Leverage Dynamics without Commitment

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March 2017

We analyze equilibrium leverage dynamics in a dynamic tradeoff model when the firm is unable to commit to a leverage policy ex ante. We develop a methodology to characterize equilibrium equity and debt prices in a general jump-diffusion framework, and apply our approach to the standard Leland (1998) setting. Absent commitment, the leverage ratchet effect (Admati et al. 2015) distorts capital structure decisions, leading shareholders to take on debt gradually over time and never voluntarily reduce debt. On the other hand, countervailing effects of asset growth and debt maturity cause leverage to mean-revert towards a long run target. In equilibrium, bond investors anticipate future leverage increases and require significant credit spreads even for firms with currently large distance-to-default. As a result, the tax benefits of future debt increases are fully dissipated, and equilibrium equity values match those in a model where the firm commits not to issue new debt.

In our model, leverage is dependent on the full history of the firm's earnings. Despite the absence of transactions costs, an increase in profitability causes leverage to decline in the short-run, but the rate of new debt issuance endogenously increases so that leverage ultimately mean-reverts. The target level of leverage, and the speed of adjustment depends critically on debt maturity; nonetheless, in equilibrium shareholders are indifferent toward the debt maturity structure.

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* DeMarzo: Stanford University, Graduate School of Business, and NBER. He: University of Chicago, Booth School of Business, and NBER. We thank Anat Admati, Harry DeAngelo, Douglas Diamond, Mike Fishman, Konstantin Milbradt, Paul Pfleiderer, Adriano Rampini, Martin Szydlowski, Josef Zechner, and seminar participants at Duke, London School of Economics, MIT, NYU, Princeton, Stanford, USC, WFA, and FMA 2017 for helpful comments. Zhiguo He thanks the Fama-Miller Center for Research in Finance at the University Of Chicago Booth School Of Business for research support and (name of RA) for research assistance.

1. Introduction

Understanding the determinants of a firm's capital structure, and how its leverage is likely to evolve over time, is one of the central questions in corporate finance. Leverage and its expected dynamics are crucial to valuing the firm, assessing its credit risk, and pricing its financial claims. The optimal response of leverage to shocks, such as the 2007-2008 financial crisis, are important to forecasting the likely consequences of the crisis and its aftermath, and to evaluate alternative policy responses.

Despite its importance, a fully satisfactory theory of leverage dynamics has yet to be found. Many models assume the absolute level of debt is fixed; for example, in the traditional framework of Merton (1974), as well as Leland (1994, 1998), the firm is committed not to change its outstanding debt before maturity, irrespective of the evolution of the firm's fundamentals. As a result, the dynamics of firm leverage is driven solely by the stochastic growth in value of the firm's assets-in-place. More recent work that allows the firm to restructure its debt over time typically assumes that all existing debt must be retired (at a cost) before any new debt can be put in place.[1] These assumptions are neither innocuous, as the constraints on leverage generally bind in the model, nor are they consistent with practice, where firms often borrow incrementally over time. See, for example, Figure 1, which shows how debt levels for American and United Airlines changed over time in response to fluctuations in their enterprise values (market value of equity plus book value of debt).

In contrast, we study a model in which equity holders lack the ability to commit to their future leverage choices, and can issue or buyback debt at the current market price at any time. Aside from corporate taxes and bankruptcy costs, there are no other frictions or transactions costs in our model. In such a setting, when debt can be freely adjusted over time, it is feasible for the firm to avoid the standard leverage "tradeoff" by simply increasing debt to exploit tax shields when cash flows are high, and reducing debt to avoid distress costs when cash flows fall.

[1] See e.g. Fischer, Heinkel, Zechner (1989), Titman and Tsyplakov (2007), Goldstein, Ju, and Leland (2001) and Strebulaev (2007), Dangl and Zechner (2016).

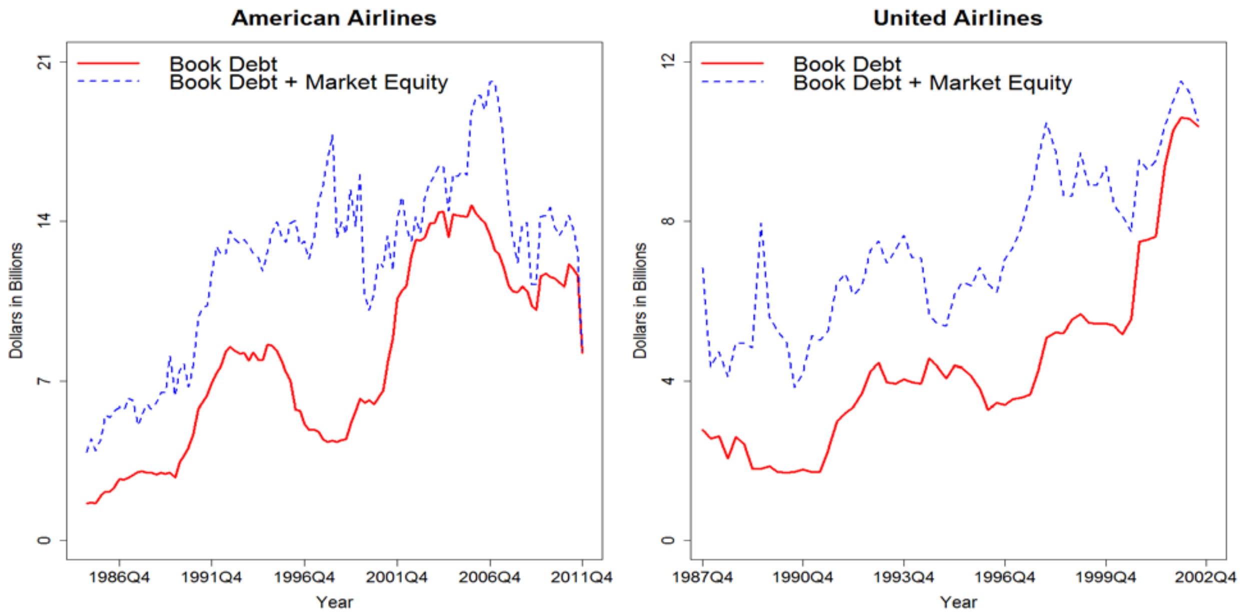

Figure 1. Time-series of outstanding book debt and enterprise value for American and United Airlines, for fifteen years before their bankruptcies in 2011 and 2002, respectively. Book debt is calculated as the sum of "long-term debt" and "debt in current liabilities", and market equity is calculated as "stock price" multiplying "common shares outstanding." Data source: WRDS.

But although such an ideal policy is feasible, absent commitment an important agency friction emerges with regard to the firm's future leverage choices. As emphasized by Admati et al. (2016), equity holders will adjust leverage to maximize the current share price rather than total firm value. They demonstrate a "leverage ratchet effect," in which equity holders are never willing to voluntarily reduce leverage, but always have an incentive to borrow more -- even if current leverage is excessive and even if new debt must be junior to existing claims. While the leverage ratchet effect is itself quite general, they calculate numerically a dynamic equilibrium only for a specialized model in which debt is perpetual and the firm does not grow but is subject to Poisson shocks.

Solving the dynamic tradeoff model without commitment is challenging because of the dynamic interdependence of competitive debt prices today and equity's equilibrium leverage/default policies in the future. In this paper we develop a methodology to solve for such an equilibrium in a general setting that allows for finite maturity debt, asset growth, investment, and both Brownian and Poisson shocks. In this equilibrium, equity holders increase debt gradually over time, at a rate which increases with the current profitability of the firm. On the other hand, following negative shocks, equity holders never voluntarily reduce leverage, but only allow it to decline passively via debt maturity and asset growth.

In our model, equity holders keep issuing debt to exploit tax benefits even after the firm's leverage passes above the "optimal" level with commitment, leading to excessive inefficient default. This result holds even when there is no dilution motive to issue debt (either because debt is prioritized or there is zero recovery value in bankruptcy). Interestingly, we show that equity holders obtain the same value in equilibrium as if they *commit not to issue any debt in the future,* but over-borrowing raises the probability of default and so lowers the price of debt in equilibrium. This low debt price offsets the tax advantage of leverage sufficiently so that, on the margin, equity holders are indifferent to leverage increases. In other words, the extra tax shield benefits that tempt equity holders are exactly dissipated by the bankruptcy cost caused by excessive leverage.

We apply our methodology to the special case of geometric Brownian motion (as in Leland 1994) and solve for the equilibrium debt price and issuance policy in closed form. Because equity holders refuse to buy back debt once it is issued, debt issuance is effectively irreversible in equilibrium, which, as a result, slows its initial adoption. Debt accumulates over time at a rate that increases with profitability, while if profits decline sufficiently, new issuance falls below the rate of debt maturity and the debt level falls. Leverage thus becomes path dependent, and we show explicitly that the firm's outstanding debt at any point of time can be expressed as a weighted average of the firm's earnings history. The endogenous adjustment of leverage leads the firm's interest coverage ratio to mean revert gradually in equilibrium, with the speed of adjustment decreasing with debt maturity and asset volatility. These dynamics differ from abrupt adjustment to a "target" leverage level, a common implication from models with an exogenous adjustment cost (for instance, Fischer, Heinkel, Zechner, 1989; Goldstein, Ju, Leland, 2001; and Strebulaev, 2007; etc.).

We compare our model without commitment to two benchmarks with full commitment. In the first case equity holders commit not to issue any debt in the future, and in the second case they commit to maintain a constant outstanding debt obligation (i.e., always issue the same amount of new debt to replace debt that is maturing) as in Leland (1998). The central difference between our model and these two benchmark models is the endogenous mean-reverting firm leverage and its implications on equilibrium debt prices, as bond investors anticipate future leverage changes. Recall that in response to positive profitability shocks the firm issues more debt, pulling the firm back toward default. In fact, the mean-reversion is sufficiently strong that credit spreads remain strictly bounded away from zero even when the firm's current interest coverage ratio is arbitrarily large. In contrast, in standard models such as Leland (1998), the firm's interest coverage ratio follows a geometric Brownian motion and hence implied credit spreads vanish as the firm's "distance to default" increases.

We also study the optimal debt maturity structure, which we model in terms of a constant required repayment (or amortization) rate stipulated in the debt contract. Our model without commitment to future leverage policies provides a fresh perspective on this question. We show that equity holders are indifferent to the maturity structure of the firm's *future* debt issuance. Short maturity debt leads to higher leverage on average, as equity holder issue debt more aggressively knowing leverage can be reversed when debt matures. Nevertheless, the gain from additional tax shields is offset by increased default costs. Thus, the agency costs associated with the leverage ratchet effect persist even as debt maturity becomes instantaneous.

The choice of debt maturity structure does affect the value of equity if the firm is forced to borrow a fixed amount upfront. Indeed, this question has been studied in the Leland (1998) setting, and often long-term debt, which minimizes rollover risk, is preferred (He and Xiong, 2012; Diamond and He, 2014). In contrast, we show that without commitment, firms prefer short-term debt for any positive targeted debt financing. Shareholders of a firm with shorter-term debt are more willing to allow leverage to decline following negative shocks, and this future equilibrium leverage policy lowers the required default premium today. Of course, longer-term debt is preferred from a social perspective, which lowers the bankruptcy cost.

Finally, we consider the interaction of the firm's leverage and investment policies. When the firm cannot commit to its investment policy, leverage distorts investment due to debt overhang. Compared to a fixed debt policy as in Leland (1998), the no commitment leverage policy leads to greater under-investment (due to debt overhang) when profitability is high, but less under-investment when profitability is low. Even more interesting is the effect of agency frictions in the investment decision on the firm's leverage policy. We show that when the firm does not commit to its future investment decisions, the leverage ratchet effect becomes more severe and the rate of debt issuance increases: Shareholders are even more tempted to increase debt if they can simultaneously cut investment later on if they wish.

Our paper is most closely related to Admati et al. (2016). They demonstrate the leverage ratchet effect in the context of a one-time leverage adjustment, and then numerically evaluate a dynamic equilibrium in a stationary model with regime shocks and perpetual debt. Our paper studies leverage dynamics in a much richer continuous-time framework that allows for both asset growth and debt maturity, as well as both Brownian and Poisson shocks. We develop a general methodology to solve for an important class of equilibria, and for the standard workhorse model of Leland (1994), we solve for the equilibrium in closed-form, allowing for deeper analysis.

In Dangl and Zechner (2016), the firm can choose how much maturing debt to rollover, but as stipulated by covenants it cannot increase the aggregate face value of debt without first repurchasing all existing debt (at par plus a call premium and a proportional transaction cost). Rolling over debt maintains the firm's tax shields, as in in our model, and dilutes current creditors given their setting with a strictly positive recovery rate and pari-passu debt (which we analyze in Section 3.4). They show that when debt maturity is long, equity holders will rollover existing debt fully as it comes due, except for when leverage is so low that recapitalization to a higher face value of debt is imminent (in which case it is not worthwhile to issue debt that will be shortly replaced at a cost). If debt maturity is sufficiently short, however, then when facing high leverage shareholders may rollover only a portion of the maturing debt so that the total face value of debt gradually declines. This behavior abruptly reverses when the firm approaches default as shareholders maximize dilution (and minimize equity injections) by again rolling over debt fully.[2] Importantly, they show that firm value is not monotonic in debt maturity; depending on parameters, an interior optimal maturity may exist that trades off the transactions costs of debt rollover (which favors long maturities) with the benefit from debt reductions given high leverage (which favors short maturities).[3] As in our model, the choice of debt maturity becomes an important commitment device that allows for future debt reductions in the face of negative shocks.

[2] In our model, because there is no constraint on the rate of new issuance, this effect is even more pronounced, with debt issuance only increasing at the moment of default.

[3] The same trade-off would apply in our model if we were to adopt the same assumption on transaction costs.

In a somewhat different context, Brunnermeier and Yogo (2009) stress the advantage of short-term debt in providing the firm with flexibility to adjust debt quickly in the face of shocks to firm value, but that long-term debt is more effective at reducing costs from rollover risk. Abel (2016) considers a dynamic model with investment in which firms adjust leverage by issuing debt with instantaneous maturity. He assumes i.i.d. regime shocks to profitability and shows that in response to a shock, (i) shareholders never reduce the amount of debt, and (ii) only firms that are borrowing constrained (i.e. have borrowed an amount equal to 100% of firm value) choose to increase debt.

Our paper proceeds as follows. In Section 2 we introduce a general continuous-time model of the firm and develop our methodology for solving for a continuous equilibrium. Section 3 applies our general results to the special case when cash flows are lognormal with possible jumps and derives a closed-form solution for security prices and debt issuance. Section 4 analyzes debt dynamics and shows that the firm gradually adjusts leverage towards a target level. We also compare our equilibrium with standard benchmarks such as Leland (1998), and evaluate the firm's choice of debt maturity. Section 5 extends the model to include agency costs of investment, and Section 6 concludes.

2. A General Model

We begin by outlining a general jump-diffusion model of cash flows that encompasses typical settings used in the literature. We include both taxes and bankruptcy costs as in a standard tradeoff model. We depart from the existing literature by assuming equity holders can issue or repurchase debt at any time at the current market price, and analyze the optimal no-commitment leverage policy in equilibrium. Clearly, this policy depends on equilibrium debt prices; but equilibrium debt prices depend on the firm's *future* leverage choices, which determines the likelihood of default. Despite this interdependence, we are able to characterize the time-consistent leverage policy explicitly and show that the rate of debt issuance depends on the ratio of tax benefits to the price sensitivity of debt to new issues. We also show that equity values can be computed as though the firm committed not to issue new debt.

2.1. The Firm and Its Securities

All agents are risk neutral with an exogenous discount rate of $r>0$.[4] The firm's assets-in-place generate operating cash flow (EBIT) at the rate of Y_t which evolves according to

$$dY_t = \mu(Y_t)\,dt + \sigma(Y_t)\,dZ_t + \zeta(Y_{t-})\,dN_t, \tag{1}$$

where the drift $\mu(Y_t)$ and the volatility $\sigma(Y_t)$ are general functions that satisfy regularity conditions; dZ_t is the increment of standard Brownian motion; dN_t is Poisson increment with intensity of $\lambda(Y_t)>0$; and $\zeta(Y_{t-})$ is the jump size given the Poisson event.[5]

Denote by F_t the aggregate face value of outstanding debt. The constant coupon rate of the debt is $c>0$, so that over $[t,t+dt]$ debt holders receive coupon payments of $cF_t dt$ in total.[6] Equity holders pay tax $\pi(Y_t - cF_t)dt$, where $\pi(\cdot)$ is a non-decreasing function of the firm's profit net of interest. When the marginal tax rate is positive ($\pi'>0$), the net after tax cost to the firm of the marginal coupon payment is $1-\pi'$, reflecting the debt tax shield subsidy.

For simplicity we assume that debt takes the form of exponentially maturing coupon bonds with a constant amortization rate $\xi>0$. More specifically, each instant there are $\xi F_t dt$ units of required principal repayments from maturing bonds, corresponding to an average bond maturity of $1/\xi$. Together with the aggregate coupon of $cF_t dt$, over $[t,t+dt]$ equity holders are required to pay debt holders the flow payment of $(c+\xi)F_t\,dt$ in order to avoid default.

In the main analysis we assume investors recover zero value from the assets-in-place when equity holders default. The key implication of this assumption, which will simplify our analysis, is that *debt seniority becomes irrelevant*. In addition, because there are no claims to divide in default, old debt holders do not get "diluted" by new debt holders in bankruptcy even if

[4] Alternatively, we can interpret the model as written under a fixed risk-neutral measure that is independent of the firm's capital structure decision.

[5] We have simplified notation by assuming the jump size $\zeta(Y_t)$ conditional on cash flow Y_t is deterministic. We can easily generalize the model to allow a random jump size $\tilde{\zeta}(Y_t)$, as long as the law of $\tilde{\zeta}(Y_t)$ depends on Y_t only.

[6] The coupon rate c is fixed and arbitrary in our model; in practice, there may be limits/adjustments to the tax deductibility of the coupon if it is far from the par coupon rate.

the new debt has equal (or higher) priority. We make the zero recovery value assumption to emphasize that our results are not driven by the "dilution" effect which often arises when issuing pari-passu debt (e.g., Brunnermeier and Oehmkhe, 2014; Dangl and Zechner, 2016). Section 3.4 studies the case with positive recovery and pari passu debt, and demonstrates that the threat of dilution reduces the equilibrium level of debt issued by the firm prior to default.[7]

Equity holders control the outstanding debt F_t through endogenous issuance/repurchase policy $d\Gamma_t$, where Γ_t represents the cumulative debt issuance over time. We focus our main analysis on a class of equilibria in which along the equilibrium path equity holders find it optimal to adjust the firm's outstanding debt smoothly with order dt. More specifically, we conjecture that at each instant the adjustment to existing debt is $d\Gamma_t = G_t dt$, where G_t specifies the rate of issuance at date *t*, and verify later that other issuance policies, including discrete ones, are suboptimal in equilibrium. From now on, we call this equilibrium "smooth" equilibrium, and call G_t the equity holders' issuance policy, which could be issuing new debt if $G_t > 0$ or repurchasing existing debt in which case $G_t < 0$. Given our debt maturity assumption, the evolution of outstanding face value of debt F_t is given by

$$dF_t = \left(G_t - \xi F_t\right) dt. \tag{2}$$

Thus, the face value of debt will grow only if the rate of issuance more than offsets the contractual retirement rate. To highlight the economic forces at play, and in contrast to the bulk of the literature, we assume zero transaction costs in issuing or repurchasing debt.[8]

Given the equity holders' expected issuance/repurchase policy $\{G_t\}$, debt holders price the newly issued or repurchased debt in a competitive market. Denote by p_t the endogenous

[7] If any new debt is restricted to be junior to existing claims, our qualitative results still hold with a positive recovery rate. This setting adds significant complexity, however, as debt securities issued at different times have distinct prices. In contrast, given zero recovery or pari passu debt, all debt is identical independent of the timing of issuance.
[8] It is common in the dynamic capital structure literature, e.g. Fischer, Heinkel, and Zechner (1989) and Leland, Goldstein, and Ju (2000), to assume that firms--in order to adjust their capital structure--have to buy back all of their existing debt and then reissue new debt, and that there is a positive adjustment cost associated with this transaction. We eliminate this artificial constraint to highlight equity holders' intrinsic incentives to adjust leverage at any time.

debt price per unit of promised face value. Then over $[t, t+dt]$ the net cash flows to equity holders are equal to

$$\left(\underbrace{Y_t}_{\text{operating cash flow}} - \underbrace{\pi(Y_t - cF_t)}_{\text{tax payment}} - \underbrace{(c+\xi)F_t}_{\text{debt interest \& principal}} + \underbrace{G_t p_t}_{\text{debt issuance/repurchase}} \right) dt . \tag{3}$$

The firm continues to operate until the operating cash flow Y_t drops below some sufficiently low level (which may depend on the outstanding debt level F_t), at which point equity holders find it optimal to default on their contractual payment to debt holders (we will later characterize the optimal default boundary). As in the literature (Leland 1994, 1998), shareholders cannot commit to a certain default policy, but instead default strategically. After default, debt holders take over the firm but recover zero by assumption (for the positive recovery case, see Section 3.4).

2.2. Equilibrium Analysis

We focus on Markov perfect equilibria in which the two payoff-relevant state variables are: the firm's *exogenous* operating cash flow Y_t, and the outstanding aggregate debt face value F_t, which is an *endogenous* state variable. We will analyze the equity's value function $V(Y_t, F_t)$ and the debt price $p(Y_t, F_t)$. Denote by τ_B the equilibrium default time; presumably, this is the first time that the state pair (Y_t, F_t) falls into the endogenous default region, which we denote by $\mathscr{B}$. For the value of equity, given future issuance policies and debt prices $\{(G_s, p_s) : s > t\}$, when the firm is in the survival region, i.e., $(Y_t, F_t) \notin \mathscr{B}$, we have

$$V(Y,F) = \mathrm{E}_t \left[\int_t^{\tau_B} e^{-r(s-t)} \left[Y_s - \pi(Y_s - cF_s) - (c+\xi) F_s + G_s p_s \right] ds \middle| Y_t = Y, F_t = F \right]. \tag{4}$$

For debt prices, similarly we have

$$p(Y,F) = \mathrm{E}_t \left[\int_t^{\tau_B} e^{-(r+\xi)(s-t)} (c+\xi) dt \middle| Y_t = Y, F_t = F \right]. \tag{5}$$

An Optimality Condition

Recall that we are interested in an equilibrium when there is no commitment of equity holders to future leverage policies. Thus at any point in time, the issuance policy G_s for $s > t$ has to be optimal in solving the equity holders' instantaneous maximization problem at time s, given the equity's value function and equilibrium debt prices.

In this section we consider the necessary and sufficient conditions for the optimality of the debt issuance policy G_t. The Hamilton-Jacobi-Bellman (HJB) equation for equity holders is

$$\underbrace{rV(Y,F)}_{\text{required return}} = \max_G \left[\underbrace{Y - \pi(Y - cF)}_{\text{after-tax cash flow}} - \underbrace{(c+\xi)F}_{\text{coupon\&principal payment}} + \underbrace{Gp(Y,F)}_{\text{new debt issuance}} + \underbrace{(G-\xi F)V_F(Y,F)}_{\text{evolution of } dF} \right] \\ + \underbrace{\mu(Y)V_Y(Y,F) + \frac{1}{2}\sigma(Y)^2 V_{YY}(Y,F) + \lambda(Y)\left[V\left(Y+\zeta(Y)\right) - V(Y)\right]}_{\text{evolution of } dY} \tag{6}$$

In the first line, the objective is linear in G with a coefficient of $p(Y,F)+V_F(Y,F)$ which represents the (endogenous) marginal benefit of increasing debt. If equity holders find it optimal to adjust debt smoothly, then it must be that this coefficient equals zero everywhere, i.e.

$$p(Y,F)+V_F(Y,F)=0. \tag{7}$$

This first-order condition (FOC) must hold for any (Y,F) along the equilibrium path.[9] But to be sure that this policy is globally optimal, we must verify that there is no discrete adjustment to the debt level that shareholders would prefer. We show next that global optimality holds if and only if the debt price is weakly decreasing in the firm's total debt, i.e., $p_F(Y,F) \le 0$.

PROPOSITION 1 (OPTIMALITY OF SMOOTH DEBT POLICY). Suppose the debt price $p = -V_F(Y,F)$ is weakly decreasing in the total face value F of the firm's debt, i.e., the

[9] This relation holds trivially in the default region $\mathcal{B}$, as for defaulted firm the debt price $p=0$ and $V(Y,F)=0$ implies $V_F(Y,F)=0$ as well. It is worth pointing out that the zero-bankruptcy-recovery assumption is not necessary for $p=0$ at default. Even with a strictly positive recovery, as long as newly issued debt is junior to any pre-existing debt, newly issued debt is worthless at the moment of default as existing debt gets all the recovery. (Alternatively, if the firm can issue pari passu debt, then right before default it will issue new debt and dilute existing creditors to the point that their recovery value becomes zero, as we show in Section 3.4.)

value of equity $V(Y,F)$ is convex in F. Then condition (7) implies that the policy G_t is an optimal debt issuance policy for shareholders. Conversely, if the policy G_t is optimal, then $p = -V_F\left(Y,F\right)$ is weakly decreasing in F for all equilibrium states $\left(Y,F\right)$.

PROOF. Equity holders are solving the following problem each moment along the optimal path

$$\max_{\Delta} V(Y,F+\Delta)+\Delta\cdot p(Y,F+\Delta)-V(Y,F). \tag{8}$$

For the proposed smooth policy to be optimal, $\Delta = 0$ must be optimal in (8). This problem has the first-order condition $V_F + p + \Delta\, p_F = 0$ at $\Delta = 0$, which implies that $p = -V_F$. To check for global optimality, suppose that equity holders choose any $\Delta > 0$. Then equity's gain is

$$\begin{aligned} V\left(Y,F+\Delta\right)-V\left(Y,F\right)+\Delta\cdot p\left(Y,F+\Delta\right) &= \int_0^{\Delta} V_F\left(Y,F+\delta\right)d\delta + \int_0^{\Delta} p\left(Y,F+\Delta\right)d\delta \\ &\le \int_0^{\Delta} V_F\left(Y,F+\delta\right)d\delta + \int_0^{\Delta} \underbrace{p\left(Y,F+\delta\right)}_{\text{monotonicity of } p}d\delta \\ &= \int_0^{\Delta} \underbrace{\left[V_F\left(Y,F+\delta\right)+p\left(Y,F+\delta\right)\right]}_{\text{FOC in Equation (7)}}d\delta = 0 \end{aligned} \tag{9}$$

where in the second line we have used the condition that p is weakly monotone in F. Note the above inequalities still hold if $\Delta < 0$; in this case $p\left(F+\Delta\right) \ge p\left(F+\delta\right)$ but $d\delta < 0$. Finally, the condition that the debt price is weakly decreasing in F implies that equity's value is convex in F, i.e. $p_F \le 0 \iff V_{FF} \ge 0$.

Now we prove the second part. First of all, it is easy to see $p\left(Y,F\right)$ is continuous in F from (5). If $p\left(Y,F\right)$ is weakly increasing in F, the above argument in (9) implies that any Δ is a profitable deviation. But if $p\left(Y,F\right)$ is not monotone in F, then for some Y, due to continuity there must exist two face values F_1 and F_2 with $F_1 < F_2$, so that $p\left(Y,\hat{F}\right) < p\left(Y,F_1\right) = p\left(Y,F_2\right)$ for $\hat{F} \in \left[F_1,F_2\right]$. Now setting $\Delta = F_2 - F_1 > 0$ and $F = F_1$ in (9) leads to a strictly positive deviation gain. ■

Equity Valuation

The First-Order Condition (FOC) in (7), which implies a zero-profit condition for equity holders in adjusting the debt burden instantaneously, has deep implications for the equilibrium in our model. Plugging condition (7) into the equity HJB equation (6), the sum of terms involving G equals zero and we have the following revised HJB equation for equity:

$$\begin{aligned} rV(Y,F) &= Y-\pi(Y-cF)-(c+\xi)F+\mu(Y)V_Y(Y,F)-\xi F V_F(Y,F) \\ &\quad +\tfrac{1}{2}\sigma(Y)^2 V_{YY}(Y,F)+\lambda(Y)\big[V(Y+\zeta(Y),F)-V(Y,F)\big]. \end{aligned} \tag{10}$$

This equation says that in the no-commitment equilibrium, the equity value can be solved as if there is no debt adjustment $G_t = 0$, except for the natural retirement at rate ξ.

Intuitively, because equity holders gain no marginal surplus from adjustments to debt, their equilibrium payoff must be the same as if they never issue/repurchase any debt. The important implication of this observation is that we can solve for the equilibrium equity value $V(Y,F)$, even without the knowledge of the equilibrium debt price – $p(Y,F)$ does not enter equation (10)). Therefore, we have the following key result:

Proposition 2 (No-Trade Equity Valuation). Let $V^0(Y,F)$ be the value of equity that solves (10) in which the firm were committed not to issue or buyback debt ($G_t = 0$). Then the value V of equity in any smooth equilibrium is equal to V^0.

Proof. Immediate from (10) and the fact that the boundary conditions are unchanged. ■

This result, while perhaps striking at first, is analogous to the Coase (1972) conjecture for durable goods monopoly: the firm is a monopolist issuer of its own debt. When the firm is unable to commit to restricting its future sales, it trades sufficiently aggressively to dissipate any surplus from trading.[10]

[10] A closely related result appears in DeMarzo and Urosevic (2001) in a model of trade by a large shareholder trading off diversification benefits and price impact due to reduced incentives. In equilibrium, share prices are identical to those implied by a model with no trade. Similarly, the monopolist buyer in Daley and Green (2016) who cannot commit to his/her future strategy gains nothing from the ability of screening (sellers with different types).

Optimal Debt Issuance

Given the equity value V, we can invoke the FOC in (7) to obtain the equilibrium debt price $p(Y,F) = -V_F(Y,F)$. Finally, to confirm that this outcome indeed represents an equilibrium, we must verify whether $p(Y,F) = -V_F(Y,F)$ is weakly decreasing in F, or equivalently that the equity value is convex in F.

Now that we have determined the equilibrium values for equity and debt, how can we find the optimal leverage policy *G*? Note that we have determined the equilibrium debt price using the optimality $p = -V_F$, where the equity value *V* is equivalent to the no-trade value from (10) and therefore independent of *G*. On the other hand, we can also calculate the debt price directly based on debt holders expected cash flows using (5). This calculation will depend on the timing of default, which does depend on the rate of debt issuance. Thus the leverage policy *G* must be chosen to make these two methods of valuing the debt consistent.[11]

First, let us consider the HJB equation that should hold for the debt price from (5), which is given by

$$\begin{aligned}\underbrace{rp(Y,F)}_{\text{required return}} = \underbrace{c}_{\text{coupon}} + \underbrace{\xi\left(1-p(Y,F)\right)}_{\text{debt retirement}} + \underbrace{(G^{*}-\xi F)\,p_F(Y,F)}_{\text{evolution of debt } dF} + \\ \underbrace{\mu(Y)\,p_Y(Y,F) + \tfrac{1}{2}\sigma(Y)^2\,p_{YY}(Y,F) + \lambda(Y)\left[p\left(Y+\zeta(Y),F\right) - p(Y,F)\right]}_{\text{evolution of cash flow } dY}.\end{aligned} \tag{11}$$

Next, starting with the HJB equation (10) for equity value $V(Y,F)$, if we differentiate by F and use the optimality condition $p = -V_F$, we obtain

$$\begin{aligned}-rp(Y,F) = \pi'(Y-cF)c - (c+\xi) + \xi\,p(Y,F) + \xi F p_F(Y,F) \\ -\mu(Y)\,p_Y(Y,F) - \tfrac{1}{2}\sigma(Y)^2\,p_{YY}(Y,F) + \lambda(Y)\left[-p\left(Y+\zeta(Y),F\right) + p(Y,F)\right].\end{aligned} \tag{12}$$

Although equation (12) is written in terms of the debt price p, we emphasize that it follows mechanically from the valuation equation (10) for equity, together with the FOC (7) for the

[11] Intuitively, if $G = 0$ and the firm never issues additional debt, the debt price would exceed its marginal cost to shareholders, $-V_F$, due to the incremental tax shield. By increasing the rate of issuance, the likelihood of default will increase and the price of debt will fall to the point that (7) holds.

optimal issuance policy. Finally, adding (12) to (11), we obtain a simple expression for G shown below:

PROPOSITION 3 (EQUILIBRIUM DEBT ISSUANCE). Let $V(Y,F)$ be the no-trade value of equity. If V is convex in F, then there exists a unique smooth equilibrium with debt issuance policy

$$G^*\left(Y,F\right)=\frac{\pi'(Y-cF)c}{-p_F\left(Y,F\right)}=\frac{\pi'(Y-cF)c}{V_{FF}\left(Y,F\right)}. \tag{13}$$

Under this policy, the debt price given by (5) satisfies $p=-V_F$.

PROOF. For a smooth policy to be optimal, (7) is necessary. But then (6) and (7) imply (12), which combined with (11) imply (13). Then $p=-V_F$ follows since their HJB equations and boundary conditions are equivalent, and the global optimality of the policy (13) follows from Proposition 1 and convexity of V. ■

Note that the convexity of equity value V implies that, no matter how high the current level of debt, the rate of issuance G^* is always positive provided a strictly positive tax benefit $\pi'>0$. We can interpret the policy (13) as follows. The rate of issuance of debt is such that the rate of devaluation of the debt induced by new issuances just offsets the marginal tax benefit associated with the coupon payments:

$$G^*\left(Y,F\right)\cdot p_F\left(Y,F\right)=c\cdot\pi'\left(Y-cF\right) \ .$$

Thus, if there were no tax subsidy $\pi'=0$, then $G^*\left(Y,F\right)=0$, i.e. without a tax subsidy equity holders choose not to increase debt. On the other hand, they choose not to actively reduce debt via buybacks either, despite that fact that there are deadweight costs of bankruptcy. This result is consistent with the leverage ratchet effect of Admati et al. (2016) – even if the firm's current leverage is excessive, equity holders never actively reduce debt but always have an incentive to increase debt when it provides a marginal tax benefit.

Summary

In sum, for the general model in which equity holders are free to issue or repurchase any amount of debt at the prevailing market price, one can solve for the no commitment equilibrium as follows:

(i) Use (10) to solve for the equity holder's value function $V(Y,F)$ by setting $G=0$, i.e. as if equity holders commit to not issue any future debt;

(ii) Set the debt price $p(Y,F) = -V_F(Y,F)$;

(iii) Check the global second-order condition by verifying the debt price $p(Y,F)$ is weakly decreasing in aggregate debt F, or equivalently $V(Y,F)$ is convex in F;

(iv) Finally, given $p(Y,F)$ we can solve for the optimal time consistent issuance policy $G^*(Y,F)$ from (13).

In the remainder of the paper we will use this methodology to analyze several standard settings.

3. A Closed-Form Solution

We now apply the general methodology developed in the previous section to the widely used framework of a lognormal cash flow process.[12] The results from Section 2 allow us to fully characterize an equilibrium in closed form, and evaluate the corresponding leverage dynamics. We also extend the model to allow for jumps to cash flows and asset values, and show that the solution is qualitatively unchanged. Finally, Section 3.4 studies the case of a positive recovery, and shows that when equity holders have the option to dilute existing creditors, the equilibrium level of debt is reduced.

3.1. Log-Normal Cash Flows

In the special case of lognormal operating cash flow, Y_t follows a geometric Brownian motion so that

[12] This setting is consistent with e.g. Merton (1974), Fischer et al. (1989), Leland (1994), Leland and Toft (1996), and follows the development of starting from cash flows rather than firm value as in Goldstein et al. (2001).

$$\mu(Y_t) = \mu Y_t \text{ and } \sigma(Y_t) = \sigma Y_t \text{, with } r > \mu. \tag{14}$$

Given the scale invariance in this special case, we analyze the model using a unidimensional state variable equal to operating cash flow scaled by the outstanding face value of debt F_t , i.e.

$$y_t \equiv \frac{Y_t}{F_t}. \tag{15}$$

As an interpretation, note that y_t / c equals the firm's *interest coverage ratio*, i.e. the ratio of operating income Y_t to total interest expense cF_t , a widely used measure of leverage and financial soundness. Alternatively, $1/ y_t$ expresses the amount of debt as a multiple of the firm's cash flow (or EBIT).

To maintain homogeneity, we make the common assumption of a constant tax rate, i.e.

$$\pi(Y_t - cF_t) = \overline{\pi} \cdot (y_t - c) \cdot F_t, \tag{16}$$

where the positive constant $\overline{\pi} > 0$ is the marginal corporate tax rate that applies to both losses and gains.[13] With this setting, we conjecture and verify that the equity value function $V(Y,F)$ and debt price $p(Y,F)$ are homogeneous so that

$$V(Y,F) = V\left(\frac{Y}{F},1\right)F \equiv v(y)F \text{ and } p(Y,F) = p\left(\frac{Y}{F},1\right) \equiv p(y). \tag{17}$$

We will solve for the (scaled) equity value function $v(y)$ and debt price $p(y)$ in closed form.

Given the evolution of our state variables Y_t and F_t:

$$dY_t = \mu Y_t dt + \sigma Y_t dZ_t, \text{ and } dF_t = (G_t - \xi F_t)dt, \tag{18}$$

the scaled cash-flows evolve as

$$\frac{dy_t}{y_t} = (\mu + \xi - g_t)dt + \sigma dZ_t, \text{ where } g_t \equiv G_t / F_t. \tag{19}$$

[13] The methods developed here could also be applied with different marginal tax rates for losses versus gains, e.g. $\pi(y-c) = \overline{\pi} \cdot \max(y-c,0)$, though we do not pursue that here.

As (19) shows, the scaled cash flow growth has the same volatility as total cash flow, as the outstanding debt face value F_t in (2) grows in a locally deterministic way. The drift of the scaled cash flow growth corresponds to the growth in operating cash flows less the net growth rate of the debt $g_t - \xi$, where ξ is the debt amortization rate and $g_t = G_t / F_t$ is the endogenous growth rate from debt issuance. The more new debt the firm issues, the faster the scaled cash-flows shrink.

When the scaled cash flow y_t falls below some endogenous default boundary y_b, equity holders are no longer willing to service the debt, and therefore choose to strategically default. At that event, equity holders walk away and debt holders recover nothing by the assumption of a zero liquidation value.

3.2. Model Solution

Recall from Section 2 that we can solve for the equilibrium equity value *as if* $g_t = 0$ and equity holders do not actively adjust the firm's outstanding debt F_t, even though they do in equilibrium. Using the fact that

$$V_Y(Y,F) = v'(y),\ V_F(Y,F) = v(y) - yv'(y), \text{ and } FV_{YY} = v''(y), \tag{20}$$

we can rewrite (10) with lognormal cash flows in terms of scaled cash-flow y as follows:

$$(r+\xi)v(y) = (y-c-\xi) - \overline{\pi}(y-c) + (\mu+\xi)yv'(y) + \frac{1}{2}\sigma^2 y^2 v''(y). \tag{21}$$

There are two boundary conditions for the above Ordinary Differential Equation (ODE) (21). First, when $y \to \infty$, default becomes unimportant and we can treat the debt as riskless, and hence the equity value should converge to

$$\overline{v}(y) \equiv \underbrace{\frac{y(1-\overline{\pi})}{r-\mu}}_{\text{unlevered asset-in-place value}} + \underbrace{\frac{\overline{\pi}c}{r+\xi}}_{\text{tax shield}} - \underbrace{\frac{c+\xi}{r+\xi}}_{\text{bond value}} = \frac{y(1-\overline{\pi})}{r-\mu} - \frac{c(1-\overline{\pi})+\xi}{r+\xi}. \tag{22}$$

On the other hand, when $y = y_b$, equity is worthless so $v(y_b) = 0$. Solving (21) with these two boundary conditions, we obtain

PROPOSITION 4. Given a constant tax rate $\bar{\pi}$, the equity value function with no debt issuance is given by

$$v(y)=\frac{y(1-\bar{\pi})}{r-\mu}-\frac{c\left(1-\bar{\pi}\right)+\xi}{r+\xi}\left(1-\frac{1}{1+\gamma}\left(\frac{y}{y_b}\right)^{-\gamma}\right). \tag{23}$$

where the constant γ is defined as

$$\gamma\equiv\frac{\left(\mu+\xi-0.5\sigma^2\right)+\sqrt{\left(\mu+\xi-0.5\sigma^2\right)^2+2\sigma^2\left(r+\xi\right)}}{\sigma^2}>0, \tag{24}$$

and the optimal default boundary is given by

$$y_b=\frac{\gamma}{1+\gamma}\left(\frac{r-\mu}{1-\bar{\pi}}\right)\left(\frac{c\left(1-\bar{\pi}\right)+\xi}{r+\xi}\right). \tag{25}$$

PROOF. We can write the value function as

$$v(y)=\quad \bar{v}(y)+E\left[e^{-(r+\xi)\tau_b}\right]\left(0-\bar{v}(y_b)\right)=\underbrace{\bar{v}(y)}_{\substack{\text{equity value}\\ \text{without}\\ \text{default option}}}\quad\underbrace{-\bar{v}(y_b)\left(\frac{y}{y_b}\right)^{-\gamma}}_{\text{option value of default}}, \tag{26}$$

where the expression for $E\left[e^{-(r+\xi)\tau_b}\right]$ and γ follows by solving the ODE

$$(r+\xi)f(y)=(\mu+\xi)yf'(y)+\tfrac{1}{2}\sigma^2y^2f''(y)$$

with boundary conditions $f(y_b)=1$ and $f(\infty)=0$. Finally, the optimal default boundary y_b is determined by the smooth-pasting condition, $v'\left(y_b\right)=0$. ■

Having solved for the value of equity, recall from (7) that we can determine the equilibrium debt price from the FOC $p\left(y\right)=-V_F\left(Y,F\right)$. Then from (20), and using (23),

$$p\left(y\right)=-V_F=yv'\left(y\right)-v\left(y\right)=\frac{c\left(1-\bar{\pi}\right)+\xi}{r+\xi}\left(1-\left(\frac{y}{y_b}\right)^{-\gamma}\right). \tag{27}$$

Recall we need to verify the optimality of the issuance policy by checking the monotonicity of the equilibrium debt pricing function. It is easy to see that $p'\left(y\right)>0$ in (27), i.e.

the greater the scaled cash flow the higher the debt price. As a result, the key condition in **PROPOSITION 1** – that the debt price decreases with total debt – follows because

$$p_F\left(Y,F\right)=p'\left(y\right)\cdot\left(-\frac{Y}{F^2}\right)=-\frac{y^2v''(y)}{F}<0. \tag{28}$$

Finally, we can apply **PROPOSITION 3** combined with (28) to derive the equilibrium debt issuance policy:

PROPOSITION 5. Given a constant tax rate $\overline{\pi}$, the equity value function and debt price are given by (26)-(27), and the equilibrium issuance policy is

$$g^*\left(y\right)=\frac{G^*}{F}=\frac{\overline{\pi}c}{-Fp_F\left(Y,F\right)}=\frac{\overline{\pi}c}{yp'\left(y\right)}=\frac{\overline{\pi}c}{y^2v''\left(y\right)}=\frac{\left(r+\xi\right)\overline{\pi}c}{c\left(1-\overline{\pi}\right)+\xi}\cdot\frac{1}{\gamma}\left(\frac{y}{y_b}\right)^{\gamma}. \tag{29}$$

In equilibrium, the firm's new debt issuance $g^*\left(y\right)$ is always positive, and is increasing in the scaled cash flow *y*.

Thus, with lognormal cash flows, we can fully characterize equilibrium debt dynamics and security pricing in closed form. Based on the equilibrium values for both equity and debt, total firm value (or total enterprise value, TEV) can be expressed as a multiple of the firm's cash flow (i.e. TEV to EBIT) as

$$\frac{v(y)+p(y)}{y}=v'(y)=\frac{1-\pi}{r-\mu}-\frac{1}{y_b}\frac{c\left(1-\pi\right)+\xi}{r+\xi}\frac{\gamma}{1+\gamma}\left(\frac{y}{y_b}\right)^{-\gamma-1}=\frac{1-\pi}{r-\mu}\left[1-\left(\frac{y}{y_b}\right)^{-\gamma-1}\right], \tag{30}$$

where the first equality follows from the equilibrium condition for the debt price, and the last equality uses the expression of y_b in (25). Note that the firm's TEV multiple is strictly increasing with the scaled cash flow y. Consequently, holding the level of cash flows *Y* fixed, total firm value decreases with the debt face value F, implying no gain to total firm value from leverage. We will discuss this implication further when we compare TEV multiples across different benchmark models with commitment.

3.3. Upward Jumps

In the standard lognormal setting, cash flows and asset values evolve continuously. Suppose, however, that cash flows occasionally jump discontinuously, for example in response to new product development. In that case, one might expect debt to adjust discontinuously as well. In this section, we extend our prior model to allow for upward jumps and show that our prior solution, in which shareholders issue debt smoothly, is essentially unchanged.

Consider a jump-diffusion model in which cash flows occasionally jump from Y_t to θY_t for some constant $\theta > 1$. Specifically,

$$dY_t = \mu Y_t dt + \sigma Y_t dZ_t + (\theta - 1) Y_{t-} dN_t, \qquad (31)$$

where dN_t is a Poisson process with constant intensity $\lambda > 0$.[14] In this extension, due to upward jumps, the effective expected asset growth rate becomes

$$\hat{\mu} \equiv \mu + \lambda(\theta - 1), \qquad (32)$$

and we continue to assume $\hat{\mu} < r$ to ensure that the unlevered firm value is bounded.

As before, we can solve for the equity value as if shareholders commit not to issue any new debt. Because (31) still maintains scale-invariance, $V(Y,F) = F \cdot v(y)$ continues to hold, and the HJB equation for the equity value becomes

$$(r + \xi + \lambda) v(y) = (1 - \overline{\pi})(y - c) - \xi + (\mu + \xi) y v'(y) + \frac{1}{2}\sigma^2 y^2 v''(y) + \lambda v(\theta y). \qquad (33)$$

The last term in equation (33) captures upward jumps. The usual boundary conditions apply: When $y \to \infty$ so leverage is negligible, default risk disappears and $v(y) \to \overline{v}(y)$; while at the point of default, we have value-matching $v(y_b) = 0$ and smooth-pasting $v'(y_b) = 0$.

Somewhat surprisingly, even with jumps, equilibrium security prices and debt dynamics have exactly the same form derived before in the diffusion-only case:

[14] While we focus on upward jumps with a fixed size, allowing the upward jump to be stochastic is straightforward. Downward jumps introduce an extra complication due to jump-triggered default, in addition to diffusion-triggered default. See footnote 16 for more details.

PROPOSITION 6. Suppose cash flows evolve as a log-normal diffusion with upward jumps as in (31). Then the equilibrium equity and debt values, default boundary, and debt issuance policy are given by (23), (27), (25), and (29) respectively, with μ replaced by $\hat{\mu}$, given by (32), and γ replaced by $\hat{\gamma}$, which is the unique positive root of

$$W(\hat{\gamma}) = \lambda\theta^{-\hat{\gamma}} + \frac{\sigma^2}{2}\hat{\gamma}^2 - \left(\mu + \xi - \frac{\sigma^2}{2}\right)\hat{\gamma} - \left(r + \xi + \lambda\right) = 0 . \tag{34}$$

PROOF. Note that the HJB equation (33) has the linear solution

$$\bar{v}\left(y\right) = \frac{y\left(1-\bar{\pi}\right)}{r-\hat{\mu}} - \frac{c\left(1-\bar{\pi}\right)+\xi}{r+\xi} . \tag{35}$$

The homogenous delayed differential equation

$$(r+\xi+\lambda)\, f(y) = (\mu+\xi)\, y f'(y) + \tfrac{1}{2}\sigma^2 y^2 f''(y) + \lambda f(\theta y)$$

has solutions of the form $y^{-\hat{\gamma}}$ where $\hat{\gamma}$ solves the characteristic equation (34). In (34), because W is convex, $W\left(\infty\right) = W\left(-\infty\right) = \infty$, $W\left(0\right) < -r-\xi < 0$, and $W\left(-1\right) = \hat{\mu} - r < 0$, W has a unique positive real root (as well a unique negative real root $\hat{\eta} < -1$ that can be ruled out by the upper boundary condition). The remainder of the analysis follows exactly as in Section 3.2. ■

Consequently, although the firm's profitability (i.e., cash-flow Y_t) may jump up discretely, the equilibrium debt issuance policy continues to be smooth in the sense that it remains of order dt. In response to positive jumps in the firm's profitability, shareholders increase the speed of debt issuance, but do not issue a discrete amount of debt immediately. Consequently, leverage falls discretely before gradually mean-reverting. This property holds even we set $\sigma^2 = 0$ so that the firm's cash flows only grow with discrete jumps.[15,16]

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[15] When the diffusion term vanishes, we must impose $\mu + \xi < 0$ so that cash flows decline faster than debt matures between jumps. Otherwise, it is optimal for the firm to sustain 100% debt financing without risking default.

[16] If we allow negative jumps, there is an additional complication that jumps may trigger default. Nonetheless, the analysis in Chen and Kou (2009), with certain special assumptions on jump distributions, suggests that one can still solve for the equity valuation in closed-form. As long as the equity value function remains convex, the key qualitative property of smooth debt issuance policy continues to hold in general diffusion-jump models, and we leave to future research the exploration of such models.

3.4. Positive Recovery Value

Thus far we have assumed that in the event of default the liquidation value of the firm is zero. Under this assumption, there is no difference between junior or senior debt, which rules out any dilution motive for issuing debt.

What if the firm has a positive recovery value in default, and the firm can issue pari passu or even senior debt, so that there is a dilution motive for debt issuance? Specifically, suppose that given cash flows Y, the firm has a liquidation value $L(Y)$ equal to a fraction of its unlevered value:

$$L(Y) \equiv \alpha \frac{1-\bar{\pi}}{r-\mu} Y \text{ for } \alpha \in [0,1). \tag{36}$$

These liquidation proceeds are paid to the firm's creditors. But if the firm can issue senior or pari passu debt without restriction, then by issuing new debt shareholders can dilute the claim of existing creditors in default. Indeed, at the moment of default, shareholders have an incentive to issue new debt to dilute the existing creditors fully, so that as a result existing creditors earn a zero recovery. Because shareholders receive the proceeds from the new debt issued, this scenario is equivalent to equity holders having the option to default on existing creditors and recover $L(Y)$.[17]

Interestingly, we can show that in this case the resulting equilibrium is equivalent to one in which only a fraction $(1-\alpha)$ of the firm's cash flows can be pledged to creditors (with shareholders owning the rest α fraction of non-pledgeable cash flows separately). The optimal default and issuance policies derived above continue to apply with the relevant measure of cash flow equal to just the pledgeable component $(1-\alpha)y$. The value of equity is then the sum of the value from the pledgeable and non-pledgeable parts. The following proposition formalizes this result:

[17] In other words, it is as if there is a complete violation of absolute priority so that equity holders receive the entire recovery value of the firm (while debt holders recover nothing).

PROPOSITION 7. Suppose cash flows evolve as a log-normal diffusion as in (14), all debt is pari passu, and in the event of default the firm is worth $L(Y)$ as in (36). Then the equilibrium equity and debt values, default boundary, and debt issuance policy are:

$$v^L(y) = v(y(1-\alpha)) + L(y)\,,\ y_b^L = \frac{y_b}{1-\alpha}\,, \tag{37}$$

$$p^L(y) = p(y(1-\alpha))\,,\text{ and } g^L(y) = g^*(y(1-\alpha))\,. \tag{38}$$

PROOF. It is straightforward to check that (37) satisfies the HJB equation (21) or (33) with the boundary condition $v^L(y_b^L) = L(y_b^L)$ and smooth pasting condition $v^{L\prime}(y_b^L) = L'(y_b^L)$. The expressions for p^L and g^L follow from $p(y) = yv' - v$ and $g\left(y\right) = \dfrac{\bar{\pi}c}{yp'\left(y\right)}$ as in (27) and (29). ■

In this scenario, the smooth issuance policy g^L applies only up to the default boundary y_b^L. At the moment of default, the firm issues an infinite amount of debt to dilute existing creditors. The main qualitative effect of a positive recovery rate is to raise the value of equity (i.e. $v^L(y) > v(y)$) and *reduce* the equilibrium level of debt prior to default (because debt issuance will match that of a firm with proportionally lower cash flows).

It is interesting to observe that $\partial g^L / \partial y > 0$, so that shareholders issue less debt when the firm edges closer to default (but before actual default). Even more surprising, we have $\partial g^L / \partial \alpha < 0$ so that a firm with a dilution motive issues less debt prior to default relative to the baseline case without a dilution motive. The extra dilution motive is instead reflected by the more aggressive default policy (and the associated infinite dilution at default), whereas the reduced pledgeability of cash flows lowers debt capacity prior to default. This result squares nicely with Dangl and Zechner (2016) who consider similar pari passu debt and positive recovery, but shareholders are constrained by an upper bound on the rate of debt issuance. Because of this constraint, shareholders in Dangl and Zechner (2016) raise debt at the maximum speed possible for some period prior to default.

4. Debt Dynamics

Now that we have solved for the equilibrium debt issuance policy and security pricing, we can analyze the implications for observed debt dynamics. Although lack of commitment leads the firm to always have a positive rate of debt issuance, the countervailing effect of debt maturity and asset growth cause leverage to mean-revert towards a target. We begin by characterizing this target as well as the speed of adjustment. We then compare the equilibrium without commitment to two benchmark cases: the first is that the firm can fully commit not to issue/repurchase any debt in the future, i.e. $G_t = 0$; and the second is the Leland (1998) assumption that the firm commits to keep the aggregate face value constant, i.e. $G_t = \xi F_t$. Finally, we consider the implications of alternative debt maturities.

4.1. Target Leverage and Adjustment Speed

From **Proposition 5**, we see that the firm will issue debt at a faster rate when cash flows are high, and the rate of issuance slows as the firm approaches default. Figure 2 illustrates the net debt issuance rate given different debt maturities as a function of the firm's current leverage.

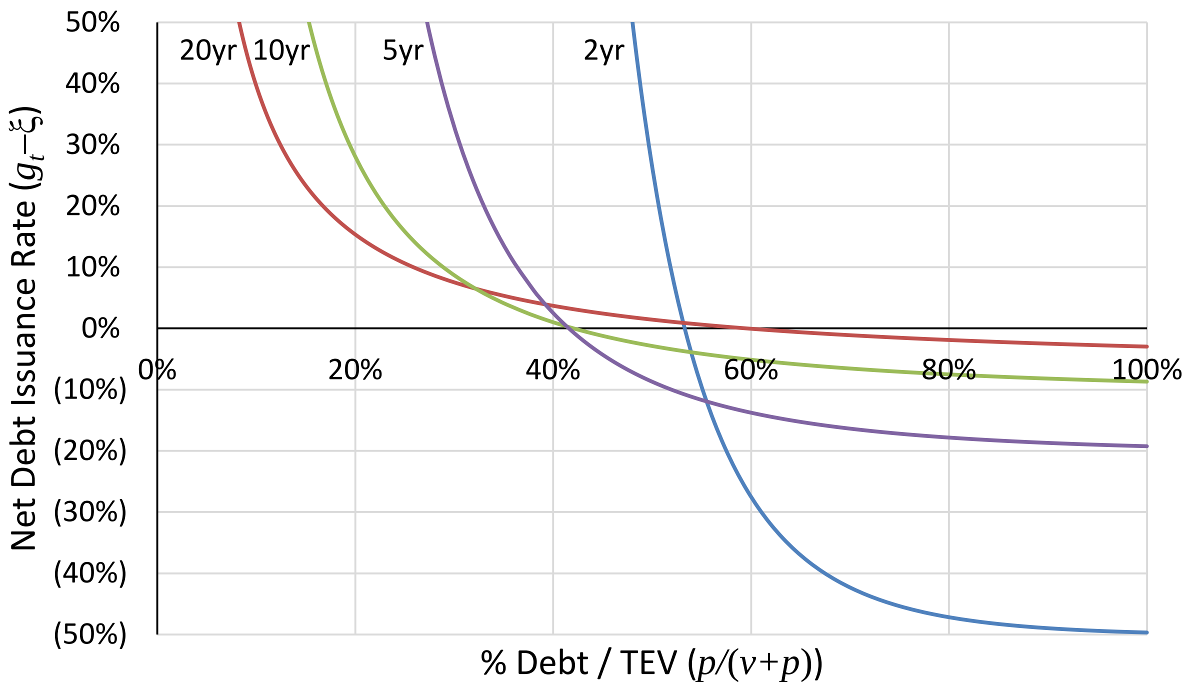


Figure 2: Net Debt Issuance versus Firm Leverage for Different Debt Maturities

Parameters: $\mu = 2\%, \sigma = 40\%, \bar{\pi} = 30\%, c(1-\bar{\pi}) = r = 5\%$

Because the mapping from the cash flow to leverage is strictly monotonic, there is a unique level of the scaled cash flow $\hat{y}_g$ such that new net debt issuances occur at any given rate g. We can compute $\hat{y}_g$ from (29) as follows:

$$\hat{y}_g \equiv y_b \left(\frac{\gamma\left(c\left(1-\bar{\pi}\right)+\xi\right)}{\left(r+\xi\right)\bar{\pi}c} \cdot g \right)^{1/\gamma} . \tag{39}$$

We can interpret $\hat{y}_\xi$ (i.e., when the new debt issuance exactly balances with retiring of existing debt) as the "neutral" level of the scaled cash flow: at this point, the firm is neither accumulating nor retiring debt, corresponding to the intersection points with *x*-axis in Figure 2. If cash flows were fixed at this level, the firm's debt would remain constant.[18]

Without commitment, the firm's debt is path dependent, with the current level of debt equal to the cumulative past issuance net of debt retirement. Because the issuance rate varies with the level of cash flows, this path dependence can be quite complex. Surprisingly, using our expression for g^* in **PROPOSITION 5**, we can derive the evolution of the firm's debt explicitly as a function of the firm's initial debt position and its earnings history, as shown next.

PROPOSITION 8. Given the debt issuance policy g^* and initial debt face value $F_0 \geq 0$, the firm's debt on date t given the cash-flow history $\{Y_s : 0 < s < t\}$ is

$$F_t = \left[F_0^{\gamma} e^{-\gamma\xi t} + \int_0^t \gamma\xi \left(\frac{Y_s}{\hat{y}_\xi} \right)^{\gamma} e^{\gamma\xi(s-t)} ds \right]^{1/\gamma} \tag{40}$$

PROOF: Using (29) the change in the face value of debt is (where we denote $\dot{F} \equiv \frac{dF_t}{dt}$)

[18] Recall though that the cash flows are expected to grow at rate μ, and therefore $\hat{y}_{\mu+\xi}$ is the scaled cash flow level at which debt is expected to grow at the same rate as the cash flows. Therefore at $\hat{y}_{\mu+\xi}$, absent any cash flow shocks, y_t, as well as the firm's interest coverage ratio y_t / c, would be expected to remain constant (as can be seen from (19)).

$$\dot{F} = (g^*(Y/F) - \xi)F = \left(\frac{(r+\xi)\bar{\pi}c}{\gamma\left(c(1-\bar{\pi})+\xi\right)y_b^\gamma} Y^\gamma \right) F^{1-\gamma} - \xi F \,. \tag{41}$$

Let $H = F^\gamma$, then $\dot{H} = \gamma F^{\gamma-1}\dot{F}$ and so (41) implies

$$\dot{H} = \gamma F^{\gamma-1}\dot{F} = \gamma\left(\frac{(r+\xi)\bar{\pi}c}{\gamma\left(c(1-\bar{\pi})+\xi\right)y_b^\gamma} Y^\gamma \right) - \gamma\xi F^\gamma = \gamma\xi\left(\frac{Y}{\hat{y}_\xi}\right)^\gamma - \gamma\xi H \,.$$

Given H_0, this equation is a linear differential equation with general solution

$$H_t = e^{-\gamma\xi t}\left[H_0 + \int_0^t \gamma\xi\left(\frac{Y_s}{\hat{y}_\xi}\right)^\gamma e^{\gamma\xi s} ds \right].$$

(40) then follows from $F = H^{1/\gamma}$. ■

Equation (40) implies that the firm's debt at today equals the sum of discounted initial debt F_0, and an appropriately "discounted" average of the firm's intervening cash flows $\{Y_s : 0 < s < t\}$. This point becomes transparent in the special case of $F_0 = 0$, i.e., when the firm starts with no debt. Setting $F_0 = 0$ in (40), we have

$$F_t = \frac{1}{\hat{y}_\xi}\left(\gamma\xi\int_0^t e^{\gamma\xi(s-t)}\, Y_s^\gamma ds\right)^{1/\gamma} \tag{42}$$

Since $\gamma\xi\int_0^t e^{\gamma\xi(s-t)}\, ds = 1 - e^{-\gamma\xi t}$, (42) implies that in equilibrium debt starts at 0 and grows gradually with order of $t^{1/\gamma}$, with the long run debt level dependent on the weighted average of the firm's historical earnings. The weight put on recent cash flows relative to more distant ones depends on the product $\gamma\xi$. Intuitively, faster maturity allows leverage to shrink more quickly in the face of declining cash flows. From (24) one can show that higher γ is associated with lower volatility, which makes the firm more aggressive about adding leverage when cash flows are high. Finally, note that the only impact of the tax rate $\bar{\pi}$ is to rescale the debt level through $\hat{y}_\xi$.

Equation (42) demonstrates clearly that once the firm is free to adjust leverage over time, equilibrium debt dynamics depart strongly from the standard predictions of tradeoff theory.

Figure 3 simulates the evolution of debt for an initially unlevered firm, issuing debt with a five-year maturity, using the same parameters as Figure 2. Leverage initially increases until it approaches approximately 40% of firm value. Once it exceeds that level, the firm issues new debt at a slower rate than its existing debt matures, and the total amount of debt declines. Overall, the firm's debt level evolves gradually based on a weighted average of past earnings. This gradual adjustment of debt towards a target level, as simulated in Figure 3, resembles the evolution of debt observed in practice (see, example, Figure 1).

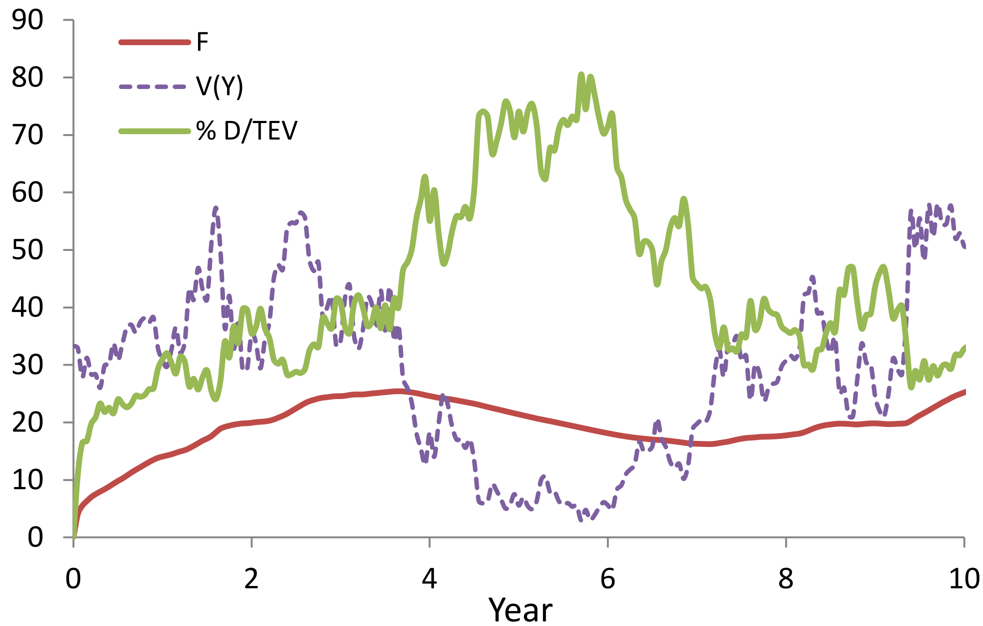


Figure 3: Simulation of Debt Evolution

($\mu = 2\%, \sigma = 40\%, \bar{\pi} = 30\%, c(1-\bar{\pi}) = r = 5\%, \xi = 20\%$)

4.2. Full Commitment Benchmarks

To facilitate discussion, we now solve two other cases that serve as benchmark with commitment. In each case, the firm's relevant state variables evolve exactly as in (18), except that the firm has the ability to commit to a certain future debt issuance policy.

No Future Debt Issues

We first consider the benchmark case in which the firm commits not to issue debt in the future, i.e. $g_t = 0$ always. We call this case the "No Future Debt" case, and indicate the corresponding solutions with the superscript "0" (representing the commitment to $g = 0$).

Recall that our methodology developed in Section 2 first called for solving the equity value function *as if* there will be no future debt issues, even though the firm will choose to add debt equilibrium, because the lack of commitment dissipates the benefits of the debt tax shield. Therefore, from **PROPOSITION 2**, we have the equity value $v^0(y) = v(y)$ with the same default boundary y_b as calculated in **PROPOSITION 4.**

Indeed, the only change in this setting will be the debt pricing. Intuitively, new debt issues (despite the absence of dilution motive given zero recovery) harm existing creditors by accelerating default, and thus debt holders are willing pay more for the same promise today if the firm can commit not to issue more debt in the future. With such a commitment, the firm's scaled cash flow y evolves according to

$$\frac{dy_t}{y_t} = (\mu + \xi)\,dt + \sigma dZ_t\,, \tag{43}$$

and thus the HJB equation for debt price can be written as

$$\underbrace{rp^0(y)}_{\text{required return}} = \underbrace{c}_{\text{coupon}} + \underbrace{\xi\left(1 - p^0(y)\right)}_{\text{gain from principal repayment}} + \underbrace{(\mu+\xi)\,yp^{0\prime}(y) + \tfrac{1}{2}\sigma^2 y^2 p^{0\prime\prime}(y)}_{\text{evolution of } dy}, \tag{44}$$

with boundary conditions:

1. No recovery value: $p^0\left(y_b^0\right) = 0$, and

2. Risk-free pricing as the distance to default grows: $p^0(y) \to \dfrac{c+\xi}{r+\xi}$ as $y \to \infty$.

Using standard methods (see e.g. the proof of **PROPOSITION 4**) the solution for the debt price is

$$p^0(y) = \frac{c+\xi}{r+\xi}\left[1 - E\left[e^{-(r+\xi)\tau_b}\right]\right] = \frac{c+\xi}{r+\xi}\left[1 - \left(\frac{y}{y_b}\right)^{-\gamma}\right], \tag{45}$$

where the constant γ is again given by (24). We summarize these results as follows:

PROPOSITION 9. If the firm can commit not to issue future debt, i.e.. $g = 0$, then the equity value is unchanged, as is the default boundary, relative to the no commitment case. Default is delayed, however, and thus the debt price improves by the value of the debt tax shield

$$p^0(y) - p(y) = \underbrace{\frac{\overline{\pi} c}{r+\xi}\left[1 - \left(\frac{y}{y_b}\right)^{-\gamma}\right]}_{\text{debt tax shield value}}. \tag{46}$$

PROOF: Equation (46) follows immediately from (27), (45) and $y_b^0 = y_b$. ■

From (46), we observe that when the firm can fully commit not to issue any debt in the future and hence is less likely to default, its debt will trade at a higher price than that of firms who cannot commit. More interestingly, the premium is equal to the value of the tax shield, consistent with the observation that, in the no commitment case, the firm issues new debt at a rate so that expected bankruptcy costs offset the expected tax benefit. Thus, commitment to $g_t = 0$ does not benefit equity holders, but does improve the value of the debt due to the reduction in bankruptcy costs, which is just the expected tax benefit.

Fixed Face Value (Leland 1998)

Another relevant benchmark for our model without commitment is Leland (1998), who assumes that firm commits to keep a fixed total face value *F*. Specifically, in Leland (1998), the firm commits to replace the maturing debt (with intensity ξ) by the same amount of newly issued debt with the same coupon, principal, and maturity. We denote this case using the superscript "ξ", which requires $g_t = \xi$ always.

The solution with constant face value is as follows. The scaled cash-flow y_t in this case follows

$$\frac{dy_t}{y_t} = \mu\, dt + \sigma dZ_t, \tag{47}$$

and equity holders in equilibrium will default at a threshold y_b^ξ to be derived shortly. Then, using the same logic as we did to compute p^0, we have the analogous solution to (45):

$$p^\xi(y)=\frac{c+\xi}{r+\xi}\left(1-\left(\frac{y}{y_b^\xi}\right)^{-\gamma_p^\xi}\right), \tag{48}$$

where the constant γ_p^ξ is defined by effectively lowering the drift by ξ (the rate of new debt issues) in (24):

$$\gamma_p^\xi=\frac{\mu-0.5\sigma^2+\sqrt{\left(\mu-0.5\sigma^2\right)^2+2\sigma^2\left(r+\xi\right)}}{\sigma^2}>0\ . \tag{49}$$

Next, the equity value $v^\xi(y)$ must solve

$$\underbrace{rv^\xi(y)}_{\text{required return}}=\underbrace{(1-\overline{\pi})(y-c)}_{\text{net after-tax earnings}}+\underbrace{\xi\left(p^\xi(y)-1\right)}_{\text{debt rollover gains/losses}}+\underbrace{\mu y v^{\xi\prime}(y)+\tfrac{1}{2}\sigma^2y^2v^{\xi\prime\prime}(y)}_{\text{evolution of state variable y}}, \tag{50}$$

with boundary conditions $v^\xi\left(y_b^\xi\right)=v^{\xi\prime}\left(y_b^\xi\right)=0$ and $v^\xi(y)\to\frac{y(1-\overline{\pi})}{r-\mu}+\frac{\overline{\pi}c}{r}-\frac{c+\xi}{r+\xi}$ as $y\to\infty$. Note that the second term in (50) captures the rollover gains/losses when equity holders refinance the maturing debt, as emphasized by He and Xiong (2012): per dollar of face value, the firm must repay principal at rate ξ, while equity holders commit to replace the maturing debt by issuing ξ new bonds at price p^ξ.

In the Appendix, we follow the approach of Leland (1998) to derive the equity value function as

$$v^\xi(y)=\frac{y(1-\overline{\pi})}{r-\mu}+\frac{\overline{\pi}c}{r}-\left(\frac{y_b^\xi(1-\overline{\pi})}{r-\mu}+\frac{\overline{\pi}c}{r}\right)\left(\frac{y}{y_b^\xi}\right)^{-\gamma_v^\xi}-\frac{c+\xi}{r+\xi}\left(1-\left(\frac{y}{y_b^\xi}\right)^{-\gamma_p^\xi}\right) \tag{51}$$

where the constant γ_v^ξ is defined as

$$\gamma_v^\xi\equiv\frac{\mu-0.5\sigma^2+\sqrt{\left(\mu-0.5\sigma^2\right)^2+2\sigma^2r}}{\sigma^2}>0\,,$$

and the endogenous default boundary y_b^ξ satisfying the smooth-pasting condition $v^{\xi\prime}\left(y_b^\xi\right)=0$ is

$$y_b^\xi = \frac{r-\mu}{(1+\gamma_v^\xi)(1-\bar{\pi})}\left[\frac{c+\xi}{r+\xi}\gamma_p^\xi - \frac{\bar{\pi}c}{r}\gamma_v^\xi\right].$$

In this case, the valuation multiple for the firm (TEV/EBIT) is given by

$$\frac{v^\xi(y)+p^\xi(y)}{y} = \frac{1-\bar{\pi}}{r-\mu} + \frac{\bar{\pi}c}{ry} - \left(\frac{1-\bar{\pi}}{r-\mu} + \frac{\bar{\pi}c}{ry_b^\xi}\right)\left(\frac{y}{y_b^\xi}\right)^{-\gamma_v^\xi - 1}. \qquad (52)$$

Since $\bar{\pi} > 0$, this multiple decreases in y for y sufficiently large, in which case the lost tax benefits outweigh the gains from lower default costs.

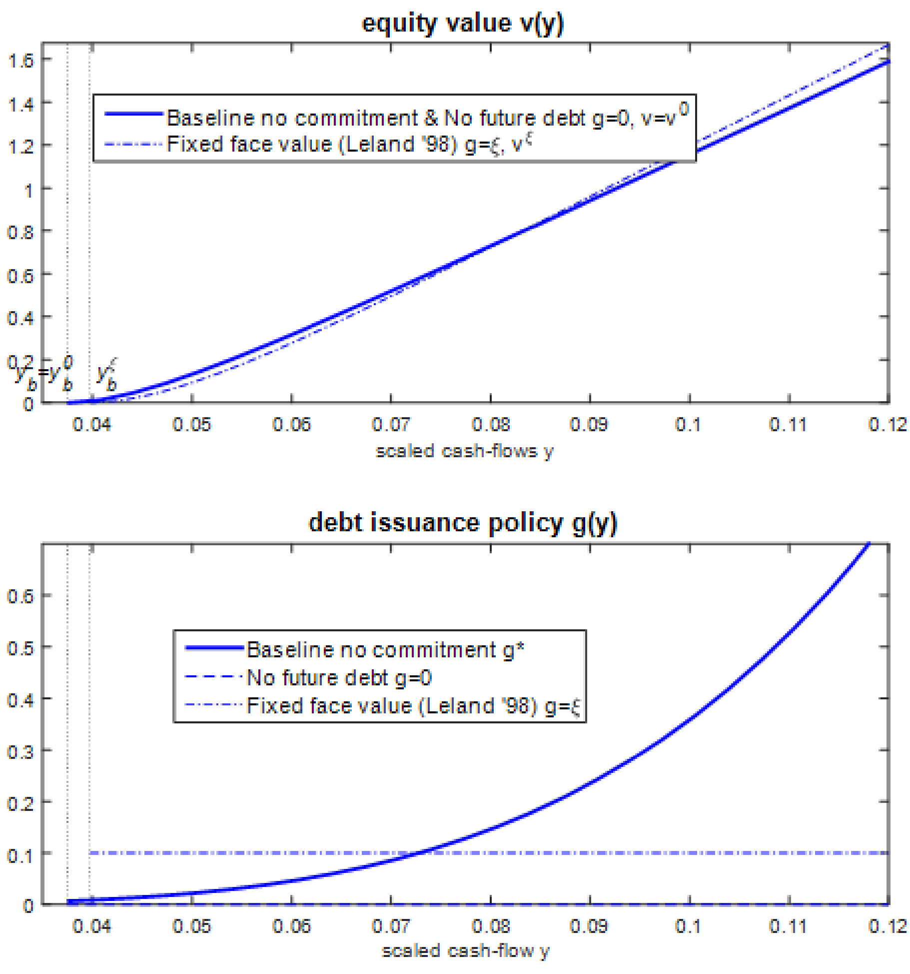


Figure 4: Equity Values for Alternative Debt Issuance Policies

Parameters are $r = 5\%$, $c = 8\%$, $\xi = 0.1$, $\mu = 2\%$, $\sigma = 25\%$, $\overline{\pi} = 35\%$.

4.3. Model Comparisons and Implications

In this section we illustrate the implications of our model by comparing the no commitment solution to other benchmarks with commitment.

Value Functions and Debt Issuance Policy

Figure 4 plots the equity value and debt issuance policies for the three models: the base model without commitment ($g = g^*$, solid thick line), full commitment to no future debt ($g = 0$, dashed thin line), and Leland (1998) commitment to a fixed face value ($g = \xi$, dash-dotted thin line).

As explained, the equity value in the no-commitment case coincides with the setting when there are no future debt issues. With a fixed face value policy, the equity value is lower when cash flows are low, as the firm is committed to continuing to issue debt even in the face of large rollover losses. This effect gives rise to a higher default boundary y_b^ξ than the default boundary in the other two cases $y_b = y_b^0$, as indicated in the top plot of Figure 4.[19] On the other hand, when cash flows are high, the equity value in the fixed face value case is higher than that in either case with $g = 0$ or $g = g^*$. Relative to the $g = 0$ case where tax benefits are lost as debt matures, the firm in the fixed face value case maintains its debt and so enjoys greater tax benefits. On the other hand, the firm in the fixed face value case commits to a debt policy that is much less aggressive than the no-commitment case, hence incurring a much lower bankruptcy cost.

Figure 5 illustrates the debt price and valuation multiple for each policy. Not surprisingly, as shown in **PROPOSITION 9**, the debt price with “no future debt issuance” $g = 0$ dominates that without commitment, simply because future debt issuance pushes the firm closer to the default boundary. This also explains why in the bottom panel, the TEV multiple without commitment is

[19] However, $y_b^\xi < y_b$ could potentially occur, especially when the tax benefit $\overline{\pi}$ is high. Recall y_b is the default policy as if equity holder obtains no tax benefit, while for y_b^ξ, the firm with fixed debt face value indeed captures some tax benefit (hence, a high $\overline{\pi}$ pushes equity holders to default later).

always lower than that under commitment of $g = 0$ (recall equity values are the same under these two cases).

From (46), we see that the debt price premium due to "no future debt issuance" grows with the firm's distance to default. This implies that the debt of firms that cannot commit will exhibit large credit spreads even when the firm's current total leverage is very low--but, of course, the future leverage might be high. In fact, even for almost zero current leverage, the credit spreads for firms without commitment are non-zero. In contrast, the credit spreads for almost zero leverage firms are zero for the other two benchmark cases.

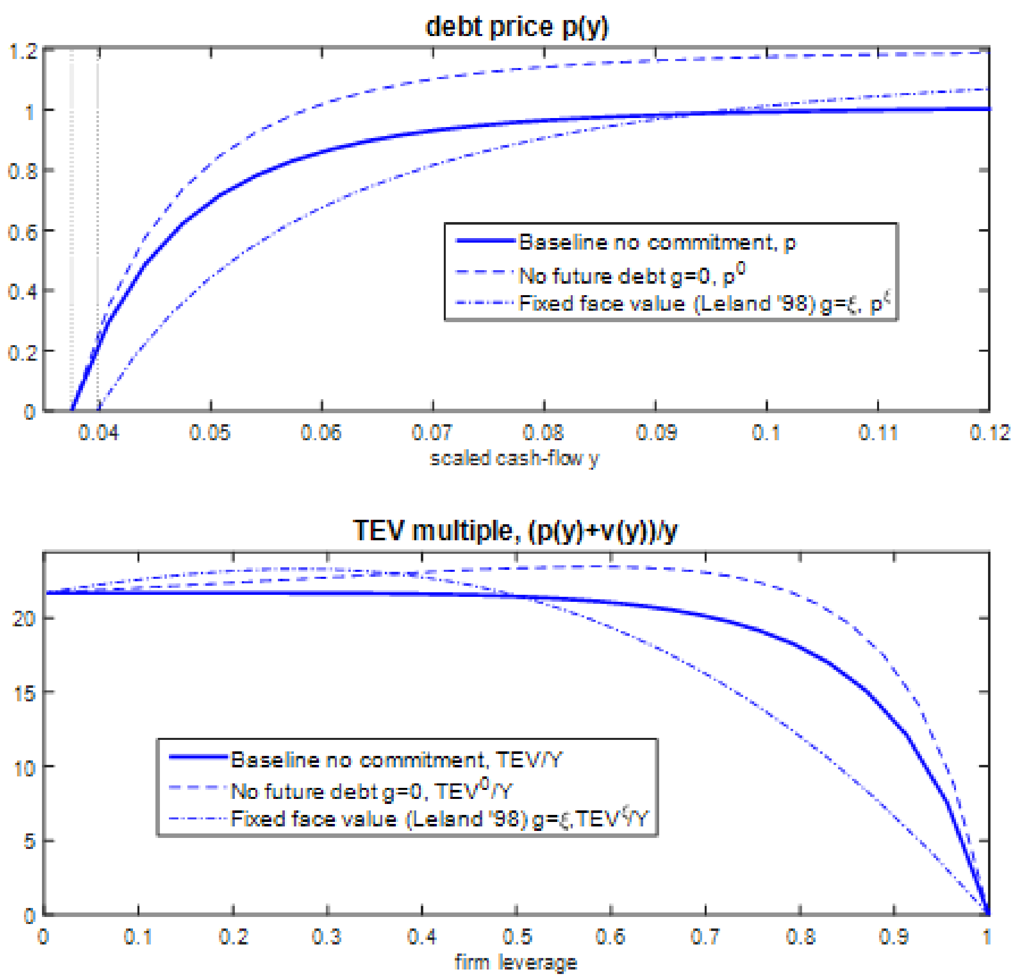


Figure 5: Debt Price and TEV Multiple for Alternative Debt Issuance Policies

Parameters are $r = 5\%$, $c = 8\%$, $\xi = 0.1$, $\mu = 2\%$, $\sigma = 25\%$, $\overline{\pi} = 35\%$.

Relative to our base case, the fixed face value (Leland 1998) case generates a lower debt price for low y but higher debt price for high y. This is due to the endogenous issuance policy g^* plotted in Figure 4. There, we observe that the debt issuance policy without commitment is increasing in y, and slower (faster) than the fixed face value policy when y is low (high), and investors price the debt in anticipation of these future leverage polices.

The next proposition summarizes the comparison of debt values across three models, depending on the firm's profitability state y. Figure 5 corresponds to the case of $y_b^\xi > y_b$, so for sufficiently low y, the debt price in the case of fixed face value p^ξ drops below the other two cases.

PROPOSITION 10. We always have $p(y) < p^0(y)$. For $y \to \infty$ we have

$$p(y) < p^\xi(y) < p^0(y)$$

For sufficiently low y so that $y \to \min(y_b, y_b^\xi)$, we have

$$\begin{aligned} & p^\xi(y) < p(y) < p^0(y) \text{ if } y_b^\xi > y_b \\ & p(y) < p^0(y) < p^\xi(y) \text{ if } y_b^\xi < y_b \end{aligned}.$$

PROOF: $p(y) < p^0(y)$ is implied by (46). When $y \to \infty$, $p^\xi(y) = \dfrac{c+\xi}{r+\xi}\left(1-\left(\dfrac{y}{y_b^\xi}\right)^{-\gamma_p^\xi}\right) \to \dfrac{c+\xi}{r+\xi}$

which exceeds $p(y) = \dfrac{(1-\bar{\pi})c+\xi}{r+\xi}\left(1-\left(\dfrac{y}{y_b}\right)^{-\gamma}\right) \to \dfrac{(1-\bar{\pi})c+\xi}{r+\xi}$. To show that $p^\xi(y) < p^0(y)$,

we need to show that $\left(\dfrac{y}{y_b}\right)^{-\gamma} < \left(\dfrac{y}{y_b^\xi}\right)^{-\gamma_p^\xi}$ holds when $y \to \infty$, which is equivalent to $\gamma_p^\xi < \gamma$; but the latter holds by comparing (49) and (24). The second part of result is obvious as the debt price drops to zero at the default boundary. ■

Finally, as indicated in (30), in our no commitment case the firm's TEV multiple is strictly increasing in the scaled cash flow y. Consequently, holding the level of cash flows Y fixed, total firm value decreases with the debt face value F. In other words, in the no commitment equilibrium there is always a loss to total firm value from leverage – the tax benefits of debt more than offset the resulting bankruptcy costs due to future debt increases. This result is shown in the TEV-multiple-against-leverage plot in the bottom panel of Figure 5: there, the solid line (i.e., the no commitment case) achieves its maximum at zero leverage. In contrast, TEV multiples in both commitment cases have an interior maximum.[20] This interior maximum is often viewed as the "optimal" leverage in the traditional trade-off theory, though of course in a dynamic context it may be far from optimal ex-post once shocks are realized (see, e.g., Fischer, Heinkel, Zechner (1989) and Strebulaev (2007)).

A Comparison of Leverage Dynamics

In two benchmark cases with commitment, the scaled cash-flows follow a geometric Brownian motion with exogenous drifts, i.e., $dy_t/y_t = (\mu+\xi)dt + \sigma dZ_t$ for the case of no future debt issuance, and $dy_t/y_t = \mu dt + \sigma dZ_t$ for the fixed face value case. In our model with no commitment, the equilibrium evolution of the firm's scaled cash-flows is:

$$dy_t = \left(\mu + \xi - g^*(y_t)\right) y_t dt + \sigma y_t dZ_t = \left[\mu + \xi - \frac{(r+\xi)\bar{\pi} c y_t^{\gamma}}{\gamma\left((1-\bar{\pi})c + \xi\right) y_b^{\gamma}}\right] y_t dt + \sigma y_t dZ_t . \tag{53}$$

The equilibrium debt issuance policy $g^*(y_t)$ in (29) is increasing in y_t, implying that y_t grows slower when y_t is higher. In fact, the firm's scaled cash-flows are mean-reverting towards the steady-state value (recall the definition of $\hat{y}_g$ in (39)).[21]

$$\hat{y}_{g=\mu+\xi} \equiv y_b \left(\frac{\gamma(\mu+\xi)\left(c(1-\bar{\pi})+\xi\right)}{(r+\xi)\bar{\pi} c} \right)^{1/\gamma} \tag{54}$$

---

[20] This is evident in the bottom panel of Figure 5, as both dashed and dash-dotted lines have a positive slope at zero leverage, and drop to zero when leverage reaches 100%.

[21] Strictly speaking, to ensure y_t to mean revert over its equilibrium region $y_t \in [y_b, \infty)$, one has to show that $\hat{y}_{\mu+\xi} > y_b$ so that the drift of y_t is positive when $y_t = y_b$, which is indeed the case for our baseline parameters. However, $\hat{y}_{\mu+\xi} < y_b$ could occur for sufficiently large σ (so $\gamma \to 0$ in (54)).

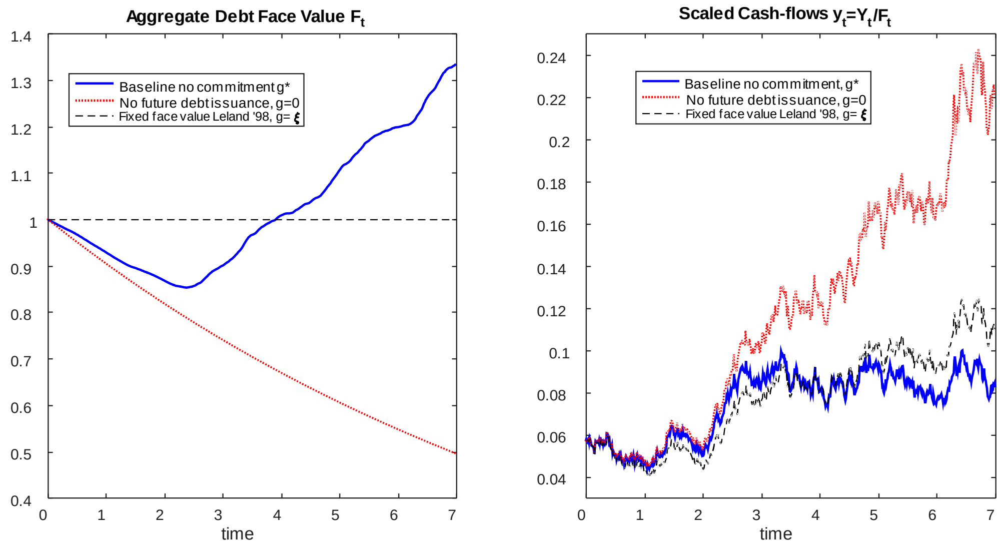


Figure 6: Aggregate debt face values (left panel) and scaled cash-flows dynamics (right panel) for three models.

With fixed cash-flow shocks $\{dZ\}$ and $r = 5\%$, $c = 8\%$, $\xi = 0.1$, $\mu = 2\%$, $\sigma = 25\%$, $\overline{\pi} = 35\%$.

We are interested in the firm leverage dynamics implied by three different models. For given underlying cash-flow shocks $\{dZ_t\}$, the left panel of Figure 6 plots the debt face value F_t, while the right panel shows the dynamics of scaled cash-flow y_t, which tracks one-to-one to the firm's interest-coverage-ratio (or book leverage). Because the underlying shocks are the same, the differences across these three different models are purely due to their different debt issuance policies. In this sample path, negative shocks in the early years cause the firm in our baseline no commitment case to issue less debt compared to the fixed face value case which commits to $g = \xi$, but of course since $g^* > 0$ the firm has more debt than it would in the no issuance case. Later, when the firm has positive shocks, the firm issues debt even faster than it matures and the debt level grows. As a result, y_t in the no commitment case (blue solid line) has a larger upward drift initially, but this reverses near the end of the sample path.

4.4. Debt Maturity Structure

In our model, the firm commits to a constant debt maturity structure, i.e., all debt has an expected maturity of $1/\xi$. This assumption is common in much of the dynamic capital structure literature which treats the debt maturity structure as a parameter.[22] It is beyond the scope of this paper to relax the full commitment assumption on the firm's debt maturity structure policy; for some recent research, see He and Milbradt (2016).

We can consider, however, the consequences of the initial maturity choice. What if the firm can choose the debt maturity ξ to commit to for the future? In this section, we first show if the firm does not need to borrow a fixed amount upfront, then it is indifferent between all debt maturity choices. We then show that if the firm is restricted to borrow a fixed amount initially, then debt with shortest maturity becomes optimal in maximizing the expected tax shield net of bankruptcy cost. However, the planner who only concerns about bankruptcy cost prefers the longest debt maturity.

Optimal Debt Maturity without Fixed Borrowing

We have seen that from (30) that the firm's TEV multiple is increasing in $y = Y / F$. Consequently, value-maximizing firms should set the optimal initial debt face value to be $F_0^* = 0$. Given this choice, (30) implies that the firm's TEV multiple no longer depends on the debt maturity structure ξ. This indifference result is deeper than it appears: Although the firm starts with no initial debt, recall that (42) says the firm will begin issuing debt immediately, and the debt maturity ξ does affect those future debt contracts. Nevertheless, this dynamic consideration has no bite on the optimal maturity choice by equity holders: Although different maturity structures lead to very different future leverage dynamics (as illustrated shortly), any gains from tax savings are offset by increased bankruptcy costs.

This irrelevance result can be generalized further in our model. Consider the following thought experiment, in which equity -- facing the state pair (Y_t, F_t) and the existing debt maturity structure ξ -- is offered with a one-time chance of choosing ξ' for the firm's future debt. That is, the firm's existing debts continue to retire at the old speed ξ, but the newly issued debts are with the new maturity and hence will retire at the new speed ξ'. We have the following proposition.

[22] To mention a few, Leland (1998), Leland and Toft (1996), He and Xiong (2012), and Diamond and He (2014).

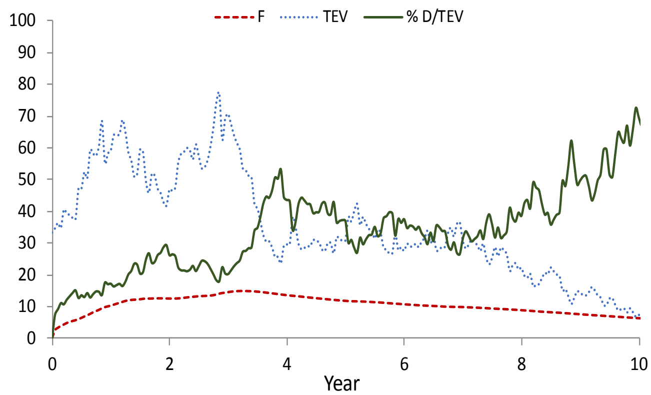
PROPOSITION 11. In a no-commitment equilibrium with smooth debt issuance, the firm's equity value is independent of the maturity structure ξ' of new debt.

PROOF: For equilibria in which equity holders are taking smooth debt issuance polices, equity holders obtain zero profit by issuing future debt, and their value will be the same as if equity does not issue any future debt. As a result, the equity value depends on the maturity structure ξ of existing debt, but not on the maturity structure ξ' of future debt. ■

The logic of Proposition 9 and hence the indifference result can be further generalized to a setting in which the firm is free to choose any maturity structure for its newly issued debt any time. Again, the equity value only depends on the maturity structure of existing debt.

While equity holders are indifferent between alternative maturity structures, different maturity choices will lead to very different patterns and levels of future leverage. For example, **Figure 7** shows the total enterprise value, debt amount, and leverage (debt value/TEV) for the firm given identical productivity shocks, but financed either using five-year or one-year debt. In both cases the initial TEV and equity value is the same, but leverage evolves quite differently. With longer-term debt, debt changes gradually, as the firm issues debt more slowly, and leverage is lower on average. With shorter-term debt, the firm issues debt more rapidly knowing it can decrease debt quickly by not rolling over maturing debt. Because it can adjust debt more quickly, the firm has higher leverage on average.[23]

[23] Of course, in our model we have assumed away transactions costs associated with issuing or rolling over debt. Such considerations would make long-term debt less costly, as in Dangl and Zechner (2016).



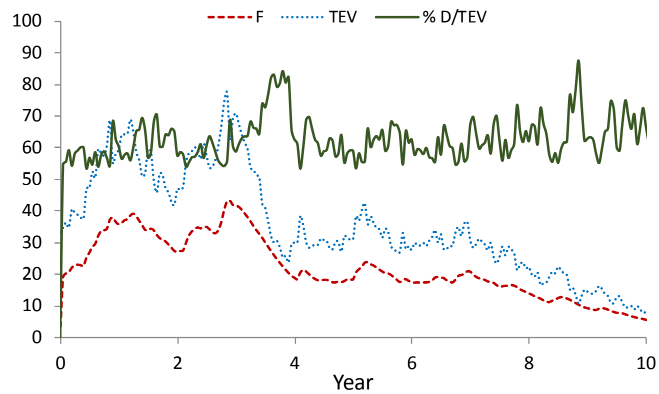


Figure 7: Debt and Leverage with Differing Maturities.

Left panel shows TEV, debt face value, and market leverage with 5-year average debt maturity. Right panel shows 1-yr average debt maturity. In either case, initial firm value is unchanged, but leverage is higher and adjusts more quickly with shorter-term debt. Parameters are $\mu = 2\%$, $\sigma = 40\%$, $\bar{\pi} = 30\%$, $c(1-\bar{\pi}) = r = 5\%$, $\xi = 0.2$ (5-year debt) or 1 (1-year debt).

Optimal Debt Maturity with Fixed Borrowing

Suppose the firm *must* raise some amount of funds initially through debt. Issuing a discrete amount of debt in our model is not optimal – shareholders would be better off issuing debt gradually – but suppose the firm must raise funds quickly and equity capital is not available in the short run. In that case we can show that short-term debt maximizes not only the firm value, but also the debt capacity, i.e., the maximum amount of debt that the firm can raise.

Our model highlights one advantage of short-term debt which allows firms to adjust their leverage burden in response to fundamental shocks in a more flexible way. This point has been neglected in the Leland-style literature which often assumes the firm is committing to a future leverage policy with fixed debt face value. For instance, He and Xiong (2012) show that the longest possible debt maturity structure minimizes the rollover risk. As we will explain, the difference is driven by different assumptions on leverage policies.

Given initial cash-flow Y_0, the firm sets the initial debt face value F_0 to raise D_0 from debt holders. From (30) we know that the firm value is (recall $y_0 = Y_0 / F_0$)

$$Y_0 \frac{v(y_0) + p(y_0)}{y_0} = Y_0 \frac{1-\bar{\pi}}{r-\mu}\left[1-\left(\frac{y}{y_b}\right)^{-\gamma-1}\right] < Y_0 \frac{1-\bar{\pi}}{r-\mu} \qquad (55)$$

Hence, both the firm value TEV_0 and the debt value D_0 cannot exceed the upper bound $Y_0 \dfrac{1-\bar{\pi}}{r-\mu}$.

PROPOSITION 12. Suppose the firm starts with initial cash flows Y_0.

i) For any target debt value $D_0 < Y_0 \frac{1-\bar{\pi}}{r-\mu}$, the optimal debt maturity structure that maximizes the levered firm value (and hence the equity value) is $\xi^* = \infty$.

ii) The debt capacity $\sup D_0$, which is the highest debt value that the firm is able to raise, equals $Y_0 \frac{1-\bar{\pi}}{r-\mu}$ by setting $F_0 \uparrow \frac{Y_0}{y_b}$ and $\xi^* \to \infty$.

PROOF: The first claim follows by showing that $\xi = \infty$ always achieves the upper bound of $Y_0 \frac{1-\bar{\pi}}{r-\mu}$. Because

$$\gamma = \frac{\mu + \xi - 0.5\sigma^2 + \sqrt{\left(0.5\sigma^2 - \mu - \xi\right)^2 + 2\sigma^2 \left(r + \xi\right)}}{\sigma^2} \to \infty \text{ as } \xi \to \infty$$

So given $y > y_b$ we have $\left(y / y_b\right)^{-\gamma-1}$ to vanish in (55) as $\xi \to \infty$, which proves the claim. For the second claim, from (27) we have

$$D_0 = F_0 p\left(\frac{Y_0}{F_0}\right) = F_0 \frac{c\left(1-\bar{\pi}\right)+\xi}{r+\xi}\left(1-\left(\frac{y_0}{y_b}\right)^{-\gamma}\right)$$

By setting $\xi \to \infty$ the term in the parentheses vanishes, while $F_0 \uparrow \frac{Y_0}{y_b}$ delivers the upper bound debt value $D_0 \uparrow Y_0 \frac{1-\bar{\pi}}{r-\mu}$. ■

Figure 8 illustrates Proposition 12 by plotting the firm value TEV_0 (left panel) and debt value D_0 (right panel), both as a function of initial face value F_0. We consider three debt maturities: long-term debt with a 100-year maturity ($\xi = 0.01$, dash-dotted); medium-term debt with a 10-year maturity ($\xi = 0.1$, dashed), and short-term debt with 3-day maturity ($\xi = 100$, solid). The left panel shows that the firm value is maximized by using 3-day maturity debt. This is simply because $\xi \to \infty$ implies that $\gamma \to \infty$, and hence for any $y > y_b$ the firm value achieves its upper bound $Y_0 \frac{1-\bar{\pi}}{r-\mu}$ in (55). Of course, a too high F_0 pushes the scaled cash-flow y_0 below

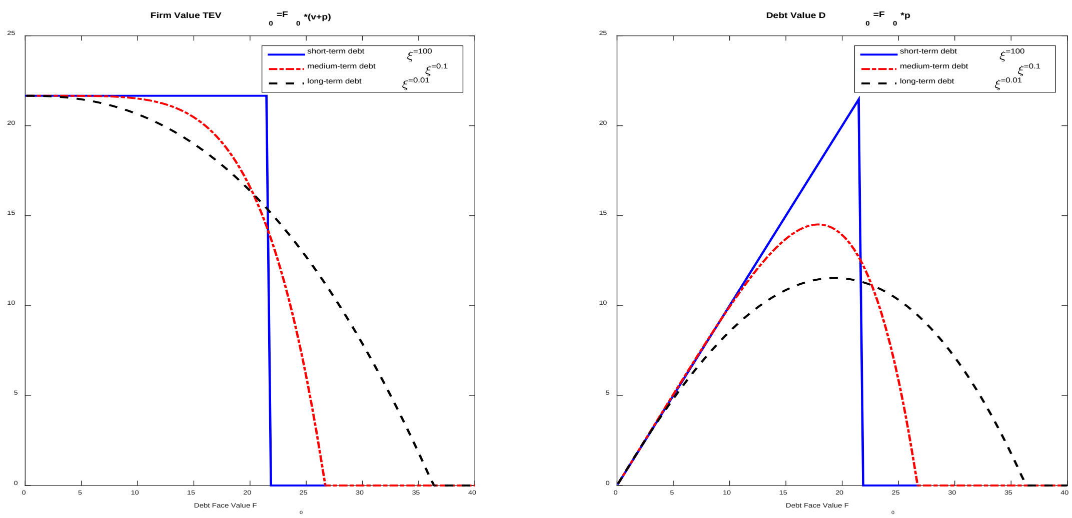


Figure 8. Firm value and debt value as a function of initial debt face value F_0 , for three levels of debt retiring rate ξ 's. Blue solid line is for shortest debt maturity $1/\xi = 0.01$, red dash-dotted line is for medium debt maturity $1/\xi = 10$, and black dashed line is for longest debt maturity $1/\xi = 100$ Both firm and debt values hit zero when $F_0 = Y_0 / y_b$. Parameters are r = 5% , c = 8% , ξ = 0.1 , μ = 2% , σ = 25% , $\overline{\pi}$ = 35% .

y_b, triggering default---and the firm value drops to zero. Although firms with the shortest-term debt---with the highest default threshold y_b---face the tightest constraint in setting a high F_0,[24] the right panel of Figure 8 shows that they achieve the highest market value of debt, thanks to the "Laffer" curve effect: Because long-term debt makes it difficult for the firm to reduce leverage in response to shocks, at some point an increase in the face value is more than offset by the increase in credit risk. As shown, the upper bound of D_0, $Y_0 \dfrac{1-\overline{\pi}}{r-\mu}$, is achieved by setting F_0 at (slightly below) the default boundary for firms using "overnight" debt.

Our model differs from Leland type models in the firm's future leverage policy. In both settings, the debt maturity structure ξ captures the speed of debt retiring. In Leland, committing to refinance those retiring bonds (to maintain the aggregate debt face at a constant) leads equity to bear rollover gains/losses $\xi\left(p^{\xi}\left(y\right)-1\right)$ -- see Equation (50). When the debt price is low (because leverage is already high), rollover losses lower the option value to equity holders of

[24] Under a mild sufficient condition $r > c(1-\overline{\pi})$, one can show that the firm defaults earlier with shorter-term debt, i.e., default threshold y_b in (25) is increasing in debt maturity ξ . This is a general result in Leland-style models.

keeping the firm alive and thus they will default earlier, creating rollover risk. The higher the ξ, the stronger the rollover risk.[25] To mitigate this rollover risk, a value-maximizing firm will set $\xi = 0$, corresponding to a consol bond that is free of rollover concerns.

When the firm can change its future debt burden freely, there emerges an important benefit of short-term debt. Short-term debt retires quickly, and the firm swiftly adjusts its leverage in response to profitability shocks, in that more (less) debt is issued following positive (negative) performance. As shown in **Figure 7**, this flexibility allows the firm to borrow more on average and hence capturing a higher debt tax shield. We will come back shortly to this issue when we discuss the optimal debt maturity choice from a social perspective.

We caution that **PROPOSITION 12** heavily relies on a strong assumption that underlies all of these models: the firm is facing a frictionless equity market through which equity could inject liquidity at any time in a costless way. Following a sequence of negative shocks, to repay the mounting debts that are maturing instantaneously, firms issue equity as needed. This assumption allows a firm with very short-term debt to hold high leverage (as in the right panel of Figure 7) without risking a liquidity induced default. We expect that modelling an equity market with realistic frictions may effect these qualitative results, an interesting question left for future research.

Ultra-Short-term Debt and Commitment

The previous section shows that in our model the firm prefers to set the debt maturity to be ultra-short-term if the firm is forced to raise some amount of funds initially through debt. The ultra-short-term debt matures instantaneously every dt, much like demand deposits.

In the literature, some papers (e.g. Tserlukevich, 2008) suggest that when the firm can adjust its leverage freely in response to the cash-flows shocks, then it is optimal to set $F_t \approx Y_t / c$ always so that the firm avoids default while capturing the entire debt tax shield. In particular, it seems that ultra-short-term debt, which at any moment matures entirely and allows the firm to

[25] See He and Xiong (2012) who show this result by having the debt face value fixed, and Diamond and He (2014) who strengthen this result by have the debt market value fixed.

reset its leverage to its favorable level in response to cash-flow shocks, can achieve this goal. For instance, setting issuance policy $d\Gamma_t$ as

$$d\Gamma_t = \left(\mu + \xi\right) F_t dt + \sigma F_t dZ_t \tag{56}$$

could potentially prevent the scaled cash flow y_t from fluctuating over time. Essentially, this captures the flexibility advantage offered by short-term debt.

Although the flexibility benefit does apply in our model, the above argument implicitly assumes that equity holders can commit to the first best leverage policy. In our model, the inability to commit to certain future leverage policy matters in a significant way – equity holders continue to raise debt until the likelihood of default impacts its price. This point is highlighted in (56) that the firm repurchases debt following negative cash-flow shocks $dZ_t < 0$, while in our model shareholders never find debt repurchases optimal. In the limit, even with instantaneously maturing debt, there is always a risk of bankruptcy in our model, so that the implied bankruptcy cost offsets the tax benefit.

Optimal Debt Maturity from a Social Perspective

As shown in Figure 7, a shorter debt maturity leads firms to take on more debt and hence capturing a larger tax shield, but at a cost of a higher expected bankruptcy cost. Therefore, though firms who treat the tax shield as a tool to boost earnings are indifferent to debt maturity, from a social perspective there is a clear *ex ante* ranking. Because only bankruptcy results in a real deadweight cost, a planner should favor debt with the longest possible maturity, as long-term debt leads firms to take on less debt on average (see **Figure 7**).

We can calculate the expected bankruptcy cost $BC\left(y\right)$, as a fraction of unlevered firm value, and study how it varies with debt maturity ξ . Suppose the firm stands at $\left(Y_0 = yF_0, F_0\right)$. Using $dF / F = \left(-\xi + g\left(y\right)\right)dt$ in (41) and $Y_{\tau_b} = y_b F_{\tau_b}$, we have

$$BC\left(y\right) = \frac{1}{Y_0} E\left[e^{-r\tau_b} Y_{\tau_b} \left| Y_0 = yF_0, F_0\right.\right] = \frac{y_b}{y} \cdot \underbrace{E\left[\exp\left(-\int_0^{\tau_b} \left(r + \xi - g\left(y_t\right)\right)dt\right) \middle| y_0 = y\right]}_{H(y)}$$

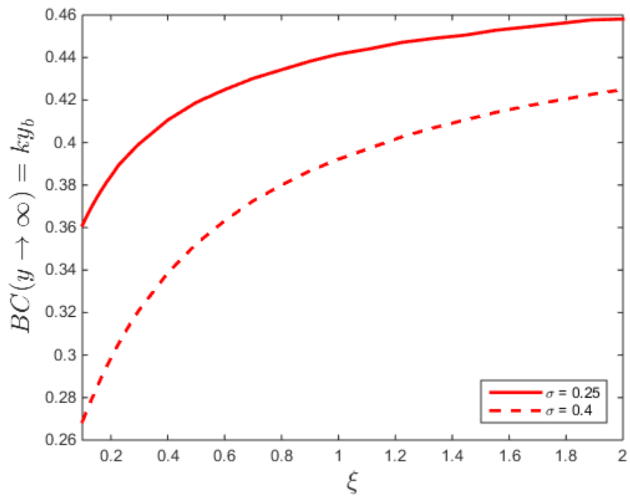


Figure 9. Expected bankruptcy cost *BC* (as a fraction of unlevered firm value when $y = \infty$) as a function of debt retiring rate ξ. Bankruptcy cost is increasing in ξ, implying that the planner who minimizes bankruptcy cost prefers the longest-term debt. Two volatility levels are considered. Other parameters are $r = 5\%$, $c = 8\%$, $\xi = 0.1$, $\mu = 2\%$, $\bar{\pi} = 35\%$.

We then calculate $H(y)$ numerically based on the following ODE

$$[r+\xi-g(y)]H(y) = (\mu+\xi-g(y))yH'(y)+0.5\sigma^2y^2H''(y)$$

with boundary conditions $H(y_b)=1$ and $H(y)\to ky$ when $y\to\infty$ for some constant $k>0$.[26]

For illustration, we focus on the case of $y=\infty$; as we mentioned, in our model value-maximizing firms should set the optimal initial debt face value to be $F_0^*=0$. In this case, we have $BC=ky_b$, where both endogenous constants k and y_b depend on debt maturity ξ. **Figure 9** plots BC against ξ for two different levels of cash-flow volatility. First, the planner who aims to minimize bankruptcy cost prefers longer-term debt (lower ξ). Second, interestingly, a lower volatility gives rise to a greater bankruptcy cost. The key to this counter-intuitive result is the endogenous debt issuance policy. Equity in a firm with lower volatility is more aggressive in leveraging the firm up, so much so that we have a greater overall expected bankruptcy cost.

[26] Bankruptcy cost cannot exceed the first-best firm value which is linear in y. Strictly speaking, $H(y)=ky+o(y)$ for $y\to\infty$.

Another important advantage of studying $y=\infty$ is that, in this case, one can show that expected bankruptcy costs move in tandem with expected tax shields.[27] This illustrates the sharp contrast between the objectives of the firm and that of the planner, as the shortest debt maturity maximizes tax shield but in the same time yields the largest bankruptcy cost. Note however, that this ex ante ranking is reversed ex post, if the firm is close to default (y near y_b). Once the firm is near distress, shorter-term debt maturity (a larger ξ) would result in a lower bankruptcy costs $BC(y)$ because a firm with a larger ξ can delever more quickly when close to default (see **Figure 2**).

5. Endogenous Investment and Debt Overhang

In this setting we extend our model by adding an endogenous investment decision also under the control of shareholders. Including investment allows us to explore the interaction of shareholder-creditor conflicts with regard to both investment and leverage choices. As expected from Myers (1977), debt overhang leads firms to underinvest. We show that when shareholders are unable to commit to future leverage decisions, debt overhang is more severe when profitability is high, but less severe when profitability is low, than when the debt level is fixed as in the Leland (1998) case. In addition, we show that when shareholders cannot commit to future investment decisions, the leverage ratchet effect becomes more severe, and the firm will issue debt more rapidly, than the benchmark case in which investment is fixed.

5.1. General Analysis

Now we allow equity holders to choose the firm's endogenous investment policy i_t, which affects the drift of cash-flow process by $\mu(Y_t,i_t)$ at a cost of $K(Y_t,i_t)$. Both functions are smooth with $\mu_i(Y_t,i_t)>0$, $K_i(Y_t,i_t)>0$ and $K_{ii}(Y_t,i_t)>0$.

---

[27] When $y=\infty$, the total enterprise value is $(1-\bar{\pi})y/(r-\mu)$, while the unlevered asset value is $y/(r-\mu)$. Hence, the sum of bankruptcy cost and tax shield equals to $\bar{\pi}y/(r-\mu)$, independent of the debt maturity structure ξ. As a result, bankruptcy cost and tax shield have to move in the same direction when we vary ξ.

For illustration purposes, we carry out the general analysis with investment under the general cash-flow diffusion process without jumps; the analysis for a cash-flow process as in (1) with jumps is similar.

The HJB equation for equity holders, who are choosing the firm's debt issuance G and investment i, can be written as

$$\underbrace{rV\left(Y,F\right)}_{\text{required return}} = \underbrace{\left(1-\bar{\pi}\right)Y-\left(\left(1-\bar{\pi}\right)c+\xi\right)F}_{\text{after-tax cash-flows net coupon-principal payment}} +$$

$$\max_{G,i}\left[\underbrace{p\left(Y,F\right)G+(G-\xi F)V_F\left(Y,F\right)}_{\text{evolution of } dF}+\underbrace{\mu(Y,i)V_Y\left(Y,F\right)+\frac{\sigma(Y)^2}{2}V_{YY}\left(Y,F\right)}_{\text{evolution of } dY}-\underbrace{K\left(Y,i\right)}_{\text{investment cost}}\right]$$

As before we focus on the equilibrium where G takes interior solutions, which implies that $p=-V_F$, and we can solve for the equity value function as if there is no issuance, i.e., $G=0$. Suppressing the arguments for $V\left(Y,F\right)$, the HJB equation with $G=0$ becomes

$$rV=\max_i\left[\left(1-\bar{\pi}\right)Y-\left(\left(1-\bar{\pi}\right)c+\xi\right)F-\xi FV_F-K\left(Y,i\right)+\mu(Y,i)V_Y+\frac{1}{2}\sigma(Y)^2V_{YY}\right] \tag{1}$$

The first-order condition for the optimal investment policy i^* is

$$K_i\left(Y,i^*\right)=\mu_i\left(Y,i^*\right)V_Y \tag{2}$$

This condition characterizes the optimal investment policy i^* under the assumption that $\mu(Y,i)V_Y-K\left(Y,i\right)$ is strictly concave in i.

The equilibrium issuance policy can be derived as before. Plugging the optimal investment policy i^* into (1), and taking derivative with respect to F further, we have

$$rV_F=-\left(\left(1-\bar{\pi}\right)c+\xi\right)-\xi V_F-\xi FV_{FF}+\mu(Y,i^*)V_{YF}+\frac{1}{2}\sigma(Y)^2V_{YYF}$$

As before, we have used the Envelope theorem that we can ignore the dependence of the optimal policy i^* on F.[28] Using $p=-V_F$, we have

$$-rp\left(Y,F\right)=-((1-\bar{\pi})c+\xi)-\mu(Y,i^*)p_Y\left(Y,F\right)+\xi p\left(Y,F\right)+\xi Fp_F\left(Y,F\right)-\tfrac{\sigma(Y)^2}{2}p_{YY}\left(Y,F\right) \qquad (3)$$

The valuation equation for debt price $p\left(Y,F\right)$ in (5), with the equilibrium evolution of state variables $\left(Y,F\right)$, is

$$\underbrace{rp\left(Y,F\right)}_{\text{required return}}=\underbrace{c+\xi\left(1-p\left(Y,F\right)\right)}_{\text{coupon and principal payment}}+\underbrace{(G-\xi F)p_F\left(Y,F\right)}_{\text{evolution of }dF}+\underbrace{\mu(Y,i^*)p_Y\left(Y,F\right)+\tfrac{\sigma(Y)^2}{2}p_{YY}\left(Y,F\right)}_{\text{evolution of }dY} \qquad (4)$$

Combining (3) and (4) gives rise to the exact same equilibrium debt issuance policy as in PROPOSITION 3**:**

$$G\left(Y,F\right)=\frac{\bar{\pi}c}{-p_F\left(Y,F\right)}=\frac{\bar{\pi}c}{V_{FF}\left(Y,F\right)}>0.$$

Again, the equilibrium requires the condition in PROPOSITION 1, i.e. $-p_F\left(Y,F\right)=V_{FF}\left(Y,F\right)>0$ holds always.

5.2. Log-normal Cash Flows and Quadratic Adjustment Costs

Consider the setting with a log-normal cash-flow process studied in Section 3, with $\mu(Y,i)=\left(\mu+i\right)Y$ and $K\left(Y,i\right)=0.5\kappa i^2Y$, where $\kappa>0$ is a positive constant. Here, the investment i increases the cash-flow growth rate linearly, and the cost is proportional to cash-flow size Y but quadratic in investment i. Without debt, our model is similar to Hayashi (1982), and the optimal investment policy for unlevered firm, denoted by $i_{unlever}$, is given by:

$$i_{unlever}=r-\mu-\sqrt{\left(\mu-r\right)^2-2\left(1-\pi\right)/\kappa}. \qquad (5)$$

Now we derive the solution to our model with leverage without commitment. Denote the optimal investment rate by i_t^* so that the evolution of scaled cash-flow $y_t=Y_t/F_t$ is

[28] The envelope theorem readily applies if i^* takes interior solutions always. Even if the optimal investment policy takes a binding solution, our logic goes through as long as the constraint does not depend on F (hence there is no first-order gain by changing the firm's debt).

$$\frac{dy_t}{y_t} = \left(\mu + i_t^* + \xi - g_t\right)dt + \sigma dZ_t .$$

Equity holders default when y_t hits the endogenous default boundary y_b. The scaled equity value $v(y)$ without debt issuance satisfies

$$(r+\xi)v(y) = \max_i (1-\bar{\pi})(y-c) - \xi - \frac{\kappa i^2}{2} y + (\mu + i + \xi) y v'(y) + \frac{1}{2}\sigma^2 y^2 v''(y) . \tag{6}$$

Given the optimal investment $i^*(y) = v'(y)/\kappa$, the above equation becomes

$$(r+\xi)v(y) = (1-\bar{\pi})(y-c) - \xi + \frac{y\left[v'(y)\right]^2}{2\kappa} + (\mu+\xi) y v'(y) + \frac{1}{2}\sigma^2 y^2 v''(y), \tag{7}$$

with two boundary conditions: $v(y) = \kappa i_{unlever} y - \frac{(1-\bar{\pi})c + \xi}{r+\xi}$ for $y \to \infty$, and $v(y_b) = 0$. The default boundary y_b is determined by the smooth-pasting condition $v'(y_b) = 0$.

The equity value function $v(y)$ and default boundary y_b are readily solvable by standard Matlab built-in ODE solver. The debt price is then given by $p(y) = yv'(y) - v(y)$, and the debt issuance policy is $g(y) = \frac{\bar{\pi} c}{yp'(y)} > 0$.

Finally, we need to verify the key condition $v''(y) > 0$ (or equivalently $p'(y) > 0$) in Proposition 1. Although we no longer have closed-form solution in the model with investment, in the next proposition we show that $v''(y) > 0$ holds by analyzing the ODE (7) satisfied by the equity value.

PROPOSITION 13. In the log-normal cash-flow model with quadratic investment costs, equity value is strictly convex, i.e., $v''(y) > 0$, so that the debt price is decreasing in debt face value. This guarantees the optimality of smooth issuance policy and hence the investment policy $i^*(y) = v'(y)/\kappa$ and issuance policy $g(y) = \dfrac{\bar{\pi} c}{yp'(y)} > 0$, together with the debt price $p(y) = yv'(y) - v(y)$, constitute an equilibrium.

PROOF: Define a constant $B \equiv \kappa i_{unlever}$ and $w(y) \equiv v(y) - By + \dfrac{(1-\bar{\pi})c + \xi}{r+\xi}$; then $w(\cdot)$ is concave if and only if $v(\cdot)$ is concave. Using (7) and $v'(y) = w'(y) + B$, we have:

$$
\begin{aligned}
(r+\xi)w(y) &= (r+\xi)v(y) - (r+\xi)By + (1-\bar{\pi})c + \xi \\
&= (1-\bar{\pi})y + \frac{y(w'(y)+B)^2}{2\kappa} + (\mu+\xi)y(w'(y)+B) + \frac{\sigma^2 y^2}{2}w''(y) - (r+\xi)By \\
&= \underbrace{y\left[1-\bar{\pi} + \frac{B^2}{2\kappa} + (\mu+\xi)B - (r+\xi)B\right]}_{0,\text{ because } B=\kappa(r-\mu)-\sqrt{\kappa^2(r-\mu)^2-2\kappa(1-\bar{\pi})}} \\
&\quad + y\frac{2Bw'(y) + (w'(y))^2}{2\kappa} + (\mu+\xi)yw'(y) + \frac{\sigma^2 y^2}{2}w''(y) \\
&= yw'(y)\left[\frac{B}{\kappa} + \mu + \xi\right] + y\frac{(w'(y))^2}{2\kappa} + \frac{\sigma^2 y^2}{2}w''(y).
\end{aligned}
$$

Hence $w(y)$ satisfies the following ODE:

$$
(r+\xi)w(y) = yw'(y)\left[\frac{B}{\kappa} + \mu + \xi\right] + y\frac{(w'(y))^2}{2\kappa} + \frac{\sigma^2 y^2}{2}w''(y) \qquad (8)
$$

We need two steps to show that $w''(y) > 0$ for all $y > y_b$.

__**Step 1**__. $w(y) > 0$ for all $y \geq y_b$. We know that at default $w'(y_b) = v'(y_b) - B = -B < 0$, and $w(\infty) = 0$. This implies that if $w(y) \leq 0$ ever occurs, then the global minimum must be nonpositive and interior. Pick that global minimum point y_1; we must have $w'(y_1) = 0$ and $w''(y_1) > 0$. Suppose that $w(y_1) < 0$; evaluating (8) at y_1, we find that the LHS is strictly negative while the RHS is positive, contradiction. Suppose that $w(y_1) = 0$; then there must exist some local maximum point $y_2 > y_1$, so that $w(y_2) > 0$, $w'(y_2) = 0$ and $w''(y_2) < 0$. But the same argument of evaluating (8) at y_2 leads to a contradiction.

<u>Step 2</u>. Because $w(y)$ approaches 0 from above when $y \to \infty$, we know that for y to be sufficiently large $w(y)$ is convex. Suppose counterfactually that $w(y)$ is not convex globally; we can take the largest inflection point y_2 with $w''(y_3) = 0$. We must have $w'(y_3) < 0$ and $w'''(y_3) > 0$ (it is because for $y > y_3$ the function $w(y)$ is convex and decreasing to zero from above). At this point, differentiate (8) and ignore the term with $w''(y_3) = 0$, and we have

$$\left(r - \mu - \frac{B}{\kappa}\right) w'(y_3) - \frac{\left(w'(y_3)\right)^2}{2\kappa} = \frac{\sigma^2 y_3^2}{2} w'''(y). \tag{9}$$

Recall $B = \kappa(r-\mu) - \sqrt{\kappa^2 (r-\mu)^2 - 2\kappa}$ which implies

$$\left(r - \mu - \frac{B}{\kappa}\right) w'(y_3) = \sqrt{\kappa^2 (r-\mu)^2 - 2\kappa} \cdot w'(y_3) < 0$$

As a result, the LHS of (9) is negative while the RHS of (9) is positive, contradiction. This implies that $w(y)$ is convex globally.

Combining all the results above, we have shown that $v''(y) > 0$ for $y > y_b$. ■

5.3. Optimal Investment and Leverage

The extension considered above with endogenous investment $i^*(y)$ and debt issuance $g^*(y)$ allows us to study implications on the endogenous firm growth and its interaction with the firm's leverage policies. To facilitate discussion, we consider the following three benchmark cases which features either a constant investment policy i (i.e., independent of the state y_t), or a constant debt issuance policy g. In each case, we isolate either endogenous investment or endogenous debt issuance separately, allowing us to study the interactions between these two policies.

In the first case, the firm commits to the socially optimal investment policy which is a constant:

$$i_{social} = r - \mu - \sqrt{\left(\mu - r\right)^2 - 2/\kappa} \ . \tag{10}$$

In the second case, the committed constant investment policy is optimal to an unlevered firm who pays taxes, as in equation (5). Both cases are relevant because equity holders in our model are maximizing the levered firm value, which includes debt tax shields.

Importantly, in each benchmark case, equity holders---given the respective constant investment policy---are making endogenous debt issuance decisions as in our base model. In contrast, in the third Leland (1998) benchmark, the firm commits to fully rollover always its debt so that $g\left(\cdot\right) = \xi$, but the investment decision is taken by equity holders endogenously. Appendix 7.2 gives the details of solving these three benchmark cases.

Figure 10 plots the investment policies (left panel) and debt issuance polices (right panel) for our extension with both endogenous investment and debt issuance polices (which is indicated by "i^*, no commitment g^*," solid lines), together with three benchmark cases. Start with the left panel. The general take-away there is that the well-known debt overhang effect (Myers 1977; Hennessey, 2004) causes the firm to underinvest; to the extreme, equity holders stop investing when the firm is close to default. Because all debt tax shields get dissipated in our no-commitment model, the relevant benchmark is "$i_{unlever}$, no commitment g^*" plotted in dashed line; and as expected, the firm in our model underinvests.

Similarly, the Leland (1998) firm, red dash-dotted line "i^*, Leland '98 $g = \xi$", underinvests compared to the socially optimal level (black solid line with circle). Interestingly, relative to the Leland benchmark, the solid line of our model sits below (above) the dash-dotted line of Leland (1998) for low (high) y. In words, no commitment leverage policy leads to more severe debt overhang when profitability is high, but the overhang effect is less severe when profitability is low. This is a result of history dependent leverage policy discussed in Section 1---equity holders allow leverage to decline following negative shocks.

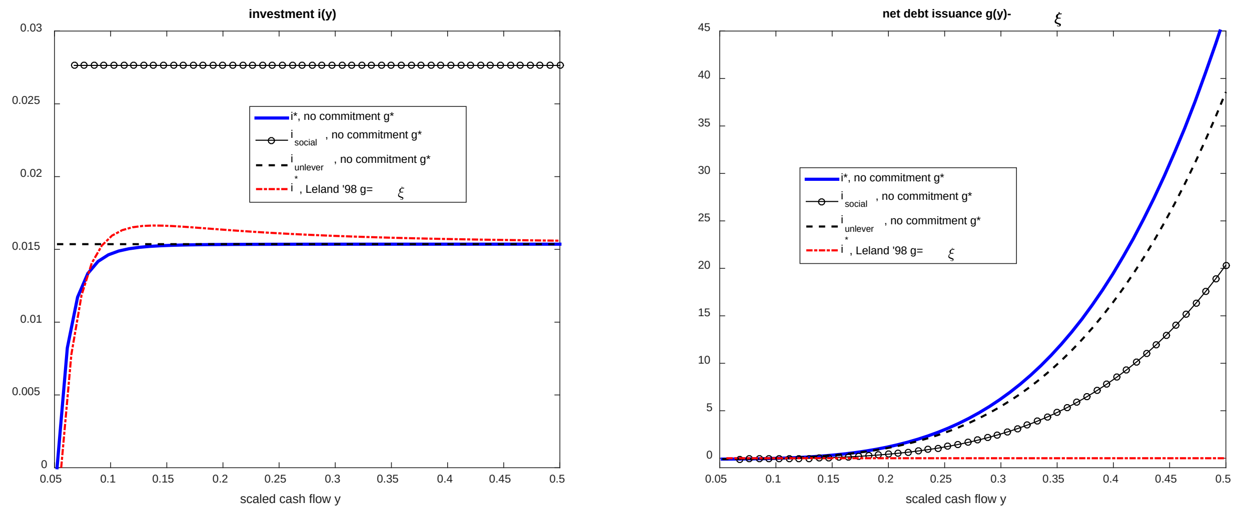


Figure 10: Endogenous investment and debt issuance policies, and comparison to three benchmarks. Our model extension with endogenous investment is denoted by "i*, no commitment g*," with solid lines. "i_{social}, no commitment g*" with solid-circle lines and "$i_{unlever}$, no commitment g*" with dashed lines refer to the cases of constant investment $i(y)=i_{social}$ and $i(y)=i_{unlever}$, but shareholders has no commitment on the debt issuance policy. The third benchmark is Leland (1998) with constant debt issuance policy $g(y)=\xi$ but equity chooses investment endogenously. Parameters are $r=8\%, c=8\%, \sigma=25\%, \mu=0, \bar{\pi}=35\%, \xi=0.1, \kappa=10^3$.

The differences in net debt issuance polices $g(y)-\xi$ in the right panel of Figure 10 reveal an additional economic insight. By assumption, the net debt issuance is identically zero in the benchmark Leland (1998) case (dash-dotted line). For other cases, we observe that the solid line of "i^* , no commitment g^*" sits above both constant investment benchmarks (solid-circle line and dashed line). In words, equity holders are more aggressive in issuing new debt when they are also in charge of the endogenous investment policies of the firm. For intuition, recall that in deciding how much new debt to issue, equity holders are trading off the tax benefit against the losses caused by higher leverage that are borne by themselves. Compared to the case in which equity holders are forced to implement an (ex ante) optimal investment policy, the losses are mitigated when equity holders are also in charge of future investment policies that maximize equity value ex post, which explains the more aggressive leverage policy in this case.

6. Conclusions

When the firm cannot commit ex ante to future leverage choices, shareholders will adjust the level of debt to maximize the firm's current share price. As shown by Admati et al. (2015), capital structure decisions are then distorted and a leverage ratchet effect emerges: shareholders will choose to issue new debt gradually over time even if leverage is already excessive relative to the standard tradeoff theory optimum. This endogenous rate of debt issuance decreases as the firm approaches default, and is offset by the rate of asset growth and debt maturity, so that the firm's equilibrium leverage is ultimately mean-reverting.

We develop a general methodology to solve for equilibrium debt dynamics in this setting, and apply it to several standard models. When earnings evolve as geometric Brownian motion (including possible upward jumps), we explicitly solve for the firm's debt as a weighted average of past earnings, with the speed of adjustment decreasing with debt maturity and volatility. When shareholders also control investment decisions, we show that debt overhang exacerbates the leverage ratchet effect, leading the firm to take on leverage more aggressively.

Because creditors expect the firm to issue new debt in the future, credit spreads are wider in our model than in standard models with fixed debt, and remain wide even when firms are arbitrarily far from default. Lower debt prices dissipate the tax shield benefits of leverage, so that the share price of the firm is identical to its value if debt were fixed. Finally, while shortening the maturity of future debt issues raises the average level of leverage as well as its speed of adjustment, it has no impact on the share price. As a result, even "instantaneous" debt does not resolve the agency problem, and equity holders have no incentive what-so-ever to adjust the firm's debt maturity structure.

Firms may try to reduce the agency costs resulting from future debt issuance by agreeing to covenants that restrict future debt issuance. Equity and debt issuance may also incur transactions costs or expose the firm to other market imperfections. We leave for future work an exploration of the leverage dynamics that arise from the interaction of these additional forces with the leverage ratchet effects explored here.

7. Appendix.

7.1. Appendix for Leland (1998)

Following Leland (1998), we can first solve for the scaled levered firm value TEV as

$$\frac{TEV}{F} = \frac{(1-\bar{\pi})\,y}{r-\mu} + \frac{\bar{\pi}c}{r} - \left(\frac{\bar{\pi}c}{r} + \frac{(1-\bar{\pi})\,y_b^{\xi}}{r-\mu}\right)\left(\frac{y}{y_b^{\xi}}\right)^{-\gamma_v^{\xi}} \tag{11}$$

where the constant $\gamma_v^{\xi} \equiv \frac{\mu - 0.5\sigma^2 + \sqrt{\left(0.5\sigma^2-\mu\right)^2 + 2\sigma^2 r}}{\sigma^2}$. Then, the equity value $v^{\xi}\left(y\right) = \frac{TEV}{F} - p^{\xi}\left(y\right)$ equals:

$$v^{\xi}\left(y\right) = \frac{(1-\bar{\pi})\,y}{r-\mu} + \frac{\bar{\pi}c}{r} - \left(\frac{\bar{\pi}c}{r} + \frac{(1-\bar{\pi})\,y_b^{\xi}}{r-\mu}\right)\left(\frac{y}{y_b^{\xi}}\right)^{-\gamma_v^{\xi}} - \frac{c+\xi}{r+\xi}\left(1-\left(\frac{y}{y_b^{\xi}}\right)^{-\gamma_p^{\xi}}\right) \tag{12}$$

7.2. Appendix for Section 5.3

With slight abuse of notation, consider a constant investment policy i, which could take either the constant value of $i_{unlever}$ in equation (5) or i_{social} in equation (10). The flow payoff to equity holders is $\left(1-\bar{\pi}\right)\left(y-c\right) - \frac{\kappa i^2}{2}y$, and the method in our base model allows us to derive the equity holders' value to be

$$v(y) = \frac{y(1-\bar{\pi}-\frac{\kappa i^2}{2})}{r-\mu} - \frac{c\left(1-\bar{\pi}\right)+\xi}{r+\xi}\left(1-\frac{1}{1+\gamma}\left(\frac{y}{y_b}\right)^{-\gamma}\right),$$

with endogenous default boundary

$$y_b = \frac{\gamma}{1+\gamma}\frac{r-\mu}{r+\xi}\left[\frac{c\left(1-\bar{\pi}\right)+\xi}{1-\bar{\pi}-0.5\kappa i^2}\right].$$

Then we can solve for the endogenous debt issuance policy g^* as in (29).

The solution to the Leland (1998) model with endogenous investment is characterized by a pair of ODE, one for the equity value $v^{\xi}\left(y\right)$ and the other for the debt price $p^{\xi}\left(y\right)$. For equity value, we have

$$rv^{\xi}(y) = \max_i (1-\overline{\pi})(y-c) + \xi\left(p^{\xi}(y)-1\right) + (\mu+i)\,yv^{\xi\prime}(y) - \frac{\kappa i^2}{2}y + \tfrac{1}{2}\sigma^2 y^2 v^{\xi\prime\prime}(y)$$

With optimal investment $i^*(y) = \dfrac{v^{\xi}{}'(y)}{\kappa}$, the above ODE becomes

$$rv^{\xi}(y) = (1-\overline{\pi})(y-c) + \xi\left(p^{\xi}(y)-1\right) + \mu yv^{\xi\prime}(y) + \frac{v^{\xi\prime}(y)^2}{2\kappa}y + \tfrac{1}{2}\sigma^2 y^2 v^{\xi\prime\prime}(y) \quad (13)$$

With boundary conditions

$v^{\xi}(y_b) = 0, v^{\xi}{}'(y_b) = 0, v^{\xi}{}'(y) = \kappa i_{FB} = \kappa\left(r-\mu-\sqrt{(\mu-r)^2 - 2(1-\pi)/\kappa}\right)$ for sufficient large y,

For debt price $p^{\xi}(y)$, we have

$$(r+\xi)\,p^{\xi}(y) = c + \xi + \left(\mu + \frac{v^{\xi\prime}(y)}{\kappa}\right) yp^{\xi\prime}(y) + \tfrac{1}{2}\sigma^2 y^2 p^{\xi\prime\prime}(y) \quad (14)$$

with boundary conditions $p^{\xi}(y_b) = 0, p^{\xi}{}'(y) = 0$ for sufficiently large y. One can easily solve for $\left\{v^{\xi}(y), p^{\xi}(y)\right\}$ by solving the ODE system (13)-(14), with respective boundary conditions.

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