Leverage Dynamics without Commitment

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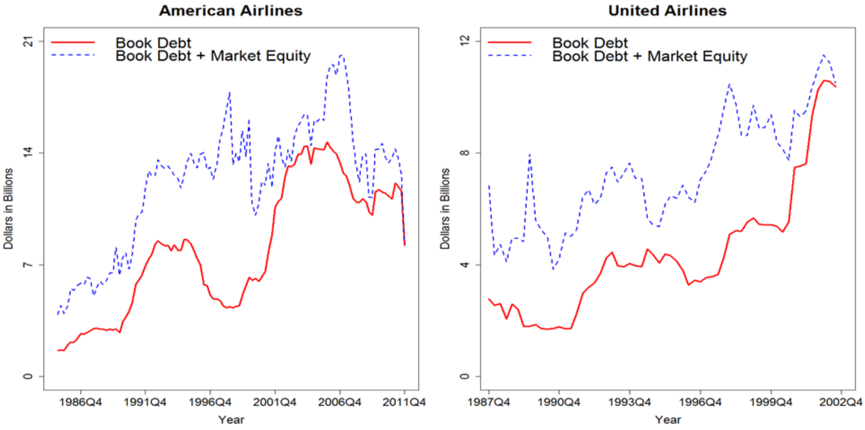
Introduction (1)

- Leverage dynamics is at the heart of dynamic corporate finance, but hard to be analyzed in models
  - Equilibrium debt prices interact with future equilibrium leverage polices
- Existing literature relies on some ad hoc "commitment" of future debt policy
  - Refinance to keep outstanding debt face value constant (Leland 1994 1998)
  - Whenever adjusting debt, the firm has to retire the existing debt first, with some transaction costs (Fischer, Heinkel, and Zechner 1989; Goldstein, Leland, Ju 2001)
    - Abrupt adjustment to "target" leverage
- Empirically counterfactual: firms are actively managing their debt, and often incrementally

Introduction (2)



This Paper (1)

- **The firm cannot commit to future debt policies**
  - Otherwise, standard trade-off setting (tax shield vs bankruptcy cost) with stochastic asset growth; no transaction cost
  - Rather than exogenous frictions to adjust leverage, we consider no commitment at all (so a more endogenous "friction")
- Two cases
  - Zero recovery: seniority structure irrelevant
  - Positive recovery: pari-passu debt
- Leverage may go down via asset growth and debt maturing, but equity never reduces debt voluntarily
  - Repurchase debt is never optimal—**leverage ratchet effect** (Admati DeMarzo Hellwig Pfleiderer, 2016)
  - Robust to the canonical setting and debt overhang (endogenous investment)

This Paper (2)

- A general method to solve this class of models
  - A result reminiscent of Coase conjecture
- Closed-form solutions for work-horse log-normal cash-flow setting
- History-dependent leverage dynamics: issue more (less) following good (bad) shocks
  - Leverage dynamics tend to be mean-reverting; no immediate adjustment to leverage "target"

General Model: Environment

Preferences

- Risk-neutral world, with common discount rate r

Assets

- Assets in place generate operating income (could allow for jumps):

$$dY_t = \mu(Y_t)\,dt + \sigma(Y_t)\,dZ_t$$

- Focus on zero recovery now (debt seniority irrelevant); relaxed later

Debt contract: aggregate face value F_t (endogenous)

- Each debt with coupon rate c, face value 1
- Exponentially retiring (Poisson maturing) with rate ξ

Corporate tax: $\pi \cdot (Y_t - cF_t)$

General Model: Debt Issuance and Default

Evolution of debt

- Equity issues/repurchases debt $d\Gamma_t$, so aggregate debt face value evolves as

$$dF_t = \underbrace{-\xi F_t dt}_{\text{contractual debt maturing}} + \underbrace{d\Gamma_t}_{\text{active debt managment}}$$

Timing within $[t, t+dt]$ & lack of commitment

- Cash flow realizes; either default or pay coupon/principal; announce $d\Gamma_t$; debt price is set and transaction is implemented; next period
- Unable to commit on future $d\Gamma_{t+s}$ for $s > 0$

Smooth equilibrium: issuance/repurchase policy $d\Gamma_t = G_t dt$

- G_t / F_t: endogenous debt adjustment speed
- Equity could adjust debt discretely, but not optimal in equilibrium

Equity default at endogenous stopping time τ_b

General Model: Equity Value

State variables (Markov Perfect Equilibrium)

- Exogenous cash-flows Y_t, and endogenous debt obligation F_t

Equity's problem, taking debt prices p as given

- Equity receives cash-flows (if negative, covered by issuing equity)

$$\underbrace{Y_t}_{\text{cash-flows}} - \underbrace{\pi\left(Y_t - cF_t\right)}_{\text{corproate taxes}} - \underbrace{\left(c+\xi\right)F_t}_{\text{interest \& principal}} + \underbrace{p_t G_t}_{\text{issuance/repurchase}}$$

  - Endogenous debt price p_t determined later
- Given $Y_t = Y$ and $F_t = F$, equity is solving

$$V\left(Y,F\right) \equiv \max_{\{G_s\},\tau_b} \mathbb{E}_t\left\{\int_t^{\tau_b} e^{-r(s-t)}\left[Y_s - \pi\left(Y_s - cF_s\right) - \left(c+\xi\right)F_s + p_s G_s\right] ds\right\}$$

  - Controlling 1) debt evolution $dF_t = F_t dt + G_t dt$; and 2) when to default

General Model: Debt Price

Debt price:

- Competitive risk neutral debt investors price debt rationally
- Given equity default decision τ_b, equilibrium debt price

$$p(Y, F) \equiv \mathbb{E}_t \left\{ \int_t^{\tau_b} e^{-(r+\xi)(s-t)} (c+\xi)\, ds \,|\, Y_t = Y, F_t = F \right\}$$

Why does commitment matter?

- p_t depends on equilibrium default time τ_b
- τ_b depends on firm's future debt policy—the more the future debt, the more likely the default

General Model: Value Equivalence of No-Issuance

- Hamilton-Jacobi-Bellman equation for equity

$$rV(Y,F) = \max_{G} \Bigg[\underbrace{G\,p(Y,F)}_{\text{issuance/repurchase}} + \underbrace{(G - \xi F)\,V_F(Y,F)}_{\text{evolution of debt}} \Bigg]$$

$$Y - \pi(Y - cF) - (c+\xi)F + \mu(Y)V_Y(Y,F) + \frac{\sigma^2(Y)}{2}V_{YY}(Y,F)$$

- Objective linear in G. Optimal $G \Rightarrow$ First-Order Condition

$$\underbrace{p(Y,F)}_{\text{MB of issuance}} + \underbrace{V_F(Y,F)}_{\text{MC on future value}} = 0$$

- Under FOC, equity indifferent at any G (given equilibrium p)
  - Linear control with interior solution (smooth policy $G_t dt$)
- Equity value can be solved by setting $G = 0$ always. No gain in equilibrium by issuance/repurchase
  - Any potential tax shield gain is dissipated by bankruptcy cost caused by future excessive leverage
  - Reminiscent of Coase conjecture; DeMarzo and Urosevic (2006)

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- **Get equity value $V(Y,F)$ without knowing debt price**

Sufficiency of Local FOC

Proposition 1: Global optimality of local FOC holds if and only if debt price $p(Y,F) = -V_F(V,F)$ is non-increasing in debt F

- **Proof:** $\underbrace{V(Y,F+\Delta) - V(Y,F)}_{\text{Equity value change}} + \underbrace{\Delta p(Y,F+\Delta)}_{\text{Debt proceeds}}$ equals

$$
\begin{aligned}
& \int_0^\Delta V_F(Y,F+\delta)\,d\delta + \int_0^\Delta p(Y,F+\Delta)\,d\delta \\
\leq \quad & \int_0^\Delta V_F(Y,F+\delta)\,d\delta + \int_0^\Delta p(Y,F+\delta)\,d\delta \\
= \quad & \int_0^\Delta \left[\underbrace{V_F(Y,F+\delta) + p(Y,F+\delta)}_{\text{local FOC}} \right] d\delta = 0
\end{aligned}
$$

- Debt price decreasing in $F \Rightarrow$ a bad idea to repurchase by paying a higher price
- Leverage ratchet effect (Admati et al 2015)

Equilibrium Policies

Basic idea

- Debt price $p(Y,F)$ must satisfy the valuation equation

$$p(Y,F) = \mathbb{E}_t \left\{ \int_t^{\tau_b} e^{-(r+\xi)(s-t)} (c+\xi)\, ds \right\}$$

- $V(Y,F)$ gives $-V_F(V,F) = p(Y,F)$ using equity's FOC
- How to make both match? Via debt management $G(Y,F)$
  - HJB for p, which depends on $G(Y,F)$
  - ODE for $V_F(V,F)$ (HJB for V), which does not depend on $G(Y,F)$

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Equilibrium debt issuance policy

$$G^*(Y,F) = \frac{c \cdot \pi'(Y-cF)}{-p_F(Y,F)}$$

- $\pi'(Y-cF) \geq 0$, tax benefit $\Rightarrow$ always issuing debt
- Recall $-p_F(Y,F) = V_{FF}(Y,F) > 0$, capturing the price impact

Summary of General Model

1. Solve for equity value $V(Y,F)$ by setting $G(Y,F)=0$
2. Set the equilibrium debt price $p(Y,F)=-V_F(Y,F)$
3. Check the equity holders' global optimality condition
   - Verifying $p(Y,F)$ is non-increasing in F (or $V(Y,F)$ is convex in F)
4. Equilibrium debt issuance $G^*(Y,F)=\frac{\pi'(Y-cF)\cdot c}{-p_F(Y,F)}>0$

Strict Optimality in Discrete Time

- Taking the value function at $t+h$ as given, consider equity's problem at t, where $h>0$
- Denote the debt issuance by $G \cdot h$. Equity is maximizing

$$\max_{G} \underbrace{\pi ch \cdot Gh}_{\text{DTS}} + \underbrace{Gh \cdot p\left(F+Gh, Y\right)}_{\text{New Debt Proceeds}} + \underbrace{V\left(F+Gh, Y\right) - V\left(F, Y\right)}_{\text{Impact on Equity Value}}$$

- Taylor expansion of the objective upto h^2

$$\pi cGh^2 + Gh \cdot \left[p\left(F, Y\right) + p_F Gh\right] + V_F\left(F, Y\right) Gh + \frac{1}{2} V_{FF}\left(F, Y\right) G^2 h^2$$

$$= \underbrace{\left[p\left(F, Y\right) + V_F\left(F, Y\right)\right]}_{\text{FOC}=0} G \cdot h + \underbrace{\left[\pi cG + \left[p_F + \frac{1}{2} V_{FF}\left(F, Y\right)\right] G^2\right]}_{\text{Second-order Terms}} \cdot h^2$$

- Since $p\left(F, Y\right) + V_F\left(F, Y\right) = 0 \Rightarrow p_F\left(F, Y\right) = -V_{FF}\left(F, Y\right)$, the optimal issuance is

$$G^*\left(Y, F\right) = -\frac{\pi c}{2p_F\left(F, Y\right) + V_{FF}\left(F, Y\right)} = \frac{\pi c}{-p_F\left(F, Y\right)}$$

Log-Normal Cash-flows Model

- Scale-invariance, cash-flows $dY_t / Y_t = \mu dt + \sigma dZ_t$
  - The work-horse model of dynamic corporate finance
- One-dimensional state variable: scaled cash-flow $y_t \equiv Y_t / F_t$
  - Equity value $V(Y, F) = F \cdot v(y)$, debt price $p(Y, F) = p(y)$
  - Total Enterprise Value (TEV) multiple $\frac{TEV}{Y} = \frac{v(y)+p(y)}{y}$
- Let $g^*(y_t) \equiv G^*(Y_t, F_t) / F_t$, then

$$\frac{dy_t}{y_t} = \left(\underbrace{\mu}_{\text{CF growth}} + \underbrace{\xi}_{\text{debt maturing}} - \underbrace{g_t^*}_{\text{debt issuance}} \right) dt + \underbrace{\sigma dZ_t}_{\text{CF shocks}}$$

  - Debt grows at the rate of $g_t^* - \xi = \frac{dF_t}{F_t dt}$, with endogenous g_t^*

Model Solution

- **Equity value** and default threshold (endogenous constant γ)

$$v(y) = \underbrace{\frac{(1-\pi)\,y}{r-\mu}}_{\text{asset in place}} + \underbrace{\frac{\pi c}{r+\xi}}_{\text{tax shield}} - \underbrace{\frac{c+\xi}{r+\xi}}_{\text{debt value}} + \underbrace{\frac{c\,(1-\pi)+\xi}{(1+\gamma)\,(r+\xi)}\left(\frac{y}{y_b}\right)^{-\gamma}}_{\text{default option}}$$

$$y_b = \frac{\gamma}{1+\gamma}\frac{r-\mu}{r+\xi}\left[c+\frac{\xi}{1-\pi}\right]$$

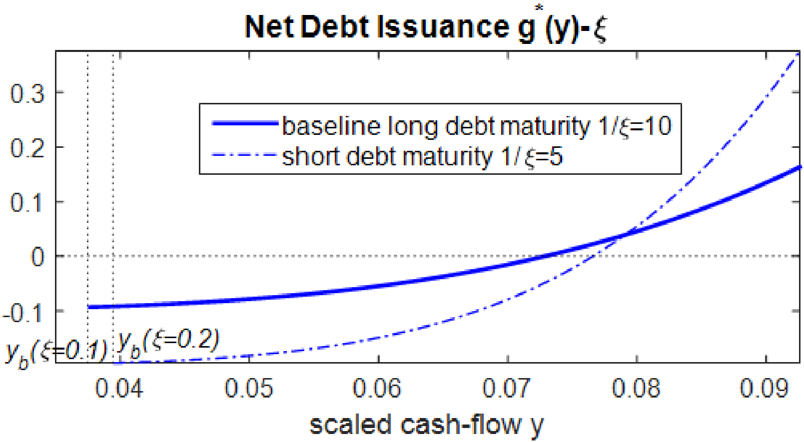
- **Debt price**, increasing in y hence decreasing in F

$$p(y) = -V_F(V,F) = yv(v) - v'(y) = \frac{c\,(1-\pi)+\xi}{r+\xi}\left(1-\left(\frac{y}{y_b}\right)^{-\gamma}\right)$$

- **TEV multiple** $\frac{1-\pi}{r-\mu}\left(1-\left(\frac{y}{y_b}\right)^{-\gamma-1}\right)$, lower than unlevered TEV multiple $\frac{1-\pi}{r-\mu}$
  - Default cost more than offsets tax shield
- **Debt issuance**, increasing in y

$$g^*(y) = \frac{(r+\xi)\,\pi c}{c\,(1-\pi)+\xi}\frac{1}{\gamma}\left(\frac{y}{y_b}\right)^{\gamma} > 0$$

Net Debt Issuance $g^*(y) - \xi$, Debt Maturity



Two Benchmarks with Commitment

No future debt issuance:

- The firm commits to set $g_t = 0$ always (superscript 0)
- Equity value is the same (so does y_b), debt price is higher (by the tax shield)

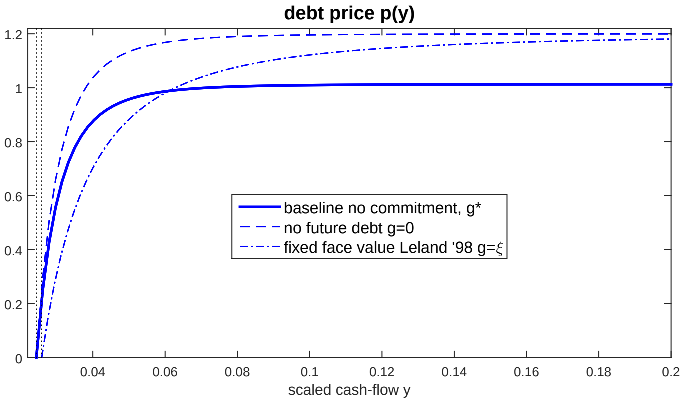
$$p^0(y) = p(y) + \frac{\pi c}{r + \xi}\left(1 - \left(\frac{y}{y_b}\right)^{-\gamma}\right)$$

- Less debt $\Rightarrow$ less likely to default (same y_b but y has a higher drift)

Fixed future debt:

- The firm commits to set $g_t = \xi$ always; Leland 1998

Model Comparisons: Debt Prices and Credit Spreads



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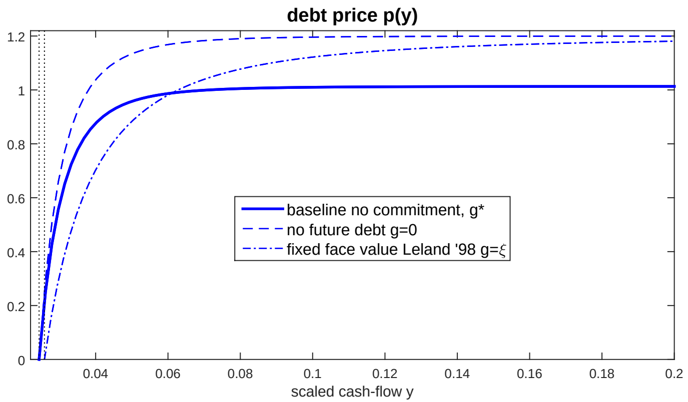

Implication of credit spreads: $y \to \infty$ i.e. zero **current** leverage

- $p^{\xi}(y)$ and $p^{0}(y) \to \frac{c+\xi}{r+\xi}$, with zero credit spread
- $p(y) \to \frac{c(1-\pi)+\xi}{r+\xi}$, **non-zero credit spreads (high future excessive leverage!)**

Equilibrium Debt Dynamics

- **Proposition.** Given cash-flow history $\{Y_s : 0 \leq s \leq t\}$, time-$t$ debt is ($\hat{y}_\xi$ is a constant)

$$F_t = \frac{1}{\hat{y}_\xi}\left[\int_0^t \gamma\xi Y_s^\gamma e^{-\gamma\xi(s-t)} ds\right]^{1/\gamma}$$

  - Start from $t = 0$ debt grows at the order of $t^{1/\gamma}$
  - Outstanding debt is average past earnings, with decaying weights $\gamma\xi$
- High mean-reverting speed, or more aggressive in adding leverage given high cash flows, when
  - shorter debt maturity $\Rightarrow$higher ξ

Leverage Ratchet Effect

- What is the impact of debt repurchase on **equity value**?
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- Reducing current debt will alleviate future default—but does equity benefit strictly from it? **No**
  - Yes less debt $\Rightarrow$ default less often
  - But equity optimizes default decision ex post already $\Rightarrow$ zero indirect impact on equity value now (**envelope theorem**)

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- **"Leverage Ratchet Effect" Admati et al. (2016)**
  - The same logic to debt overhang—equity is optimizing investment decisions ex post

Positive Recovery

- Now suppose the recovery in bankruptcy is (baseline $\alpha = 1$)

$$(1-\alpha)\frac{1-\pi}{r-\mu}Y_{\tau_b}, \text{ for } \alpha \in (0,1)$$

  - For tractability, assume **pari-passu** debt
- Equity as if has the option to capture recovery value ($1-\alpha$ fraction of TEV) by issuing infinite new debt right before default
  - Why not wait till the end?
- The equilibrium is as if the firm can only pledge out α fraction of cash-flow Y (and grab $1-\alpha$ at default)

$$v^{\alpha}(y) = v(\alpha y), p^{\alpha}(y) = p(\alpha y), \text{ and } g^{\alpha}(y) = g(\alpha y)$$

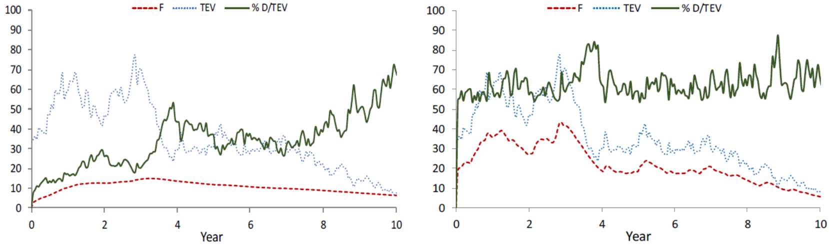
  - Similar debt issuance policy, but with infinite issuance at (right before) $y_b^{\alpha} = \frac{y_b}{\alpha}$
- Interestingly, $g^{\alpha}(y) > g^{\alpha=1}(y)$, so the final diluting option $\Rightarrow$ **less debt issuance before default**

Optimal Debt Maturity Structure

- So far the debt maturity structure ξ is taken as a parameter
- Say the firm gets a one-time chance to set ξ optimally together with leverage decision
- At time zero, $F_0^* = 0$; but the choice of ξ affects the maturity of **future** debt issuances
- **Proposition**: Equity holders are **indifferent** at any ξ
  - Why? Because equity value is as if there is no **future** debt issuance...
- This indifference result holds even if the firm gets a chance of readjusting ξ after time-0

Long-term vs. Short-term Debt

- Two firms start with zero debt, with different debt maturities (both being optimal)—but have different leverage dynamics/targets



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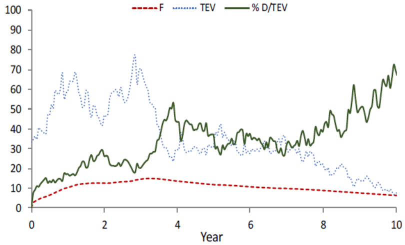

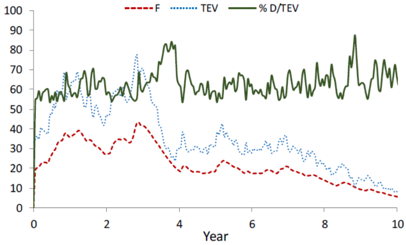

- With flexibility of shorter-term debt, the firm borrows more for higher debt tax shield
- But tax shield is a transfer from social perspective—so long-term debt is preferred to minimize bankruptcy cost

Investment

- Special case of log-normal process. Capital K_t evolves as

$$\frac{dK_t}{K_t} = (i_t - \delta)\,dt + \sigma dZ_t$$

with quadratic investment cost $\frac{\kappa i_t^2}{2} K_t$, and output $Y_t = AK_t$

  - Prove $v(\cdot)$ is strictly convex, so always an equilibrium

- Leverage ratchet effect prevails despite debt overhang considerations

- Equity issues debt more aggressively when controlling investment endogenously, compared to exogenous investment
  - Endogenous investment offers equity more protection later

Conclusion and Future Work

What we have done

- A general methodology solving dynamic corporate finance model without commitment
- Leverage policy depending on the entire earnings history, new insight on debt maturity and investment
- Slow initial adoption of leverage, but leads ultimately to excess

Future extensions

- Covenant of no debt issuance once in distress (say for $y < \hat{y}$)
  - Discrete debt issuance (jump to $\hat{y}$) may occur in equilibrium
- What changes when one has internal cash (with some equity issuance cost)? How about liquidity-driven default?