in the setting of additive functionals by extracting additive martingale components. The increments to these additive martingale components are the permanent shocks.[6]

My initial investigation of additive functionals is consistent with the common practice of building models that apply to logarithms of macroeconomic or financial times series. While there are alternative ways to decompose time series,

[4]This latter implication gives the key ingredient of a more general definition of a multiplicative functional.

[5]Interestingly, Box and Tiao (1977) anticipated the potentially important notion of long-run co-movement in their method of extracting canonical components of multivariate time series.

[6]Martingale extraction of additive functionals has been used in other ways. Following Gordin (1969), by extracting a martingale, we can establish a more general characterization of central limit approximations for additive functionals. Specifically, an implication of the martingale central limit theorem is that

$$\lim_{t\to\infty} \frac{1}{\sqrt{t}}(Y_t - \nu t) \approx \frac{1}{\sqrt{t}}\widetilde{Y}_t \Rightarrow \text{normal}$$

is normally distributed with mean zero. See Billingsley (1961) for a discrete-time martingale central limit theorem. Moreover, there are well known functional extensions of this result; for instance, see Hall and Heyde (1980).

what follows is the most germane to my interest in the valuation of macroeconomic risk. An additive functional can be decomposed into three components:

$$Y_t = \underset{\text{linear trend}}{\underset{\uparrow}{\nu t}} + \underset{\text{martingale}}{\underset{\uparrow}{\widetilde{Y}_t}} - \underset{\text{stationary difference.}}{\underset{\uparrow}{[g(X_t) - g(X_0)]}} \tag{4}$$

This decomposition gives a way to identify shocks with "permanent" consequences. Recall that the best forecast of the future values of a martingale is the current value of that martingale. Thus permanent shocks are reflected in the increment to the martingale component of (4). Isolating such shocks allow me to identify the exposure of economic time series to macroeconomic risk that dominates the fluctuation of Y over long time horizons. By contrast, $g(X_t) - g(X_0)$ evolves as a stationary process translated by the term $-g(X_0)$, which is time invariant. Thus this term contributes only transitory variation to the process Y.

The remainder of this section is organized as follows. First, I verify formally the martingale property for $\widetilde{Y}$ and then I show how to construct this decomposition. I finish the section with two illustrative examples that are designed to be pedagogically revealing with sufficient richness to capture the dynamic structure of many existing models. One example considers a model in which stochastic volatility is introduced into a continuous-time counterpart of a vector autoregression. This example allows for volatility to fluctuate over time in a manner that can be highly persistent. The second example considers a mixture-of-normals model, in which the increments of Y are conditional normal, but the mean and the exposure to a vector of Brownian increments change over time in accordance with a finite-state Markov process. Both examples include particular forms of nonlinearity that have received considerable attention in both the macroeconomics and the asset-pricing literature.

A reader may be concerned that these chosen examples rely too heavily on normal distributions. While the instantaneous increments are conditional normal over finite time intervals, these normal distributions get "mixed" over time. This mixing is a common device for modeling distributions with fat tails.[7]

My first formal statement of decomposition (4) is the following theorem.

THEOREM 3.1: *Suppose that Y is an additive functional with increments that have finite second moments. In addition, suppose that*

$$\lim_{\tau\to\infty} \frac{1}{\tau} E(Y_\tau | X_0 = x) = \nu$$

[7]As I mentioned earlier, some of the methods I describe have direct extensions to more general Levy processes.

and

$$\lim_{\tau\to\infty} E(Y_\tau - \nu\tau | X_0 = x) = g(x),$$

where the convergence is in mean square. Then Y *can be represented as*

$$Y_t = \nu t + \widetilde{Y}_t - g(X_t) + g(X_0),$$

where $\{\widetilde{Y}_t\}$ *is an additive martingale.*

See Appendix A for the proof.

I next show how to use the local evolution of the additive functional to construct the components of this decomposition. Recall the representation given in (2),

$$Y_t = \int_0^t \beta(X_u)\,du + \int_0^t \xi(X_{u-})\cdot dW_u + \sum_{0<u\le t} \chi(X_u, X_{u-}),$$

and the construction of $\tilde{\beta}$ in formula (3). Then

$$\tilde{Y}_t = \int_0^t \xi(X_{u-})\cdot dW_u + \sum_{0<u\le t} \chi(X_u, X_{u-}) - \int_0^t \tilde{\beta}(X_u)\,du \tag{5}$$

is a local martingale. In what follows, let

$$\kappa(x) = \beta(x) + \tilde{\beta}(x) = \beta(x) + \int_{\mathcal{E}} \chi(x^*, x)\eta(dx^*|x),$$

where $\kappa(x)$ is the drift or local mean of Y when the Markov state is x. Thus

$$Y_t = \tilde{Y}_t + \int_0^t \kappa(X_u)\,du.$$

I now have one of the ingredients for the decomposition: $\nu = E[\kappa(X_t)]$.

To obtain a solution g to a long-run forecasting problem, I construct an equation from the local evolution of the Markov process X. I presume that $\tilde{Y}_t$ is itself a martingale. Thus from Theorem 3.1,

$$g(x) = \lim_{\tau\to\infty} E\left[\int_0^\tau \kappa(X_u)\,du - \tau\nu \,\middle|\, X_0 = x\right]. \tag{6}$$

To derive an alternative operational formula for g, I equate two alternative expressions for the local mean of Y,

$$\kappa(x) = \nu - \lim_{t\downarrow 0} \frac{1}{t} E[g(X_t) - g(x)|X_0 = x], \tag{7}$$

where the second one uses the additive decomposition. Combining this formula with the local evolution for the Markov state X gives an equation that can be solved for g.[8] In the case of a multivariate diffusion, this equation is a second-order differential equation as an implication of Ito's formula. There are well known extensions to accommodate jumps. Given a function g that solves equation (7), this same function translated by a constant also satisfies the equation. I resolve this multiplicity conveniently by choosing g that has mean zero when integrated with respect to a stationary distribution.

The following theorem is a summary.

THEOREM 3.2: *Suppose the following statements hold*:
(i) *X is a stationary, ergodic Markov process.*
(ii) *$\tilde{Y}$ given in (5) is a square integrable martingale.*
(iii) *$\kappa(X_t)$ has a finite second moment.*
(iv) *There is a solution g for which $g(X_t)$ has a finite second moment to the equation*

$$\lim_{t\downarrow 0}\frac{1}{t}E[g(X_t)-g(x)|X_0=x]=\eta-\kappa(x).$$

Then $\widetilde{Y}$ given by $Y_t-\nu t+g(X_t)-g(X_0)$ is a martingale with stationary, square integrable increments, where $\nu=E[\kappa(X_t)]$.

The function g plays a central role in the additive decomposition of Theorems 3.1 and 3.2. As we see from Theorem 3.2, it suffices to solve the equation in part (iv). Much is known about such an equation. As argued by Bhattacharya (1982) and Hansen and Scheinkman (1995), when X is ergodic, this equation has at most one solution. These and other authors have established general conditions for the existence of a solution when X is exponentially ergodic.[9]

I do not wish to oversell the novelty of the additive decomposition characterized in Theorems 3.1 and 3.2. They give a continuous-time Markov version of a result that is familiar in other literatures. My interest in this decomposition is that it identifies a martingale increment that has permanent consequences for the process under consideration. A nice feature of the decomposition of an additive functional is that even when nonlinearity is introduced, the sum of two additive functionals is an additive functional. Moreover, the sum of the martingale components is the martingale component for the sum of the additive

[8]See, for instance, Hansen and Scheinkman (1995) for a discussion showing when this local condition implies the infinite-horizon relation in (6).

[9]These references suppose that X is stationary. Hansen and Scheinkman (1995) used an L^2 notion of exponential ergodicity built with the implied stationary distribution of X as a measure. Bhattacharya (1982) established a functional counterpart to the central limit theorem using these methods. Under strong dependence in X, existence of a solution to the equation in part (iv) of Theorem 3.2 is still possible, but it is less generic.

functionals provided that the constructions use a common information structure. When there are multiple additive functionals under consideration, and they have a common growth term and a common martingale component, then one obtains a special case of the co-integration model of Engle and Granger (1987) and others.

Recall that the permanent shock is the increment to the martingale component of Y. For the diffusion model, local evolution of this martingale is given by

$$\xi(X_t) \cdot dW_t + \left[\frac{\partial g(X_t)}{\partial x}\right]' \sigma(X_t)\, dW_t, \tag{8}$$

where the first term is contributed by the local evolution of $\tilde{Y}$ and the second term is contributed by the local evolution of $g(X)$. The state-dependent weighting

$$\xi(X_t) + \left[\frac{\partial g(X_t)}{\partial x}\right]' \sigma(X_t)$$

of the Brownian increment dW_t measures the *exposure* of the permanent shock to Brownian motion risk. The state dependence reflects the impact of stochastic volatility: volatility that evolves randomly over time as a function of the Markov state. From Ito's formula the function g solves

$$\frac{\partial g(x)}{\partial x} \cdot \mu(x) + \frac{1}{2} \operatorname{trace}\left[\sigma(x)\sigma(x)' \frac{\partial^2 g(x)}{\partial x\, \partial x'}\right] = \nu - \kappa(x). \tag{9}$$

For a Markov process with a jump component, the martingale increment includes

$$[\chi(X_t, X_{t-}) - \tilde{\beta}(X_{t-})\, dt] + \left[g(X_t) - g(X_{t-}) - \int g(x^*)\eta(dx^*, X_{t-}) \right.$$
$$\left. + g(X_{t-}) \int \eta(dx^*, X_{t-})\right],$$

which are the terms analogous to those in (8) for the diffusion specification. The first term is the direct contribution from the unanticipated jump component in Y_t and the second term is the unanticipated jump component in $g(X_t)$. The equation for g must be altered to accommodate the jumps.

As an illustration of a nonlinear diffusion model, I introduce stochastic volatility with diffusion dynamics. In this example, macroeconomic volatility evolves over time with a continuous sample path. I use a specific model with affine dynamics in the sense of Duffie and Kan (1994) so that I that can conveniently illustrate constructions that interest me.

EXAMPLE 3.3: Suppose that X and Y evolve according to

$$dX_t^{[1]} = A_{11}X_t^{[1]}\,dt + A_{12}\big(X_t^{[2]} - 1\big)\,dt + \sqrt{X_t^{[2]}}B_1\,dW_t,$$
$$dX_t^{[2]} = A_{22}\big(X_t^{[2]} - 1\big)\,dt + \sqrt{X_t^{[2]}}B_2\,dW_t,$$
$$dY_t = \nu\,dt + H_1X_t^{[1]}\,dt + H_2\big(X_t^{[2]} - 1\big)\,dt + \sqrt{X_t^{[2]}}F\,dW_t.$$

Both $X^{[2]}$ and Y are scalar processes. The process $X^{[2]}$ is an example of a Feller square root process, which I use to model the temporal dependence in volatility. I restrict $B_1B_2' = 0$ implying that $X^{[1]}$ and $X^{[2]}$ are conditionally uncorrelated. The matrix A_{11} has eigenvalues with strictly negative real parts and A_{22} is negative. Moreover, to prevent zero from being attained by $X^{[2]}$, I assume that $A_{22} + \frac{1}{2}|B_2|^2 < 0$.

I have parameterized this process to have mean 1 when initialized in its stationary distribution, which for my purposes is essentially a normalization. In this example, g solves the partial differential equation

$$\begin{aligned}&\frac{\partial g(x_1, x_2)}{\partial x}\begin{bmatrix} A_{11}x_1 + A_{12}(x_2 - 1) \\ A_{22}(x_2 - 1)\end{bmatrix} \\ &\qquad + \frac{x_2}{2}\operatorname{trace}\left(\begin{bmatrix} B_1B_1' & 0 \\ 0 & |B_2|^2\end{bmatrix}\frac{\partial^2 g(x_1, x_2)}{\partial x\,\partial x'}\right) \\ &\quad = -H_1x_1 - H_2(x_2 - 1),\end{aligned}$$

which is a special case of (9). The solution is

$$g(x_1, x_2) = -H_1(A_{11})^{-1}x_1 - [H_2 - H_1(A_{11})^{-1}A_{12}](A_{22})^{-1}(x_2 - 1).$$

The local contribution of the Brownian increment to $g(X_t)$ is

$$\sqrt{X_t^{[2]}}\big[-H_1(A_{11})^{-1}B_1 - [H_2 - H_1(A_{11})^{-1}A_{12}](A_{22})^{-1}B_2\big]\,dW_t.$$

Thus in this example, the martingale component for Y is given by

$$\begin{aligned}\widetilde{Y}_t = \int_0^t &\sqrt{X_u^{[2]}}\big[F - H_1(A_{11})^{-1}B_1 \\ &- [H_2 - H_1(A_{11})^{-1}A_{12}](A_{22})^{-1}B_2\big]\,dW_u.\end{aligned}$$

The next example allows for the local mean and the variance in the additive functional to depend on a finite-state Markov chain. This example induces rich dynamics by allowing for a large number of discrete states and gives rise to a dynamic version of a "mixture-of-normals" model.

EXAMPLE 3.4: When a Markov process has n states, the mathematical problem that we study can be formulated in terms of matrices. To model a jump process, consider a matrix $\mathbb{N}$ with all nonnegative entries as a way to encode the conditional measure $\eta(dx^*|x)$. Recall that this measure encodes both the jump intensity (the likelihood of a jump) of the underlying Markov state X and the jump distribution (conditioned on a jump, where the process will jump to). The matrix of transition probabilities for X over an interval of time t is known to be given by $\exp(t\mathbb{A})$, where

$$\mathbb{A} = \mathbb{N} - \operatorname{diag}\{\mathbb{N}\mathbf{1}_n\},$$

where $\mathbf{1}_n$ is an n-dimensional vector of 1's and $\operatorname{diag}\{\cdot\}$ is a diagonal matrix with the entries of the vector argument located in the diagonal positions. Notice in particular that $\mathbb{A}$ has only nonnegative entries in the off-diagonal positions and it satisfies $\mathbb{A}\mathbf{1}_n = \mathbf{0}_n$. This property is the local counterpart to the requirement that the entries in any row of $\exp(t\mathbb{A})$ are the transition probabilities conditioned on the state associated with the selected row; that is, $\exp(t\mathbb{A})\mathbf{1}_n = \mathbf{1}_n$. The matrix $\mathbb{A}$ is typically referred to as an intensity matrix. Then a vector Q of stationary probabilities satisfies

$$\mathbb{A}'Q = 0.$$

Construct an additive functional

$$dY_t = X_t \cdot (H\,dt + F\,dW_t) + \nu\,dt,$$

where H is a vector in $\mathbb{R}^n$, F is a matrix with n rows, and W is a multivariate standard Brownian motion that is independent of X. Then a vector Q of stationary probabilities satisfies

$$\mathbb{A}'Q = 0.$$

I restrict H so that $H \cdot Q = 0$, implying that $X_t \cdot H$ has mean zero under the Q distribution.

The process Y in this example has continuous sample paths, but both its conditional mean and its conditional variance depend on the finite-state Markov process X. We now seek a function $g(x) = G \cdot x$ such that

$$\widetilde{Y}_t = \int_0^t (X_{u-}) \cdot F\,dW_u + G \cdot X_t - G \cdot X_0 + \int_0^t X_u \cdot H\,du$$

is a martingale.[10] Now

$$G \cdot X_t - G \cdot X_0 - \int_0^t (\mathbb{A}G) \cdot X_u\,du$$

[10]The martingale is adapted to the filtration generated jointly by X and W.

and

$$\int_0^t (X_{u-}) \cdot F\, dW_u\, du$$

are martingales. For $\widetilde{Y}$ to be a martingale,

$$\mathbb{A}G + H = 0.$$

The matrix $\mathbb{A}$ is singular so there will typically be multiple solutions for G. Any such solution works, but I choose one solution so that $Q \cdot G$ is zero. In contrast to Y, the martingale $\widetilde{Y}$ does not have continuous sample paths. The permanent shock is

$$(X_{u-}) \cdot F\, dW_u + G \cdot (X_u - X_{u-}) + (X_{u-}) \cdot H\, du$$

and thus there is a jump component to the permanent shock to Y.

A nice feature of the decomposition of an additive functional is that even when nonlinearity is introduced, the sum of two additive functionals is an additive functional. Moreover, the sum of the martingale components is the martingale component for the sum of the additive functionals provided that the constructions use a common information structure.

In what follows, I will explore what impact such a component has for valuation. For the purposes of interpretation, I will scale these shocks so that their unconditional variance is proportional to the gap of time used in forming the increment, while preserving the contribution of state dependence. Of course, a structural stochastic model of the economy may well provide alternative guidance for how to identify structural shocks. I am also interested in the decomposition because it provides a comparison for a related result that follows.[11]

Prior to our development of an alternative decomposition, I discuss some limiting characterizations that will interest us.

4. LIMITING BEHAVIOR OF GROWTH OR DISCOUNTING

As a warmup to providing more complete characterizations of valuation and pricing in the presence of stochastic growth, I investigate the limiting behavior of valuation through the construction of a dominant eigenvalue. In this section, I derive the *limiting* counterpart to a risk–return trade-off in contrast to the *local* or one-period trade-offs familiar from the asset-pricing literature. When are these limits empirically relevant? While a precise answer to this question

[11] See Alvarez and Jermann (2005) for a discussion of why this type of decomposition may have intriguing links to a similar factorization that I will establish.

depends on the specific economic model under scrutiny, model builders nevertheless can only benefit from a better understanding how model components affect long-term valuation. This long-term extrapolation expands our understanding by revealing the model-induced valuation over long investment horizons.[12] In Section 8, I will show that these same methods are informative about economic models in which investor beliefs about the long-term future, including uncertainty, are important.

4.1. *Alternative Growth Rates*

I will distinguish between a *local* growth rate of a multiplicative functional M and its *long-term* or asymptotic counterpart. For the purposes of this discussion, M can be a stochastic growth functional, be a stochastic discount functional, or the product of the two. The *local* growth rate of M at $t = 0$ is defined as

$$\lambda(M)(x) = \lim_{\varepsilon \downarrow 0} \frac{E(M_\varepsilon | X_0 = x) - 1}{\varepsilon}, \tag{10}$$

provided that this limit exists. Recall that $M_0 = 1$. Shifting the analysis forward, it may be shown that

$$\lambda(M)(X_t) = \lim_{\varepsilon \downarrow 0} E\left(\frac{M_{t+\varepsilon} - M_t}{\varepsilon M_t} \middle| \mathcal{F}_t \right).$$

Since $M_t = \exp(Y_t)$ and

$$Y_t = \int_0^t \beta(X_u)\, du + \int_0^t \xi(X_{u-}) \cdot dW_u + \sum_{0 \le u \le t} \chi(X_u, X_{u-})$$

as in (2), the local growth rate is computed to be

$$\lambda(M)(x) = \beta(x) + \frac{1}{2}|\xi(x)|^2 + \int \big(\exp[\chi(y, \cdot)] - 1\big) \eta(dy|x), \tag{11}$$

using, when necessary, continuous-time stochastic calculus. Notice that direct exposure to Brownian motion risk, reflected in ξ, and jump risk, reflected in χ, contribute to this local growth rate. This growth rate is state dependent.

By contrast, define the *long-term* growth (or decay) rate as

$$\rho(M) = \lim_{t \to \infty} \frac{1}{t} \log E[M_t | X_0 = x], \tag{12}$$

[12] These limiting characterizations are valuation counterparts to steady-state relations extended to accommodate stochastic growth.

provided that this limit is well defined. Compounding has nontrivial consequences for the long-term growth rate when the local growth rate is state dependent. I will characterize this asymptotic limit and explore the relation between the local growth rate and the asymptotic growth rate. Here I am interpreting growth liberally so as to include discounting as well. What I develop in this section is also germane to the study of long-term implications of compounding of short-term discount rates that are state dependent. While characterizations of this compounding are straightforward for log-normal models and, more generally, for term-structure models, in the study of valuation it is, in fact, the co-dependence of macroeconomic growth and stochastic discounting that becomes of central interest, even more so than when the two components are studied separately.[13]

4.2. *Risk–Return Trade-Offs*

Prior to developing further some mathematical tools, let me represent continuous-time risk–return relations using the notation that I just developed. To accomplish this, I use alternative constructions of M, including ones with multiple components. The stochastic process components have explicit economic interpretations, including stochastic discount factor processes or growth processes used to represent hypothetical financial claims to be priced. My use of stochastic discount factor processes to reflect valuation is familiar from empirical asset pricing; for instance, see Hansen and Richard (1987), Cochrane (2001), and Singleton (2006). A stochastic discount factor for a given payoff horizon discounts the future and adjusts for risk when used to assign values to a future payoff. A stochastic discount factor functional assigns values to a cash flow process. Cash flows may be a consumption process that will be realized in future dates or a dividend process on an infinitely lived security. This process typically decays asymptotically. Such decay is needed for an infinitely lived security with a growing cash flow to have a finite value as in the case of equity.

For an investment horizon t, a cash flow or payoff G_t, and a stochastic discount factor S_t, the logarithm of the expected return relative to a riskless benchmark is

$$\begin{array}{ccccc} \frac{1}{t}\log E[G_t|X_0=x] & - & \frac{1}{t}\log E[S_tG_t|X_0=x] & + & \frac{1}{t}\log E[S_t|X_0=x]. \\ \text{log} & & \text{log} & & \text{log} \\ \text{expected payoff} & - & \text{price} & - & \text{riskfree return} \end{array} \tag{13}$$

The term

$$\frac{E[G_t|X_0=x]}{E[S_tG_t|X_0=x]}$$

[13]See, for instance, Alvarez and Jermann (2005) for a study of the impact of compounding stochastic discount factors over multiple investment horizons.

is the expected return on the investment over the horizon t, and the term

$$\frac{1}{E[S_t|X_0 = x]}$$

is the expected return on a riskless investment. Thus formula (13) measures the risk premium on the investment. I divide by t to facilitate comparisons across investment horizons.

I now consider two limiting cases. The first calculation essentially reproduces the continuous-time risk–return trade-off familiar from the asset-pricing literature, but uses the mathematical formulation that I have laid out. The instantaneous expected excess return takes limits of (13) as t declines to zero:

$$\text{instantaneous excess return} = \lambda(G)(x) - \lambda(SG)(x) + \lambda(S)(x). \tag{14}$$

The first two terms together give the continuous-time limiting formula for the instantaneous expected rate of return. The instantaneous riskless rate is given by $-\lambda(S)(x)$. Thus the three terms on the right-hand side of (14) give the excess expected return. By altering the exposures to risk capture by ξ and χ, we trace out a set of feasible excess returns implied by an economic model of the stochastic discount factor S. This local relation is best conceived of as an approximation, because asset trading at sufficiently high frequencies is typically altered by trading details associated with the market microstructure.

Consider now the long-term limiting counterpart. We have counterparts to all three terms:

$$\text{long-term excess return} = \rho(G) - \rho(SG) + \rho(S). \tag{15}$$

This formula takes limits of (13) as the investment horizon made arbitrarily long and, as a consequence, state dependence is absent. By changing the risk exposures, Hansen and Scheinkman (2009) suggested this as a way to build a long-term counterpart to a risk–return trade-off.

Asset-pricing models also have implications for the intermediate investment horizons, but the two calculations given in (14) and (15) provide an interesting starting point to study the term structure of risk premia for growth-rate risk.

Covariances play a prominent role in representing risk premia in asset valuation. This is reflected most directly in the instantaneous risk–return relation (14) when I abstract from the jump component. Let ξ_g be the loading of G on the Brownian increment and let ξ_s be the loading of S on the Brownian increment. Then a simple calculation shows that instantaneous expected excess return is $-\xi_g \cdot \xi_s$, which can be verified by using formulas (11) and (14). This dot product is the instantaneous covariance between $\log G$ and $-\log S$. Since ξ_g is the local exposure to risk, $-\xi_s$ can be interpreted as the vector of prices to exposure to alternative components to the multivariate Brownian increment.

Consider next a long-run counterpart reflected in (15). While the product of two multiplicative functionals is multiplicative, the growth rate of the products is not simply the sum of the growth rates; that is,

$$\rho(SG) \neq \rho(S) + \rho(G).$$

Co-dependence is important when characterizing even the limiting behavior of the product SG or any other multiplicative functionals. As we have seen from (15), the discrepancy gives a long-term notion of risk premium. If S and G happen to be jointly log normal for each t, then the long-term excess return is

$$\rho(G) - \rho(GS) + \rho(S) = \lim_{t\to\infty} \frac{1}{t} \operatorname{Cov}(-\log S_t, \log G_t).$$

While this illustrates that co-dependence plays a central role in characterizing risk premia, we will not require log-normality in what follows. Compounding even locally normal models with state dependence can lead to important departures from analogous calculations premised on log-normality.

Prior to proceeding, I comment a bit on the previous literature. The study of the dynamics of risk premia is familiar from the work of Wachter (2005), Lettau and Wachter (2007), Hansen, Heaton, and Li (2008), and Bansal, Dittmar, and Kiku (2009). Hansen, Heaton, and Li (2008) characterized the resulting limiting risk premia and the associated risk prices in a log-linear environment. Hansen and Scheinkman (2009) extended this approach to fundamentally nonlinear models with a Markov structure.

4.3. *Long-Term Entropy of Stochastic Discount Factors*

In their study of the behavior of stochastic discount factors, Alvarez and Jermann (2005), Backus, Chernov, and Zin (2011), and Backus, Chernov, and Martin (2012) constructed a one-period measure of "entropy" of the stochastic discount factor. Alvarez and Jermann (2005) motivated this measure as a generalized notion of discount factor volatility, and Backus, Chernov, and Zin (2011) connected the one-period version of this measure to a measure of relative entropy in applied mathematics. This notion of entropy is a relative discrepancy between the actual probability distribution and the so-called risk-neutral probability distribution from mathematical finance. A conditional version of their measure for investment horizon t is

$$\text{entropy} = \log E(S_t|X_0 = x) - E(\log S_t|X_0 = x),$$

where the entropy comparison is between the probability distribution over investment horizon t and a probability distribution that uses the random variable $\frac{S_t}{E(S_t|X_0)}$ to induce a change conditional distribution. The nonnegativity of this measure of entropy is a direct consequence of Jensen's inequality. The local counterpart to entropy is given by

$$\begin{aligned}
&\text{instantaneous entropy}\\
&\quad = \lim_{\varepsilon \downarrow 0} \frac{\log E(S_\varepsilon | X_0 = x)}{\varepsilon} - \lim_{\varepsilon \downarrow 0} \frac{E(\log S_\varepsilon | X_0 = x)}{\varepsilon}\\
&\quad = \lambda(S)(x) - \kappa(\log S)(x)\\
&\quad \geq 0,
\end{aligned}$$

where $\kappa(\log S)$ is the local mean of $\log S$ computed in Section 3.[14] I include the local growth rate λ on the right-hand side because

$$\lim_{\varepsilon \downarrow 0} \frac{\log E(S_\varepsilon | X_0 = x)}{\varepsilon} = \lim_{\varepsilon \downarrow 0} \frac{E(S_\varepsilon | X_0 = x) - 1}{\varepsilon} = \lambda(S)(x).$$

In the calculation of instantaneous entropy, $-\lambda(S)$ is the instantaneous short-term interest rate. For the diffusion specification,

$$\text{instantaneous entropy} = \frac{1}{2} |\xi_s(x)|^2,$$

which is one-half times the local variance of $\log S$ in Markov state x. Introducing jumps leads to an adjustment that includes higher moments as was emphasized by Backus, Chernov, and Zin (2011), and Backus, Chernov, and Martin (2012):

$$\begin{aligned}
&\text{instantaneous entropy}\\
&\quad = \frac{1}{2} |\xi_s(X_t)|^2 + \int \big(\exp[\chi(x^*, x)] - 1 - \chi(x^*, x)\big) \eta(dx^* | x).
\end{aligned}$$

The contribution of higher moments is evident in the power series expansion of the exponential. Bansal and Lehmann (1997) argued that an upper bound for the discrete-time counterpart to the term $\kappa(\log S)$ is the negative of the maximal growth rate on a portfolio return. Empirical researchers often find it convenient to average out the state dependence. Notice, in particular, that $E[\kappa(\log S)] = \nu(\log S)$, which is the coefficient on the time trend in the decomposition of Y. These calculations are meant to extend the volatility bounds on stochastic discount factors deduced by Shiller (1982) and Hansen and Jagannathan (1991) using asset market data. An empirical measure of the entropy

[14]The analysis in Section 3 applies to a general additive functional. I now modify the notation to make it clear which additive functional is being used in the computation.

bound can be constructed from historical data on portfolio returns and risk-free rates. This measure provides a way to evaluate alternative asset-pricing models.

The long-term counterpart to this notion of entropy is given by

$$\text{long-term entropy} = \rho(S) - \nu(\log S) \geq 0,$$

where $\nu(\log S)$ is the coefficient on the time trend in the additive decomposition of $\log S$. While $\nu(\log S)$ is the unconditional mean of $\kappa(S)(X_t)$, as we have already argued, $\rho(S)$ is not the mean of $\lambda(S)(X_t)$.

To relate the entropy measure to the measure of risk premia, notice that we express the finite-horizon risk premium in formula (13) as having three component parts:

$$\begin{aligned}
&\frac{1}{t}\log E[G_t|X_0 = x] - \frac{1}{t}\log E[S_t G_t|X_0 = x] + \frac{1}{t}\log E[S_t|X_0 = x] \\
&\quad = \frac{1}{t}[\log E(G_t|X_0 = x) - E(\log G_t|X_0 = x)] \\
&\qquad + \frac{1}{t}[\log E(S_t|X_0 = x) - E(\log S_t|X_0 = x)] \\
&\qquad - \frac{1}{t}[\log E(G_t S_t|X_0 = x) - E(\log G_t + \log S_t|X_0 = x)].
\end{aligned}$$

The right-hand side compares the entropies of G_t and S_t to the entropy of the product $S_t G_t$, and can be viewed as an entropy-based measure of co-dependence.[15] Since this equality holds for every investment horizon t, it also holds in limiting cases as t tends to zero or infinity.

Why are the various limits explored in this section interesting? Say that the approximations require compounding growth or discounting over an extremely long time span. Then one might worry that resulting impacts are of little consequence in applications. Even in this case, however, comparing the short-term risk–return trade-off to the long-term counterpart helps us to understand when this trade-off has an important horizon component to it. If the long-term measures exceed the instantaneous measures, then there is a potential for risk adjustments to be more pronounced over longer investment horizons. Therefore, I use these long-run risk and variation measures to amplify Fisher's (1930) conjecture about the importance of the dynamic valuation of risk over the whole prospective investment horizon. My comparison of long-term valuation mea-

[15] I thank Ian Martin for suggesting this connection. This measure of co-dependence is distinct from a conditional version of what is called *mutual information*, a concept that is used in coding and information theory.

sures to their short-term counterparts provides a starting point for understanding the full dynamics.[16]

In the next two sections, I lay out a mathematical structure that will provide support for a more refined analysis.

5. A REVEALING EXAMPLE WITH DISCRETE STATES

Prior to a more general development, I illustrate calculations by reconsidering the specification in Example 3.4 in which the dynamics are governed by a Markov chain that visits only a finite number of states. I extend the example to allow the multiplicative functional to jump with the Markov state. The resulting mathematical problem can be formulated in terms of transition matrices for valuation applicable to a discrete state space.

I characterize long-run stochastic growth (or decay) by posing and solving an approximation problem using what is called a *principal* eigenvector and eigenvalue. The principal eigenvector has only positive entries. As I will illustrate, there is a well defined sense in which this eigenvector dominates over long valuation horizons. The approximation problem that I will study more generally has its origins in what is known as the Perron–Frobenius theory of matrices.

For a multiplicative functional associated with an n-state jump process, state-dependent growth or decay rate is modeled using β and χ, where

$$\beta(x) = H \cdot x + \nu \mathbf{1}_n.$$

Recall that β dictates the growth or decay absent any jump and χ dictates how the multiplicative functional jumps as a function of the jumps in the underlying Markov process. I represent function $\exp[\chi(x^*, x)]$ as an $n \times n$ matrix $\mathbb{K}$ with positive entries and unit entries in the diagonal positions. Also, I construct a diagonal matrix $\mathbb{D}$, where the jth diagonal entry is given by the jth entry of H plus the jth diagonal entry of $\frac{1}{2}FF'$ plus ν. Form an $n \times n$ matrix

$$\mathbb{B} = \mathbb{K} \circ \mathbb{A} + \mathbb{D},$$

where $\circ$ used in the matrix multiplications denotes element-by-element (Hadamard) multiplication. This construction of $\mathbb{B}$ modifies $\mathbb{A}$ to include state-dependent growth (or decay) associated with the corresponding multiplicative functional. In particular, I exploit local normality which allows me to compute the expectation of the exponential of a normally distributed random variable by making a variance adjustment. The off-diagonal entries of $\mathbb{B}$ are all positive, but typically $\mathbb{B}\mathbf{1}_n$ is not equal to $\mathbf{0}_n$.

[16]For alternative intriguing uses of long-term growth measures of multiplicative functionals constructed from cumulative portfolio returns, see Stutzer (2003), and from stochastic discount factors and cumulative returns, see Martin (2011).

I form a family of operators, in this case matrices, indexed by the time horizon by exponentiating the matrix $\mathbb{B}: \mathbb{M}_t = \exp(t\mathbb{B})$, where the exponential of a matrix is constructed as a power series. Note that

$$\mathbb{M}_t \mathbf{f}(x) = E[\exp(Y_t)\mathbf{f}(X_t)|X_0 = x].$$

One possible choice of $\mathbf{f} = \mathbf{1}_n$, which is of interest because

$$\frac{1}{t}\log \mathbb{M}_t \mathbf{1}_n(x),$$

is the expected growth rate in $\exp(Y)$ over a time horizon t. Then conveniently

$$\mathbb{M}_t = \exp(t\mathbb{B}).$$

The date t matrix $\mathbb{M}_t$ reflects the expected growth, discounting, or the composite of both over an interval of time t. The entries of $\mathbb{M}_t$ are all nonnegative, and I presume that for some time horizon t, the entries are strictly positive. In our applications, the matrix $\mathbb{M}_t$ is typically not a probability matrix. (Columns do not sum to a vector of 1's.) Instead $\mathbb{M}_t$ reflects continuous compounding of stochastic growth or discounting over a horizon t. The matrix $\mathbb{B}$ encodes the instantaneous contributions to growth or discounting, and it *generates* the family of matrices $\{\mathbb{M}_t : t \geq 0\}$. Specifically, the vector

$$\lambda = \mathbb{B}\mathbf{1}_n$$

contains the state-dependent growth rates. In (10), I used the notation $\lambda(M)(x)$ to denote these growth rates as a function of the state realization x. I represent this function of x as a vector by letting component i denote the value of the growth rate for state i since there are only a finite number of states.

Given an $n \times 1$ vector $\mathbf{f}$, Perron–Frobenius theory characterizes limiting behavior of $\frac{1}{t}\log \mathbb{M}_t \mathbf{f}$ by first solving

$$\mathbb{B}\mathbf{e} = \rho\mathbf{e},$$

where $\mathbf{e}$ is a column eigenvector restricted to have strictly positive entries and the corresponding ρ is a real eigenvalue. Importantly, ρ is larger than the real part of any other eigenvalue of the matrix $\mathbb{B}$. This eigenvalue gives the long-term growth rate of the corresponding multiplicative functional (denoted by $\rho(M)$ in construction (12)), and this justifies my repeated use of the notation ρ.

Depending on the application, ρ can be positive or negative. In what follows, I will also need the row eigenvector, and thus I also consider the transpose problem

$$\mathbb{B}'\mathbf{e}^* = \rho\mathbf{e}^*, \tag{16}$$

where $\mathbf{e}^*$ also has positive entries.

Exponentiating a matrix preserves the eigenvectors. The new eigenvalues are the exponentials of the eigenvalues of the initial matrix. As a consequence, $\mathbb{M}_t$ has an eigenvector given by $\mathbf{e}$ and an associated eigenvalue equal to $\exp(\rho t)$. The multiplication by t implies that the magnitude of $\exp(\rho t)$ relative to the other eigenvalues of $\mathbb{M}_t$ becomes arbitrarily large as t gets large. As a consequence,

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{M}_t \mathbf{f} = \rho \mathbf{1}_n, \tag{17}$$

$$\lim_{t\to\infty} (\log \mathbb{M}_t \mathbf{f} - t\rho \mathbf{1}_n) = \log(\mathbf{f} \cdot \mathbf{e}^*)\mathbf{1}_n + \log \mathbf{e} \tag{18}$$

for any vector $\mathbf{f}$ for which $\mathbf{f} \cdot \mathbf{e}^* > 0$, where I have normalized $\mathbf{e}^*$ so that $\mathbf{e}^* \cdot \mathbf{e} = 1$. Formally, (17) defines ρ as the long-run growth rate of the family of matrices $\{\mathbb{M}_t : t \geq 0\}$. Backus, Gregory, and Zin (1989) used such a result to construct the long-term forward rate as $-\rho$, where M is the stochastic discount factor functional.[17] The logarithm of the eigenvector $\mathbf{e}$ exposes the impact of state-dependent compounding of growth or discounting over long horizons. As I illustrate in Section 6.5, this state dependence will be reflected in long-term holding period returns.

In the following section, I will give a more general statement of this approximation that allows for Markov states to be continuous and use this approximation to build a factorization of a multiplicative functional.

6. A MULTIPLICATIVE FACTORIZATION

So far, I have focused on the behavior of long-term growth or decay rates related to valuation. I turn now to extracting more information by refining the analysis. I do so in three ways. First, to accommodate continuous Markov states, I use operators in place of the matrices (Section 6.1). Following Hansen and Scheinkman (2009), I represent a family of valuation operators (indexed by the investment horizon) using a multiplicative functional. This multiplicative functional is the product of a *stochastic discount factor functional* and a *stochastic growth functional*. Second, I factor the multiplicative functional to extract the long-term growth or decay component as an exponential function of the investment horizon and to extract the long-horizon risk component as a positive martingale (Section 6.2). Roughly speaking, this factorization is the multiplicative counterpart to Section 3's additive decomposition, but there are important differences. These differences require that I adopt a Perron–Frobenius approach that generalizes the discussion of Section 5. I use the martingale component to provide an alternative representation of the valuation operators, which is convenient for long-term approximation (Sections 6.3–6.5).

[17]See their Proposition 2 for their application of Perron–Frobenius theory. They built a discrete-time specification of which $\log M_{t+1} - \log M_t$ is a function of the date $t+1$ state that evolves as a finite-state Markov chain.

Third, I characterize "risk-price" elasticities for varying investment horizons, which show how valuation changes when I change the risk exposure of stochastically growing cash flows (Section 6.6). These methods give the core ingredients for what I call dynamic value decompositions (DVD's).

6.1. *Operator Families*

Asset values are fruitfully depicted using valuation operators that assign (state-dependent) prices to future payoffs or consumption at alternative investment horizons. These operators must be linked when there is trading at intermediate dates and they are necessarily recursive. My interest is in economies with Markov characterizations.

Following Hansen and Scheinkman (2009), a central ingredient in my analysis is the construction of a family of operators from a multiplicative functional M. Formally, with any multiplicative functional M, we associate a family of operators

$$\mathbb{M}_t f(x) = E[M_t f(X_t) | X_0 = x], \tag{19}$$

indexed by t. When M has finite first moments, this family of operators is at least well defined on the space of bounded functions.[18]

Notice that I have used a multiplicative functional M to represent the operators for alternative investment horizons t. Why feature multiplicative functionals? The operator families that are of interest must satisfy the law of iterated values, which imposes a recursive structure on valuation. In the case of models with frictionless trade at all dates, this structure is enforced by the absence of arbitrage opportunities. Since I wish to accommodate macroeconomic growth in the payoff, I construct $M = SG$, where $G_t f(X_t)$ is the priced payoff.

Formula (19) depicts values at date zero. The corresponding formula for trading at date $\tau < t$ for investment horizon $t - \tau$ is

$$\mathbb{M}_{t-\tau} f(x) = E\left[\frac{M_t}{M_\tau} f(X_t) \Big| X_\tau = x\right].$$

At date zero, we could purchase the payoff $G_\tau \mathbb{M}_{t-\tau} f(X_\tau)$ at date τ and then use this payoff at date τ to purchase a claim to $G_t f(X_t)$. The date zero price of this transaction is

$$\begin{aligned} \mathbb{M}_\tau \mathbb{M}_{t-\tau} f(x) &= E\left(S_\tau G_\tau E\left[\frac{M_t}{M_\tau} f(X_t) \Big| X_\tau\right] \Big| X_0 = x\right) \\ &= E\left(M_\tau E\left[\frac{M_t}{M_\tau} f(X_t) \Big| \mathcal{F}_\tau\right] \Big| X_0 = x\right) \end{aligned}$$

[18] See Hansen and Scheinkman (2009) for a more general and explicit formulation of the domain of such operators.

$$= E[M_t f(X_t)|X_0 = x]$$
$$= \mathbb{M}_t f(x).$$

Notice that $\mathbb{M}_t f(x)$ is the date zero purchase price of the payoff $G_t f(X_t)$ without resorting to an intermediate trade. On the right-hand side of the first equation, S_τ is used to represent the date zero prices of payoffs at date τ, where the payoff is $G_\tau \mathbb{M}_{t-\tau} f(X_\tau)$. An outcome of these relations is the recursive structure of the valuation that links $\mathbb{M}_t$ to the sequential application of $\mathbb{M}_{t-\tau}$ and $\mathbb{M}_\tau$. Important to my analysis is that this recursive valuation embeds macroeconomic growth in the consumption payoffs.

The law of iterated values in a Markov setting is captured formally as a statement that the family of operators should be a semigroup.[19]

DEFINITION 6.1: A family of operators $\{\mathbb{M}_t\}$ is a (one-parameter) semigroup if (a) $\mathbb{M}_0 = \mathbb{I}$ and (b) $\mathbb{M}_t \mathbb{M}_\tau = \mathbb{M}_{t+\tau}$ for $t \geq 0$ and $\tau \geq 0$.

I now explain why I use multiplicative functionals to represent operator families. I do so because a multiplicative functional M guarantees that the resulting operator family $\{\mathbb{M}_t : t \geq 0\}$ constructed using (19) is a one-parameter semigroup.[20]

6.2. *Martingale Extraction*

I now propose a multiplicative factorization of stochastic growth or discount functionals with three components: (a) an exponential function of time, (b) a positive martingale, and (c) a ratio of functions of the Markov process at zero and t:

$$M_t = \underbrace{\exp(\rho t)}_{\text{(a) growth or decay}} \; \underbrace{\widetilde{M}_t}_{\text{(b) martingale change in probability}} \; \underbrace{\left[\frac{e(X_0)}{e(X_t)}\right]}_{\text{(c) ratio state dependence}}. \qquad (20)$$

Component (a) governs the long-term growth or decay. It is constructed from a principal eigenvalue. I will use component (b), the strictly positive martingale, to build an alternative probability measure.[21] Component (c) is the function of the Markov state that is most important over long valuation horizons.

[19] Garman (1985) proposed the use of semigroups as a valuable mathematical framework for studying asset pricing.

[20] Lewis (1998), Linetsky (2004), and Boyarchenko and Levendorskii (2007) use operator methods of this type to study the term structure of interest rates and option pricing while abstracting from stochastic growth.

[21] As an alternative approach, Rogers (1997) proposed a convenient parameterization of a martingale from which one can construct examples of multiplicative functionals of the form (20).

I build the factorization as follows. First I solve

$$\mathbb{M}_t e(x) = E[M_t e(X_t)|X_0 = x] = \exp(\rho t) e(x) \tag{21}$$

for any t, where e is strictly positive as in (16). The function e can be viewed as a *principal eigenfunction* of the semigroup with ρ being the corresponding eigenvalue.[22] Notice that for alternative values of t, I have used the same positive eigenfunction e. The associated eigenvalues are related via an exponential formula. Since equation (21) holds for any t, it can be localized by computing

$$\lim_{t\downarrow 0} \frac{\mathbb{M}_t e(x) - \exp(\rho t) e(x)}{t} = 0, \tag{22}$$

which gives an equation in e and ρ to be solved. The local counterpart to this equation is

$$\mathbb{B} e = \rho e, \tag{23}$$

where

$$\lim_{t\downarrow 0} \frac{\mathbb{M}_t e(x) - e(x)}{t} = \mathbb{B} e(x).$$

The operator $\mathbb{B}$ is the so-called *generator* of the semigroup constructed with the multiplicative functional M. It is an operator on a space of appropriately defined functions. Heuristically, it captures the local evolution of the semigroup. In the case of a diffusion model, this generator is known to be a second-order differential operator:

$$\mathbb{B} f = \left(\beta + \frac{1}{2}|\xi|^2\right) f + (\sigma \xi' + \mu) \cdot \frac{\partial f}{\partial x} + \frac{1}{2} \operatorname{trace}\left(\sigma\sigma' \frac{\partial^2 f}{\partial x \, \partial x'}\right).$$

It is convenient to express the corresponding eigenvalue equation in terms of $\log e$ after dividing the equation by e:

$$\rho = \left(\beta + \frac{1}{2}|\xi|^2\right) + (\sigma \xi' + \mu) \cdot \frac{\partial \log e}{\partial x} + \frac{1}{2} \operatorname{trace}\left(\sigma\sigma' \frac{\partial^2 \log e}{\partial x \, \partial x'}\right) + \frac{1}{2}\left(\frac{\partial \log e}{\partial x}\right)' \sigma\sigma' \left(\frac{\partial \log e}{\partial x}\right).$$

Given my interest in structural economic models of asset pricing and in products of stochastic discount factor and growth functionals, I will instead explore factorizations of prespecified multiplicative functionals.

[22]My reference to e as an eigenfunction is a bit loose because I have not prespecified a space of functions that it resides in. Instead I use the formalization of Hansen and Scheinkman (2009) that defines an eigenfunction using a martingale approach.

We have seen the finite-state counterpart to this equation in Section 5.

Typically it suffices to solve the local equation (23) to obtain a solution to (21); see Hansen and Scheinkman (2009) for a more detailed discussion of this issue. In the finite-state Markov model of Section 5, convenient and well known sufficient conditions exist for there to be a unique (up to scale) positive eigenfunction satisfying (21). More generally, however, this uniqueness will not hold. Instead I establish uniqueness from additional considerations.

Given a solution to (21), I construct a martingale via

$$\widetilde{M}_t = \exp(-\rho t) M_t \left[\frac{e(X_t)}{e(X_0)} \right],$$

which is itself a multiplicative functional. The multiplicative decomposition (20) follows immediately by letting $\widetilde{e} = \frac{1}{e}$ and solving for M in terms of $\widetilde{M}$, ρ, and $\widetilde{e}$.

6.3. *Martingale and a Change in Probability*

Why might a multiplicative martingales be of interest? A positive martingale scaled to have unit expectation is known to induce an alternative probability measure.[23] This trick is familiar from asset pricing, but it is valuable in many other contexts. Since $\widetilde{M}$ is a martingale, I form the distorted or twisted expectation

$$\widetilde{E}[f(X_t)|X_0] = E[\widetilde{M}_t f(X_t)|X_0]. \tag{24}$$

For each time horizon t, I define an alternative conditional expectation operator. The martingale property is needed so that the resulting family of conditional expectation operators obeys the law of iterated expectations. It insures consistency between the operators defined using $\widetilde{M}_{t+\tau}$ and $\widetilde{M}_t$ for expectations of random variables that are in the date t conditioning information sets. Moreover, with this (multiplicative) construction of a martingale, the process remains Markov under the change in probability measure. While (24) defines conditional expectations, I also use the notation $\widetilde{E}$ to define the corresponding unconditional expectations when a stationary distribution exists and the process is ergodic under the change of measure.

I present a method for long-term approximation where a martingale component of M is used to change the probability measure. This alternative probability measure gives me a framework for a formal study of long-term approximation on which I can use the existing toolkit for the study of Markov processes that are "stochastically stable."

[23] While I use multiplicative martingales to change the probability measure, Martin (2011) featured long-term sample path properties of a multiplicative martingale constructed from the product of cumulative returns and stochastic discount factors.

DEFINITION 6.2: The process X is *stochastically stable* under the measure $\widetilde{\cdot}$ if

$$\lim_{t\to\infty} \widetilde{E}[f(X_t)|X_0 = x] = \widetilde{E}[f(X_t)]$$

for any f for which $\widetilde{E}(f)$ is well defined and finite, and computed using a stationary distribution associated with the $\widetilde{\cdot}$ Markov transition.[24]

THEOREM 6.3: *Given a multiplicative functional M, suppose that e and ρ satisfy equation (22) and that X is stochastically stable under the $\widetilde{\cdot}$ probability measure. Then*

$$E[M_t f(X_t)|X_0 = x] = \exp(\rho t)\widetilde{E}\left[\frac{f(X_t)}{e(X_t)}\Big|X_0 = x\right]e(x).$$

Moreover,

$$\lim_{t\to\infty} \exp(-\rho t)E[M_t f(X_t)|X_0 = x] = \widetilde{E}[f(X_t)\widetilde{e}(X_t)]e(x)$$

provided that $\widetilde{E}[f(X_t)\widetilde{e}(X_t)]$ is finite where $\widetilde{e} = 1/e$.

It follows from Theorem 6.3 that once we scale by the growth rate ρ, I obtain a one-factor representation of long-term behavior. Changing the function f simply changes the coefficient on the function e. Thus the state dependence is approximately proportional to e as the horizon becomes large. For this method to justify our previous limits, we require that $f\widetilde{e}$ have a finite expectation under the $\widetilde{\cdot}$ probability measure. The class of functions f for which this approximation works depends on the stationary distribution for the Markov state of the $\widetilde{\cdot}$ probability measure and the function $\widetilde{e}$.

As I noted previously, there is an extensive set of tools for studying the stability of Markov processes that can be brought to bear on this problem. For instance, see Meyn and Tweedie (1993) for a survey of such methods based on the use of Foster–Lyapunov criteria. See Rosenblatt (1971), Bhattacharya (1982), and Hansen and Scheinkman (1995) for alternative approaches based on mean-square approximation. While there may be multiple representations of the form (20), Hansen and Scheinkman (2009) showed that there is at most *one* such representation for which the process X is stochastically stable.

The valuation dynamics are best understood in terms of this alternative probability measure as reflected in the formula

$$\log \mathbb{M}_t f(x) - \rho t = \log e(x) + \log \widetilde{E}[f(X_t)\widetilde{e}(X_t)|X_0 = x], \tag{25}$$

[24]This is stronger than ergodicity because it rules out periodic components. Ergodicity requires that time series averages converge, but not necessarily that conditional expectation operators converge. The limit used in this definition could be state by state or it could be in mean square under the change of measure.

where $\widetilde{E}$ is used to denote the expectation operator under the twisted measure, $\widetilde{e} = \frac{1}{e}$, and f is a positive function of the Markov state. Thus after adjusting for the growth (or decay) rate ρ, the implied values of a cash flow (as a function of the investment horizon) are represented conveniently in terms of the dynamics under the *twisted* probability measure. This alternative probability measure gives me a framework for a formal study of long-term approximation on which I can use the existing toolkit for the study of Markov processes that are stochastically stable.

In light of the stochastic stability, the limit as the investment horizon t becomes large ((25)) is

$$\lim_{t\to\infty}[\log \mathbb{M}_t f(x) - t\rho] = \log e(x) + \log \widetilde{E}[f(X_t)\widetilde{e}(X_t)].$$

The second term on the right-hand side computes the unconditional expectation of $f\widetilde{e}$ under the twisted probability measure, provided of course that this expectation is finite and positive. This generalizes result (18) for the finite-state Markov chain example given in Section 5. By applying this approximation to $M = SG$, I have an operational method for creating a more refined characterization of the behavior of long-term values. As is evident in this approximation, component (c) of (20) is built directly from the principal eigenfunction e. It captures state dependence and it provides a way to characterize the convergence to the limiting growth (or decay) rate ρ. The choice of f contributes a constant term $\log \widetilde{E}[f(X_t)\widetilde{e}(X_t)]$, while the principle eigenfunction e determines the state dependence.

6.4. *Change of Measure Revisited*

Mathematical finance uses a different change of measure based on a different factorization. Let M be a multiplicative functional. Following Ito and Watanabe (1965), a multiplicative supermartingale can be factored as

$$M_t = \exp\left[\int_0^t \lambda(M)(X_u)\,du\right]\widehat{M}_t,$$

where $\lambda(M)(x)$ is given by (10) in Section 4 and is the local growth rate for M, and $\widehat{M}$ is a local martingale. When the local martingale $\widehat{M}$ is a martingale, then this martingale gives an alternative change in measure. If the local growth (or decay) rate $\lambda(M)$ does not depend on the Markov state, this factorization coincides with the one that we have been using. On the other hand, if this rate is state dependent, the two decompositions differ.

Mathematical finance uses the Ito and Watanabe (1965) style factorization applied to a stochastic discount factor process S. Then $\lambda(S)(X_t)$ is the negative of the instantaneous interest rate and the change of measure is the so-called "risk-neutral measure." When interest rates vary over time, in effect they adjust for risk for finite investment horizons. This risk-neutral adjustment only eliminates the need for local risk adjustments, complicating its interpretation for characterizing asset-pricing dynamics.

6.5. *Holding-Period Returns on Cash Flows*

A return to equity with cash flows or dividend that have stochastic growth components can be viewed as a bundle or portfolios of holding-period returns on cash flows with alternative payout dates. (See Lettau and Wachter (2007) and Hansen, Heaton, and Li (2008).) As an application of the multiplicative factorization, I now extend a result obtained by Hansen, Heaton, and Li (2008) for log-normal models applied to holding-period returns on cash flows. Let $M = SG$ and consider the date t value of the payoff $G_{t+\tau} f(X_{t+\tau})$ given by

$$E\left[\frac{S_{t+\tau}}{S_t} G_{t+\tau} \middle| \mathcal{F}_t\right] = G_t \mathbb{M}_\tau[f(X_t)].$$

To construct a holding-period return between dates 0 and t, I assign a value to the constructed payoff $G_t \mathbb{M}_\tau[f(X_t)]$,

$$\begin{aligned} E\big[S_t G_t \mathbb{M}_\tau[f(X_t)] | X_0 = x\big] &= E[S_{t+\tau} G_{t+\tau} f(X_{t+\tau}) | X_0 = x] \\ &= \mathbb{M}_{t+\tau}[f(X_0)], \end{aligned}$$

where I am using the law of iterated values in conjunction with the underlying Markov specification. Thus the gross holding-period return over horizon t is

$$\frac{G_t \mathbb{M}_\tau[f(X_t)]}{\mathbb{M}_{t+\tau}[f(X_0)]}.$$

To characterize this return for large τ, apply the change in measure and represent this return as

$$\exp(-\rho t) G_t \left[\frac{e(X_t)}{e(X_0)}\right] \left(\frac{\widetilde{E}[f(X_{t+\tau}) \widetilde{e}(X_{t+\tau}) | X_t]}{\widetilde{E}[f(X_{t+\tau}) \widetilde{e}(X_{t+\tau}) | X_0]}\right).$$

The last term converges to unity as the payoff horizon τ increases, and the first two terms do not depend on τ. Thus the limiting return is

$$\exp(-\rho t) G_t \left[\frac{e(X_t)}{e(X_0)}\right]. \tag{26}$$

This limit has a cash flow component G_t and a state-dependent valuation component $\frac{e(X_t)}{e(X_0)}$. Both of these terms contribute to the return exposure to risk. Finally, there is an exponential adjustment ρ, which is, in effect, a value-based measure of duration of the cash flow G and is independent of the Markov state. When ρ is near zero, the cash flow values deteriorate very slowly as the investment horizon is increased.

The holding-period return process, (26), is itself a multiplicative functional with the same martingale component as G. Since

$$-\rho(SG) = -\rho(G) + [\rho(G) - \rho(SG)]$$

and $-\rho(G)$ offsets the growth in G, the growth rate of the limiting holding-period return process is $\rho(G) - \rho(SG)$, which is minus the logarithm of the expected long-term return.

6.6. *Perturbations and Elasticities*

Derivatives computed at a baseline configuration of parameters reveal sensitivity of a valuation model to small changes in a parameters. For instance, in Hansen, Heaton, and Li (2008), the pricing implications of a parameterized family of valuation models depend on the intertemporal elasticity of the investors. They compute derivatives as an alternative to solving the model for the alternative parameter configurations. A risk price is also a derivative. It is a marginal change in a risk premium induced by a marginal change in risk exposure. Thus a key to constructing a risk price is to parameterize the risk exposure of a hypothetical cash flow as in Hansen and Scheinkman (2010) and Borovička, Hansen, Hendricks, and Scheinkman (2011). These elasticities are the ingredients for dynamic value decompositions. Other perturbations would also be of interest.

In both of these applications, the multiplicative functionals used in constructing the semigroup depend on a model parameter. Thus I consider $M(\varepsilon)$ as a parameterized family of multiplicative functionals and analyze value implications in the vicinity of $\varepsilon = 0$. The parameter can be a preference parameter as in the work of Hansen, Heaton, and Li (2008) or it can be a parameter that governs the exposure to a source of risk on a cash flow that is to be valued. With a perturbation analysis, it is possible to exploit a given solution to a model in the study of sensitivity to model specification. Changing the parameter ε of $M(\varepsilon)$ allows me to perturb the valuations associated with this process. My choice of a *scalar* parameterization is made for notational convenience. The multivariate extension is straightforward.

In what follows, I explore the pricing implications for alternative shock exposures for cash flows that grow stochastically as in Hansen and Scheinkman (2010) and Borovička et al. (2011). While I focus on diffusion models, Borovička et al. (2011) discussed related calculations for some models with jump components. It is convenient for me to exploit the multiplicative construction by writing

$$M(\varepsilon) = MH(\varepsilon),$$

where $H(0) = 1$. Since I feature diffusion models, I suppose that $H(\varepsilon)$ can be expressed as

$$\log H_t(\varepsilon) = \varepsilon \int_0^t \xi_h(X_u)\,dW_u - \frac{\varepsilon^2}{2}\int_0^t |\xi_h(X_u)|^2\,du,$$

implying that H is a (local) martingale. I normalize the perturbation by imposing

$$E[|\xi_h(X_t)|^2] = 1.$$

The calculation that interests me is the elasticity:

$$\text{elasticity } = \frac{1}{t}\frac{d}{d\varepsilon}\log E[M_t H_t(\varepsilon)|X_0 = x]\bigg|_{\varepsilon=0}.$$

By varying ξ_h, I explore implications of changing exposures. By letting M be a stochastically growing cash flow G, I obtain measures of how the exposure changes for different investment horizons. By computing elasticities in conjunction with the corresponding expected returns,

$$\text{(27)} \qquad \text{risk-price elasticity} = \frac{1}{t}\frac{d}{d\varepsilon}\log E[G_t H_t(\varepsilon)|X_0 = x]\bigg|_{\varepsilon=0} - \frac{1}{t}\frac{d}{d\varepsilon}\log E[S_t G_t H_t(\varepsilon)|X_0 = x]\bigg|_{\varepsilon=0},$$

I infer which exposures are of most concern to investors as reflected by pricing implications of an underlying economic model. To support these calculations, Hansen and Scheinkman (2010) justified the equality

$$\frac{d}{d\varepsilon}\log E[M_t H_t(\varepsilon)|X_0 = x]\bigg|_{\varepsilon=0} = \frac{E[M_t \Delta_t|X_0 = x]}{E[M_t|X_0 = x]},$$

where Δ is the additive (local) martingale

$$\Delta_t = \int_0^t \xi(X_u)\cdot dW_u.$$

The mathematical finance literature includes calculations of the local sensitivity of prices of derivative securities to underlying parameters. The finite-horizon risk prices that interest me have the same structure as some of the calculations in this literature.[25] The specific calculation that interests me is

[25]Such derivatives are often referred to as the Greeks in the option pricing literature.

closely related to a formula in Fournié, Lasry, Lebuchoux, Lions, and Touzi (1999).[26] The basic approach described here is justified formally and extended in Hansen and Scheinkman (2010), who used methods very similar to those of Bismut (1981).[27]

To study limiting properties, I first solve the principal eigenvalue problem for $\varepsilon = 0$ and use the solution to construct a probability measure $\widetilde{\cdot}$ as we described previously. Recall that in the stochastic evolution under the twisted probability measure, dW_t becomes a multivariate standard Brownian motion with an explicit drift distortion that depends on the Markov state. With this change of measure,

$$\frac{1}{t}\frac{d}{d\varepsilon}\log E[M_t H_t(\varepsilon)|X_0 = x]\bigg|_{\varepsilon=0} = \frac{\widetilde{E}[\widetilde{e}(X_t)\Delta_t|X_0 = x]}{\widetilde{E}[\widetilde{e}(X_t)|X_0 = x]},$$

where $\widetilde{e}$ is the Perron–Frobenius eigenfunction for the semigroup constructed from M. With the change of measure, we may show that the limiting derivative as the investment horizon becomes long is

$$\lim_{u\to\infty}\frac{1}{u}\frac{d}{d\varepsilon}\log E[M_u H_u(\varepsilon)|X_0 = x]\bigg|_{\varepsilon=0} = \frac{1}{t}\widetilde{E}\Delta_t,$$

where the right-hand side can be evaluated for any choice of t. Equivalently, the limit on the right-hand side is

$$\lim_{u\to\infty}\frac{1}{u}\frac{d}{d\varepsilon}\log E[M_u H_u(\varepsilon)|X_0 = x]\bigg|_{\varepsilon=0} = \widetilde{E}[\xi_h(X_t)\cdot\widetilde{\xi}(X_t)],$$

where $\widetilde{\xi}$ is the drift of dW_t under the change of measure. Notice that this elasticity is linear in the exposure ξ_h used to construct the perturbation.

7. ADDITIVE DECOMPOSITION VERSUS MULTIPLICATIVE FACTORIZATION

Recall the additive decomposition and multiplicative factorization from previous sections,

$$Y_t = \nu t + \widetilde{Y}_t + [g(X_0) - g(X_t)],$$
$$M_t = \exp(\rho t)\times\widetilde{M}_t\times\left[\frac{e(X_0)}{e(X_t)}\right],$$

[26] See their Proposition 3.1. The results in Fournié et al. (1999) have been extended to include some specifications of jumps in Davis and Johansson (2006) with corresponding modifications of the additive functional Δ.

[27] Bismut (1981), Hansen and Scheinkman (2010), and Borovička et al. (2011) explored other specifications of the perturbation process $H(\varepsilon)$.

where $\widetilde{Y}$ is an additive martingale and $\widetilde{M}$ is a multiplicative martingale. When $M = \exp(Y)$, they appear to be related. There are, however, important differences in the study of additive and multiplicative functionals and their corresponding decompositions and factorizations. The exponential of a martingale is not a martingale and the principal eigenfunction e in the multiplicative factorization is not the exponential of the function g used in the additive decomposition:

$$\exp[g(x)] \neq e(x).$$

In this section, I start by considering an example that illustrates the difference between additive decomposition and multiplicative factorization. I then discuss how they are related in interesting ways for characterizing long-term valuation.

7.1. *They Are Different...*

Consider again Example 3.3 and recall the dynamic evolution for the stochastic volatility model:

$$dX_t^{[1]} = A_{11}X_t^{[1]}\,dt + A_{12}\big(X_t^{[2]} - 1\big)\,dt + \sqrt{X_t^{[2]}}B_1\,dW_t,$$

$$dX_t^{[2]} = A_{22}\big(X_t^{[2]} - 1\big)\,dt + \sqrt{X_t^{[2]}}B_2\,dW_t,$$

$$dY_t = \nu\,dt + H_1X_t^{[1]}\,dt + H_2\big(X_t^{[2]} - 1\big)\,dt + \sqrt{X_t^{[2]}}F\,dW_t.$$

Form

$$M_t = \exp(Y_t).$$

Guess a solution $e(x) = \exp(\alpha \cdot x)$, where

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \text{and} \quad \alpha = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}.$$

To compute $\rho(M)$, I solve a special case of (23),

$$\nu + x_1'(A_{11}'\alpha_1 + H_1') + (x_2 - 1)(A_{12}'\alpha_1 + A_{22}\alpha_2 + H_2) + \frac{1}{2}x_2|\alpha'B + F|^2 = \rho.$$

Thus the coefficients on x_1 and x_2 are zero when

$$A_{11}'\alpha_1 + H_1' = 0, \tag{28}$$

$$A_{12}'\alpha_1 + A_{22}\alpha_2 + H_2 + \frac{1}{2}|\alpha_1'B_1 + \alpha_2B_2 + F|^2 = 0.$$

I solve the first equation for α_1 and the second equation for α_2 given α_1. The solution to the first equation is

$$\alpha_1 = -(A'_{11})^{-1}H'_1.$$

The second equation is quadratic in α_2, so there may be two solutions. Specifically,

$$\begin{aligned}\alpha_2 = {} & -\left(\frac{B_2 \cdot F + A_{22}}{|B_2|^2}\right) \\ & \pm \big(|B_2 \cdot F + A_{22}|^2 - |B_2|^2(|F - H_1(A_{11})^{-1}B_1|^2 \\ & + 2H_2 - 2H_1(A_{11})^{-1}A_{12})\big)^{1/2}/|B_2|^2,\end{aligned} \tag{29}$$

provided that the term under the square root sign is positive. Notice, in particular, that this term will be positive for sufficiently small $|B_2|$. I select the "minus" solution to achieve stochastic stability (see Appendix B). Finally,

$$\rho = \nu - (A'_{12}\alpha_1 + A_{22}\alpha_2 + H_2).$$

The function e is not the exponential of the function g used in the additive martingale construction in Example 3.3.

In the case of a log-normal model in which $B_2 = 0$ and the volatility state variable $X^{[2]}$ is identically 1, there is a tight connection between the additive decomposition and the multiplicative factorization. Notice from (28) that

$$\alpha_2 = -\frac{1}{A_{22}}\left(A'_{12}\alpha_1 - H_2 - \frac{1}{2}|\alpha'_1 B_1 + F|^2\right).$$

As a consequence,

$$\rho = \nu + \frac{1}{2}|\alpha'_1 B_1 + F|^2.$$

For this example, it straightforward to verify that

$$g(x) = \alpha_1 \cdot x_1$$

and hence $e(x) \propto \exp[g(x)]$. While the exponential of a martingale is not a martingale, in this case, the exponential of the additive martingale will become a martingale provided that we multiply the additive martingale by an exponential function of time. This simple adjustment exploits the log-normal specification that

$$\widetilde{M}_t = \exp\left(\widetilde{Y}_t - \frac{t}{2}|F - HA^{-1}B|^2\right)$$

is a martingale. See Hansen, Heaton, and Li (2008) for a more complete discussion of such results for log-normal models and applications to asset pricing.

7.2. … *but Related*

While an additive decomposition is fundamentally different from a multiplicative factorization, there is an interesting connection. Consider two additive functionals that have the same linear growth and additive martingale components. Thus we may write

$$\exp\left(Y_t^{[1]}\right) = \exp(\nu t)\exp(\widetilde{Y}_t)\left(\frac{\exp[g^{[1]}(X_0)]}{\exp[g^{[1]}(X_t)]}\right),$$

$$\exp\left(Y_t^{[2]}\right) = \exp(\nu t)\exp(\widetilde{Y}_t)\left(\frac{\exp[g^{[2]}(X_0)]}{\exp[g^{[2]}(X_t)]}\right).$$

While we know that $\exp(\widetilde{Y})$ is not a martingale, the existence of this common component remains informative.

THEOREM 7.1: *Consider two additive functionals $Y^{[1]}$ and $Y^{[2]}$ that have the same linear growth component and additive martingale component $\widetilde{Y}$. Then $M^{[1]} = \exp(Y^{[1]})$ and $M^{[2]} = \exp(Y^{[2]})$ share the same exponential growth rate and multiplicative martingale component, and this martingale component is given by the martingale component of $\widetilde{Y}$. Conversely, suppose that $M^{[1]}$ and $M^{[2]}$ share the same exponential growth rate and the same multiplicative component $\widetilde{M}$. Provided that $\log \widetilde{M}$ has a well defined additive decomposition, and $\log e^{[1]}$ and $\log e^{[2]}$ have finite second moments, $\log M^{[1]}$ and $\log M^{[2]}$ have a common linear growth component and a common additive martingale component.*

Thus the additive martingale decomposition used to identify the permanent shock to $\log M$ identifies multiplicative functionals with common multiplicative components and growth components. For instance, the valuation of two cash flows with the same permanent components identified via an additive decomposition have long-term values that are approximately the same. Analogously, two stochastic discount factor processes whose logarithms share the martingale component have the same limiting valuation implications.

I now suggest what it means for there to be temporary components to stochastic discount factors and hence to their assignment of values. Consider a benchmark valuation model represented by a stochastic discount factor $M = S$ or the product of a stochastic discount functional and a reference growth functional $M = SG$. I ask, "What modifications have transient implications for valuation?" Given an M implied by a benchmark valuation model, recall our additive decomposition

$$\log M_t = \eta t + \widetilde{Y}_t - g(X_t) + g(X_0),$$

where $\widetilde{Y}$ is an additive martingale, and our multiplicative factorization (20),

$$M_t = \exp(\rho t)\widetilde{M}_t \frac{e(X_0)}{e(X_t)},$$

where $\widetilde{M}$ is a multiplicative martingale. Moreover, suppose that under the associated probability measure, X satisfies a stochastic stability condition (Definition 6.2).

Consider an alternative model of the form

$$(30) \qquad M_t^* = M_t \frac{f^*(X_t)}{f^*(X_0)}$$

for some f^*, where M is used to represent a benchmark model and M^* is an alternative model. As argued by Bansal and Lehmann (1997) and others, a variety of asset pricing models can be represented like this with the time-separable power utility model used to construct M for a given process in aggregate consumption. Function f^* may be induced by changes in the preferences of investors such as habit persistence or social externalities. I will illustrate such representations in Section 8. In light of (30), a candidate factorization for M^* is

$$M_t^* = \exp(\rho t)\widetilde{M}_t \frac{e(X_0)f^*(X_t)}{e(X_t)f^*(X_0)}.$$

The exponential and martingale components remain the same as for the factorization of M. This gives me an operational notion of a *transient* change in valuation.

DEFINITION 7.2: The difference in the semigroups associated with M^* and M is *transient* if their multiplicative factorizations share the same exponential growth (or decay) component $\exp(\rho t)$ and the same martingale component $\widetilde{M}$ for which the process X is stochastically stable under the implied change in probability measure.

When the difference between the semigroups associated with M and M^* is transient, long-term components to the additive decomposition and multiplicative factorization remain the same. The first conclusion requires that

$$E\big[[\log f^*(X_t)]^2\big] < \infty.$$

Under this restriction, we may form the additive decomposition for $\widetilde{M}$ by first constructing the additive decomposition for M and then replacing g with

$$g^*(x) = g(x) - \log f^*(x).$$

Consider next the multiplicative factorization. I will illustrate such representations in Section 8. In light of (30), a candidate factorization for M^* is

$$M_t^* = \exp(\rho t)\widetilde{M}_t \frac{e(X_0) f^*(X_t)}{e(X_t) f^*(X_0)}.$$

The exponential and martingale components remain the same as for the factorization of M. The principal eigenfunctions, however, can be different. For instance, the principal eigenfunction for M^* given by (30) is

$$e^* = \frac{e}{f^*}.$$

The difference between e and e^* alters the family of transient functions, because for f to be transient for M^*,

$$\widetilde{E}[f(X_t)\widetilde{e}(X_t) f^*(X_t)] < \infty.$$

In particular, this restriction depends on f^*. Thus the collection of functions for which the long-term approximation methods are applicable is altered, although the change of probability measure remains the same.

With these qualifications, consider two models with stochastic discount factors S and S^*. Provided that the semigroups associated with these are transient, then the semigroups associated with SG and S^*G are transient and the difference in the long-term risk premium vanishes as the investment horizon becomes large for the stochastically growing cash flow G:

$$\begin{aligned}\text{long-term excess return} &= \rho(G) - \rho(SG) + \rho(S) \\ &= \rho(G) - \rho(S^*G) + \rho(S^*).\end{aligned}$$

Moreover,

$$\text{long-term entropy} = \rho(S) - \eta(\log S) = \rho(S^*) - \eta(\log S^*).$$

Alternatively, consider two stochastically growing cash flows G and G^*, where $\log G$ and $\log G^*$ have a common time trend and martingale component. Then the semigroups associated with SG and SG^* have transient differences. Since their long-term risk exposure is approximately the same, they have the same long-term risk prices. Again the long-term excess returns will be the same for the two cash flows.

8. RECURSIVE UTILITY AND SENTIMENTS ABOUT THE FUTURE

In this section, I do two things. First I use a multiplicative factorization to provide a novel characterization of the impact of recursive utility on asset valuation. I obtain this characterization by exploring parameter configurations that magnify the impact of sentiments about the future of the macroeconomy in the valuation of macroeconomic growth and its volatility. After establishing this characterization, I show how to use the apparatus of dynamic value decompositions to compare two models: a model in which investors have power utility preferences for which the forward-looking channel is absent and a model with recursive utility as featured in Kreps and Porteus (1978) and Epstein and Zin (1989), where the forward-looking channel can be potent.[28]

8.1. *Continuation Values in Discrete Time*

By examining continuation values, I explore the impact on stochastic discount factors computed as shadow prices of an underlying consumption process fit to time series data. This recursive utility specification gives a potential role for forward-looking sentiments to be present in even short-term risk return trade-offs, a point featured by Bansal and Yaron (2004). While there are well known formulations of recursive utility in continuous time, for pedagogical reasons I initiate my discussion in discrete time. Consider a discrete-time specification with grid points spaced distance ε apart from one another. For instance, when $\varepsilon = 2^{-j}$, I envision a sequence of time intervals that are continually sliced in half as j increases. I use the homogeneous-of-degree-1 aggregator specified in terms of current-period consumption C_t and the continuation value

$$V_t = \left[\varepsilon(\mathsf{r}C_t)^{1-\theta} + \exp(-\delta\varepsilon)[\mathcal{R}_t(V_{t+\varepsilon})]^{1-\theta}\right]^{1/(1-\theta)},$$

where $\frac{1}{\theta}$ is the elasticity of intertemporal substitution and

$$\mathcal{R}_t(V_{t+\varepsilon}) = \left(E[(V_{t+\varepsilon})^{1-\gamma}|\mathcal{F}_t]\right)^{1/(1-\gamma)}$$

adjusts the continuation value $V_{t+\varepsilon}$ for risk. The parameter r does not alter preferences, but simply scales the continuation values. I include it to allow me to choose r in a convenient manner when I take limits. Next I divide by C_t to obtain

$$\frac{V_t}{C_t} = \left[\varepsilon\mathsf{r}^{1-\theta} + \exp(-\delta\varepsilon)\left[\mathcal{R}_t\left(\frac{V_{t+\varepsilon}}{C_{t+\varepsilon}}\frac{C_{t+\varepsilon}}{C_t}\right)\right]^{1-\theta}\right]^{1/(1-\theta)}, \tag{31}$$

which exploits the aggregator function's homogeneity of degree 1. In what follows, I will consider infinite-horizon specifications, which lead me to solve a fixed-point problem. Thus I will explore the construction of the continuation value V_t as a function of $C_t, C_{t+\varepsilon}, C_{t+2\varepsilon}, \ldots$.

[28]The origins of the recursive utility model are in Koopmans (1960).

The corresponding intertemporal marginal rate of substitution over an interval ε is

$$(32)\qquad \frac{S_{t+\varepsilon}}{S_t} = \exp(-\varepsilon\delta)\left(\frac{C_{t+\varepsilon}}{C_t}\right)^{-\theta}\left[\frac{V_{t+\varepsilon}}{\mathcal{R}_t(V_{t+\varepsilon})}\right]^{\theta-\gamma}$$
$$= \exp(-\varepsilon\delta)\left(\frac{C_{t+\varepsilon}}{C_t}\right)^{-\gamma}\left[\frac{\dfrac{V_{t+\varepsilon}}{C_{t+\varepsilon}}}{\mathcal{R}_t\left(\dfrac{V_{t+\varepsilon}}{C_{t+\varepsilon}}\dfrac{C_{t+\varepsilon}}{C_t}\right)}\right]^{\theta-\gamma}.$$

Absent market frictions, when I evaluate this intertemporal marginal rate of substitution at the equilibrium consumption, I have a discrete-time specification of the stochastic discount factor between dates t and $t+\varepsilon$. This formula reproduces the representation of valuation used by Rubinstein (1976) and Lucas (1978) when $\gamma = \theta$, but it contains a forward-looking adjustment reflected in the ratio of the continuation value to its risk-adjusted counterpart as in the first line of (32)

In what follows, I will take a "Lucas (1978) type approach" by positing a process for aggregate consumption, without formally modeling production via the accumulation of capital. This will allow me to feature the impact of changing investor preferences on prices, although, in general, we expect these changes to have an impact on consumption outcomes. More generally, we may envision the consumption process that I prescribe below as the equilibrium outcome of an economy in which consumption is endogenously determined.

Let consumption C be a multiplicative functional. Such functionals are initialized at 1. For notational convenience, I will maintain this specification, but it is straightforward to change the initialization. Given the Markov specification for consumption and the homogeneity of the utility recursion, I consider continuation values that can be depicted as

$$\frac{V_t}{C_t} = h(X_t).$$

Then from (31), the fixed-point equation for infinite-horizon valuation is

$$(33)\qquad h(x) = \Bigg(\varepsilon \mathrm{r}^{1-\theta} + \exp(-\delta\varepsilon)$$
$$\times\left[E\left([h(X_{t+\varepsilon})]^{1-\gamma}\left(\frac{C_{t+\varepsilon}}{C_t}\right)^{1-\gamma}\Big| X_t = x\right)\right]^{(1-\theta)/(1-\gamma)}\Bigg)^{1/(1-\theta)}.$$

The next subsection shows a close relation between this fixed-point problem and a principal eigenvalue problem.

8.2. *An Interesting Limit*

In this subsection, I explore the range of allowable discounting, through the parameter δ, in an infinite horizon. To allow δ to attain its limit value, I have to adjust the scale r appropriately so that the limiting continuation value is positive yet finite.[29] This exercise has a clear interpretation: I vary δ and thereby alter the forward-looking channel on continuation values in the recursive utility specification. As I have already emphasized, these continuation values can be important contributors to the stochastic discount factors used to represent asset values. As a consequence, even short-term risk pricing will be maximally affected by beliefs about future growth rates in the macroeconomy and the accompanying uncertainty. More generally, the continuation values are a mechanism through which sentiments about the future matter in asset valuation. The limit I characterize is more than an idle curiosity: empirical investigations often target the forward-looking preference component as a way to enhance risk prices as a way to magnify risk premia.

In what follows, I assume that $\gamma > 1$ and consider the limit as $(\mathsf{r})^{1-\theta}$ decreases to zero and δ tends to a conveniently chosen limit. Later I will be more precise about this limit. To explore this limit, I use the multiplicative functional $C^{1-\gamma}$ to construct

$$\mathbb{V}_t f(x) = E\big(\exp[(1-\gamma)\log C_t] f(X_t) | X_0 = x\big).$$

Suppose that

$$\mathbb{V}_t e(x) = \exp(\rho t) e(x) \tag{34}$$

for some positive function e and some scalar ρ. Choose e so that the associated martingale

$$\widetilde{M}_t = \left(\frac{C_t}{C_0}\right)^{1-\gamma} \frac{e(X_t)}{e(X_0)} \exp(-t\rho)$$

induces a change of probability measure with stochastically stable dynamics (Definition 6.2).[30] By solving equation (34), I also solve a limiting version of equation (33) provided that I also set

$$\delta = \rho \frac{1-\theta}{1-\gamma}.$$

[29]The mathematical characterization that follows is very similar to that of Runolfsson (1994), who studied ergodic risk-sensitive control problems using eigenfunction methods. Our analysis differs from that of Runolfsson because we include stochastic growth in our specification.

[30]In this and some other formulas, I include a division by C_0, even though I stipulated that C is a multiplicative functional that is assumed to be 1 at date zero. The assumption that $C_0 = 1$ is essentially a normalization. Instead, I could have started more generally by assuming that C scaled by C_0 is a multiplicative functional. My division by C_0 in some formulas allows for this more general starting point.

The limiting continuation value satisfies

$$\frac{V_t}{C_t} = h(X_t) = e(X_t)^{1/(1-\gamma)}.$$

This statement is too casual. Eigenfunctions are only defined up to scale in contrast to value functions. Notice, however, that by changing r, I alter the scale of the value function. Thus by choosing r for alternative values of δ, I can achieve a scale normalization comparable to that used for an eigenfunction. A formal approximation argument along these lines is beyond the scope of this paper, although some additional discussion follows. Prior to this discussion, I study pricing using this limit.

THEOREM 8.1: *Suppose that C is a multiplicative functional and that there is a stochastically stable multiplicative martingale representation for $C^{1-\gamma}$ constructed using an eigenfunction e and eigenvalue ρ. Then the limiting stochastic discount factor is*

$$S_t = \exp(-t\rho)\left(\frac{C_t}{C_0}\right)^{-\gamma}\left[\frac{e(X_t)}{e(X_0)}\right]^{(\theta-\gamma)/(1-\gamma)}.$$

Moreover, SC is a multiplicative martingale when $\theta = 1$.

Notice that with the theorem, I have illustrated, or perhaps more accurately asserted, a link between the construction of a value function for the recursive utility model and a Perron–Frobenius eigenfunction. This eigenfunction contributes what I called in Section 7 a transient model component relative to a power utility specification of a Rubinstein (1976), Lucas (1978), or Breeden (1979) model with power $1-\gamma$ (or, equivalently, to a preference specification in which $\theta = \gamma$). The parameter θ, the inverse of the intertemporal elasticity of substitution, contributes to the transient term

$$\left[\frac{e(X_t)}{e(X_0)}\right]^{(\theta-\gamma)/(1-\gamma)}$$

and can have important consequences for short-term risk pricing. It is well known that recursive utility models with large values of γ can have substantially different implications for short-term risk pricing and interest rates when $\gamma = \theta$ versus when $\theta = 1$, because of the impact of the intertemporal elasticity of substitution (IES).

I add some formality to this discussion by first defining

$$\zeta = \frac{1-\gamma}{1-\theta}.$$

I presume that $\theta \neq 1$ in what follows because the $\theta = 1$ case requires a separate argument. Rewrite (33) as

$$h(x)^{1-\gamma} = \Bigg(\varepsilon \mathsf{r}^{1-\theta} + \exp(-\delta\varepsilon) \times \left[E\left([h(X_{t+\varepsilon})]^{1-\gamma}\left(\frac{C_{t+\varepsilon}}{C_t}\right)^{1-\gamma} \Big| X_t = x\right)\right]^{1/\zeta}\Bigg)^{\zeta}.$$

While the function h depends on the parameter r, this parameter does not alter the underlying preferences of the investor. Consider the limiting equation as $(\mathsf{r})^{1-\theta}$ declines to zero:

$$h(x)^{1-\gamma} = \exp(-\delta\zeta\varepsilon)\left[E\left([h(X_{t+\varepsilon})]^{1-\gamma}\left(\frac{C_{t+\varepsilon}}{C_t}\right)^{1-\gamma} \Big| X_t = x\right)\right]. \tag{35}$$

Notice that (35) is a special case of the Perron–Frobenius eigenvalue problem considered in (34) provided that

$$\delta = \frac{\rho}{\zeta}.$$

The rate ρ will often be negative, as it is the growth rate of $C^{1-\gamma}$ and $\gamma > 1$. In this case, the restriction on δ is positive when $\theta < 1$ and negative when $\theta > 1$. However, there are examples in which ρ becomes positive for large γ because of the role of cumulative impact of stochastic volatility over the long run. For this limit equation, h and the stochastic discount factor functional are those given by Theorem 8.1.

Both of these formulas have direct extensions to continuous time. Duffie and Epstein (1992) provided a general treatment of representing recursive preferences in continuous time. For our focal parameterization, the continuous-time counterpart to fixed-point equation (33) is

$$-\zeta \mathsf{r}^{1-\theta} h(x)^{\theta-\gamma} + \delta\zeta h(x)^{1-\gamma} = \lim_{\varepsilon \downarrow 0} \frac{\mathbb{V}_\varepsilon(h^{1-\gamma})(x) - h(x)^{1-\gamma}}{\varepsilon}$$

(see Appendix C).

To understand better why the conclusion of Theorem 8.1 is a limiting case, use the change-of-probability measure associated with the Perron–Frobenius eigenfunction to write

$$\exp(-\delta\varepsilon)\left[\mathcal{R}_t\left(\frac{V_{t+\varepsilon}}{C_{t+\varepsilon}}\frac{C_{t+\varepsilon}}{C_t}\right)\right]^{1-\theta} = \exp(-\delta\varepsilon)\exp\left(\frac{\rho\varepsilon}{\zeta}\right)e(x)^{\zeta}\left[\widetilde{E}\left([h(X_{t+\varepsilon})]^{1-\gamma}e(X_{t+\varepsilon})^{-1}|X_t = x\right)\right]^{1/\zeta}.$$

Let

$$g(x) = h(x)^{1-\gamma} e(x)^{-1}.$$

Then the fixed-point equation used to construct continuation values as a function of the Markov state is

$$g(x) = \left(\varepsilon \mathrm{r}^{1-\theta} e(x)^{-1/\zeta} + \exp\left[-\left(\delta - \frac{\rho}{\zeta} \right) \varepsilon \right] \times \left[\widetilde{E}\big([g(X_{t+\varepsilon})] | X_t = x\big) \right]^{1/\zeta} \right)^{\zeta}. \tag{36}$$

This transformed recursion has absorbed growth into a conveniently chosen change of measure. More can be done with the equation. Hansen and Scheinkman (2011) used moment restrictions on e under the change of probability measure in conjunction with Jensen's inequality to establish the existence of an "interesting" fixed point to equation (36) for g. Thus, by establishing the existence of a solution to the eigenvalue equation (34), Hansen and Scheinkman (2011) also established the existence of a solution to equation (36) provided that

$$\delta > \frac{\rho}{\zeta} = \rho \frac{1-\theta}{1-\gamma}$$

in addition to moment restrictions. The resulting bound on δ extends the analysis of Kocherlakota (1990), who considered the special case in which $\theta = \gamma$.

8.3. *An Example Economy*

I now produce calculations to illustrate the impact of risk aversion embedded over alternative investment horizons. Thus I show how "the manner in which risk operates upon time preference will differ… according to the particular periods in the future to which the risk applies," the challenge mentioned by Fisher (1930). I base these calculations on a parameterization of consumption dynamics due to Bansal and Yaron (2004). I use the continuous-time specification from Hansen, Heaton, Lee, and Roussanov (2007) designed to approximate the discrete-time model of Bansal and Yaron (2004). Applying the specification in Example 3.3, I study a parameterized model of the consumption dynamics:

$$dX_t^{[1]} = -0.021 X_t^{[1]}\,dt + \sqrt{X_t^{[2]}}\,[0.00031 \quad -0.00015 \quad 0]\,dW_t,$$

$$dX_t^{[2]} = -0.013\big(X_t^{[2]} - 1\big)\,dt + \sqrt{X_t^{[2]}}\,[0 \quad 0 \quad -0.038]\,dW_t,$$

$$dY_t = 0.0015\,dt + X_t^{[1]}\,dt + \sqrt{X_t^{[2]}}\,[0.0034 \quad 0.007 \quad 0]\,dW_t.$$

The first component of the state vector is the state-dependent component to the conditional growth rate; the second component is a volatility state. In constructing this example, I configure the shocks so that the first one is the "permanent shock" identified using the methods from Section 3, normalized to have a unit standard deviation. The second shock is a so-called temporary shock, which by construction is uncorrelated with the permanent shock and also has a unit standard deviation. The third shock alters the volatility.

In applications, it is common to assume a large value of the risk aversion parameter γ, even larger than the value of $\gamma = 8$ that I use here. I find that an attractive alternative justification is to interpret this parameter as revealing investors' concern about robustness to model misspecification. This link appeals to insights from both the control theory and the economics literature; see, for instance, the control theory papers of Jacobson (1973), Whittle (1990), Hansen and Sargent (1995), and Petersen, James, and Dupuis (2000) or the economics papers of Anderson, Hansen, and Sargent (2003), Maenhout (2004), and Hansen, Sargent, and Turmuhambetova (2006). While I applaud a serious discussion of the calibration of such an essential parameter, given space constraints, I send interested readers to the literature that I referenced.

I report the price elasticities for the power utility model ($\theta = \gamma$) and for the recursive utility model for three different choices of $\theta = \{0.5, 1, 1.5\}$ and $\gamma = 8$. These formulas are based on (27), with the growth functional given by $G = C$. Figure 1 compares risk-price elasticities for the $\theta = \gamma$ specification to those for the $\theta = 1$ specification. Since $\theta = \gamma$ preferences coincide with the familiar time-separable power utility preference specification, I think of the resulting instantaneous risk-price elasticities as those that emerge from a (special case of) the Breeden (1979) model. Breeden (1979) focused on these instantaneous elasticities, while my interest is in the entire term structure of such prices. Figure 1 also explores the interquartile range for the elasticities as a function of the initial volatility state.

By design, the two specifications have the same limiting risk-price elasticities, but for short investment horizons there are pronounced differences. (See Theorem 8.1.) The forward-looking contribution to the stochastic discount factor induces the permanent shock to have a much more pronounced impact on the risk prices than the temporary shock. Under this parameterization, the volatility shock has modest prices elasticities. The magnitude differences in the price elasticities are evident in the changes in the vertical axes across the three panels of Figure 1. The primary impact of stochastic volatility is to induce fluctuations over time in the price elasticities as is reflected in the quartiles. The recursive utility specification results in a flat trajectory for the risk-price elasticities for the permanent shock, in contrast to those for the power utility model. Since the impact of the permanent shock on consumption takes time to build, the elasticities are eventually large for the growth-rate shock under power utility. In contrast, the trajectory is large at the outset for the recursive utility model. Figure 2 shows that changing θ from 0.5 to 1.5 has little consequence for the

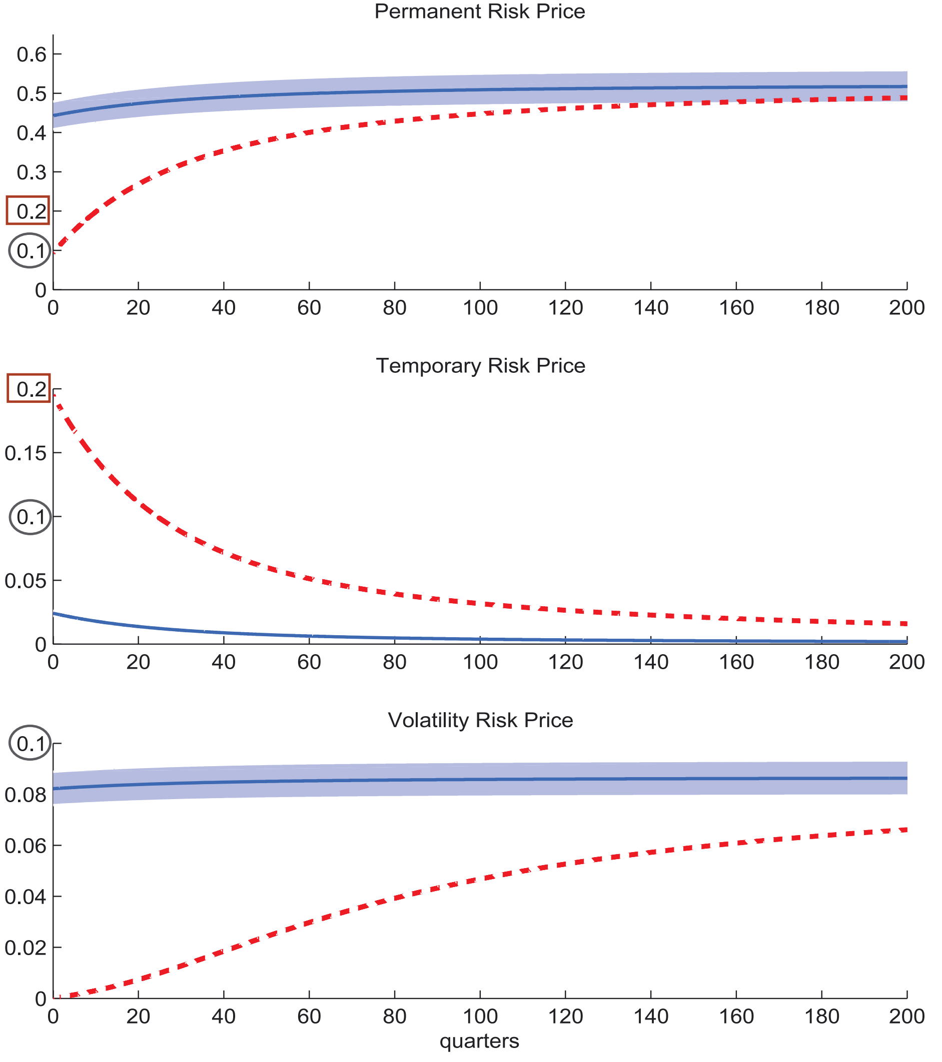

FIGURE 1.—Risk-price elasticities as a function of the investment horizon for each of the three shocks. The numbers depicted on the vertical axis are annualized. The dashed curves denote the implied prices when investors have expected utility preferences ($\gamma = \theta$), and the solid curves denote the implied prices when investors have recursive utility preferences with an IES $= 1$. The shaded region gives the interquartile range constructed from the stationary distribution for the state vector. I set the risk aversion parameter $\gamma = 8$ ($\delta = 0$) in generating these curves. The squares and circles are provided to show how the scaling of the vertical axes changes across panels.

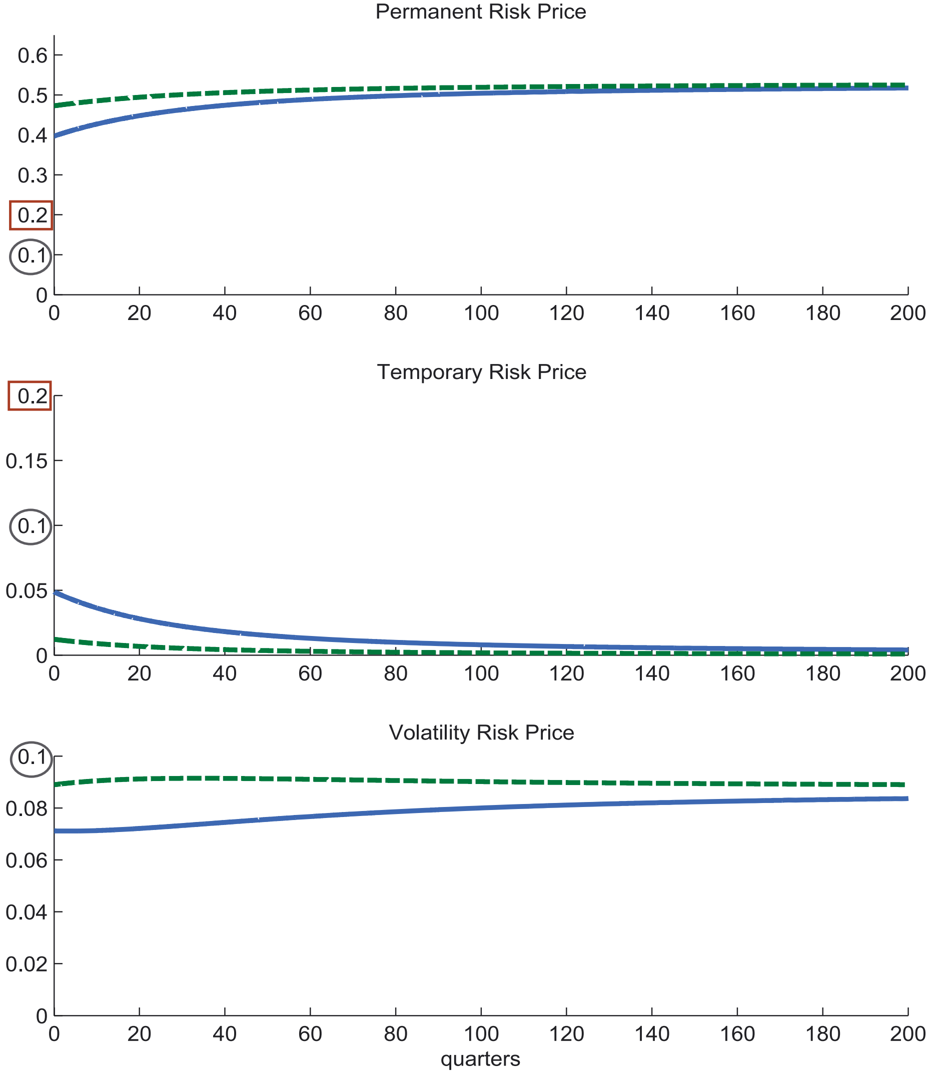

FIGURE 2.—Risk prices as a function of the investment horizon for each of the three shocks. The numbers depicted on the vertical axis are annualized. The solid curves depict the implied prices when investors have recursive utility preferences with an IES $= 0.5$ ($\theta = 2$, $\delta = -0.00033$) and the dashed curves denote the implied prices when investors have recursive utility preferences with an IES $= 2$ ($\theta = 0.5$, $\delta = 0.00016$). I set the risk aversion parameter $\gamma = 8$ in generating these curves. The squares and circles are provided to show how the scaling of the vertical axes change across panels.

prices. Altering this parameter has more important consequences for the real term structure of interest and for how the wealth consumption ratio responds to shocks.[31]

In summary, Figures 1 and 2 give a dynamic characterization of (a) the impact of risk aversion over alternative investment horizons and (b) the impact of risk prices on risk premia. They display the impact of the forward-looking channel embedded in recursive utility, which is especially prominent for "permanent shocks" to aggregate consumption. This same prominence shows up when $\gamma = \zeta$ only for long-investment horizons.

These calculations deliberately extrapolate value implications beyond the support of the data by looking at pricing implications for hypothetical cash flows at different horizons. In this sense, I am using the models as "structural." These pricing calculations are of direct interest and they are informative for welfare cost calculations using the methods in Hansen, Sargent, and Tallarini (1999) and Alvarez and Jermann (2004). To feature the role of preferences, I hold fixed the consumption dynamics when I make comparisons across specifications. I do not mean to preclude interest in other approaches such as in Hansen, Sargent, and Tallarini (1999) and Tallarini (2000) that explicitly introduce capital accumulation into the analysis. In these examples, the more primitive starting point has only minor implications for risk pricing.

Motivated by the work of Alvarez and Jermann (2005), Backus, Chernov, and Martin (2012), and Backus, Chernov, and Zin (2011), I also report entropy measures for the stochastic discount factors for alternative investment horizons. In contrast to the localization method I used to construct elasticities, these measures are global. They depend on all of the shocks and they abstract from the interaction between growth and discounting. The top panel of Figure 3 compares power utility and recursive utility. Like the risk-price elasticities, the entropy measures are much less sensitive to the investment horizon under the recursive utility specification than under power utility. The bottom panel of Figure 3 shows that the impact of changing the intertemporal elasticity parameter θ between 0.5 and 1.5 is small.

9. CONCLUSION

Decompositions of additive functionals have proved valuable in macroeconomic time series as an aid in identifying shocks and quantifying their impact. The increments to the martingale components of these decompositions are the permanent shocks. In this paper, I have considered an alternative decomposition. To support a dynamic value decomposition (DVD), I featured multiplicative factorizations of stochastic discount and growth functionals. These factorizations allowed me to

(a) characterize a long-term risk–return relation

[31] See Hansen et al. (2007) for one discussion of this second point.

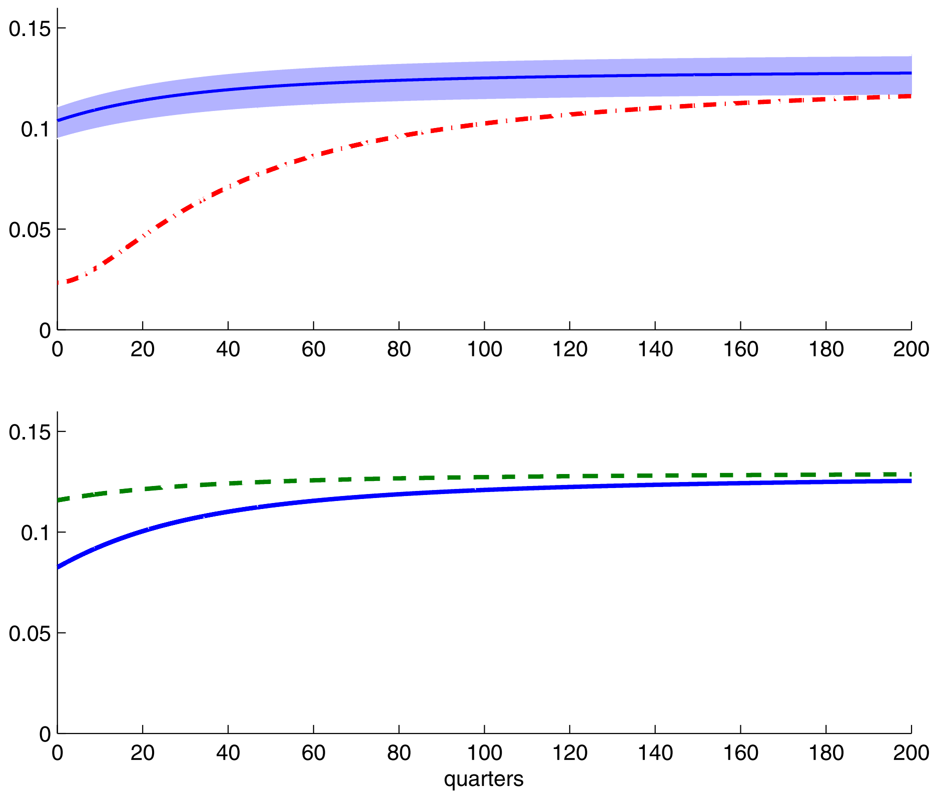

FIGURE 3.—Conditional entropy of the stochastic discount factor as a function of the investment horizon. The numbers depicted on the vertical axis are annualized. In the top panel, the dashed–dotted curve denotes the implied prices when investors have expected utility preferences ($\theta = \gamma$), and the solid curve denotes the implied prices when investors have recursive utility preferences with an IES = 1. I set the risk aversion parameter $\gamma = 8$ in generating these curves. The shaded region gives the interquartile range constructed from the stationary distribution for the state vector. In the bottom panel, the dashed curve depicts the entropies when investors have recursive utility preferences with an IES = 0.5 ($\theta = 2$) and the solid curve denotes the entropies when investors have recursive utility preferences with an IES = 2 ($\theta = 0.5$).

(b) construct risk prices for alternative investment horizons and characterize their long-term behavior

(c) compare implications for valuation of alternative structural economic models.

The methods I described require a "structural" model because they extrapolate value implications by featuring the pricing of synthetically constructed martingale cash flows. Valuation of such cash flows reveals the dynamics of risk prices. While local risk prices are familiar in the literature on asset pricing, my ambition was to explore the entire term structure of such prices. To support a DVD, I produced dynamic risk-price elasticities that measure the impact of pricing over alternative investment horizons. These methods have been extended by Borovička et al. (2011) to produce "shock price" elasticities that are the valuation counterparts to impulse response functions used in economic dynamics. While I have featured risk-price dynamics, DVD methods have other applications. For instance, consumption or cash flows will typically have different risk exposures at alternative investment horizons. The methods suggested here provide a way to measure such exposure elasticities.

While the examples that I have studied feature the role of a forward-looking channel in recursive utility preferences, Borovička et al. (2011) have used DVD methods to study asset-pricing models with other specifications of investor preferences. They include an analysis of a model with long-lasting consumption externalities in preferences of the type featured by Campbell and Cochrane (1999) and others. Similar analyses could be applied to some equilibrium models with market frictions. The solvency constraint models of Luttmer (1992), Alvarez and Jermann (2000), and Lustig (2007) have the same multiplicative martingale components as the corresponding representative consumer models without market frictions. While suggestive, a formal study of the type I have just presented for other models would reveal the precise nature of the transient modification to stochastic discount factors induced by market imperfections.[32] Applying DVD methods to the disaster-recovery models of Rietz (1988), Barro (2006), and Gourio (2008) could expand on the comparisons made across specifications of the consumption dynamics. Gourio (2008) showed that adding recoveries following disasters has an important impact on local (one-period) risk premia. Recoveries make the consequences of disasters "transient": specification changes such as those reported in Gourio (2008) could have important consequences for the entire term structure of risk prices.

To conclude, I want to be clear on two matters. First, while a concern about the role of economics in model specification is a prime motivator for this analysis, I do not mean to shift focus exclusively on the limiting characterizations. Specifically, my analysis of long-term approximation in this paper is not meant to remove discussions of transient implications off the table. Instead, I mean to add some clarity into our understanding of how valuation models work by understanding better which model levers move which parts of the complex machinery. As I showed in two examples from the asset-pricing literature, the initial points in the risk-price elasticity trajectories—the local risk prices—can be far from their limits. Moreover, the outcome of the analysis is informative even if it reveals that some models *blur* the distinction between permanent and transitory model components.

Second, while my discussion of statistical approximation has been notably absent, I do not have to remind time series econometricians of the particular measurement challenges associated with the long run. Indeed, there is a

[32]See Hansen and Renault (2010) for a discussion of alternative specifications of stochastic discount factors.

substantial literature on such issues. My aim is to suggest a framework for the use of such measurements, but I understand that some of the measurement challenges remain and I suspect that prior information about the underlying economic model will be required for sensible applications. I believe some of the same statistical challenges with which we econometricians struggle should be passed along to the hypothetical investors that populate our economic models. When decision-making agents within an economic model face difficulties in making probabilistic extrapolations of the future, the associated ambiguities in statistical inferences or concerns about model misspecification may well be an important component of the behavior of asset prices.

APPENDIX A: PROOF OF THEOREM 3.1

Let $Y_t^* = Y_t - \nu t$. Let $\mathcal{F}_t$ be the sigma algebra generated by the X process between time 0 and time t. As a consequence of the law of iterated expectations and the mean-square convergence,

$$\begin{aligned} g(X_t) + Y_t^* &= \lim_{\tau\to\infty} E(Y_{t+\tau}^* - Y_t^* | X_t) + Y_t^* \\ &= \lim_{\tau\to\infty} E(Y_{t+\tau}^* - Y_t^* | \mathcal{F}_t) + Y_t^* \\ &= \lim_{\tau\to\infty} E(Y_{t+\tau}^* | \mathcal{F}_t) \\ &= \lim_{\tau\to\infty} E[E(Y_{t+\tau}^* - Y_{t+\varepsilon}^* | \mathcal{F}_{t+\varepsilon}) + Y_{t+\varepsilon}^* | \mathcal{F}_t] \\ &= E[g(X_{t+\varepsilon}) + Y_{t+\varepsilon}^* | \mathcal{F}_t]. \end{aligned}$$

Thus $\{Y_t^* + g(X_t)\}$ is a martingale with initial value $g(X_0)$. After subtracting $g(X_0)$,

$$\hat{Y}_t = Y_t^* + g(X_t) - g(X_0)$$

remains a martingale, but it has initial value zero as required for an additive functional. *Q.E.D.*

APPENDIX B: IMPOSING STOCHASTIC STABILITY

Consider the computations in Section 7.1. I sketch how to use stochastic stability to select the eigenfunction and eigenvalue of interest. As an implication of the Girsanov theorem, associated with each solution is an alternative probability measure under which

$$dW_t = \sqrt{X_t^{[2]}}(F' + B_1'\alpha_1' + B_2'\alpha_2)\,dt + d\widetilde{W}_t,$$

where $\widetilde{W}_t$ is a multivariate standard Brownian motion under the twisted measure. The implied *twisted* evolution equation for $X^{[1]}$ is

$$\begin{aligned} dX_t^{[1]} &= A_{11}X_t^{[1]}\,dt + A_{12}\big(X_t^{[2]} - 1\big)\,dt \\ &\quad + (B_1F' + |B_1|^2\alpha_1)X_t^{[2]}\,dt + B_1\,d\widetilde{W}_t \end{aligned}$$

and for $X^{[2]}$ is

$$\begin{aligned} dX_t^{[2]} &= A_{22}\big(X_t^{[2]} - 1\big)\,dt + (B_2F' + |B_2|^2\alpha_2)X_t^{[2]}\,dt + \sqrt{X_t^{[2]}}B_2\,d\widetilde{W}_t \\ &= -\big(|B_2F' + A_{22}|^2 - |B_2|^2(|F - H_1(A_{11})^{-1}B_1|^2 \\ &\quad + 2H_2 - 2H_1(A_{11})^{-1}A_{12})\big)^{1/2}X_t^{[2]}\,dt \\ &\quad - A_{22}\big(X_t^{[2]} - 1\big)\,dt + \sqrt{X_t^{[2]}}B_2\,d\widetilde{W}_t, \end{aligned}$$

where, in the second representation, I have substituted from solution (29). I select the "minus" solution to achieve stochastic stability.

APPENDIX C: CONTINUOUS-TIME VALUE FUNCTION RECURSION

Recall the forward expectational difference equation for recursive utility:

$$\begin{aligned} h(x)^{1-\gamma} &= \Bigg(\varepsilon \mathsf{r}^{1-\theta} + \exp(-\delta\varepsilon) \\ &\quad \times \left[E\left([h(X_{t+\varepsilon})]^{1-\gamma}\left(\frac{C_{t+\varepsilon}}{C_t}\right)^{1-\gamma}\Big|X_t = x\right)\right]^{1/\zeta}\Bigg)^{\zeta}. \end{aligned}$$

Raise both sides to the power $\frac{1}{\zeta}$ and rearrange terms:

$$\begin{aligned} & h(x)^{1-\theta} - \varepsilon \mathsf{r}^{1-\theta} \\ &\quad = \exp(-\delta\varepsilon)\left[E\left([h(X_{t+\varepsilon})]^{1-\gamma}\left(\frac{C_{t+\varepsilon}}{C_t}\right)^{1-\gamma}\Big|X_t = x\right)\right]^{1/\zeta}. \end{aligned}$$

Next raise both sides to the power ζ:

$$[h(x)^{1-\theta} - \varepsilon \mathsf{r}^{1-\theta}]^{\zeta} = \exp(-\delta\zeta\varepsilon)\mathbb{V}_\varepsilon(h^{1-\gamma})(x).$$

Finally, subtract $[h(x)]^{1-\gamma}$ from both sides, divide by ε, and take limits:

$$-\zeta(\mathsf{r})^{1-\theta}[h(x)]^{\theta-\gamma} = -\zeta\delta[h(x)]^{1-\gamma} + \lim_{\varepsilon\downarrow 0}\frac{\mathbb{V}_\varepsilon(h^{1-\gamma})(x) - [h(x))]^{1-\gamma}}{\varepsilon}.$$

APPENDIX D: RISK PREMIA FOR FINITE PAYOFF HORIZONS

In this appendix, I give differential equations I solve to compute the risk-price trajectories for the model with consumption predictability. The analytical tractability is familiar from the literature on affine models (e.g., see Duffie and Kan (1994)).

Consider Example 3.3. The additive functional is

$$dY_t = \nu\, dt + H_1 X_t^{[1]}\, dt + H_2\big(X_t^{[2]} - 1\big)\, dt + \sqrt{X_t^{[2]}}\, F\, dW_t.$$

Form

$$M_t = \exp(Y_t).$$

My aim is to compute

$$\mathbb{M}_t 1(x) = E[M_t | X_0 = x],$$

where the left-hand side notation reflects the fact that the operator is evaluated at the unit function and this evaluation depends on the state x. For this computation, I use the formula

$$\mathbb{B}\mathbb{M}_t f = \frac{d}{dt}[\mathbb{M}_t f(x)]. \tag{37}$$

Guess a solution

$$\mathbb{M}_t 1(x) = E[M_t | X_0 = x] = \exp[\alpha(t) \cdot x + \varrho(t)],$$

where

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \text{and} \quad \alpha(t) = \begin{bmatrix} \alpha_1(t) \\ \alpha_2(t) \end{bmatrix}.$$

Notice that

$$\begin{aligned} &\frac{d}{dt} \exp[\alpha(t) \cdot x + \varrho(t)] \\ &\quad = \exp[\alpha(t) \cdot x + \varrho(t)] \left(\left[\frac{d}{dt} \alpha(t) \right] \cdot x + \frac{d}{dt} \varrho(t) \right). \end{aligned}$$

Moreover,

$$\begin{aligned} &\frac{\mathbb{B} \exp[\alpha(t) \cdot x + \varrho(t)]}{\exp[\alpha(t) \cdot x + \varrho(t)]} \\ &\quad = \nu + H_1 x_1 + H_2(x_2 - 1) \end{aligned}$$

$$+ x_1' A_{11}' \alpha_1(t) + (x_2 - 1)[A_{12}' \alpha_1(t) + A_{22} \alpha_2(t)]$$
$$+ \frac{x_2}{2} |F + \alpha_1(t)' B_1 + \alpha_2(t) B_2|^2.$$

First use (37) to produce a differential equation for $\alpha_1(t)$,

$$\frac{d}{dt} \alpha_1(t) = H_1' + A_{11}' \alpha_1(t),$$

by equating coefficients on x_1. This differential equation has an initial condition $\alpha_1(0) = 0$. Similarly, by equating coefficient on x_2,

$$\frac{d}{dt} \alpha_2(t) = H_2 + A_{12}' \alpha_1(t) + A_{22} \alpha_2(t) + \frac{1}{2} |F + \alpha_1(t)' B_1 + \alpha_2(t) B_2|^2.$$

This uses the solution for $\alpha_1(t)$ as an input. The initial condition is $\alpha_2(0) = 0$. Finally,

$$\frac{d}{dt} \varrho(t) = \nu - H_2 - A_{12}' \alpha_1(t) - A_{22} \alpha_2(t).$$

The initial condition is $\varrho(0) = 0$.

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