Fragility of Purely Real Macroeconomic Models

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Abstract

Over the past thirty years, a great deal of business cycle research has been based on purely real models that abstract from the presence of nominal rigidities, and so (at least implicitly) assume that the Phillips curve is vertical. In this paper, I show that such models are fragile, in the sense that their implications change significantly when the Phillips curve is even slightly less than vertical. I consider a wide class of purely real macroeconomic models and perturb them by introducing a non-vertical Phillips curve. I show that in the perturbed models, if there is a lower bound on the nominal interest rate, then current outcomes necessarily depend on agents' beliefs about the long-run level of economic activity. The magnitude of this dependence becomes arbitrarily large as the slope of the Phillips curve becomes arbitrarily large in absolute value (closer to vertical). In contrast, the limiting purely real model ignores this form of monetary non-neutrality and macroeconomic instability. I conclude that purely real models are too incomplete to provide useful guides to questions about business cycles. I describe what elements should be added to such models in order to make them useful. I argue that useful models need to incorporate significant nominal rigidities or explicit descriptions of the evolution of beliefs about the long run (or both).

1   Introduction

In the wake of the classic papers by Kydland and Prescott (1982) and Long and Plosser (1983), a great deal of macroeconomic research has focused on the implications of *purely real* models. By construction, such models abstract from nominal rigidities and the Phillips curve is vertical, in the sense that a given real outcome is thought to be consistent with any level of inflation. Presumably, this class of models is intended to provide useful approximations to a more complicated real world, in which nominal rigidities are small but not non-existent.

In this paper, I demonstrate that this presumption is wrong. I consider a wide class of purely real models. I perturb them by assuming that the Phillips curve is almost, but not exactly, vertical. I show that, if the nominal interest rate is pegged, then the perturbed models' predictions for current economic activity are radically different from those of the original purely real models. In particular, in the perturbed models, *current* real outcomes are highly sensitive to agents' beliefs about the future level of economic activity. The degree of this sensitivity becomes arbitrarily large as the slope of the Phillips curve gets arbitrarily large in absolute value, even though there is no such feedback from the future to current real outcomes in the limiting (original) model with a vertical Phillips curve. I conclude that the predictions of a purely real macroeconomic model are highly *fragile* to apparently small deviations in assumptions about nominal rigidities.[1]

This result is seemingly paradoxical, but the intuition is simple. Suppose that agents believe at date t that there will be a negative output gap at some future date $(t+T)$ (in the sense that output at date $(t+T)$ is expected to be below its level in a hypothetical purely real world with a vertical Phillips curve). If the Phillips curve is nearly vertical, the expectation of relatively low output translates into a (very) low anticipated inflation

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[1] De Long and Summers (1986) argue that it is possible in some circumstances for increased price flexibility to increase macroeconomic instability. This paper generalizes their argument considerably by emphasizing the role of a lower bound on the nominal interest rate.

rate in period $(t + T)$, and (since the nominal interest rate is pegged) into a very high real interest rate between $(t + T - 1)$ and period $(t + T)$. Agents in period t expect that the high expected real interest rate will push aggregate demand downward, so that the output gap in period $(t + T - 1)$ will be markedly below its level in period $(t + T)$. But we can now apply this logic iteratively to conclude that if agents expect a negative output gap in period $(t + T)$, there will be a negative output gap in period t that is much larger in absolute value. The amplification, as we go backwards in time, is provided by the connection between output and expected inflation. Hence, the magnitude of amplification increases as the slope of the Phillips curve increases in absolute value, and converges to infinity as the Phillips curve converges to vertical.

The above logic relies on the assumption that the nominal interest rate is pegged. I demonstrate that the fragility result disappears if we restrict the central bank to use an *active* monetary policy in which the nominal interest rate changes more than one-for-one with changes in the current inflation rate. However, active monetary policies are intrinsically unrealistic, because they assume that there is no lower bound on the nominal interest rate. I show that, if there is some lower bound on the nominal interest rate, the predictions of purely real models are again fragile. With a lower bound, current real outcomes are sensitive to long-run pessimism - that is, to perceived *downside* risks to long-run economic activity. The degree of sensitivity that approaches infinity in absolute value as the slope of the Phillips curve approaches infinity in absolute value.

I see the fragility result as being highly consequential for the construction of macroeconomic models. All models are necessarily approximations to reality. However, their fragility means that purely real models are insufficiently robust to be useful in interpreting the world. In the second part of the paper, I describe three elements that need to be added to a purely real model so that it will be sufficiently robust to be empirically useful. First, and perhaps most importantly, the model needs to have a Phillips curve that describes the connection between the level of real activity and inflation. Second,

the model needs to have an explicit description of the central bank's reaction function. Finally, if the Phillips curve is sufficiently vertical, then the model needs to include a description of how households form their expectations of long-run equilibrium outcomes, and how policy choices can influence those expectations.[2] This description likely will include some specification of households' beliefs about the nature of the long-run fiscal regime.

The class of models that I consider has two distinctive features relative to the models usually analyzed in the literature. First, I use a somewhat unusual formulation of the Phillips curve, in which the current marginal utility of consumption, averaged across agents, is related to the current rate of inflation. This formulation allows for a clearer connection between goods and asset markets.[3]

The second (and more important) distinctive feature is that the models are finite horizon. This feature matters a great deal for the results. Infinite horizon models typically impose tight implicit restrictions on long run outcomes. These implicit restrictions mean that the models are unable to capture the dependence of current outcomes on long run outcomes that occurs in models with long, but finite, horizons. This dependence lies at the core of my analysis.

Mathematically, my paper is related to recent work using New Keynesian models to study the impact of a zero lower bound on the nominal interest rate. Werning (2012) and Cochrane (2015) both consider the impact of price flexibility on equilibrium outcomes at the zero lower bound. As I do, Werning (2012) proves that the behavior of real equilibrium outcomes is very different when nominal rigidities are small, as opposed to

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[2]See Garcia-Schmidt and Woodford (2015) for one approach along these lines.

[3]As is well-known, there are many ways to rationalize Phillips curves. My goal is to provide a set of results that is valid for any of these particular rationalizaitons. However, in Appendix A, I do provide an example micro-foundation. In that micro-foundation, the slope of the Phillips curve is increasing, in absolute value, in the fraction of firms that are able to set prices flexibly in a given period. The slope converges to infinity in absolute value as this fraction converges to one. In Appendix B, I describe how the main results can be generalized to the case in which the Phillips curve incorporates the impact of expectations of future inflation.

literally non-existent.[4] Cochrane (2015) shows that this discontinuity result disappears if agents' post-zero lower bound expectations vary appropriately with the degree of price flexibility. Del Negro, Giannoni, and Patterson (2013), McKay, Nakamura, and Steinsson (2015) and Werning (2015) show that economic outcomes at the zero lower bound are highly sensitive to the central bank's forward guidance about interest rates in the distant future.

My paper does differ from the prior literature in an important respect. The earlier papers follow Eggertsson and Woodford (2003) by treating the lower bound on the nominal interest rate as a restriction on the ability of the central bank to respond to exogenous macroeconomic shocks. In the class of models that I study, the lower bound on the nominal interest rate restricts the ability of the central bank to eliminate undesirable equilibria that emerge because of self-fulfilling pessimistic beliefs about the long run. The impact of these beliefs on current outcomes grows as the slope of the Phillips curve converges to infinity.

The remainder of the paper is organized as follows. In the next section, I describe a large class of finite horizon real model economies. Then, I explain how to add a Phillips curve to these models. I next examine the properties of equilibrium outcomes when the Phillips curve is near-vertical. Finally, I discuss the implications of the results, in terms of what ingredients are needed in models if they are to be non-fragile. The main conclusion is that, to be non-fragile, models must include significant nominal rigidities or explicitly model the formation of beliefs about the long run.

2   Real Models

I consider a class of *real models* in which time is discrete and indexed $1, 2, ..., T$. All agents in the economy live for T periods, and are indexed by ψ in a set Ψ. A real model

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[4]Christiano, Eichenbaum, and Rebelo (2011) use numerical simulations to illustrate this same result. See also Christiano and Eichenbaum (2012).

has four components: (X, M, Γ, g). The first two components (X, M) are sets, and the last two components are mappings between the sets:

$$\Gamma \quad : \quad M \to X \tag{1}$$

$$g \quad : \quad X \to M \tag{2}$$

I impose no restrictions on the set X. A typical element x in X is a description of economic characteristics. The flexibility in the specification of X means that it can include, for example, the laws of motion of the aggregate capital stock, the cross-sectional distribution of wealth, or labor market matches.

The set M is more specific. Let $(\Omega, F, \Pr)$ be a probability space and $(F_t)_{t=1}^T$ be a filtration of F. The filtration is intended to represent the information encoded in the history of *aggregate shocks* that have affected the economy, including potential sunspots. Let M be a set of all m such that:

$$m : \Omega \to R_+^T \tag{3}$$

$$m_t \text{ is } F_t\text{-measurable}$$

(Here, R_+ represents the set of positive real numbers.) I view the typical element m in M as being an *average marginal utility stochastic process*. More particularly, let $MU_t(\psi)$ represent agent ψ's marginal period 0 utility[5] of additional units of a fixed consumption bundle $\bar{c}$ in period t. This marginal utility $MU_t(\psi)$ is random and may not be the same for all ψ. Define m_t to be the average of $MU_t(\psi)$ over ψ. This cross-sectional average is assumed to be measurable with respect to F_t (the history of aggregate shocks).

An *equilibrium* to a real model (X, M, Γ, g) is defined to be (x^{REAL}, m^{REAL}) in

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[5]The definition of m_t implies that it equals $\beta^t u'(c_t)$, in models with a representative agent with time-additive preferences.

$X \times M$ such that:

$$\Gamma(m^{REAL}) \ = \ x^{REAL} \tag{4}$$

$$g(x^{REAL}) \ = \ m^{REAL} \tag{5}$$

I focus on real models (X, M, Γ, g) in which an equilibrium exists and in which that equilibrium is unique.

Economically, I think of the average marginal utility process m^{REAL} as being a description of the evolution of aggregate (consumption) demand. (Aggregate demand is high in dates and states in which m^{REAL} is low, and vice versa.) In an equilibrium, an aggregate demand process m^{REAL} generates an economic outcome x^{REAL} that is, in turn, consistent with that same average marginal utility process m^{REAL}. The second equilibrium restriction is, essentially, Say's Law: the structure x^{REAL} of production and consumption in the economy generates sufficient aggregate demand m^{REAL} to clear markets.

The following example illustrates how this abstract formulation works. Note that, while the example assumes the existence of a representative agent, the overall framework makes no such assumption.

Example 1. Consider a T period representative agent model. The represent agent has momentry utility $ln(c) - v(n)$ over consumption and labor, with a discount factor β. In this example, X contains processes labor $(n_t)_{t=1}^{T}$ that are adaptable to a filtration $(F_t)_{t=1}^{T}$. Output is produced using labor as an input, with exogenous labor productivity $(A_t)_{t=1}^{T}$. The mapping Γ from stochastic processes for m_t to X is then defined (recursively) by:

$$n_t = \frac{\beta^t}{A_t m_t}, t = 1, ..., T \tag{6}$$

The mapping g is defined by setting output equal to its efficient level in every date and

state. More formally:

$$(g(n))(A^t) = \beta^t v'(n_t)/A_t, t = 1, .., T \tag{7}$$

$\triangle$

The following concepts will prove useful in the ensuing discussion. First, I define the period t (gross) *natural real interest rate* as:

$$r_t^{NAT} = \frac{m_t^{REAL}}{E\{m_{t+1}^{REAL}|F_t\}} \tag{8}$$

Note that r_t^{NAT} is F_t-measurable. As well, I define the *natural risk-neutral probability measure Q^{NAT}* to be:

$$Q^{NAT}(A) = E\{r_1^{NAT}...r_{T-1}^{NAT} \frac{m_T^{REAL}}{m_1^{REAL}} 1_A\} \tag{9}$$

where A is any element of the space F of events. I assume that Q^{NAT} is absolutely continuous with respect to the physical probability measure Pr. It is readily shown that, if z_{t+1} is F_{t+1}-measurable, then the Q^{NAT}-expectation of z_{t+1}, conditional on F_t, is given by:

$$E^{NAT}(z_{t+1}|F_t) = r_t^{NAT} E\{\frac{z_{t+1}m_{t+1}^{REAL}}{m_t^{REAL}}|F_t\} \tag{10}$$

3   Models with a Phillips Curve

I now describe a way to convert an arbitrary real model to a model with money and a simple Phillips curve, meant to capture the effects of imperfect price level adjustment.

3.1 Monetary Policy and Fiscal Policy

Each agent is given M dollars at the beginning of period 1. I let p_t denote the *price level* - that is, the price of the consumption bundle $\bar{c}$ in terms of dollars. I let π_t denote the gross rate of inflation p_t/p_{t-1}. The gross rate of return on money is governed by an *interest rate rule* R that depends on the (gross) inflation rate in period t and on the natural real interest rate in period t. More specifically, money pays a gross rate of return $R(\pi_t, r_t^{NAT})$ (in dollars) in period $t = 2, ..., T$.

Fiscal policy is quite simple. There is no initial government debt, so that the only initial outstanding government liability is money. In period T, agents pay a lump sum tax equal to $M[\prod_{s=2}^{T} R(\pi_s, r_s^{NAT})]$. This lump sum tax extracts all of the outstanding money in the economy. It serves to give money its value in earlier periods. Note that this form of fiscal policy is Ricardian (in the language of Woodford (1995)) and passive (in the language of Leeper (1991)), in the sense that the government's budget constraint is satisfied for any sequence of equilibrium price levels.[6]

3.2 Aggregate Euler Equations

Money is an asset. If agents are to be marginally indifferent about further exchanges of the consumption bundle $\bar{c}$ for money, then the price level (the reciprocal of money's asset price) has to obey a set of *aggregate Euler equations*:

$$m_t/p_t = R(\pi_t, r_t^{NAT})E(m_{t+1}/p_{t+1}|F_t), t = 1, ..., (T-1) \tag{11}$$

[6]This model of monetary policy is motivated by the current situation of the Federal Reserve. As of this writing, banks hold a large amount of excess reserves and the Federal Reserve intends to affect financial market conditions by varying the interest rate on those reserves. It is traditional in macroeconomics to assume instead that the central bank influences the short-term interest rate by exchanging non-interest-bearing (relatively liquid) bank reserves for interest-bearing (relatively illiquid) short-term government debt. The results in the paper rely only on the specification of the aggregate Euler equation and so would generalize to this more traditional (but more complicated) model.

This first order condition assumes that no agent faces a binding borrowing constraint with respect to money. The formulation of fiscal policy, which requires all agents to hold sufficient wealth at each date to know that they will be able to pay their taxes in the final period, implies that this assumption is valid in equilibrium.

3.3 Inverse Phillips Curves

I now add a set of short-term *inverse Phillips curves* to the model. (In Appendix A, I describe how these Phillips curves can be micro-founded. In Appendix B, I describe how they could be modified to allow for more forward-looking elements.) They describe how deviations in the current and future inflation rates, from a benchmark gross inflation rate π^*, are associated with deviations of aggregate economic activity from the equilibrium level in the real model. I use the average marginal utility of $\bar{c}$ as the measure of aggregate economic activity. More specifically, I define $\widehat{m}_t$ to be m_t/m_t^{REAL} and $\widehat{\pi}_t = \pi_t/\pi^*$ and define the inverse Phillips curves to be:

$$\widehat{m}_t = h(\widehat{\pi}_t; \phi), t = 1, ..., T \tag{12}$$

where $\phi \geq 0$.

I impose the following restrictions on h. I assume that h is twice continuously differentiable with respect to ϕ, and that:

$$h(\widehat{\pi}; 0) = 1 \text{ for any } \hat{\pi} > 0 \tag{13}$$

As we will see, this restriction corresponds to assuming that prices adjust instantaneously when $\phi = 0$. More substantively, I assume that, if inflation equals the benchmark π^*, then aggregate economic activity lines up with that in the real model.

$$h(1; \phi) = 1 \text{ for any } \phi \geq 0, t = 1, ..., T \tag{14}$$

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This restriction links the model with price rigidities back to the original real model.

The final assumptions about h deal with the association of inflation and economic activity. I assume that, if $\phi > 0$, the function h is strictly decreasing in $\widehat{\pi}$. This assumption implies that if inflation is above the benchmark level today, then aggregate economic activity is above its equilibrium level (so that $\widehat{m}$ is low). I assume too that h satisfies Inada conditions:

$$\lim_{\widehat{\pi} \to 0} h(\widehat{\pi}; \phi) = \infty \text{ if } \phi > 0 \tag{15}$$

$$\lim_{\widehat{\pi} \to \infty} h(\widehat{\pi}; \phi) = 0 \text{ if } \phi > 0 \tag{16}$$

Intuitively, the parameter ϕ governs the elasticity of the Phillips curve (drawn with π on the vertical axis and m on the horizontal axis). Thus, when $\phi = 0$, m is independent of π. This is a situation in which the inverse Phillips curve is horizontal, and the Phillips curve is vertical.. When ϕ is near zero, the differentiability assumption that I made about h ensures that the slope of the inverse Phillips curve is nearly zero, so that the Phillips curve is nearly vertical. The main point in this paper is that equilibrium outcomes are very different when the Phillips curve is nearly vertical as opposed to vertical.

3.4   Normalized Aggregate Euler Equations

It will be useful to rewrite the aggregate Euler equations in terms of the deviations $\widehat{m}$ and $\widehat{\pi}$ from the original real equilibrium m^{REAL} and the benchmark gross inflation rate π^*. We can rewrite the aggregate Euler equations as follows:

$$\widehat{m}_t = \widehat{R}(\widehat{\pi}_t, r_t^{NAT}) E^{NAT}(\frac{\widehat{m}_{t+1}}{\widehat{\pi}_{t+1}} | F_t) \tag{17}$$

where E_t^{NAT} is the risk-neutral conditional expectation described above. Here, I define a normalized interest rate rule $\widehat{R}$ by:

$$\widehat{R}(\widehat{\pi}, r^{NAT}) = \frac{R(\widehat{\pi}\pi^*, r^{NAT})}{r^{NAT}\pi^*} \tag{18}$$

3.5   Summary of Equilibrium Conditions

Given the unique equilibrium (x^{REAL}, m^{REAL}) to a real model (X, M, Γ, g), we can define an equilibrium (x^{MON}, m^{MON}, π) to a model $(X, M, \widehat{R}, \Gamma, \phi)$ with price rigidities to be any solution to the equations:

$$x^{MON} = \Gamma(m^{MON}) \tag{19}$$

$$m_t^{MON} = \widehat{m}_t m_t^{REAL}, t = 1, ..., T \tag{20}$$

$$\pi_t = \widehat{\pi}_t \pi^*, \ t = 1, ..., T \tag{21}$$

$$\widehat{m}_t = \widehat{R}(\widehat{\pi}_t, r_t^{NAT}) E^{NAT}\left(\frac{\widehat{m}_{t+1}}{\widehat{\pi}_{t+1}} \Big| F_t\right), t = 1, .., T - 1 \tag{22}$$

$$\widehat{m}_t = h(\widehat{\pi}_t; \phi), t = 1, ..., T \tag{23}$$

$$\widehat{\pi}_t \text{ is } F_t\text{-measurable} \tag{24}$$

where E^{NAT}, $\widehat{R}$, and r^{NAT} are defined as above.

This definition of an equilibrium assumes that the restrictions between x and m embedded in Γ are not altered by the addition of money. Verbally, this means that any *real* rigidities/frictions embedded in Γ are not affected by the introduction of money.

4   Basic Properties of Equilibrium

In this section, I describe the basic properties of equilibrium in a model with nominal rigidities. In models with a vertical Phillips curve, all equilibria have the same real allocation of resources as the original, real, economy. In models with a non-vertical

Phillips curve, there is a continuum of equilibrium real allocations, indexed by the (random) rate of inflation in the final period T on which agents co-ordinate.

4.1 Vertical Phillips Curve: Unique Real Allocation

The first case that I consider is one in which the Phillips curve is vertical, so that $\phi = 0$.

Proposition 1. *Suppose (x^{MON}, m^{MON}, π) is an equilibrium to a model $(X, M, \widehat{R}, \Gamma, \phi)$ with price rigidities in which the parameter $\phi = 0$ and $\hat{\pi}_t > 0$ with probability one for all t. Then, for all t, $m_t^{MON} = m_t^{REAL}$ with probability 1, and $x^{MON} = x^{REAL}$.*

Proof. The proof is trivial. Consider any equilibrium. If $\phi = 0$, then $h(\widehat{\pi}_t; \phi) = 1$ for any $\widehat{\pi}_t$, and so $\widehat{m}_t = 1$. Since $m_t^{MON} = \widehat{m}_t m_t^{REAL}$, it follows that $m_t^{MON} = m_t^{REAL}$. The equilibrium value of $x^{MON} = \Gamma(m^{MON}) = \Gamma(m^{REAL}) = x^{REAL}$. $\square$

Thus, if $\phi = 0$ and the gross inflation rate is positive, then nominal rigidities are irrelevant, in the sense that the equilibrium marginal utility process is m^{REAL}.

4.2 Non-Vertical Phillips Curve

In this subsection, I describe the set of equilibrium real allocations in models with a non-vertical Phillips curve ($\phi > 0$). These allocations are indexed by the random inflation rate in the final period.

Proposition 2. *Consider a model $(X, M, \widehat{R}, \Gamma, \phi)$ with price rigidities in which $\phi > 0$. Pick an arbitrary F_T-measurable $\widetilde{\pi}_T > 0$. Then, there exists an equilibrium (x^{MON}, m^{MON}, π), in which $\pi_T = \widetilde{\pi}_T$ with probability 1.*

Proof. Define the function $\Psi : \mathbb{R}^2 \to \mathbb{R}$ to be the solution to the equation:

$$\frac{h(\Psi(w, r^{NAT}; \phi); \phi)}{\widehat{R}(\Psi(w, r^{NAT}), r^{NAT})} = w \qquad (25)$$

Ψ is well-defined and strictly decreasing in w because of the Inada conditions on h, because h is strictly decreasing and because $\widehat{R}$ is strictly increasing. Then, recursively define a stochastic process for (normalized) inflation by:

$$\widehat{\pi}_t = \Psi(E^{NAT}(h(\widehat{\pi}_{t+1}; \phi)\widehat{\pi}_{t+1}^{-1}|F_t), r_t^{NAT}; \phi), t = 2, ..., (T-1) \tag{26}$$

where:

$$\widehat{\pi}_T = \frac{\widetilde{\pi}_T}{\pi^*} \tag{27}$$

Define period t (normalized) marginal utility to be:

$$\widehat{m}_t = h(\widehat{\pi}_t; \phi), t = 1, ..., T \tag{28}$$

For any t, the random variable $\widehat{\pi}_t$ is F_t-measurable, and $(\widehat{m}, \widehat{\pi})$ satisfies the asset pricing equation:

$$\widehat{m}_{t-1} = \widehat{R}(\widehat{\pi}_{t-1}, r_{t-1}^{NAT})E^{NAT}(\widehat{m}_t\widehat{\pi}_t^{-1}|F_{t-1}) \tag{29}$$

We can complete the construction of the equilibrium by defining:

$$\pi = \widehat{\pi}_t\pi^* \tag{30}$$

$$m^{MON} = \widehat{m}_t m^{REAL} \tag{31}$$

$$x^{MON} = \Gamma(m^{MON}) \tag{32}$$

$\square$

Regardless of whether the Phillips curve is vertical or not, there is a continuum of equilibria. The set of equilibria is indexed by the (random) final period inflation rate on which agents co-ordinate. When the Phillips curve is vertical, real allocations are the same in all of these equilibria. When the Phillips curve is non-vertical, the different inflation rate processes translate into different real allocation processes.

5   Equilibria Under Different Interest Rate Rules

In this section, I show how the nature of equilibrium depends on the nature of the interest rate rule used by the central bank.

5.1   Interest Rate Peg

I first consider an interest rate peg, in which the nominal interest rate evolves so as to be equal to the sum of the inflation target π^* and the natural real interest rate r_t^{NAT}. (I refer to this policy rule as a peg, because the evolution of the nominal interest rate R over time is independent of any variables affected by monetary policy.) This specification of monetary policy implies that if households believe that inflation equals π^* in the long run, then the real outcomes in any model with price rigidities are same as those in the real model.

Proposition 3. *Consider a model* $(X, M, \widehat{R}, \Gamma, \phi)$ *with price rigidities in which* $\phi > 0$ *and* $\widehat{R}(\widehat{\pi}, r^{NAT}) = 1$ *for all* $\widehat{\pi}$ *and all* r^{NAT}. *There exists an equilibrium* (x^{MON}, m^{MON}, π), *in which* $\pi_T = \pi^*$ *with probability 1. In that equilibrium:*

$$m_t^{MON} = m_t^{REAL} \ \ w.p. \ 1 \ for \ all \ t \tag{33}$$

$$\pi_t = \pi^* \ for \ all \ t \tag{34}$$

$$x^{MON} = x^{REAL} \tag{35}$$

Proof. The existence of such an equilibrium is a direct consequence of proposition 2. We can then establish the results about (m^{MON}, π) using reverse induction. Suppose that $\pi_{t+1} = \pi^*$ with probability one and $m_{t+1}^{MON} = m_{t+1}^{REAL}$ with probability one. The asset pricing equation says that:

$$\widehat{m}_t = E(\widehat{m}_{t+1} \widehat{\pi}_{t+1}^{-1} | F_t) \tag{36}$$

Because $\widehat{m}_t = 1$ and $\widehat{\pi}_t = 1$, the equation implies that $m_t^{MON} = m_t^{REAL}$ with probability one. Since the Phillips curve is strictly decreasing in $\widehat{\pi}$, we can conclude that $\widehat{\pi}_t = 1$ with probability one. This establishes that $m_t^{MON} = m_t^{REAL}$ and $\pi_t = \pi^*$ with probability one for all t. As well:

$$x^{MON} = \Gamma(m^{MON}) \tag{37}$$

$$= \Gamma(m^{REAL}) \tag{38}$$

$$= x^{REAL} \tag{39}$$

which proves the theorem. $\qquad\qquad\square$

More generally, though, there is no reason why agents should believe that inflation will in fact equal π^* in the long run. The next proposition shows that current economic outcomes are highly sensitive to beliefs about long-run economic activity when the Phillips curve is nearly vertical. In reading the proposition, recall that h is the inverse Phillips curve. The hypothesis of the proposition imposes an upper bound on the absolute value of the elasticity of h and this imposes a lower bound on the absolute value of the elasticity of the Phillips curve itself.

Proposition 4. *Suppose that the inverse Phillips curve h satisfies the restrictions:*

$$h(\widehat{\pi}; \phi) \geqslant \widehat{\pi}^{-\phi} \text{for all } \widehat{\pi} > 1$$

$$h(\widehat{\pi}; \phi) \leqslant \widehat{\pi}^{-\phi} \text{for all } \widehat{\pi} < 1$$

Consider a model $(X, M, \widehat{R}, \Gamma, \phi)$ with price rigidities in which $\phi > 0$ and $\widehat{R}(\widehat{\pi}, r^{NAT}) = 1$ for all $\widehat{\pi}$ and all r^{NAT}. There exists an equilibrium (x^{MON}, m^{MON}, π) in which there is a positive probability event A in F_t such that:

$$(\ln m_T^{MON}(\omega) - \ln m_T^{REAL}(\omega)) > B\phi \text{ for all } \omega \text{ in } A \tag{40}$$

where B is a positive constant. In any such equilibrium:

$$(\ln m_t^{MON}(\omega) - \ln m_t^{REAL}(\omega)) > B\phi(1 + 1/\phi)^{T-t} \text{ for all } \omega \text{ in } A \qquad (41)$$

There is also an equilibrium (x^{MON}, m^{MON}, π) *in which there is a positive probability event A in* F_t *such that:*

$$(\ln m_T^{MON}(\omega) - \ln m_T^{REAL}(\omega)) < -B\phi \text{ for all } \omega \text{ in } A \qquad (42)$$

where B is a positive constant. In any such equilibrium:

$$(\ln m_t^{MON}(\omega) - \ln m_t^{REAL}(\omega)) < -B\phi(1 + 1/\phi)^{T-t} \text{ for all } \omega \text{ in } A \qquad (43)$$

Proof. I will prove the lower bound result; the upper bound result can be proved similarly. Pick some $\widetilde{\pi}_T$ such that $h(\widetilde{\pi}_T(\omega)/\pi^*; \phi) > exp(B\phi)$ for almost all ω in an event $A \in F_t$. From Proposition 2, there exists an equilibrium (m^{MON}, x^{MON}, π) in which $\pi_T = \widetilde{\pi}_T$. In that equilibrium, the inverse Phillips curve implies that:

$$(\ln m_T^{MON}(\omega) - \ln m_T^{REAL}(\omega)) > B\phi \text{ for almost all } \omega \text{ in } A \qquad (44)$$

Now consider any equilibrium in which:

$$(\ln m_T^{MON}(\omega) - \ln m_T^{REAL}(\omega)) > B\phi \qquad (45)$$

for almost all ω in A. I proceed by reverse induction. Let $\widehat{m}_s = m_s^{MON}/m_s^{REAL}$ and suppose that $\ln \widehat{m}_{s+1}(\omega) > B(1 + 1/\phi)^{T-s-1}$ for any $(s+1) > t$ and almost all ω in A. Note that since $\widehat{m}_{s+1}(\omega) > 1$ for almost all ω in A, $\widehat{\pi}_{s+1}(\omega) < 1$ for almost all ω in A. We know from the asset pricing equation and the restrictions on h that for almost all

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ω in A:

$$\widehat{m}_s(\omega) \;=\; E^{NAT}(\widehat{m}_{s+1}\widehat{\pi}_{s+1}^{-1}|F_s)(\omega) \tag{46}$$

$$\geq\; E^{NAT}(\widehat{m}_{s+1}^{1+1/\phi}|F_s)(\omega) \tag{47}$$

Hence:

$$\widehat{m}_s(\omega) > \exp(B\phi(1+1/\phi)^{T-s}) \tag{48}$$

for almost all ω in A. □

The assumption about h says that the slope of the Phillips curve (as drawn with $\ln\widehat{\pi}_t$ on the vertical axis and $\ln\widehat{m}_t$ on the horizontal axis) is more negative than $-1/\phi$, so that the Phillips curve is nearly vertical if ϕ is small. In that case, agents expect that any current marginal utility gap ($\ln\widehat{m}_t$) will vanish rapidly. But the contrapositive of this statement is that, if people believe that there will still be a marginal utility gap in many periods, the gap today must be very large. The size of the current gap is highly sensitive to variations in agents' beliefs about the long-run marginal utility gap. The sensitivity is decreasing in ϕ - that is, the current marginal utility gap is more sensitive with respect to the long run if the Phillips curve is close to vertical.

Proposition 4 stresses a lack of continuity[7] with respect to ϕ. However, the persistence of marginal utility gaps *is* continuous with respect to ϕ. When ϕ is positive, but near zero, agents expect a marginal utility gap at date t to shrink over time, with a rate of contraction that is decreasing in ϕ. When ϕ is exactly zero, then any gap disappears instantaneously. It is this kind of continuity that, for example, Golosov and Lucas (2007) have in mind when they argue that money is nearly neutral for plausible

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[7]The term "lack of continuity" is not literally true. Proposition 1 demonstrates that, with a vertical Phillips curve, the equilibrium real allocation is the same as in the purely real model. However, Proposition 1 presumes that the logged (gross) inflation rate is finite. In Proposition 4, when the nominal interest rate is pegged, logged gross inflation converges to infinity in absolute value as the slope of the Phillips curve converges to infinity in absolute value. Proposition 1 imposes no restriction on real equilibrium allocations in these limiting cases.

estimates of menu costs.

The point of Proposition 4 (and this paper) is that we cannot gauge the neutrality of money only through forward-looking metrics like the persistence of marginal utility gaps. Indeed, it is exactly when money is nearly neutral in a forward-looking sense that it is highly non-neutral in a backward-looking sense, so that current marginal utility gaps are in fact highly sensitive to long-run beliefs about real outcomes. Using purely real models will miss this latter backward-looking non-neutrality.

Following Atkeson and Ohanian (2001), a number of authors have obtained estimates indicating that, in aggregate data, the Phillips curve is nearly horizontal. If the inverse Phillips curve h is log-linear, then this characterization corresponds to setting the power ϕ to be very large. The proof of Proposition 4 demonstrates that in this case, current marginal utility gaps still depend on agents' beliefs about long-run gaps under an interest rate peg. However, the degree of dependence is typically much smaller when the Phillips curve is near-horizontal. For example, if $\widehat{R}(\widehat{\pi}, r^{NAT})$ equals one (as assumed in Proposition 4) and $1/\phi$ is set equal to zero, then the marginal utility gap process is a martingale with respect to the measure defined by Q^{NAT}. Current marginal utility gaps respond only one-for-one to changes in beliefs about long-run gaps.

5.2   Active Interest Rate Rule

A nominal interest rate peg like that studied above is a well-known example of a passive monetary policy rule. The next proposition shows that the fragility result does not apply if monetary policy is active, in the sense that the nominal interest rate adjusts more than one-for-one with changes in the inflation rate.

Proposition 5. *Suppose that the inverse Phillips curve $h(\widehat{\pi}; \phi) = \widehat{\pi}^{-\phi}$. Consider a model $(X, M, \widehat{R}, \Gamma, \phi)$ with price rigidities in which $\phi > 0$ and $\widehat{R}(\widehat{\pi}, r^{NAT}) = \widehat{\pi}^{\gamma}$ for all $\widehat{\pi}$ and all r^{NAT} for some $\gamma > 1$. There exists an equilibrium (x^{MON}, m^{MON}, π) in which there are two constants $b < 0$ and $B > 0$ such that:*

$$\Pr(b \leq \ln(m_T^{MON}) - \ln(m_T^{REAL}) \leq B) = 1 \tag{49}$$

In any such equilibrium, at any date t:

$$\Pr(b[(\phi+1)/(\phi+\gamma)]^{T-t} \leq \ln(m_t^{MON}) - \ln(m_t^{REAL}) \leq B[(\phi+1)/(\phi+\gamma)]^{T-t}) = 1 \tag{50}$$

Proof. The existence of such an equilibrium follows trivially from Proposition 2. To establish the rest of the proposition, we can again proceed by reverse induction. The claim is true for $t = T$. Suppose it is true for $t + 1$, and let $\widehat{m}_s = m_s^{MON}/m_s^{REAL}$ for all s. Then, the aggregate Euler equation and the definition of h implies:

$$\widehat{m}_t^{1+\gamma/\phi} = E^{NAT}(\widehat{m}_{t+1}^{1+1/\phi}|F_t) \tag{51}$$

$$\leq exp((1/\phi+1)[(\phi+1)/(\phi+\gamma)]^{T-t-1}) \text{ w.p. } 1 \tag{52}$$

and so:

$$\widehat{m}_t \leq exp(B[(\phi+1)/(\phi+\gamma)]^{T-t}) \text{ w.p. } 1 \tag{53}$$

We can readily prove the lower bound result in the same fashion. $\square$

With an active policy rule, both the upper and lower bounds on $\ln(\widehat{m}_t)$ converge to zero as $(T-t)$ converges to infinity. The key is that, with an active monetary policy rule, a period t gap $\widehat{m}_t$ can only survive in equilibrium if agents expect an even larger gap next period. The current gap is consequently not all that sensitive to the size of the long-run gap.

Note that this limiting behavior with respect to $(T-t)$ is independent of the slope of the Phillips curve $(-1/\phi)$. However, the rate of convergence with respect to $(T-t)$ is faster when ϕ is near zero - that is, when the Phillips curve is closer to vertical. If ϕ is near infinity (as suggested by the empirical estimates of the Phillips curve mentioned earlier), then the rate of convergence is relatively slow. In this "nearly flat" Phillips

curve case, the dependence of $\widehat{m}_t$ on long-run economic activity can be substantial, unless $(T-t)$ is very large.

This proposition underscores the potential importance of monetary policy in stabilizing the macro-economy. As we have seen, under an interest rate peg, the marginal utility gap $\widehat{m}_t$ fluctuates in response to shocks to agents' beliefs about the long-run gap $\widehat{m}_T$. This problem is actually more pronounced when prices are more flexible. Active monetary policy can eliminate the influence of the distant future on current outcomes. It is actually more effective in doing so when prices are more flexible.

5.3 Interest Rate Rule with a Lower Bound

We have seen that, if the interest rate rule is active, then the current equilibrium level of economic activity is essentially independent of agents' beliefs about long-run gaps. However, any central bank that faces a lower bound on the nominal interest rate is forced to follow a rule that is in some sense passive. In this subsection, I examine the effects of this kind of passivity on economic outcomes.

I prove two propositions about interest rate rules with a lower bound. The first proposition uses a specific equilibrium selection to rationalize the claim made by some[8] that the long-run level of inflation is *increasing* in the size of the lower bound.

Proposition 6. *Suppose that the inverse Phillips curve* $h(\widehat{\pi}; \phi) = \widehat{\pi}^{-\phi}$. *Consider a model* $(X, M, \widehat{R}, \Gamma, \phi)$ *with price rigidities in which* $\phi > 0$ *and the interest rate rule* $\widehat{R}$ *takes the following form:*

$$\widehat{R}(\widehat{\pi}, r_{nat}) = \widehat{R}_{LB} \text{ if } \widehat{\pi} \leq \pi_+ \tag{54}$$

$$\widehat{R}(\widehat{\pi}, r_{nat}) = (\widehat{\pi}/\widehat{\pi}_+)^\gamma \widehat{R}_{LB}, \text{ if } \widehat{\pi} > \widehat{\pi}_+, \text{ where } \gamma \geq 1 \tag{55}$$

where $\widehat{R}_{LB} < \widehat{\pi}_+ < 1$. *There exists an equilibrium* (x^{MON}, m^{MON}, π) *in which* $\widehat{\pi}_t = \widehat{R}_{LB}$

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[8]See, among others, Schmitt-Grohe and Uribe (2013) and Cochrane (2015).

and $\widehat{m}_t > 1$ *with probability one for all t.*

Proof. Set:

$$\widehat{\pi}_t = \widehat{R}_{LB} \text{ with probability one for all } t$$

$$\widehat{m}_t = h(\widehat{R}_{LB}; \phi) \text{ with probability one for all } t$$

The specification of $\widehat{\pi}_t$ implies that:

$$\widehat{R}(\widehat{\pi}_t, r_{nat}) = \widehat{R}_{LB}$$

with probability one for all t. It is readily verified that $(\widehat{\pi}, \widehat{m})$ satisfy the equilibrium conditions (28)-(30). We can use the other equilibrium conditions (25)-(27) to define (x^{MON}, m^{MON}, π). Since $\widehat{\pi}_t < 1$ for all t with probability one, we know that $\widehat{m}_t > 1$ for all t with probability one. $\qquad\square$

In this equilibrium, inflation is stuck at an unduly low level (because $\widehat{\pi}_t = \widehat{R}_{LB} < 1$ with probability one.). Raising $\widehat{R}_{LB}$ would raise $\widehat{\pi}$ and lower $\widehat{m}$ closer to one. In this sense, it appears desirable to raise the lower bound $\widehat{R}_{LB}$.

But this proposition relies on a particular equilibrium selection (imposing the requirement that equilibrium inflation is constant over dates and states). As the next proposition shows, there is in fact a continuum of equilibria. This set of equilibrium outcomes responds in an intuitive fashion to an increase in $\widehat{R}_{LB}$. As well, within any one of these equilibria, equilibrium outcomes are arbitrarily sensitive to beliefs about downside long-run outcomes when the Phillips curve is arbitrarily close to vertical.

Proposition 7. *Suppose that the inverse Phillips curve h satisfies the restrictions:*

$$h(\widehat{\pi}; \phi) \geq \widehat{\pi}^{-\phi} \text{ for all } \widehat{\pi} > 1$$

$$h(\widehat{\pi}; \phi) \leq \widehat{\pi}^{-\phi} \text{ for all } \widehat{\pi} < 1$$

Consider a model $(X, M, \widehat{R}, \Gamma, \phi)$ with price rigidities in which $\phi > 0$ and there exists a constant $\widehat{R}_{LB} < 1$ such that $\widehat{R}(\widehat{\pi}, r^{NAT}) \geq \widehat{R}_{LB}$ for all $\widehat{\pi}$ and r^{NAT}. There is an equilibrium (x^{MON}, m^{MON}, π) in which there is a positive probability event A in F_t such that for almost all ω in A:

$$(\ln m_T^{MON}(\omega) - \ln m_T^{REAL}(\omega)) > B\phi \tag{56}$$

where B satisfies $(B + ln\widehat{R}^{LB}) > 0$. For any such equilibrium, for almost all ω in A:

$$\ln m_t^{MON}(\omega) - \ln m_t^{REAL}(\omega) > \phi(1 + \phi^{-1})^{T-t}[B + \ln \widehat{R}^{LB}] - \phi \ln \widehat{R}^{LB} \tag{57}$$

Proof. Let $\widetilde{\pi}_T$ be an F_T-measurable random variable that satisfies the restriction $h(\widetilde{\pi}_T(\omega)/\pi^*; \phi) > \exp(B\phi)$ for almost all ω in A, where $A \in F_t$. Proposition 2 implies that there exists an equilibrium (x^{MON}, m^{MON}, π) such that $\pi_T = \widetilde{\pi}_T$ with probability one. In that equilibrium, for almost all ω in A:

$$\ln m_T^{MON}(\omega) - \ln m_T^{REAL}(\omega) > B\phi \tag{58}$$

Now consider any equilibrium of this kind, and define $\widehat{m}_s = m_s^{MON}/m_s^{REAL}$. I proceed inductively. Assume that $\widehat{\pi}_s(\omega) < 1$ and:

$$ln(\widehat{m}_s(\omega)) > \phi(1 + 1/\phi)^{T-s}[B + ln\widehat{R}^{LB}] - \phi \ln \widehat{R}^{LB} > 0 \tag{59}$$

for all $s \geqslant \tau + 1$, where $\tau \geq t$, and almost all ω in A. Note that A is in F_τ. Hence, the asset pricing equation and the restrictions on the inverse Phillips curve imply that for

almost all ω in A:

$$
\begin{aligned}
\widehat{m}_\tau(\omega) &= \widehat{R}_{LB} E^{NAT}(\widehat{\pi}_{\tau+1}^{-1}\widehat{m}_{\tau+1}|F_\tau)(\omega) \\
&\geq \widehat{R}_{LB} E^{NAT}(\widehat{m}_{\tau+1}^{1+1/\phi}|F_\tau)(\omega) \\
ln(\widehat{m}_\tau(\omega)) &\geq \ln\widehat{R}_{LB} + (1+1/\phi)E^{NAT}(\ln\widehat{m}_{\tau+1}|F_\tau)(\omega)
\end{aligned}
$$

where the last step uses Jensen's inequality. Substituting in the lower bound on $ln(\widehat{m}_{\tau+1})$, we get:

$$
\begin{aligned}
\ln\widehat{m}_\tau(\omega) &\geq \ln\widehat{R}_{LB} + (1+1/\phi)^{T-\tau}(B\phi + \phi ln\widehat{R}^{LB}) - \phi(1+1/\phi)\ln\widehat{R}^{LB} &&(60) \\
&= (1+1/\phi)^{T-\tau}(B\phi + \phi ln\widehat{R}^{LB}) - \phi\ln\widehat{R}^{LB} &&(61) \\
&&&(62)
\end{aligned}
$$

for almost all ω in A. This implies that $\ln\widehat{m}_\tau(\omega) > 0$ and that $\widehat{\pi}_\tau(\omega) < 1$ for almost all ω in A , which completes the inductive step. $\qquad\square$

For $t < T$, there is a lower bound on $\ln\widehat{m}_t$ which is *increasing* in $\ln\widehat{R}^{LB}$ and in beliefs about highly adverse (that is, positive) realizations of the long-run marginal gap $\ln\widehat{m}_T$. The degree of sensitivity of the lower bound to both $\ln\widehat{R}^{LB}$ and to long-run downside risks becomes more pronounced if ϕ is near zero (that is, the Phillips curve is near-vertical). The key to this result lies in the behavior of expected inflation. Definitionally, a near-vertical Phillips curve implies that even tiny expected future marginal utility gaps are associated with very high rates of anticipated deflation (very negative inflation). In this way, small long-run marginal utility gaps are associated with high real interest rates, and large current marginal utility gaps.

5.4 Summary

When the interest rate rule takes the form of a peg, current equilibrium outcomes in models with a near-vertical Phillips curve are highly sensitive to changes in agents' beliefs about long-run outcomes. The degree of sensitivity is, in fact, increasing as the Phillips curve becomes closer to vertical. An active interest rate rule can essentially eliminate this dependence on the long run. However, if the interest rate rule is bounded from below, then current equilibrium outcomes are highly sensitive to downside shocks to beliefs about long-run economic activity. The degree of sensitivity is, again, increasing as the Phillips curve becomes closer to vertical.

It is common in the New Keynesian literature to restrict attention to monetary policy rules that imply that equilibrium is determinate. The above analysis suggests that, under this restriction, current outcomes are essentially unaffected by beliefs about the long run. As a result, real equilibrium outcomes in a model with a near-vertical Phillips curve will be similar to those in a model with a vertical Phillips curve. However, this restriction on monetary policy rules seems unrealistic. Clarida, Gali, and Gertler (2000) argue that the Federal Reserve followed a passive monetary policy rule during 1970s. As well, as long as there is an effective lower bound on nominal interest rates, the central bank is forced to follow a rule that is, in some sense, passive.

The above analysis hinges on the aggregate Euler equation, which provides a tight connection between the distant future and current outcomes. As Del Negro, Giannoni and Patterson (2013) and MacKay, Nakamura, and Steinsson (2015) have emphasized, this tight connection implies that forward guidance is a surprisingly powerful monetary policy tool. The former authors suggest relaxing this connection by assuming that agents' lifetimes are, on average, shorter than the life of the economy. The latter authors suggest relaxing this connection via binding borrowing constraints. More recently, Gabaix (2016) has explored a model in which agents are boundedly rational. In all of these models, the authors argue that the relevant log-linearized Euler equation appears

to feature a form of "discounting". I will translate that discounting into my class of models by assuming that aggregate marginal utilities satisfy a collection of normalized Euler equations of the form:

$$\widehat{m}_t = \widehat{R}(\widehat{\pi}_t, r_t^{NAT}) E^{NAT}(\frac{\widehat{m}_{t+1}^{\delta}}{\widehat{\pi}_{t+1}} | F_t), 0 < \delta < 1 \tag{63}$$

We can readily extend the proofs of the key fragility results (Propositions 4 and 7) to cover the case in which the Euler equations take this discounted form. It is true that the discounting in (63) weakens the connection between current outcomes and beliefs about the future. Nonetheless, households still expect any current marginal utility gap to disappear exponentially quickly when the Phillips curve is near-vertical and the interest rate rule is passive. This expectation means, again, that current marginal utility gaps are highly sensitive to beliefs about long-run gaps.[9]

6 Building Useful Macroeconomic Models

The previous section shows that, for particular monetary policy regimes, the predictions of a purely real model change dramatically if we add a slight amount of nominal frictions to the model. These results imply that purely real models are too *incomplete* to provide robust predictions about the macroeconomy. In this section, I describe three modeling ingredients that we need to add to purely real models if they are to deliver robust predictions: nominal rigidities, a reaction function for the central ban, and beliefs about long-run outcomes.

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[9]In a recent paper, Kaplan, Moll, and Violante (KMV) (2015) explore the properties of what they term HANK (Heterogenous Agent New Keynesian) models. In these models, many households have zero liquid assets. As a result (and as KMV emphasize), the standard intertemporal substitution forces are much weaker. Nonetheless, I conjecture that the main fragility results in this paper (Propositions 4 and 7) would generalize to KMV's framework. In particular, it is still true in HANK models that when the Phillips curve is arbitrarily near-vertical (so that ε/θ is near infinity in KMV's notation), expectations of tiny future marginal gaps are associated with expectations of arbitrarily negative inflation. The resulting infinitely high real interest rates will have a crushing effect on consumption demand and economic activity. See also Auclert (2016).

6.1 Nominal Rigidities

A model with a near-vertical Phillips curve, and a fixed interest rate, has a near-zero impulse response from current gaps to future outcomes. This kind of relationship between the present and future is well-approximated by purely real models with a vertical Phillips curve. However, as we have seen, a model with a near-vertical Phillips curve predicts that current gaps are highly sensitive to information about future gaps. Purely real models zero out this sensitivity. Hence, we can only usefully approximate the feedback of the future to the present by using models that include an explicit treatment of nominal rigidities.

It is probably worth emphasizing that the magnitude of this feedback is especially dependent on the degree of nominal rigidity, as captured by the slope of the Phillips curve, when prices are highly flexible. In this sense, the behavior of the macroeconomy may be highly sensitive to the frequency with which firms adjust their prices.[10]

6.2 Central Bank Reaction Function

The above argument relies on the central bank's reaction function being a peg. With an active reaction function, in which the interest rate moves more than one-for-one with the inflation rate, a gap can only exist today if agents expect an even larger future gap. In this case, as we have seen in Proposition 5, the current gap displays relatively little sensitivity to information about the size of gaps in the distant future. Hence, accurate models of how information about the future affect current outcomes need to include an accurate model of the central bank's reaction function. Recently, for example, some central banks have successfully lowered their target interest rates below zero. Relaxing these constraints may matter for how downside long-run risk affects current outcomes.

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[10]Bils and Klenow (2004) initiated a large literature that provides microeconometric estimates of the frequency of price changes by firms. See Klenow and Malin (2011) for a survey of that literature.

6.3 Beliefs about Long-Run Macroeconomic Outcomes

In this subsection, I discuss when and how macroeconomic models should be explicit about the evolution of beliefs about the long run.

Propositions 4 and 7 demonstrate that current outcomes are extremely sensitive to beliefs about the long run. This finding is a consequence of two assumptions: the forward-looking behavior built into the Euler equation and the near-vertical Phillips curve. The sensitivity vanishes if we relax both assumptions in tandem.[11]

Suppose for example that the Euler equation takes the discounted form described earlier:

$$\widehat{m}_t = \widehat{R}(\widehat{\pi}_t, r_t^{NAT}) E^{NAT}(\frac{\widehat{m}_{t+1}^{\delta}}{\widehat{\pi}_{t+1}}|F_t), 0 < \delta < 1 \tag{64}$$

Suppose too that $\widehat{R}(\widehat{\pi}_t, r_t^{NAT}) = R^*$ is constant. Then, if we substitute a log-linear inverse Phillips curve into (64), we obtain:

$$\widehat{m}_t = R^* E^{NAT}(\widehat{m}_{t+1}^{\delta+1/\phi}|F_t), 0 < \delta < 1 \tag{65}$$

The nonlinear difference equation (65) has a unique solution if nominal rigidities are sufficiently severe and discounting is sufficiently strong (so that $\phi(1-\delta) > 1$). In this parametric case, current outcomes don't depend on beliefs about the long run.[12]

However, for any δ, this parametric restriction is violated if ϕ is sufficiently small. As I noted earlier, even with this "discounted" Euler equation, current outcomes are extremely sensitive to beliefs about long-run downside risks if the Phillips curve is close to vertical (that is, ϕ is sufficiently small), and there is a lower bound on the nominal interest rate. This observation means that macroeconomic frameworks with highly flexible prices need to include accurate models of the evolution of these beliefs about

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[11]Kocherlakota (2016) considers a class of models with the usual non-discounted Euler equations but with near-horizontal Phillips curves that are not log-linear. I show that in this class of models, current outcomes don't depend on beliefs about the long run.

[12]This restriction seems like it would require a very large value of ϕ to ensure the irrelevance of long-run beliefs.

long-run downside risks, and how policy choices (or communications) might be able to affect that evolution.

Models of beliefs about long-run downside risks could take many forms. In the class of models that I study, expectations are fully rational. Because fiscal policy is Ricardian, there are many possible equilibria in the final period. Agents co-ordinate on an aggregate sunspot to determine which of these equilibria get played. Over time, agents receive information about the realization of the sunspot. It is this information that generates variation in their beliefs about long-run outcomes. As Woodford (1995) argues, other kinds of fiscal policy can be used to winnow down the number of equilibria.[13] Suppose, for example, that the government commits to exchange consumption bundles $\bar{c}$ for money at a fixed price p_T in period T, where $p_T = \pi^* p_{T-1}$, and to peg the nominal interest rate $R_t = \pi^* r_t^{nat}$ in all periods $t < T$. These joint monetary/fiscal commitments ensure that $\widehat{m}_t = 1$ in any period t, and eliminates the indeterminacy at the heart of this paper.[14]Hence, under rational expectations, an accurate model of the feedback from the future to the present requires an accurate model of how fiscal policy responds to the evolution of the price level.

Of course, rational expectations is a strong assumption. It would be useful to explore the implications of other approaches to expectations formation. The recent work by Garcia-Schmidt and Woodford (2015) could be potentially useful along these lines. It would also be useful to complement theoretical models of expectation formation with empirical studies.

I conjecture that infinite horizon models are unlikely to be useful ways to model the evolution of beliefs about the long run. Users of infinite horizon models focus on equilibria in which logged outcomes are finite in absolute value. But under an interest

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[13]See also Kocherlakota and Phelan (1999).

[14]The price level peg in period T could alternatively be formulated in terms of the fiscal theory of the price level (Woodford (1995); Leeper (1991)) as a fiscal commitment to collect real taxes in period T equal to $\dfrac{M_T}{p_{T-1}\pi^*}$.

rate peg, logged gaps are expected to shrink exponentially over time (in absolute value). In an infinite horizon model, logged equilibrium outcomes can only be finite in absolute value if the long-run gap is always zero.[15] Focusing on infinite horizon models, and on finite equilibria, eliminates any variation in beliefs about long-run gaps.

To summarize: macroeconomists have two options when constructing their models. The first option is to incorporate severe forms of nominal rigidities (so that the Phillips curve is nearly horizontal). In this case, as long as other frictions generate sufficient discounting in the log-linearized Euler equation, long-run beliefs will not affect current outcomes. The second option is to allow prices to be highly flexible (so that the Phillips curve is nearly vertical). This latter class of models needs to be explicit about how households form their expectations about the long run.

7   Conclusions

George Box famously wrote that, "All models are false but some are useful." Over the past thirty-plus years, economists have written literally thousands of papers that attempt to analyze macroeconomic data using purely real models. This paper is not about the (well-understood) *empirical* point that these models are *wrong*. This paper reaches a distinct *theoretical* conclusion: purely real models are too fragile to be *useful* as a means of answering questions of interest. To take but one famous example: Kydland and Prescott (1982) conclude that their purely real model explains about 2/3 of business

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[15]To proceed somewhat more formally: suppose that $\widehat{R} = 1$ and that the inverse Phillips curve $h(\widehat{\pi}; \phi) = \widehat{\pi}^{-\phi}$. Then the aggregate Euler equation implies that:

$$\infty > \ln \widehat{m}_t \geq \lim_{T \to \infty} (1 + 1/\phi)^{T-t} E^{NAT}(\ln \widehat{m}_{t+T}|F_t)$$

It follows that

$$\lim_{T \to \infty} E^{NAT}(\ln \widehat{m}_{t+T}|F_t) = 0$$

and so:

$$Var^{NAT}(\lim_{T \to \infty} E^{NAT}(\ln \widehat{m}_{t+T}|F_t) = 0$$

There is no variability in expected long-run gaps, when the model horizon is infinite.

cycle fluctuations. But adding even a slight amount of nominal rigidity, in the form of a slightly non-vertical Phillips curve, would make the model's implied variance of output highly dependent on agents' assessments of long-run risk. Like all other purely real models, Kydland and Prescott's framework ignores key sources of monetary non-neutrality and macroeconomic instability. It is not a useful model of output fluctuations.

Why were so many misled for so long about the value of purely real models? The paper suggests an answer to this question. For a quarter-century (during the Great Moderation), most major central banks outside of Japan could be well-modeled as following an active monetary policy. Under active monetary policy, in which the central bank adjusts the interest rate more than one-for-one with movements in inflation, a model with a vertical Phillips curve is indeed a close approximation to a model with a nearly vertical Phillips curve. There is then little loss in explanatory power by restricting attention to the former class of models.

The point of this paper is that the situation changes completely once we account for the possibility that the current beliefs about long-run risks (combined with current aggregate shocks) might drive the economy into a lower bound on nominal interest rates. The lower bound automatically means that monetary policy cannot respond actively to contingencies created by sufficiently pessimistic long-run beliefs. Once we account for the lower bound, purely real models become a poor - in some sense, an infinitely poor - approximation to models with near-vertical Phillips curves. Lower bounds on nominal interest rates seem likely to be empirically relevant for many years to come.

Purely real models are simply too *incomplete* to be viewed as useful. How then do we build useful models? We need to take explicit account of nominal rigidities that will give rise to non-vertical Phillips curves. We need to take explicit account of the operating frameworks of central banks. We may need to capture the impact of shocks and macroeconomic policy choices on agents' beliefs about long-run economic activity. To my knowledge, there has as yet been no research into the empirical properties of

models that have all of these elements.

Appendix A

In this appendix, I provide an explicit micro-foundation for the kind of Phillips curves studied in this paper. The basic setup is a Tperiod version of the basic New Keynesian framework in Gali (2008, Chapter 3), with a modified form of price-setting.

Households maximize expected discounted utility of the form:

$$E_0 \sum_{t=1}^{T} \beta^{t-1}[ln(C_t) - N_t^{1+\gamma}/(1+\gamma)], 0 < \gamma < 1$$

Here, N_t represents time spent working in period t. The variable C_t is a consumption index given by:

$$C_t \equiv [\int_0^1 C_t(i)^{1-1/\epsilon}di]^{\epsilon/(\epsilon-1)}$$

where $C_t(i)$ is the household's period t consumption of good i. The household must decide how to allocate its consumption expenditures among the different goods. This requires that the consumption index C_t be maximized for any given level of expenditures $\int_0^1 P_t(i)C_t(i)di$, where $P_t(i)$ is the period t price of good i. Gali (2008) shows that the solution to that problem yields the set of demand equations:

$$C_t(i) = (\frac{P_t(i)}{P_t})^{-\epsilon}C_t$$

where $P_t = [\int_0^1 P_t(i)^{1-\epsilon}di]^{1/(1-\epsilon)}$ is an aggregate price index.

The above is the same as in Gali (2008). My treatment of price-setting is different. I assume that a fraction λ of firms (randomly distributed across the various goods) set their price in period t equal to $P_{t-1}\pi^*$, where π^* is some baseline gross inflation rate. A fraction $(1 - \lambda)$ of firms set their price in period t so as to maximize current profits. I assume that these latter firms receive an output subsidy sufficient to offset

the monopolistic distortion.

Using the arguments in Gali (2008), the gross inflation rate then evolves according to the following law of motion:

$$\pi_t^{1-\epsilon} = \lambda\pi^{*1-\epsilon} + (1-\lambda)(\frac{P_t^{flex}}{P_{t-1}})^{1-\epsilon}$$

where P_t^{flex} is the common price chosen by all of the profit-maximizing firms. These firms set their price equal to marginal cost, so that:

$$P_t^{flex} = W_t/A_t$$

where W_t is the nominal wage in period t. (Here, I've assumed that the production function is linear in N_t, with productivity in period t given by A_t.)

Wages are assumed to be perfectly flexible and so:

$$W_t/(P_tC_t) = N_t^\gamma$$

We can re-write the flexible price as:

$$P_t^{flex} = P_t m_t^{-1-\gamma} A_t^{-1-\gamma}$$

In a purely real version of this model, $m_t^{REAL} = A_t^{-1}$. We can plug this expression into the law of motion for inflation to obtain:

$$\widehat{\pi}_t^{1-\epsilon} = \lambda + (1-\lambda)\widehat{\pi}_t^{1-\epsilon}(\widehat{m}_t)^{(\epsilon-1)(1+\gamma)} \tag{66}$$

This equation implies that $\widehat{\pi}_t = (<)1$ if and only if $\widehat{m}_t = (>)1$.

Define:

$$\phi \equiv \frac{\lambda}{(1-\lambda)(1+\gamma)}$$

Then, if we apply the implicit function theorem to (66), we obtain:

$$\frac{dln(\widehat{m}_t)}{dln(\widehat{\pi}_t)} = -\frac{\widehat{m}_t^{(1-\epsilon)(1+\gamma)} - (1-\lambda)}{(1+\gamma)(1-\lambda)} \qquad (67)$$

It follows that if $\widehat{m}_t$ and $\widehat{\pi}_t$ satisfy (66), then

$$\widehat{m}_t \quad < \quad \widehat{\pi}_t^{-\phi} \text{if } \widehat{\pi}_t < 1$$

$$\widehat{m}_t \quad > \quad \widehat{\pi}_t^{-\phi} \text{if } \widehat{\pi}_t > 1$$

as is assumed in Propositions 4 and 7.

This model provides an explicit connection between the parameter ϕ in the main text to the fraction $(1 - \lambda)$ of firms that choose prices so as to maximize profits in each period. As λ converges to zero, ϕ converges to infinity - that is, the Phillips curve becomes vertical.

Intuitively, fix the size of an aggregate demand shortfall in period t, in terms of the size of the marginal utility gap $\widehat{m}_t$. The marginal utility gap translates into a real wage $w_t = \widehat{m}_t^{-\gamma} A_t$ that clears the labor market; that real wage is less than A_t and is independent of the fraction λ of inflexible firms. However, all flexible price-setters adjust their prices so that the real wage, calculated using their price, equals A_t. The low real wage is thus entirely attributable to the inflexible firms setting their prices too high relative to those of the flexible price-setters. To rationalize a given real wage w_t as the fraction of inflexible firms shrinks, the price gap between the two kinds of firms must grow without bound - and that means that the inflation rate must fall without bound below π^*.

This model does not endogenize the parameter λ. However, suppose that each firm has to pay a cost k in order to set its price flexibly (as opposed to $P_{t-1}\pi^*$), and there is a continuous density of those costs across firms. Then, the fraction λ is endogenous, but the impact of a change in the inflation rate on λ is only second-order. The slope of

the Phillips curve is still given by (67).

Appendix B

In this appendix, I describe how to extend propositions 4 and 7 to the case of a forward-looking Phillips curve similar to that used in much of the New Keynesian literature.

Suppose that we change the normalized equilibrium condition (29) to take the form:

$$\widehat{\pi}_t = E_t^{NAT} H(\widehat{m}_t, \widehat{\pi}_{t+1}; \phi)$$
$$\widehat{\pi}_T = H_T(\widehat{m}_T; \phi)$$

where H is strictly decreasing in both arguments, H_T is strictly increasing, and $H(1, 1; \phi) = H_T(1) = 1$. (Note that H and H_T are Phillips curves, not inverse Phillips curves.) Suppose too that H and H_T satisfy the restrictions:

$$H(\widehat{m}, 1; \phi) \ \le \ \widehat{m}^{-1/\phi} \text{ if } \widehat{m} \ge 1 \tag{68}$$

$$H(\widehat{m}, 1; \phi) \ \ge \ \widehat{m}^{-1/\phi} \text{ if } \widehat{m} \le 1 \tag{69}$$

$$H_T(\widehat{m}; \phi) \ \le \ \widehat{m}^{-1/\phi} \text{ if } \widehat{m} \ge 1 \tag{70}$$

$$H_T(\widehat{m}; \phi) \ \ge \ \widehat{m}^{-1/\phi} \text{ if } \widehat{m} \ge 1 \tag{71}$$

These restrictions ensure that the short-term elasticity of the Phillips curve in each period is larger than $1/\phi$ in absolute value.

The above elasticity restrictions imply that the basic inductive logic underlying Proposition 7 is still valid. Suppose that $\widehat{m}_{t+1}(\omega) > exp(B_{t+1})$, where $B_{t+1}/\phi + ln\widehat{R}_{LB} >$

0, and that $\widehat{\pi}_{t+2}(\omega) > 1$ for almost all ω in a set A in F_t. We can show that:

$$
\begin{aligned}
ln(\widehat{m}_t(\omega)) \quad &\geq \quad ln\widehat{R}_{LB} + lnE^{NAT}(\widehat{m}_{t+1}\widehat{\pi}_{t+1}^{-1}|F_t)(\omega) \\
&\geq \quad ln\widehat{R}_{LB} + lnE^{NAT}(\widehat{m}_{t+1}^{1+1/\phi}|F_t)(\omega) \\
&\geq \quad ln\widehat{R}_{LB} + (1+1/\phi)E^{NAT}(ln(\widehat{m}_{t+1})|F_t)(\omega) \\
&\geq \quad ln\widehat{R}_{LB} + (1+1/\phi)B_{t+1} \\
&> 0
\end{aligned}
$$

It follows that $\widehat{\pi}_{t+1}(\omega) = H(\widehat{m}_{t+1}(\omega), \widehat{\pi}_{t+2}(\omega); \phi) < 1$ and that $\widehat{m}_t(\omega) > exp(B_t) \equiv$ $exp(ln(\widehat{R}_{LB}) + B_{t+1}/\phi)exp(B_{t+1}) > exp(B_{t+1})$ for almost all ω in A. We can use this basic inductive argument to establish the validity of the lower bound on marginal utility gaps in Proposition 7.

We can use the same assumptions about H and H_T to prove an analog to Proposition 4. Thus, suppose that $\widehat{R}(.,.) = 1$ for all values of its arguments. Then, in equilibrium:

$$
\widehat{m}_t = E^{NAT}(\widehat{m}_{t+1}/\widehat{\pi}_{t+1}|F_t)
$$

Assume, inductively, that $\widehat{m}_{t+1}(\omega) > 1$ and $\widehat{\pi}_{t+2}(\omega) \leq 1$ for almost all ω in A, where A is in F_t. The restrictions on H imply that, for almost all ω in A:

$$
\widehat{\pi}_{t+1}(\omega)^{-1} \geq \widehat{m}_{t+1}(\omega)^{1/\phi} \geq 1.
$$

It follows that for almost all ω in A:

$$
ln(\widehat{m}_t(\omega)) \geq (1+1/\phi)ln(\widehat{m}_{t+1}(\omega)),
$$

and so $\widehat{m}_{t+1}(\omega) > 1$ and $\widehat{\pi}_{t+2}(\omega) \leq 1$ for almost all ω in A. We can use this inductive logic to derive the lower (and upper) bounds on marginal utility gaps established in

Proposition 4.

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