Sensitivity Analysis Using Approximate Moment Condition Models

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Approximate GMM

- Overidentified moment model with local misspecification
- $\mathbb{E}\left(g_i(\theta_0)\right) = n^{-1/2}c$
- GMM estimator: $\widehat{\theta}$
- Asymptotic Distribution:

$$\sqrt{n}\left(\widehat{\theta} - \theta_0\right) \to N\left(k'c, k'\Sigma k\right)$$

  where $\Sigma = \mathrm{var}\left(g_i\right)$ and k is *sensitivity*.
- AK assume we know "worst case" $|c| \leq \overline{c}$
- Proposed confidence interval

$$\widehat{\theta} \pm n^{-1/2} cv\left(\overline{c}/\sigma\right)\sigma$$

  where $cv(t)$ is critical value from folded normal $|N(t, 1)|$ distribution and $\sigma^2 = k'\Sigma k$

Efficient Weight Matrix

- The GMM weight matrix alters the sensitivity vector k
- Under correct specification we select k to minimize the variance $k'\Sigma k$
  - Lars Hansen (1982)
- Under misspecification, k also affects the bias $k'c$
- AK select k to minimize the length of the confidence interval

$$\widehat{\theta} \pm n^{-1/2} cv\left(\overline{c}/\sigma\right)\sigma$$

- This increases the variance, but decreases the bias.
- Clever idea

Example

- Consider estimation of $\theta = \mathbb{E}(y_i)$ under auxillary assumption $\mathbb{E}(x_i) = 0$
- Efficient GMM estimator is $\widehat{\theta} = \overline{y} - \overline{x}\widehat{\beta}$ where $\widehat{\beta}$ is OLS slope
- Misspecification occurs when $\mathbb{E}(x_i) \neq 0$
- Efficient GMM has conditional bias $\overline{x}\beta$
- AK estimator is $\widehat{\theta} = \overline{y} - \overline{x}\gamma$ for some γ between 0 and $\widehat{\beta}$
- Shrinks classical "efficient" GMM towards "inefficient but consistent" estimator

Challenge

- The AK confidence interval depends on known "worst case" bound $\overline{c}$
- Knowledge of $\overline{c}$ is not credible
- If worst case is unbounded then confidence interval will be unbounded
- Fundamental challenge

Alternative Possible Approach

- We can estimate $|c|$ by $|\widehat{c}| = \sqrt{n}\,|\widehat{g}|$
- Construct confidence interval $\widehat{\theta} \pm n^{-1/2} cv\left(|\widehat{c}|/\sigma\right)\sigma$
  where $cv(t)$ is critical value from $|N(t,1)|$ distribution
- Since $|\widehat{c}|$ is inconsistent this is incorrect
- $|\widehat{c}|$ has folded normal distribution with same non-centrality parameter
- We can compute the mixture distribution of $\widehat{\theta}/cv\left(|\widehat{c}|/\sigma\right)$
- Coverage error can be bounded, similar to Stock-Yogo IV method
- Discussed in my (incomplete) 2014 working paper "Robust Inference"

Biased Standard Errors

- AK use conventional GMM variance formula
- However, variance estimator is severely biased under misspecification
- Need to use alternative variance estimator
  - Alastair Hall and Atsushi Inoue (JoE, 2002)
  - Bruce Hansen and Seojeong Lee (working paper, 2019)
    - "Inference for GMM Under Misspecification"
- Intuition: Lars Hansen (1982) calculation uses assumption of correct specification $\mathbb{E}\left(g_i(\theta_0)\right) = 0$
- Correct standard errors can be found by Taylor expansion of entire first-order condition, not just the moment condition
- Correct standard errors depend on curvature of moments and weight matrix (second derivative of moments and derivatives of weight matrix)
- Simulations show that the standard error bias can be very large

Example Revisited

- Estimation of $\theta = \mathbb{E}(y_i)$ under auxillary assumption $\mathbb{E}(x_i) = 0$
- Efficient GMM estimator is $\widehat{\theta} = \overline{y} - \overline{x}\widehat{\beta}$ where $\widehat{\beta}$ is OLS slope
- Exact conditional variance
  $V = \frac{1}{n^2}\sum_{i=1}^{n} \sigma_i^2 \left(\left(\widehat{\sigma}_x^2 + \overline{x}^2\right)^2 - 2\left(\widehat{\sigma}_x^2 + \overline{x}^2\right)\overline{x}x_i + \overline{x}^2 x_i^2 \right) / \widehat{\sigma}_x^4$
- White variance estimator
  $\widehat{V}_{\mathrm{White}} = \frac{1}{n^2}\sum_{i=1}^{n} \widehat{e}_i^2 \left(\left(\widehat{\sigma}_x^2 + \overline{x}^2\right)^2 - 2\left(\widehat{\sigma}_x^2 + \overline{x}^2\right)\overline{x}x_i + \overline{x}^2 x_i^2 \right) / \widehat{\sigma}_x^4$
- Hansen (1982) GMM variance estimator $\widehat{V}_{\mathrm{gmm}} = \frac{1}{n^2}\sum_{i=1}^{n} \widehat{e}_i^2$
- Hansen-Lee misspecification-robust estimator $\widehat{V}_{\mathrm{Robust}} = \widehat{V}_{\mathrm{White}}$

Implications

- Under correct specification:
  - The Hansen (1982) variance estimator is consistent
    - But has finite sample bias
  - The Hall-Inoue-Hansen-Lee estimator is consistent
    - And accounts for finite sample effects
- Under misspecification
  - The Hansen (1982) variance estimator is inconsistent
  - The Hall-Inoue-Hansen-Lee estimator is consistent

Summary

- Misspecification under-studied
- Misspecification affects both bias and variance
  - Variance studied in Hansen and Lee (2019)
  - Bias is more difficult to handle
- AK are making a serious effort to develop methods robust to misspecification bias
- Fundamental input: worst-case bias $\overline{c}$
  - This is not reasonable for practical applications
- Any alternative will involve some sort of compromise