Sensitivity Analysis Using Approximate Moment Condition Models

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Approximate GMM

- Overidentified moment model with local misspecification
- \( \mathbb{E} (g_i(\theta_0)) = n^{-1/2} c \)
- GMM estimator: \( \hat{\theta} \)
- Asymptotic Distribution:
  \[
  \sqrt{n} \left( \hat{\theta} - \theta_0 \right) \rightarrow N \left( k' c, k' \Sigma k \right)
  \]
  where \( \Sigma = \text{var} (g_i) \) and \( k \) is sensitivity.
- AK assume we know “worst case” \( |c| \leq \bar{c} \)
- Proposed confidence interval
  \[
  \hat{\theta} \pm n^{-1/2} \text{cv} \left( \bar{c} / \sigma \right) \sigma
  \]
  where \( \text{cv}(t) \) is critical value from folded normal \( |N(t, 1)| \) distribution
  and \( \sigma^2 = k' \Sigma k \)
The GMM weight matrix alters the sensitivity vector $k$

- Under correct specification we select $k$ to minimize the variance $k'\Sigma k$
  - Lars Hansen (1982)

- Under misspecification, $k$ also affects the bias $k'c$

- AK select $k$ to minimize the length of the confidence interval

$$\hat{\theta} \pm n^{-1/2}c\nu (\bar{c}/\sigma)\sigma$$

- This increases the variance, but decreases the bias.

- Clever idea
Consider estimation of \( \theta = \mathbb{E}(y_i) \) under auxiliary assumption \( \mathbb{E}(x_i) = 0 \)

Efficient GMM estimator is \( \hat{\theta} = \bar{y} - \bar{x} \hat{\beta} \) where \( \hat{\beta} \) is OLS slope

Misspecification occurs when \( \mathbb{E}(x_i) \neq 0 \)

Efficient GMM has conditional bias \( \bar{x} \beta \)

AK estimator is \( \hat{\theta} = \bar{y} - \bar{x} \gamma \) for some \( \gamma \) between 0 and \( \hat{\beta} \)

 Shrinks classical “efficient” GMM towards “inefficient but consistent” estimator
Challenge

- The AK confidence interval depends on known “worst case” bound $\bar{c}$
- Knowledge of $\bar{c}$ is not credible
- If worst case is unbounded then confidence interval will be unbounded
- Fundamental challenge
We can estimate $|c|$ by $|\hat{c}| = \sqrt{n} |g|$

Construct confidence interval $\hat{\theta} \pm n^{-1/2} cv \left( |\hat{c}| / \sigma \right) \sigma$
where $cv(t)$ is critical value from $|N(t, 1)|$ distribution

Since $|\hat{c}|$ is inconsistent this is incorrect

$|\hat{c}|$ has folded normal distribution with same non-centrality parameter

We can compute the mixture distribution of $\hat{\theta} / cv \left( |\hat{c}| / \sigma \right)$

Coverage error can be bounded, similar to Stock-Yogo IV method

Discussed in my (incomplete) 2014 working paper “Robust Inference”
Biased Standard Errors

- AK use conventional GMM variance formula
- However, variance estimator is severely biased under misspecification
- Need to use alternative variance estimator
  - Alastair Hall and Atsushi Inoue (JoE, 2002)
  - Bruce Hansen and Seojeong Lee (working paper, 2019)
    - “Inference for GMM Under Misspecification”
- Intuition: Lars Hansen (1982) calculation uses assumption of correct specification $\mathbb{E}(g_i(\theta_0)) = 0$
- Correct standard errors can be found by Taylor expansion of entire first-order condition, not just the moment condition
- Correct standard errors depend on curvature of moments and weight matrix (second derivative of moments and derivatives of weight matrix)
- Simulations show that the standard error bias can be very large
Example Revisited

- Estimation of $\theta = \mathbb{E}(y_i)$ under auxiliary assumption $\mathbb{E}(x_i) = 0$
- Efficient GMM estimator is $\hat{\theta} = \bar{y} - \bar{x}\hat{\beta}$ where $\hat{\beta}$ is OLS slope
- Exact conditional variance
  \[ V = \frac{1}{n^2} \sum_{i=1}^{n} \sigma_i^2 \left( \left( \hat{\sigma}_x^2 + \bar{x}^2 \right)^2 - 2 \left( \hat{\sigma}_x^2 + \bar{x}^2 \right) \bar{x}_i + \bar{x}^2 \bar{x}_i^2 \right) / \sigma_x^4 \]
- White variance estimator
  \[ \hat{V}_{\text{White}} = \frac{1}{n^2} \sum_{i=1}^{n} \hat{e}_i^2 \left( \left( \hat{\sigma}_x^2 + \bar{x}^2 \right)^2 - 2 \left( \hat{\sigma}_x^2 + \bar{x}^2 \right) \bar{x}_i + \bar{x}^2 \bar{x}_i^2 \right) / \sigma_x^4 \]
- Hansen (1982) GMM variance estimator
  \[ \hat{V}_{\text{gmm}} = \frac{1}{n^2} \sum_{i=1}^{n} \hat{e}_i^2 \]
- Hansen-Lee misspecification-robust estimator
  \[ \hat{V}_{\text{Robust}} = \hat{V}_{\text{White}} \]
Implications

- **Under correct specification:**
  - The Hansen (1982) variance estimator is consistent
    - But has finite sample bias
  - The Hall-Inoue-Hansen-Lee estimator is consistent
    - And accounts for finite sample effects

- **Under misspecification**
  - The Hansen (1982) variance estimator is inconsistent
  - The Hall-Inoue-Hansen-Lee estimator is consistent
Summary

- Misspecification under-studied
- Misspecification affects both bias and variance
  - Variance studied in Hansen and Lee (2019)
  - Bias is more difficult to handle
- AK are making a serious effort to develop methods robust to misspecification bias
- Fundamental input: worst-case bias $\bar{c}$
  - This is not reasonable for practical applications
- Any alternative will involve some sort of compromise