Laws of Large Numbers for Stochastic Orders

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Abstract

We establish laws of large numbers for comparing sums of i.i.d. random variables in terms of stochastic dominance. Our results shed new light on a classic question, raised first by Samuelson (1963), on the relation between expected utility, risk aversion, and the aggregation of independent risks. In the context of statistical experiments, we answer a long-standing open question posed by Blackwell (1951): we show that generically, an experiment is more informative than another in large samples if and only if it has higher Rényi divergences.

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1 Introduction

The law of large numbers connects the long-run frequency of an event to its likelihood, and has been one of the first principles giving practical meaning to the notion of probability. It also guides our intuition in decision problems that involve multiple independent risks, as in the case of a physician treating multiple patients, or of an investor managing a large portfolio of assets.

While the law of large numbers provides exact predictions about limiting frequencies, it does not provide precise indications to decision makers who are concerned with the problem of choosing between risky prospects, even when these prospects are obtained by aggregating a large number of i.i.d. risks.

This view was articulated by Samuelson (1963), who asked under what conditions an agent could reject a gamble, but accept \( n \) independent copies of it. Samuelson deemed as inconsistent the behavior of a decision maker who is willing to accept \( n \) copies of a lottery but not one, and attributed this choice reversal to a naive interpretation of the law of large numbers. The critical point in Samuelson’s argument is that whenever a gamble \( X \) with positive expectation is aggregated into an in i.i.d. sum \( X_1 + \cdots + X_n \) consisting of \( n \) independent copies of \( X \), two effects come into play. The law of large numbers implies that the probability of incurring losses vanishes for large \( n \). At the same time, combining multiple risks exposes the decision maker to unlikely, but potentially large, losses. Since the law of large numbers does not provide information about the odds of these rare but large deviations, it is therefore insufficient as a guide for action.

In this paper we establish new laws of large numbers for comparing sums of i.i.d. random variable in terms of stochastic dominance. We say that a random variable \( X \) first-order dominates a random variables \( Y \) in the aggregate if for large \( n \), the sum \( X_1 + \cdots + X_n \) of \( n \) i.i.d. copies of \( X \) stochastically dominates the sum of \( n \) i.i.d. copies of \( Y \). Stochastic dominance in the aggregate is implied, but strictly weaker than, stochastic dominance between \( X \) and \( Y \).

Our first main result, Theorem 1, provides a characterization of aggregate stochastic dominance which renders this notion operational. For a generic pair of bounded random variables \( X \) and \( Y \), we show that \( X \) strictly dominates \( Y \) in the aggregate if and only if \( \mathbb{E}[X] > \mathbb{E}[Y] \) and, for all \( t \neq 0 \), it holds that

\[
\frac{1}{t} \log \mathbb{E} \left[ e^{tX} \right] > \frac{1}{t} \log \mathbb{E} \left[ e^{tY} \right].
\] (1)

Equivalently, if and only if every decision maker endowed with CARA utility prefers \( X \) over \( Y \). The proof relies on large deviation results that provide sharp tail estimates for sums of i.i.d. random variables in terms of their moment generating functions, resulting in condition (1).

We focus on stochastic dominance for its many applications, both in the theory of risk
and in other fields, as well as for its conceptual simplicity. In the same way the classic law of large numbers provides nonparametric predictions about long-run frequencies, stochastic dominance provides unambiguous choice predictions that are independent of the decision maker’s preferences.

In the second part of the paper, we apply aggregate stochastic dominance to Blackwell’s theory of comparison of experiments. Experiments form a general framework for modeling information: Given a set \( \Theta \) of parameters, an experiment \( P \) produces an observation distributed according to \( P_\theta \), given the true parameter value \( \theta \in \Theta \). Blackwell’s celebrated theorem (Blackwell, 1951) provides a partial order for comparing experiments in terms of their informativeness.

As is well known, requiring two experiments to be ranked in the Blackwell order is a demanding condition. Consider the problem of testing a binary hypothesis \( \theta \in \{0, 1\} \), based on random samples drawn from one of two experiments \( P \) or \( Q \). According to Blackwell’s ordering, \( P \) is more informative than \( Q \) if, for every test performed based on observations produced by \( Q \), there exists another test based on \( P \) that has lower probabilities of both Type-I and Type-II errors. This is a strong notion of informativeness which needs to apply if only one sample is produced by each experiment.

In many applications, the information produced by an experiment does not consist of a single observation but of multiple i.i.d. samples. We study a weakening of the Blackwell order that is appropriate for comparing experiments in terms of their large sample properties. Our starting point is the question, first posed by Blackwell (1951), of whether it is possible for \( n \) independent observations from an experiment \( P \) to be more informative than \( n \) observations from another experiment \( Q \), even though \( P \) and \( Q \) are not comparable in the Blackwell order. The question was answered in the affirmative by Torgersen (1970) and Azrieli (2014). However, identifying the precise conditions under which this phenomenon can occur has remained an open problem.

We say that \( P \) dominates \( Q \) in large samples if for every \( n \) large enough, \( n \) independent observations from \( P \) are more informative, in the Blackwell order, than \( n \) independent observations from \( Q \). We focus on a binary set of parameters \( \Theta = \{0, 1\} \), and show that generically \( P \) dominates \( Q \) in large samples if and only if the first is more informative in terms of Rényi divergences (Theorem 2). Rényi divergences are a one-parameter family of measures of informativeness for experiments; introduced and characterized axiomatically in Rényi (1961), we show that they capture the asymptotic informativeness of an experiment. The result crucially relies on the characterization of aggregate stochastic dominance: We associate to each experiment a new statistic (“the perfected log-likelihood ratio”) and show that the comparison of these statistics in terms of first-order stochastic dominance is in fact equivalent to the Blackwell order.

In the last part of the paper we study risk aversion. The notion of aggregate stochastic
dominance can be modified, in an obvious way, to be applied to second-order stochastic dominance. We characterize the resulting order in terms of the unanimous rankings of risk-averse CARA preferences and the opposite rankings of risk-loving CARA preferences (Theorems 3 and 4).

These results have two further implications. First, we show that if the i.i.d. sums of two gambles $X$ and $Y$ are ordered in terms of third, fourth, or higher-order stochastic dominance, then given a large enough number of repetitions they are also ordered in terms of second-order stochastic dominance. Thus, higher-order attitudes over risk (e.g. prudence, temperance, etc.) reduce to simple risk aversion when considering i.i.d. sums of gambles.

The second conclusion is that only a specific class of expected utility preferences are monotone with respect to second-order aggregate dominance: those utility functions that are mixtures of CARA utilities, also known as mixed risk-averse preferences. This is a large class which contains many utility functions used in applications, and for which our results provide a novel behavioral characterization.

The paper is organized as follows. In §2 we study first-order stochastic dominance in the aggregate. In §3 we turn to Blackwell experiments. In §4 we analyze second- and higher-order aggregate dominance. Finally, we further discuss our results and their relation to the literature in §5.

2 Aggregate Stochastic Dominance

2.1 Definition

A random variable $X$ dominates another random variable $Y$ in first-order stochastic dominance, denoted $X \succeq_1 Y$, if it holds that $E[\phi(X)] \geq E[\phi(Y)]$ for all increasing functions $\phi$ for which the expectations are well defined. Given a bounded random variable $X$ we denote by $\max[X] = \min\{a : P[X \leq a] = 1\}$ the essential maximum of $X$; this is the maximum of the support of its distribution. We define $\min[X]$ analogously.\(^1\)

The next definition is central to the paper. It introduces a notion of stochastic dominance for sums of i.i.d. random variables:

**Definition 1.** Let $X$ and $Y$ be random variables, and let $(X_i)$ and $(Y_i)$ be i.i.d. copies of $X$ and $Y$, respectively. The random variable $X$ first-order dominates $Y$ in the aggregate if for all $n$ large enough,

$$X_1 + \cdots + X_n \succeq_1 Y_1 + \cdots + Y_n.$$  \(^2\)

In §4 we extend the definition to second and higher-order stochastic dominance, thereby capturing risk aversion as well as more nuanced risk attitudes.

\(^{1}\)That is, $\min[X] = \max\{a : P[X \geq a] = 1\}$

\(^{2}\)In §4 we extend the definition to second and higher-order stochastic dominance, thereby capturing risk aversion as well as more nuanced risk attitudes.
There is an important contrast between aggregate stochastic dominance and traditional limit theorems for sums of i.i.d. variables. For example, the weak Law of Large Numbers implies that asymptotically the sum $\sum_{i=1}^{n} X_i$ will take values close to $n \cdot \mathbb{E}[X]$. In particular, if $\mathbb{E}[X] > \mathbb{E}[Y]$ then for all $n$ large enough and any value $a$ between $\mathbb{E}[X]$ and $\mathbb{E}[Y]$,

$$
\mathbb{P}[X_1 + \cdots + X_n \geq na] \geq \mathbb{P}[Y_1 + \cdots + Y_n \geq na].
$$

In comparison, first-order dominance in the aggregate is tantamount to requiring that for all $n$ large enough the inequality (3) holds for all values $a \in \mathbb{R}$, in particular, for values of $a$ that lie outside the interval $(\mathbb{E}[Y], \mathbb{E}[X])$ and correspond to unlikely, but potentially large, losses or gains. As emphasized in the introduction, the decisions of economic agents who are not risk neutral can crucially depend on the probabilities of these rare outcomes.

Now consider the strong Law of Large Numbers. It is equivalent to the statement that if $\mathbb{E}[X] > \mathbb{E}[Y]$, then there exists a random variable $N$ such that almost surely

$$
X_1 + \cdots + X_n > Y_1 + \cdots + Y_n
$$

for all $n > N$. Since the exact value of $n$ for which (4) holds is not known ex-ante, this conclusion does not provide clear guidance in comparing the two sums $\sum_{i=1}^{n} X_i$ and $\sum_{i=1}^{n} Y_i$, even for large $n$. In comparison, $X$ first-order dominates $Y$ in the aggregate if and only if for all $n$ large enough there is some coupling of the i.i.d. sequences $(X_i)_{i=1}^{n}$ and $(Y_i)_{i=1}^{n}$ with the property that (4) holds almost surely.\(^2\)

It is well-known that aggregate stochastic dominance is implied by stochastic dominance. In fact if $X \geq_1 Y$ then $\sum_{i=1}^{n} X_i \geq_1 \sum_{i=1}^{n} Y_i$ for any number $n$ of i.i.d. replicas of the two random variables.\(^3\) As we show in the next example, the converse implication is not true.

### 2.2 Example

We now illustrate a simple example of two random variables that are not comparable in terms of stochastic dominance, but can be ranked with respect to aggregate stochastic dominance. Let $X$ be a lottery that pays 1 or 0 with probability 1/2, and let $Y$ be distributed uniformly over $[-1/5, 4/5]$.

For example, $X$ might correspond to an Arrow-Debreu security, while $Y$ might correspond to an insurance contract that costs 1/5 and offers a smoothed distribution of payoff

\(^{2}\)A coupling of two random variables $X$ and $Y$ is a pair of random variables $X'$ and $Y'$ such that $X$ and $X'$ have the same distribution, as do $Y$ and $Y'$, but where the joint distribution of the pairs is not necessarily equal; by abuse of notation it is often convenient to refer to the new pair also as $X$ and $Y$. First-order stochastic dominance admits a simple definition in terms of couplings: $X \geq_1 Y$ if and only if there is some coupling of this pair such that almost surely $X \geq Y$. Likewise, $X \geq_2 Y$ if there is a coupling such that almost surely $X \geq \mathbb{E}[Y|X]$.

\(^{3}\)This follows from the coupling characterization of first-order stochastic dominance. A different proof is provided in Lemma 6 in the appendix.
that is uniform on the unit interval. The cumulative distribution functions of $X$ and $Y$ are depicted in Figure 1, from which it is clear that neither first-order dominates the other.

In fact, the two distributions are not ranked in terms of second-order stochastic dominance either. To see this, note that $Y$ has higher expected utility than $X$ for the utility function given by $u(x) = x$ for $x \leq 1/5$ and $u(x) = 1/5$ otherwise. It is also clear that $Y$ does not dominate $X$, since the latter has higher expectation.

![Figure 1: The CDFs of $X$ and $Y$, in blue and orange, respectively.](image1)

It will be an implication of our characterization theorem that $X$ first-order dominates $Y$ in the aggregate. In this example, it is not difficult to verify that replicating the two gambles makes it possible to rank them in terms of stochastic dominance: Figure 2 shows the cumulative distribution functions of the two sums $X_1 + \cdots + X_n$ and $Y_1 + \cdots + Y_n$ when setting $n = 35$, from which it is apparent that the first sum dominates the second one in terms of first-order stochastic dominance.

### 2.3 Characterization

In this section we provide necessary and sufficient conditions for aggregate stochastic dominance. To each bounded random variable $X$ we associate the function $L_X : \mathbb{R} \to \mathbb{R}$

![Figure 2: The CDFs of $X_1 + \cdots + X_n$ and $Y_1 + \cdots + Y_n$, for $n = 35$, in blue and orange, respectively.](image2)
defined as

\[ L_X(t) = \frac{1}{t} \log \mathbb{E} \left[ e^{tX} \right] \quad (5) \]

for all \( t \neq 0 \), and, to guarantee continuity,

\[ L_X(0) = \mathbb{E} [X]. \quad (6) \]

If \( X \) is a gamble, then \( L_X(t) \) is the certainty equivalent that a decision maker ascribes to \( X \), under expected utility and a utility function \( u \) whose coefficient of absolute risk aversion is constant and equal to \(-t\).\(^4\) Note that for \( t \) positive, such a decision maker is in fact risk-loving; we include these agents for the analysis of first-order stochastic dominance.

The quantity \( L_X \) is a standard tool in the theory of choice under risk, finance, probability theory, and other fields. Because it amounts to a simple normalization of the moment generating function of \( X \), the certainty equivalent \( L_X \) is known or can be easily computed for most families of distributions commonly used in applications.

Our first main result is a characterization of aggregate stochastic dominance under a mild genericity assumptions. We say that a pair \((X,Y)\) is \emph{generic} if \( \min[X] \neq \min[Y] \) and \( \max[X] \neq \max[Y] \).

\textbf{Theorem 1.} Let \( X \) and \( Y \) be a generic pair of bounded random variables. Then the following are equivalent:

(i). \( L_X(t) > L_Y(t) \) for all \( t \in \mathbb{R} \),

(ii). \( X \) first-order dominates \( Y \) in the aggregate.

Aggregate stochastic dominance ties two fundamental ideas in the theory of risk: stochastic dominance and the aggregation of independent risks. Theorem 1 establishes a equivalence between this notion and an elementary and well-known class of preferences. Whenever, as described by (i), all agents with CARA preferences unanimously prefer \( X \) over \( Y \), then, for a large enough number of repetitions, \textit{all} agents with monotone preferences will agree on this ranking. Moreover, this condition is both sufficient and necessary.

Stochastic dominance is a central tool for the nonparametric comparison of distributions, and its relevance extends to fields such as information economics, statistics, and operations research. Thus, Theorem 1 might have applications that extend beyond the theory of risk. Indeed, in §3 we apply the characterization of Theorem 1 to the comparison of repeated statistical experiments.

Figure 3 shows a simple instance of this equivalence. It depicts the certainty equivalents \( L_X \) and \( L_Y \) for the two gambles introduced in our earlier example in §2.2. As shown in the

\(^4\)The (increasing) utility function is \( u(x) = e^{tx} \) for \( t \) positive, \( u(x) = -e^{tx} \) for \( t \) negative and \( u(x) = x \) for \( t = 0 \).
figure and can be verified analytically, the certainty equivalent of \( X \) is uniformly above that of \( Y \). More generally, for many known classes of distributions (e.g. exponential families) moment generating functions have tractable closed form expressions, making aggregate stochastic dominance an operational stochastic order.

It seems difficult to obtain an applicable characterization of aggregate stochastic dominance without imposing any genericity conditions. In §F in the appendix we discuss the knife-edge case where the maxima or the minima of the supports might be equal, and show that the conclusions of Theorem 1 no longer hold. Indeed, in this knife-edge case verifying aggregate stochastic dominance involves checking for combinatorial conditions that, compared to (i) in Theorem 1, are less immediate to verify. Our genericity assumption plays an additional role. A key tool in the application of a stochastic order is a characterization in terms of a generator: a class \( V \) of functions such that \( X \) dominates \( Y \) if and only if \( E[\phi(X)] \geq E[\phi(Y)] \) for every \( \phi \in V \). The result in §F shows that without our genericity assumption, aggregate stochastic dominance does not admit a generator.

2.4 Proof Sketch

That (ii) implies (i) is intuitive and follows from a logic that can already be found in Samuelson (1963). The key observation is under constant absolute risk aversion the certainty equivalent of a sum \( X + Z \) of two independent gambles is the sum of the two certainty equivalents. That is, \( L_{X+Z}(t) = L_X(t) + L_Z(t) \) for every \( t \). In particular, the certainty equivalent of \( n \) independent copies of \( X \) is equal to \( n \cdot L_X(t) \). As a consequence, \( X_1 + \cdots + X_n \) dominates \( Y_1 + \cdots + Y_n \) in first-order stochastic dominance for some number \( n \) of repetitions only if \( L_X(t) \geq L_Y(t) \).

The converse implication is more involved and is the technical core of the paper. Assume (i) holds. The fact that \( L_X(0) > L_Y(0) \), equivalently \( E[X] > E[Y] \), guarantees that the

![Figure 3: \( L_X \) and \( L_Y \) in the example of §2.2, in blue and orange, respectively.](image)

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dominance condition
\[ P[X_1 + \cdots + X_n \geq na] \geq P[Y_1 + \cdots + Y_n \geq na]. \] (7)

holds with respect to all values of \(a\) that lie between \(E[X]\) and \(E[Y]\). This is established by applying the Berry-Esseen Theorem, a uniform version of the Central Limit Theorem. The main step in the proof Theorem 1 uses large-deviations techniques to obtain lower and upper bounds on the probabilities of the events \(\{X_1 + \cdots + X_n \geq na\}\) and \(\{Y_1 + \cdots + Y_n \geq na\}\) for the remaining values of \(a\).

Large deviation theory studies low probability events, and in particular the odds with which an i.i.d. sum deviates from its expectation. The Law of Large Numbers implies that the probability of the event \(\{X_1 + \cdots + X_n \geq na\}\) is low for \(a > E[X]\) and large \(n\). A crucial insight due to Cramér (1938) is that the order of magnitude of the probability of this event is determined by the behavior of the moment generating function \(M_X(t) = E[e^{tX}].\)

Condition (i) in Theorem 1 implies that the moment generating functions satisfy \(M_X(t) > M_Y(t)\) for all \(t > 0\). In the proof we show how this property, in turn, implies that the dominance condition (7) holds for all \(a > E[X]\). Similarly, condition (i) for negative values of \(t\) implies \(M_X(t) < M_Y(t)\) for all \(t < 0\), and that the dominance condition (7) holds for all \(a < E[Y]\).

The key difficulty is in obtaining bounds that allow, for a fixed \(n\), a comparison between the two probabilities in (7) over a whole interval of values for \(a\). This requires a careful application of uniform large deviation theorems due to Bahadur and Rao (1960) and Petrov (1965).

3 Blackwell Dominance in Large Samples

In this section we apply our characterization of aggregate stochastic dominance to the comparison of statistical experiments.

3.1 Statistical Experiments

A state of the world \(\theta\) can take two possible values, 0 or 1. A Blackwell-Le Cam experiment \(P = (\Omega, P_0, P_1)\) consists of a sample space \(\Omega\) and a pair of probability measures \((P_0, P_1)\) defined on a \(\sigma\)-algebra \(A\) of subsets of \(\Omega\), with the interpretation that \(P_\theta(A)\) is the probability of observing \(A \in A\) in state \(\theta \in \{0, 1\}\). To ease the exposition we will suppress the \(\sigma\)-algebra \(A\) from the notation. This framework is commonly encountered in simple hypothesis tests as well as in information economics. In §5 we discuss the case of experiments for more than two states.

We restrict attention to experiments where the measures \(P_0\) and \(P_1\) are mutually absolutely continuous, so that no signal realization \(\omega \in \Omega\) perfectly reveals either state.
Given two experiments $P = (\Omega, P_0, P_1)$ and $Q = (\Xi, Q_0, Q_1)$, we can form the product experiment $P \oplus Q$ given by

$$P \oplus Q = (\Omega \times \Xi, P_0 \times Q_0, P_1 \times Q_1).$$

where $P_\theta \times Q_\theta$, given $\theta \in \{0, 1\}$, denotes the product of the two measures. Under the experiment $P \oplus Q$ the realizations produced by $P$ and $Q$ are observed, and the two observations are independent conditional on the state. For instance, if $P$ and $Q$ consist of drawing samples from two different populations, then $P \oplus Q$ consists of the joint experiment where a sample from each population is drawn. We denote by

$$P^{\oplus n} = P \oplus \cdots \oplus P$$

the $n$-fold product experiment where $n$ conditionally independent observations are generated according to the experiment $P$.

Consider now a Bayesian decision maker whose prior belief assigns probability $1/2$ to the state being 1. Fixing a uniform prior simplifies the discussion, but it is without loss of generality. To each experiment $P = (\Omega, P_0, P_1)$ we can associate a Borel probability measure $\pi$ over $[0, 1]$ that represents the distribution over posterior beliefs induced by the experiment. Formally, let $p(\omega)$ be the posterior belief that the state is 1 given the realization $\omega$:

$$p(\omega) = \frac{\ell(\omega)}{1 + \ell(\omega)} \quad \text{where} \quad \ell(\omega) = \frac{dP_1}{dP_0}(\omega).$$

and define, for every Borel $B \subseteq [0, 1]$

$$\pi_\theta(B) = P_\theta(\{\omega : p(\omega) \in B\})$$

as the probability that the posterior belief will belong to $B$, given state $\theta$. We then define $\pi = (\pi_0 + \pi_1)/2$ as the unconditional measure over posterior beliefs. We say that an experiment $P$ is trivial if $P_0 = P_1$, and bounded if the support of $\pi$ is strictly included in $(0, 1)$.

### 3.2 Blackwell Theory

We first review the main concepts behind Blackwell’s order over experiments (Bohnenblust, Shapley, and Sherman, 1949; Blackwell, 1953). Consider two experiments $P$ and $Q$ and their induced distribution over posterior beliefs denoted by $\pi$ and $\tau$, respectively. The experiment $P$ Blackwell dominates $Q$, denoted $P \succeq Q$, if

$$\int_0^1 v(p) \, d\pi(p) \geq \int_0^1 v(p) \, d\tau(p)$$

for every continuous convex function $v : (0, 1) \to \mathbb{R}$. We write $P \succ Q$ if $P \succeq Q$ and $Q \npreceq P$. So, $P \succ Q$ if and only if (8) holds with a strict inequality whenever $v$ is strictly convex.
As is well known, each convex function $v$ can be seen as the indirect utility, or value function, induced by some decision problem. That is, for each convex $v$ there exists a set of actions $A$ and a utility function $u$ defined on $A \times \{0, 1\}$ such that $v(p)$ is the maximal expected utility that a decision maker can obtain in such a decision problem given a belief $p$. Hence, $P \succeq Q$ if and only if in any decision problem, a decision maker can obtain a higher payoff by basing her action on experiment $P$ rather than on the experiment $Q$.

Blackwell’s theorem shows that the order $\succeq$ can be equivalently defined by “garbling” operations: Intuitively, $P \succeq Q$ if and only if the outcome of the experiment $Q$ can be generated from the experiment $P$ by compounding the latter with additional noise, without adding further information about the state.\(^5\)

As discussed in the introduction, we are interested in understanding the large-sample properties of the Blackwell order. This motivates the next definition.

**Definition 2.** An experiment $P$ dominates an experiment $Q$ in large samples if there exists an $N \in \mathbb{N}$ such that

$$P^\oplus n \succeq Q^\oplus n \text{ for every } n \geq N. \quad (9)$$

This order was first defined by Azrieli (2014) under the terminology of eventual sufficiency. The definition captures the informal notion that a large sample drawn from $P$ is more informative than an equally large sample drawn from $Q$, provided the sample size is large enough. Consider, for instance, the case of hypothesis testing. The experiment $P$ dominates $Q$ in the Blackwell order if and only if for every test based on $Q$ there exists a test based on $P$ that has weakly lower probabilities of both Type-I and Type-II errors. Definition 2 extends this notion to large samples, in line with the standard paradigm of asymptotic statistics.

A natural alternative definition would require $P^\oplus n \succeq Q^\oplus n$ to hold for some $n$, but as we show below the resulting order is equivalent under a mild genericity assumption.

### 3.3 Rényi Divergence and the Rényi Order

Our main result relates Blackwell dominance in large samples to a well-known notion of informativeness due to Rényi (1961). Given an experiment $(\Omega, P_0, P_1)$, a state $\theta$, and parameter $t > 0$, the Rényi $t$-divergence is defined as

$$R_{P}^{\theta}(t) = \frac{1}{t-1} \log \int \left( \frac{dP_{\theta}}{dP_1-\theta}(\omega) \right)^{t-1} dP_{\theta}$$

---

\(^5\)Formally, given two experiments $P = (\Omega, P_0, P_1)$ and $Q = (\Xi, Q_0, Q_1)$, $P \succeq Q$ iff there is a measurable kernel (also known as “garbling”) $\sigma : \Omega \rightarrow \Delta(\Xi)$, where $\Delta(\Xi)$ is the set of probability measures over $\Xi$, such that for every $\theta$ and every measurable $A \subseteq \Omega$, $Q_{\theta}(A) = \int \sigma(\omega)(A) dP_\theta(\omega)$. In other terms, there is a (perhaps randomly chosen) measurable map $f$ with the property that for both $\theta = 0$ and $\theta = 1$, if $X$ is a random quantity distributed according to $P_\theta$ then $Y = f(X)$ is distributed according to $Q_\theta$. 

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when \( t \neq 1 \), and, to ensure continuity,

\[
R^\theta_P(1) = \int_{\Omega} \log \frac{dP_\theta}{dP_{1-\theta}}(\omega) \, dP_\theta(\omega).
\]

Equivalently, \( R^\theta_P(1) \) is the Kullback-Leibler divergence between the measures \( P_\theta \) and \( P_{1-\theta} \).

Intuitively, observing a sample realization for which the likelihood-ratio \( \left( \frac{dP_\theta}{dP_{1-\theta}} \right)(\omega) \) is high constitutes evidence that favors state \( \theta \) over \( 1 - \theta \). For instance, in case \( t = 2 \), a higher value of \( R^\theta_P(2) \) describes an experiment that, in expectation, more strongly produces evidence in favor of the state \( \theta \) when this is the correct state. Varying the parameter \( t \) allows to consider different moments for the distribution of likelihood ratios.

**Rényi Order.** We say that an experiment \( P \) dominates an experiment \( Q \) in the Rényi order if it holds that

\[
R^\theta_P(t) < R^\theta_Q(t)
\]

for all \( \theta \in \{0, 1\} \) and all \( t > 0 \). The Rényi order is a refinement of the (strict) Blackwell order. In the proof of Theorem 2 below, we explicitly construct a one-parameter family of decision problems with the property that dominance in the Rényi order is equivalent to higher expected utility with respect to each decision problem in this family.

A simple calculation shows that if \( P = Q \otimes T \) is the product of two experiments, then for every state \( \theta \),

\[
R^\theta_P = R^\theta_Q + R^\theta_T
\]

A key implication is that \( P \) dominates \( Q \) in the Rényi order if and only if the same relation holds for their \( n \)-th fold repetitions \( P^\otimes n \) and \( Q^\otimes n \), for any \( n \). Hence, the Rényi order compares experiments in terms of properties that are unaffected by the number of repetitions. Because, in turn, the Rényi order refines the Blackwell order, it follows that domination in the Rényi order is a necessary condition for domination in large samples.

### 3.4 Characterization

In analogy with our definition of a generic pair of random variables, we call two bounded experiments \( P \) and \( Q \) generic if the essential maxima of the log-likelihood ratios \( \log \frac{dP_\theta}{dP_0} \) and \( \log \frac{dQ_\theta}{dQ_0} \) are different, and if their essential minima are also different.

**Theorem 2.** For a generic pair of bounded experiments \( P \) and \( Q \), the following are equivalent:

(i). \( P \) dominates \( Q \) in large samples.

(ii). \( P \) dominates \( Q \) in the Rényi order.

As for the case of Theorem 1, dominance in large samples becomes less immediate to verify for non-generic pairs of experiments. In particular, while strict dominance in large samples still implies dominance in the Rényi order, the converse is not true.
3.5 Example

We apply Theorem 2 to revisit an example due to Azrieli (2014) and to complete his analysis. The example provides a simple instance of two experiments that are not ranked in Blackwell order but become so in large samples. Despite its simplicity, the analysis of this example is not straightforward, as shown by Azrieli (2014). Part of the difficulty lies in the fact that directly applying the definition of Blackwell order to repeated experiments can lead to involved calculations. As we show below, applying the Rényi order greatly simplifies the analysis, and elucidates the logic behind the example.

Consider the following two experiments $P$ and $Q$, parametrized by $\beta$ and $\alpha$, respectively. In each matrix, entries are conditional probabilities of observing each signal realization given the state $\theta$:

$P:
\begin{array}{ccc}
\theta & x_1 & x_2 & x_3 \\
0 & \beta & \frac{1}{2} & \frac{1}{2} - \beta \\
1 & \frac{1}{2} - \beta & \frac{1}{2} & \beta \\
\end{array}
$

$Q:
\begin{array}{cc}
\theta & y_1 & y_2 \\
0 & \alpha & 1 - \alpha \\
1 & 1 - \alpha & \alpha \\
\end{array}
$

The parameters satisfy $0 \leq \beta \leq 1/4$ and $0 \leq \alpha \leq 1/2$. The experiment $Q$ is a symmetric, binary experiment. The experiment $P$ with probability $1/2$ yields a completely uninformative signal realization $x_2$, and with probability $1/2$ yields an observation from another symmetric binary experiment. As shown by (Azrieli, 2014, Claim 1), the experiments $P$ and $Q$ are not ranked in the Blackwell order for parameter values $2\beta < \alpha < 1/4 + \beta$.

Azrieli (2014) points out that a necessary condition for $P$ to dominate $Q$ in large samples is that the Rényi divergences are ranked at $1/2$, that is $R_{1P}(1/2) > R_{1Q}(1/2)$.

As in his paper, this condition can be written in terms of the parameter values as

$$\sqrt{\alpha(1-\alpha)} > \sqrt{\beta(1/2 - \beta) + \frac{1}{4}}$$

Thus, when $\alpha = 0.1$ and $\beta = 0$ for example, the experiment $P$ does not Blackwell dominate $Q$ but does dominate it in large samples, as shown by Azrieli (2014).

Proposition 1. Suppose $R_{1P}(1/2) > R_{1Q}(1/2)$. Then $R_{1P}(t) > R_{1Q}(t)$ for all $t > 0$, hence $P$ dominates $Q$ in large samples.

In the rest of this section, we illustrate the main ideas behind the proof of our characterization in Theorem 2.
3.6 Repeated Experiments and Log-Likelihood Ratios

The distribution over posteriors induced by a product experiment $P^\otimes n$ can be difficult to analyze directly. A more suitable approach consists in studying the distribution of the induced log-likelihood ratio

$$\log \frac{dP_\theta}{dP_{1-\theta}}.$$  \hfill (10)

As is well known, given a repeated experiment $P^\otimes n = (\Omega^n, P^n_0, P^n_1)$, its log-likelihood ratio satisfies, for every realization $\omega = (\omega_1, \ldots, \omega_n)$ in $\Omega^n$,

$$\log \frac{dP^n_1}{dP^n_0}(\omega) = \sum_{i=1}^n \log \frac{dP_1}{dP_0}(\omega_i)$$

and moreover the random variables

$$X_i(\omega) = \log \frac{dP_1}{dP_0}(\omega_i) \quad i = 1, \ldots, n$$

are i.i.d. under $P^n_\theta$, for $\theta \in \{0,1\}$. Focusing on the distributions of log-likelihood ratios will allow us to reduce the study of repeated experiments to the study of sums of i.i.d. random variables. We can then apply the results from §2.

3.7 From Blackwell Dominance to First-Order Stochastic Dominance

Our first result is a novel characterization of the Blackwell order expressed in terms of the distributions of the log-likelihood ratios (10). Given two experiments $P = (\Omega, P_0, P_1)$ and $Q = (\Xi, Q_0, Q_1)$ we denote by $F_\theta$ and $G_\theta$, respectively, the cumulative distribution function of the log-likelihood ratios conditional on state $\theta$. That is,

$$F_\theta(a) = P_\theta\left(\left\{\log \frac{dP_\theta}{dP_{1-\theta}} \leq a\right\}\right) \quad a \in \mathbb{R}, \; \theta \in \{0,1\}$$  \hfill (11)

and $G_\theta$, $\theta \in \{0,1\}$, are defined analogously using $Q_\theta$.

We associate to $P$ a new quantity, which we call the perfect log-likelihood ratio of the experiment. Define

$$\tilde{L} = \log \frac{dP_1}{dP_0} - E$$

where $E$ is a random variable that, under $P_1$, is independent from $\log (dP_1/dP_0)$ and distributed according to an exponential distribution with support $\mathbb{R}_+$ and cdf $H(x) = 1-e^{-x}$ for all $x \geq 0$. We denote by $\tilde{F}$ the cumulative distribution function of $\tilde{L}$ under $P_1$. That is, $\tilde{F}(a) = P_1(\{\tilde{L} \leq a\})$ for all $a \in \mathbb{R}$.

More explicitly, $\tilde{F}$ is the convolution of the distribution $F_1$ with the distribution of $-E$, and thus can be defined as

$$\tilde{F}(a) = \int_{\mathbb{R}} (1-H(u-a))dF_1(u) = F_1(a) + e^a \int_{(a,\infty)} e^{-u}dF_1(u).$$  \hfill (12)
The next result shows that the Blackwell order over experiments can be reduced to first-order stochastic dominance of the corresponding perfected log-likelihood ratios:

**Proposition 2.** Let $P$ and $Q$ be two experiments, and let $\tilde{F}$ and $\tilde{G}$, respectively, be the associated distributions of perfected log-likelihood ratios. Then

$$P \succeq Q \text{ if and only if } \tilde{F}(a) \leq \tilde{G}(a) \text{ for all } a \in \mathbb{R}.$$ 

**Proof.** Let $\pi$ and $\tau$ be the distributions over posterior beliefs induced by $P$ and $Q$, respectively. As is well known, Blackwell dominance is equivalent to the requirement that $\pi$ is a mean-preserving spread of $\tau$. Equivalently, the functions defined as

$$\Lambda_\pi(p) = \int_0^p (p-q) \, d\pi(q) \quad \text{and} \quad \Lambda_\tau(p) = \int_0^p (p-q) \, d\tau(q) \quad (13)$$

should satisfy $\Lambda_\pi(p) \geq \Lambda_\tau(p)$ for every $p \in (0,1)$.

We now express (13) in terms of the distributions of log-likelihood ratios $F_\theta$ and $G_\theta$. We have

$$\Lambda_\pi(p) = p \left( \frac{1}{2} - \int_{[p,1]} \frac{1}{q} \, d\pi_1(q) \right) - \int_0^p q \, d\pi(q). \quad (14)$$

Using the fact that $q \, d\pi(q) = \frac{1}{2} \, d\pi_1(q)$, \footnote{See the appendix, equation (33), for a proof of this fact.} we obtain

$$2\Lambda_\pi(p) = p \left( 1 - \int_{[p,1]} \frac{1}{q} \, d\pi_1(q) \right) - \int_0^p \, d\pi_1(q).$$

A change of variable from posterior beliefs to log-likelihood ratios, letting $a = \log \frac{p}{1-p}$, implies

$$2\Lambda_\pi(p) = \frac{e^a}{1 + e^a} \left( 1 - \int_{(a,\infty)} \frac{1 + e^u}{e^u} \, dF_1(u) \right) - F_1(a). \quad (15)$$

Since

$$\int_{(a,\infty)} \frac{1 + e^u}{e^u} \, dF_1(u) = \int_{(a,\infty)} e^{-u} \, dF_1(u) + 1 - F_1(a),$$

(15) leads to

$$2(1 + e^a)\Lambda_\pi(p) = -F_1(a) - e^a \int_{(a,\infty)} e^{-u} \, dF_1(u) = -\tilde{F}(a),$$

where the final equality follows from (12). It then follows that $P$ dominates $Q$ if and only if $\tilde{F}(a) \leq \tilde{G}(a)$ for all $a \in \mathbb{R}$, as desired. 

Intuitively, transferring probability mass from lower to higher values of $\log(dP_\theta/dP_{1-\theta})$ leads to an experiment that, conditional on the state being $\theta$, is more likely to shift the decision maker’s beliefs towards the correct state. Hence, one might conjecture that
Blackwell dominance of the experiments $P$ and $Q$ is related to, for example, first-order stochastic dominance of the distributions $F_\theta$ and $G_\theta$. However, not all distributions over $\mathbb{R}$ can arise as the distribution of log-likelihood ratios of an experiment: Since the likelihood-ratio $dP_1/dP_0$ must satisfy the change of measure identity $\int dP_0 dP_1 = 1$, the distributions $F_1$ must satisfy

$$\int_{\mathbb{R}} e^{-u} dF_1(u) = 1.$$  

Because the function $e^{-u}$ is strictly decreasing, and the same identity must hold for $G_1$, it is then impossible for $F_1$ to dominate $G_1$ in terms of first-order stochastic dominance. Since $e^{-u}$ is additionally strictly convex, it follows that $F_1$ and $G_1$ cannot be ranked with respect to second-order stochastic dominance either.

Proposition 2 shows that subtracting an independent exponential term $E$ from the log-likelihood ratios of the two distributions leads to a formulation of the Blackwell order in terms of first-order stochastic dominance.\(^8\) Despite the fact that $F_\theta$ cannot stochastically dominate $G_\theta$, the next lemma shows that when $F_1(a)$ is smaller that $G_1(a)$ for values of $a$ above some threshold, then $\tilde{F}(a)$ is smaller than $\tilde{G}(a)$ within the same range. Likewise, when $F_0(a)$ is smaller than $G_0(a)$ for large $a$, then $\tilde{F}(-a)$ is also smaller than $\tilde{G}(-a)$. The intuition is that a dominating experiment should have higher likelihood ratios for state $\theta$, conditional on $\theta$.  

Lemma 1. Consider two experiments $P$ and $Q$. Let $F_\theta$ and $G_\theta$, respectively, be the distributions of the corresponding log-likelihood ratios, and $\tilde{F}$ and $\tilde{G}$ be the distributions of the perfected log-likelihood ratios. For $b \in \mathbb{R}$, the following hold:

(i). If $F_1(a) \leq G_1(a)$ for all $a \geq b$, then $\tilde{F}(a) \leq \tilde{G}(a)$ for all $a \geq b$.

(ii). If $F_0(a) \leq G_0(a)$ for all $a \geq b$, then $\tilde{F}(-a) \leq \tilde{G}(-a)$ for all $a \geq b$.

3.8 Rényi Order and Large Deviations

We now illustrate how dominance in the Rényi order translates into large-deviation properties of the log-likelihood ratios, and provide a sketch of the proof of Theorem 2. This proof will use as a crucial ingredient the following one-sided version of Theorem 1.

Proposition 3. Let $X$ and $Y$ be bounded random variables that satisfy $\max[X] \neq \max[Y]$ and $L_X(t) > L_Y(t)$ for all $t \geq 0$. Then there exists $N$ such that for all $n \geq N$ and $a \geq \mathbb{E}[Y]$, 

$$\mathbb{P}[X_1 + \cdots + X_n \geq na] \geq \mathbb{P}[Y_1 + \cdots + Y_n \geq na].$$  

---

\(^8\)It might appear puzzling that two distributions $F_1$ and $G_1$ that are not ranked by stochastic dominance become ranked after the addition of the same independent random variable. In a different context and under different assumptions, the same phenomenon is studied by Pomatto, Strack, and Tamuz (2019).
Consider two experiments $P = (\Omega, P_0, P_1)$ and $Q = (\Xi, Q_0, Q_1)$, with the property that $P$ dominates $Q$ in the Rényi order, and let

$$X = \log \frac{dP_1}{dP_0} \quad \text{and} \quad Y = \log \frac{dQ_1}{dQ_0}$$

be the corresponding log-likelihood ratios. We are interested in their properties conditional on $\theta = 1$, and so treat $X$ as a random variable defined on the probability space $(\Omega, P_1)$, and $Y$ as defined on $(\Xi, Q_1)$, so that their distributions are $F_1$ and $G_1$, respectively.

It follows immediately from its definition that the Rényi divergence is formally related to the certainty equivalent functional defined in (5) and (6). Indeed, for any $t > 0$,

$$R^1_P(t) = L_X(t - 1) \quad \text{and} \quad R^1_Q(t) = L_Y(t - 1)$$

and

$$R^0_P(t) = L_X(1 - t) \quad \text{and} \quad R^0_Q(t) = L_Y(1 - t).$$

Thus dominance in the Rényi order implies that

$$L_X(t) > L_Y(t) \quad \text{for all} \quad t > 1.$$

If, in addition, $P$ and $Q$ form a generic pair, then we can conclude that $X$ and $Y$ satisfy all the conditions of Proposition 3. Hence, letting $(X_i)$ and $(Y_i)$ be i.i.d. copies of $X$ and $Y$, it follows that there exists a large enough $N$ such that for all $n \geq N$ and $a \geq \mathbb{E}[Y]$, 

$$\mathbb{P}[X_1 + \cdots + X_n \geq na] \geq \mathbb{P}[Y_1 + \cdots + Y_n \geq na].$$

As discussed earlier, $X_1 + \cdots + X_n$ has the same distribution as the log-likelihood ratio $\log(dP^n_1/dP^n_0)$ of the repeated experiment. It follows, therefore, that the distribution $F^{*n}_1$ of $\log(dP^n_1/dP^n_0)$ and the distribution $G^{*n}_1$ of $\log(dQ^n_1/dQ^n_0)$ satisfy

$$F^{*n}_1(na) \leq G^{*n}_1(na) \quad \text{for all} \quad a \geq \mathbb{E}[Y].$$

In turn, Lemma 1 implies that the distributions of the perfected log-likelihood ratios of the two repeated experiments are ranked for all values above $n\mathbb{E}[Y]$.

To compare these distributions at smaller values, we apply a similar argument to the opposite pair of log-likelihood ratios $\log(dP_0/dP_1)$ and $\log(dQ_0/dQ_1)$. Combining both parts allows us to conclude that the experiments are ranked in large samples. Details are left to the appendix.

### 3.9 Connection to the Literature

Blackwell (1951, p.101) posed the question of whether dominance of two experiments is equivalent to dominance of their $n$-fold repetitions. In the statistics literature, Torgersen
(1970) provides an early example of two experiments that are not comparable in the Blackwell order, but become so after a large enough number of repetitions.

Moscarini and Smith (2002) produce an alternative criterion for comparing repeated experiments. According to their notion, an experiment $P$ dominates an experiment $Q$ if for every decision problem with finitely many actions, there exists some $N$ such that the payoff achievable from $P^{\otimes n}$ is higher than that from $Q^{\otimes n}$ whenever $n \geq N$. This order is characterized by the efficiency index of an experiment, defined, in our notation, as the minimum over $\theta$ and $t$ of the function $(t - 1)R_\theta^P(t)$.

While in Moscarini and Smith (2002) the number $n$ of repetitions is allowed to depend on the decision problem, dominance in large samples is conceptually closer to Blackwell dominance, as an objective criterion for comparing experiments that applies after a finite number of repetitions, independently of the decision problem at hand.

Azrieli (2014) shows that the Moscarini-Smith order is a strict refinement of dominance in large samples. Perhaps surprisingly, this conclusion is reversed under a modification of their definition: when extended to consider all decision problems, including problems with infinitely many actions, the Moscarini-Smith order over experiments coincides with dominance in large samples.\(^9\)

Our notion of dominance in large samples is prior-free. In contrast, several authors (Kelly, 1956; Cabrales et al., 2013) have studied a complete ordering of experiments, indexed by the expected reduction of entropy from prior to posterior beliefs (i.e. mutual information between states and signals). We note that unlike Blackwell dominance, dominance in large samples does not guarantee a higher reduction of uncertainty given any prior belief.\(^10\)

4 Risk Aversion and Higher-Order Aggregate Dominance

In this section, we extend and characterize aggregate stochastic dominance for higher-order stochastic orders. This is a natural extension which allows to study risk-aversion, prudence, and higher-order risk attitudes in the context of repeated gambles. Our main application is a new characterization of utility functions that display mixed risk-aversion, a large class that encompasses most functional forms commonly used in applications.

\(^9\)Consider the following extension of the Moscarini-Smith order: say that $P$ dominates $Q$ if for every decision problem (with possibly infinitely many actions) there exists an $N$ such that the expected utility achievable from $P^{\otimes n}$ is higher than that from $Q^{\otimes n}$ whenever $n \geq N$. Each Rényi divergence $R_\theta^P(t)$ corresponds to the indirect utility defined by a decision problem (see the proof of Theorem 2 in the appendix), and for such decision problems the ranking over repeated experiments is independent of the sample size $n$. We deduce that $P$ dominates $Q$ in this order only if $P$ dominates $Q$ in the Rényi order. By Theorem 2, $P$ must then dominate $Q$ in large samples.

\(^10\)To see this, consider the example in §3.5 with parameters $\alpha = 0.1$ and $\beta = 0$. Then Proposition 1 ensures that the experiment $P$ dominates $Q$ in large samples. However, given a uniform prior, the residual uncertainty under $P$ is calculated as the expected entropy of posterior beliefs, which is $\frac{1}{2}\log(2) \approx 0.346$. The residual uncertainty under $Q$ is $-\alpha \log \alpha - (1 - \alpha) \log(1 - \alpha) \approx 0.325$, which is lower.
Recall that a random variable $X$ dominates $Y$ in $k^{th}$-order stochastic dominance, denoted by $X \succeq_k Y$, if $E[\phi(X)] \geq E[\phi(Y)]$ for every bounded and $k$-fold differentiable function $\phi$ that is increasing and whose first $k$ derivatives alternate in sign. That is, all functions $\phi$ that satisfy $(-1)^n \phi^{(n)} \leq 0$ for all $n \leq k$.

Extending Definition 1, we say that $X$ $k^{th}$-order dominates $Y$ in the aggregate if for all $n$ large enough

$$X_1 + \cdots + X_n \succeq_k Y_1 + \cdots + Y_n,$$

where $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$ are i.i.d. copies of $X$ and $Y$, respectively.

### 4.1 Second-Order Aggregate Dominance

The next two results characterize second-order aggregate stochastic dominance. We first consider pairs of random variables with distinct expectation, and show that aggregate stochastic dominance is equivalent to the unanimous ranking generated by all risk-averse CARA preferences:

**Theorem 3.** Let $X$ and $Y$ be a generic pair of bounded random variables such that $E[X] \neq E[Y]$. Then the following are equivalent:

(i). $L_X(t) > L_Y(t)$ for all $t \leq 0$.

(ii). $X$ second-order dominates $Y$ in the aggregate.

We complete our characterization by considering random variables with equal expectations:

**Theorem 4.** Let $X$ and $Y$ be a generic pair of bounded random variables such that $E[X] = E[Y]$. Then the following are equivalent:

(i). $\text{Var}(X) < \text{Var}(Y)$, $L_X(t) > L_Y(t)$ for all $t < 0$, and $L_X(t) < L_Y(t)$ for all $t > 0$.

(ii). $X$ second-order dominates $Y$ in the aggregate.

Hence, when $X$ and $Y$ have the same expected value, $X$ dominates $Y$ in terms of second-order aggregate stochastic dominance if and only if $X$ has lower variance and is preferred to $Y$ by any risk-averse CARA agent, while $Y$ is preferred to $X$ by all CARA agents who are risk-loving.

One may wonder about the difference between the condition (i) here and the condition (i) in Theorem 1. Note that first-order aggregate dominance is equivalent to $L_X(t) > L_Y(t)$ for all $t$, whereas second-order aggregate dominance for zero mean random variables requires $L_X(t)$ to be smaller for $t > 0$. There is however no inconsistency, because the assumption $E[X] = E[Y]$ in Theorem 4 already rules out the possibility that $X_1 + \cdots + X_n$ can first-order dominate $Y_1 + \cdots + Y_n$. Furthermore, in order for $X_1 + \cdots + X_n$ to second-order
dominate $Y_1 + \cdots + Y_n$, the former sum must be a mean-preserving contraction of the latter. This suggests that the right-tail of $X_1 + \cdots + X_n$ should be less spread-out, as captured by $L_X(t) < L_Y(t)$ for $t > 0$, unlike in the case of first-order stochastic dominance.

### 4.2 Higher-Order Aggregate Dominance

Higher-order risk attitudes are properties of a decision maker’s preference over gambles that, under expected utility, are captured by the sign of the higher-order derivatives of the agent’s utility function $u$. Prudence (Kimball, 1990) is the requirement that the third derivative of $u$ is everywhere positive. Temperance (Kimball, 1991) requires the fourth derivative of $u$ to be everywhere negative. These properties can be extended to higher orders. The importance of higher-order risk attitudes is due to their implications for comparative statics in decision problems under risk, including precautionary saving problems and decisions under background risk (Gollier, 2004).

The notion of $k$-th order stochastic dominance relates to higher-order risk attitudes in the same way as second-order stochastic dominance relates to risk aversion: it is equivalent to the unanimous preference by every individual whose utility function’s first $k$ derivatives exhibit alternating signs.

Large values of $k$ capture increasingly nuanced properties of a decision maker’s preferences which might be difficult to test or to interpret.\(^{11}\) Our next result shows that when considering a sum of a sufficiently large number of i.i.d. gambles, the distinction between risk aversion and higher-order risk attitudes vanishes:

**Proposition 4.** Let $X$, $Y$ be a generic pair of bounded random variables with $\mathbb{E}[X] \neq \mathbb{E}[Y]$. Then the following are equivalent:

(i). $X \geq_k Y$ in the aggregate, for some $k \geq 2$.

(ii). $X$ second-order dominates $Y$ in the aggregate.

(iii). $X \geq_k Y$, for some $k \geq 2$.

**Proof.** To see that (i) implies (ii), suppose it holds for some $n$ and $k$ that $X_1 + \cdots + X_n \geq_k Y_1 + \cdots + Y_n$. Since the risk-averse CARA utility function $u(x) = e^{-tx}$ has derivatives that alternate signs, by definition of $\geq_k$ we know that each risk-averse CARA agent prefers $X_1 + \cdots + X_n$ to $Y_1 + \cdots + Y_n$. Thus $L_X(t) > L_Y(t)$ for all $t \leq 0$. Hence (ii) follows by Theorem 3.

That (ii) implies (iii) follows by applying Theorem 4 in Fishburn (1980), which shows that for bounded random variables $X$ and $Y$ with $\min[X] \neq \min[Y]$ and $\mathbb{E}[X] \neq \mathbb{E}[Y]$,

\(^{11}\)For instance, Eeckhoudt and Schlesinger (2006) show that a behavioral characterizations of higher-order risk attitude is possible by considering bets defined by means of an ingenious construction. These bets are progressively more complicated as $k$ increases.
$X \geq_k Y$ for some $k \geq 2$ if and only if $L_X(t) > L_Y(t)$ for all $t < 0$.\footnote{In the notation used there, $F, G \in P^*$ because $X$ and $Y$ are bounded, the condition $G <_0 F$ is satisfied since $\min[X] > \min[Y]$, and the condition $\mu_F >_L \mu_G$ is satisfied because $E[X] > E[Y]$.} Since $L_X(t) > L_Y(t)$ for all $t < 0$ holds by Theorem 3, (ii) implies (iii).

It follows from standard arguments that (iii) implies (i): Lemma 6 in the appendix shows that if $X \geq_k Y$, then the same holds if we consider the sum of any number of i.i.d. replicas.

\[\square\]

### 4.3 Mixed Risk Aversion

A utility function displays \textit{mixed risk aversion} if it is increasing and its derivatives alternate in sign. This key property is satisfied by most utility functions used in applications, as emphasized by Brockett and Golden (1987) and Caballé and Pomansky (1996), who coined the term. Despite its importance, mixed risk aversion remains a property that can be difficult to interpret.

We apply the notion of aggregate stochastic dominance to provide a new behavioral characterization of mixed risk aversion. As shown by the next proposition, this property is equivalent to monotonicity with respect to aggregate second-order stochastic dominance.

**Proposition 5.** Let $u: \mathbb{R} \rightarrow \mathbb{R}$ be a utility function. Then the following are equivalent:

(i). $u$ displays mixed risk aversion.

(ii). $E[u(X)] \geq E[u(Y)]$ for every pair of bounded random variables $X$ and $Y$ such that $X$ second-order dominates $Y$ in the aggregate.

The result provides a novel test for rejecting the hypothesis of mixed risk aversion: a decision maker violates this property if they rank $X$ as preferred to $Y$ while, for a large enough number of repetitions, any risk averse agent displays the opposite ranking.

The result follows by combining the characterization of Theorem 3 together with a well-known representation theorem for mixed risk averse utilities. By Bernstein’s Theorem (see, e.g., Choquet, 1969, Theorem 32.6), a utility function $u: \mathbb{R} \rightarrow \mathbb{R}$ displays mixed risk aversion if and only if there is a non-negative, finite Borel measure $\mu$ on $\mathbb{R}_+$, and coefficient $\alpha, \beta \geq 0$ such that for all $x \in \mathbb{R}$

\[u(x) = \alpha x + \beta - \int_0^\infty e^{-rx} \, d\mu(r).\] (20)

That is, every mixed risk averse utility is a mixture of CARA utility functions. It is then clear that mixed risk aversion implies monotonicity with respect to aggregate stochastic dominance. In the proof of Proposition 5 we establish the converse implication.
5 Discussion

Additional Related Literature. Müller and Stoyan (2002) and Shaked and Shanthikumar (2007) are comprehensive sources on stochastic orders. The ordering generated by the functionals of the form $L_X(t)$ for $t > 0$, is known in the literature as the Laplace Transform Order, and studied in Reuter and Riedrich (1981), Fishburn (1980), Alzaid et al. (1991) and Caballé and Pomansky (1996), among others.

Samuelson’s paper yielded a large literature relating his original question to the analysis of retirement decisions and insurance strategies (see, e.g., Pratt and Zeckhauser, 1987; Kimball, 1993; Gollier, 1996; Benartzi and Thaler, 1999). In addition, the paper spurred a longstanding interest in behavioral economics (Rabin and Thaler, 2001) and finance (Ross, 1999) on the extent to which preferences over gambles determine preferences over the repetitions of gambles.\(^{13}\) Whereas Ross (1999) characterizes conditions on the utility function such that every gamble (with positive expectation) will eventually be accepted when compounded sufficiently many times, we take the opposite perspective and study properties of a gamble so that it will eventually be accepted by every decision maker. Notably, our analysis applies to non-expected utility, so long as the preference obeys some version of stochastic dominance.

Compound Returns and CRRA Preferences. The notion of aggregate stochastic dominance can be naturally extended to compare compound i.i.d. returns. Two random returns $X$ and $Y$ can be ranked by requiring that for every $n$ large enough their compounded i.i.d. returns satisfy

$$X_1 \times \cdots \times X_n \geq Y_1 \times \cdots \times Y_n.$$ (21)

The resulting stochastic order amounts to aggregate stochastic dominance applied to $\log(X)$ and $\log(Y)$, and is characterized in terms of the certainty equivalents induced by all constant relative risk aversion utilities.

Other Refinements of Stochastic Dominance. Hart (2011) proposes two complete stochastic orders that refine second-order stochastic dominance: wealth-uniform dominance and utility-uniform dominance. He further shows that dominance in these orders is characterized by having a smaller riskiness index/measure given in Aumann and Serrano (2008) and Foster and Hart (2009), respectively. But since these measures of risk are distinct, an open question left by Hart (2011) is whether the two stochastic orders agree on interesting cases beyond second-order stochastic dominance. In §K, we show that the

\(^{13}\)For example, Rabin (2013) writes: “Expected-utility theory makes a powerful prediction that economic actors don’t see an amalgamation of independent gambles as significant insurance against the risk of those gambles; they are either barely less willing or barely more willing to accept risks when clumped together than when apart”.

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two uniform dominance orders both refine our second-order aggregate dominance order, which is characterized in Theorem 3 and Theorem 4.

**Experiments for Many States.** Our analysis leaves open a number of questions. The most salient is the extension of Theorem 2, our characterization of domination in large samples, to experiments with more than two states. A natural conjecture is that the ranking of the (multidimensional) moment generating function of the log-likelihood ratio—which translates to Rényi divergences in the two state case—characterizes this order for any number of states. Unfortunately, our proof technique does not straight-forwardly extend to this general case. In particular, we do not know how to extend the reduction of the Blackwell order to first-order stochastic dominance (Proposition 2).14

Nonetheless, our analysis can be applied to any finite number of states when considering the Lehmann order (Lehmann, 1988), which restricts attention to experiments that satisfy the monotone likelihood ratio property, and to decision problems defined by single-crossing utility functions.15 As Jewitt (2007) shows, Lehmann order is equivalent to Blackwell order imposed on every pair of states. That is, a family of conditional distributions \{P_\theta\}_\theta dominates another family \{Q_\theta\}_\theta if and only if for every pair of states \(\theta \neq \theta'\), the two-state experiment with conditional distributions \(P_\theta\) and \(P_{\theta'}\) Blackwell-dominates the experiment with conditional distributions \(Q_\theta\) and \(Q_{\theta'}\). Hence our Theorem 2 provides a characterization of large-sample Lehmann dominance for any number of states: \(P\) Lehmann-dominates \(Q\) in large samples if and only if the conditional distributions \(\{P_\theta, P_{\theta'}\}\) dominate \(\{Q_\theta, Q_{\theta'}\}\) in the Rényi order for every pair \(\theta \neq \theta'\).

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14 The technical difficulty that arises when studying the Blackwell order for more than two states is not new to the literature. As Jewitt (2007) writes, “the problem is the need to deal with a multivariate stochastic dominance relation for a class of functions (convex) for which the set of extremal rays is too complex to be of service.”

15 The Lehmann order has applications to the study of information acquisition in strategic environments, as illustrated by Persico (2000), Bergemann and Välimäki (2002) and Athey and Levin (2018). Quah and Strulovici (2009) show that the Lehmann order continues to apply in a bigger class of payoff functions that satisfying the “interval dominance order” property.
Appendix

A Uniform Large Deviations

We begin by reviewing some standard concepts from large deviations theory. For every bounded random variable $X$ we define $\rho_X : \mathbb{R} \to \mathbb{R}_+$ as

$$\rho_X(a) = \inf_{t \in \mathbb{R}} e^{-at} M_X(t).$$

where $M_X(t) = \mathbb{E} \left[ e^{tX} \right]$ is the moment generating function of $X$. We note that $e^{-at} M_X(t) = M_{X - a}(t)$, hence $\rho_X(a)$ is the infimum of the moment generating function of $X - a$.

We call a random variable non-degenerate if its distribution is not a point mass. In this case, as is well known, $M_X$ is strictly log-convex, and if $\min[X] < a < \max[X]$ then $M_{X - a}(t) \to \infty$ as $|t| \to \infty$. It follows that for every $a$ in the range $\min[X] < a < \max[X]$ the minimization problem in the definition of $\rho_X$, which is equivalent to minimizing the strictly convex function $-at + \log M_X(t)$, has a unique solution. We denote this minimizer by

$$t_X(a) = \arg\min_{t \in \mathbb{R}} e^{-at} M_X(t).$$

Let $K_X(t) = \log M_X(t)$ denote the “cumulant generating function” of $X$. The first-order condition gives that $t_X(a)$ solves

$$K'_X(t(a)) = \frac{M'_X(t(a))}{M_X(t(a))} = a.$$

Note that $M_X(0) = 1$ and $M'_X(0) = \mathbb{E} [X]$. So $K'_X(0) = \mathbb{E} [X]$. This, together with the convexity of $K_X$, shows that $t(a) \geq 0$ if and only if $a \geq \mathbb{E} [X]$.

Finally, for every $\min[X] < a < \max[X]$ we define

$$\sigma_X(a) = \sqrt{\frac{M''_X(t(a))}{M_X(t(a))} - a^2}.$$

Using the above formula for $t(a)$, we also have $\sigma_X(a) = \sqrt{K''_X(t(a))}$ which is strictly positive whenever $X$ is non-degenerate.

We will refer to quantities above as simply $\rho(\cdot)$, $t(\cdot)$ and $\sigma(\cdot)$ whenever $X$ is unambiguously explicit from the context. The following technical lemma relates these functions for a random variable $X$ to the corresponding functions for its negative $-X$; it will allow us to focus on large deviations “on one side” (of the expected value) and quickly deduce analogous results for the other side.
Lemma 2. Let $X$ be a bounded and non-degenerate random variable. Then $\rho_X(a) = \rho_X(-a)$ for every $a$. If in addition $\min[X] < a < \max[X]$ then $t_X(a) = -t_X(-a)$ and $\sigma_X(a) = \sigma_X(-a)$.

Proof. Notice that $M_X(t) = M_X(-t)$. Hence, given $a$, we have that for every $t$, $e^{-at} M_X(t) = e^{a(-t)} M_X(-t)$. It follows from this that $\rho_X(a) = \rho_X(-a)$ and $t_X(a) = -t_X(-a)$. $\sigma_X(a) = \sigma_X(-a)$ then follows from the definition. \hfill $\Box$

The main technical tool of this paper is the following lemma, due, in various forms, to (Bahadur and Rao, 1960, Lemma 2) and to (Petrov, 1965, Theorems 5 and 6). It is a sharp, quantitative large deviations estimate, which will be useful not only for proving our asymptotic results above, but can also be used for estimating the number $n$ of repetitions required to achieve stochastic dominance.

Lemma 3. Let $X$ be a bounded and non-degenerate random variable and let $b > 0$ satisfy $P[|X| \leq b/2] = 1$. Let $X_1, X_2, \ldots$ be i.i.d. copies of $X$.

Then for every $\mathbb{E}[X] \leq a < \max[X]$ and every $n$, it holds that

$$
P[X_1 + \cdots + X_n \geq a \cdot n] \leq \rho(a)^n. \tag{22}$$

And for every $\mathbb{E}[X] \leq a < \max[X]$ and $n \geq (10b/\sigma(a))^2$ it holds that

$$
P[X_1 + \cdots + X_n \geq a \cdot n] \geq C(a) \cdot \frac{\rho(a)^n}{\sqrt{n}} \tag{23}$$

where

$$
C(a) = \frac{e^{-10t(a)b} \cdot b}{\sigma(a)}. 
$$

Inequalities similar to (22) and (23) apply to values of $a$ that lie below the expectation of $X$. Consider the case where $\min[X] < a \leq \mathbb{E}[X]$. Then, by applying the inequality (23) to the random variable $-X$ and using Lemma 2, we obtain that for every $n \geq (10b/\sigma_X(a))^2$,

$$
\rho_X(a)^n = \rho_X(-a)^n \geq P[-X_1 - \cdots - X_n \geq -a \cdot n] = P[X_1 + \cdots + X_n \leq a \cdot n] 
\geq \frac{e^{-10t_X(-a)b} \cdot b \cdot \rho_X(-a)^n}{\sqrt{n}} = \frac{e^{10t_X(a)b} \cdot b \cdot \rho_X(a)^n}{\sqrt{n}}. \tag{24}
$$

A corollary of this lemma is a lower estimate that is uniform over $a \in [\mathbb{E}[X], \max[X] - \varepsilon]$.

Corollary 1. In the setting of Lemma 3, let $A = [a, \overline{a}] \subset [\mathbb{E}[X], \max[X])$ be a given interval. Then

$$
C_A = \inf_{a \in A} C(a) \quad \text{and} \quad n_A = \sup_{a \in A} (10b/\sigma(a))^2
$$

are positive and finite, and hence for every $a \in A$ and every $n \geq n_A$

$$
P[X_1 + \cdots + X_n \geq a \cdot n] \geq C_A \cdot \frac{\rho(a)^n}{\sqrt{n}}. \tag{25}$$
Proof. Since \( t(a) \) solves \( K_X'(t(a)) = a \) and \( K_X \) is strictly convex, \( t(a) \) must be strictly increasing in \( a \). It is thus continuous and bounded above on the compact set \( A \). Similarly \( \sigma(a) \) is continuous and strictly positive, so it is bounded above and away from zero on \( A \). Thus \( C_A > 0 \) and \( n_A < \infty \).

The next lemma is a refined version of Lemma 3, applicable to the regime of \( a \) that vanishes with \( n \).

**Lemma 4.** In the setting of Lemma 3, for every \( \mathbb{E}[X] \leq a < \max[X] \) and every \( n \) it holds that

\[
\mathbb{P}[X_1 + \cdots + X_n \geq an] \leq \frac{1 + \sqrt{2\pi \cdot t(a)b}}{\sqrt{2\pi \cdot \sigma(a)t(a)}} \cdot \frac{(\rho(a))^n}{\sqrt{n}}.
\]

And for every \( \mathbb{E}[X] \leq a < \max[X] \) and \( n \geq 2[\sigma(a)t(a)]^{-2} \) it holds that

\[
\mathbb{P}[X_1 + \cdots + X_n \geq an] \geq \frac{1 - 2\sqrt{2\pi \cdot t(a)b}}{2\sqrt{2\pi \cdot \sigma(a)t(a)}} \cdot \frac{(\rho(a))^n}{\sqrt{n}}.
\]

This, and the previous lemma 3, are proved in the rest of this section.

**A.1 Proof of Lemma 3**

We follow Bahadur and Rao (1960). For each \( a \) such that \( \mathbb{E}[X] \leq a < \max[X] \), denote

\[ p_n(a) = \mathbb{P}[X_1 + \cdots + X_n \geq an]. \]

Let \( Y^a = X - a \) and let \( F_a \) be its cumulative distribution function. Consider, in addition, a random variable \( Z^a \) whose c.d.f. is given by

\[ G(z) = \frac{1}{\rho(a)} \cdot \int_{-\infty}^{z} e^{t(a) \cdot y} dF_a(y). \]

Note that \( G(\infty) = 1 \) because by definition \( M_{Y^a}(t(a)) = \rho(a) \).

More generally, the moment generating function of \( Z^a \) is given by

\[ M_{Z^a}(r) = \frac{M_{Y^a}(r + t(a))}{\rho(a)} = \frac{M_{Y^a}(r + t(a))}{M_{Y^a}(t(a))}. \]

It follows from \( M'_{Y^a}(t(a)) = 0 \) that \( M'_{Z^a}(0) = 0 \), hence \( Z^a \) has mean 0. Moreover

\[ \sigma(a)^2 = M''_{Z^a}(t(a)) = M''_{Y^a}(0) = \text{Var}(Z^a). \]

It is clear that \( Z^a \) has the same support as \( Y^a \), which, for the entire range of values of \( a \) we consider, is contained in \([-b,b]\). Thus we further have

\[ \mathbb{E}[|Z^a|^3] \leq b \cdot \mathbb{E}[(Z^a)^2] = b \cdot \sigma(a)^2. \]
Let $Z^n_1, \ldots, Z^n_n$ be i.i.d. copies of $Z^n_a$, and define

$$U^n_a = \frac{Z^n_1 + \cdots + Z^n_n}{\sqrt{n} \cdot \sigma(a)}.$$ 

Denote by $H^n_a(z) = \mathbb{P}[U^n_a \leq z]$ the c.d.f. of $U^n_a$. Then we can apply Lemma 2 in Bahadur and Rao (1960) to obtain

$$p_n(a) = \rho(a)^n \cdot \sqrt{n} \sigma(a) t(a) \cdot \int_{0}^{\infty} e^{-\sqrt{n} \sigma(a) t(a) z} \cdot (H^n_a(z) - H^n_a(0)) \, dz.$$ 

Clearly, $H^n_a(z) - H^n_a(0) \leq 1$ for each $z$. So $p_n(a) \leq \rho(a)^n$, which yields (22), also known as the Chernoff bound.

In the other direction, for any $z_0 > 0$ we have

$$p_n(a) \geq \rho(a)^n \cdot \sqrt{n} \sigma(a) t(a) \cdot \int_{z_0}^{\infty} e^{-\sqrt{n} \sigma(a) t(a) z} \cdot (H^n_a(z_0) - H^n_a(0)) \, dz \quad = \rho(a)^n \cdot e^{-\sqrt{n} \sigma(a) t(a) z_0} \cdot (H^n_a(z_0) - H^n_a(0)).$$

(26)

By the Berry-Esseen Theorem

$$H^n_a(z_0) - H^n_a(0) \geq \int_{z_0}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \, dx - \frac{1}{n} \cdot \mathbb{E} \left[ \frac{|Z^n|^3}{\sigma(a)^3 \sqrt{n}} \right] \geq \int_{z_0}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \, dx - \frac{b}{\sqrt{n} \cdot \sigma(a)}.$$ 

Note that if $z_0 \leq 1$ then the first term on the right hand side is at least $z_0/5$. Hence, if we pick $z_0 = 10b/(\sigma(a) \sqrt{n})$, and let $n_0 = (10b/\sigma(a))^2$, then for all $n \geq n_0$ we have that $z_0 \leq 1$ and so the above yields $H^n_a(z_0) - H^n_a(0) \geq b/(\sigma(a) \sqrt{n})$. Hence from (26) it holds for all $n \geq n_0$ that

$$p_n(a) \geq \rho(a)^n \cdot e^{-\sqrt{n} \sigma(a) t(a) z_0} \cdot (H^n_a(z_0) - H^n_a(0))$$

$$\geq \rho(a)^n \cdot e^{-10t(a)b} \cdot \frac{b}{\sigma(a) \sqrt{n}},$$

which shows (23).

A.2 Proof of Lemma 4

We initially proceed as in the proof of Lemma 3, arriving at

$$p_n(a) = \rho(a)^n \cdot \sqrt{n} \sigma(a) t(a) \cdot \int_{0}^{\infty} e^{-\sqrt{n} \sigma(a) t(a) z} \cdot (H^n_a(z) - H^n_a(0)) \, dz.$$ 

Let $\Phi$ denote the c.d.f. of a standard Gaussian distribution. By the Berry-Esseen Theorem

$$H^n_a(z) - H^n_a(0) \leq \Phi(z) - \Phi(0) + \frac{b}{\sigma(a) \sqrt{n}}.$$
Hence
\[ p_n(a) \leq \rho(a)^n \cdot \sqrt{n} \sigma(a) t(a) \cdot \int_0^\infty e^{-\sqrt{n} \sigma(a) t(a) z} \cdot \left( \Phi(z) - \Phi(0) + \frac{b}{\sigma(a) \sqrt{n}} \right) \, dz. \]

Let \( c = \sqrt{n} \sigma(a) t(a) \). Then integration by parts implies
\[ c \int_0^\infty e^{-cz} \cdot (\Phi(z) - \Phi(0)) \, dz = e^{c^2/2} \cdot \Phi(-c) \] (27)

Standard bounds for \( \Phi \) assert that
\[ \frac{1}{c \sqrt{2\pi}} \left( 1 - \frac{1}{c^2} \right) \leq e^{c^2/2} \cdot \Phi(-c) \leq \frac{1}{c \sqrt{2\pi}}. \] (28)

We thus obtain from the upper bound and (27) that
\[ p_n(a) \leq \rho(a)^n \left( \frac{1}{\sqrt{2\pi} \sqrt{n} \sigma(a) t(a)} + \frac{b}{\sigma(a) \sqrt{n}} \right) \]
\[ = \rho(a)^n \frac{1}{\sqrt{2\pi} \sqrt{n} \sigma(a) t(a)} \left( 1 + \sqrt{2\pi} t(a) b \right). \]

In the other direction, applying Berry-Esseen again, we have
\[ H_n^a(z) - H_n^a(0) \geq \Phi(z) - \Phi(0) - \frac{b}{\sigma(a) \sqrt{n}}. \]

For \( n \geq 2[\sigma(a) t(a)]^{-2} \), we have \( c \geq \sqrt{2} \), and so the lower bound in (28) implies
\[ e^{c^2/2} \Phi(-c) \geq \frac{1}{2 \sqrt{2\pi} c}. \]

It follows from this estimate and (27) that
\[ p_n(a) \geq \rho(a)^n \left( \frac{1}{2 \sqrt{2\pi} \sqrt{n} \sigma(a) t(a)} - \frac{b}{\sigma(a) \sqrt{n}} \right) \]
\[ = \rho(a)^n \frac{1}{2 \sqrt{2\pi} \sqrt{n} \sigma(a) t(a)} \left( 1 - 2 \sqrt{2\pi} t(a) b \right). \]

**B Proof of Proposition 3 and Theorem 1**

It is not difficult to see that Proposition 3 implies Theorem 1. Indeed, to prove Theorem 1 we just need to show one direction, that \( L_X(t) > L_Y(t) \) for all \( t \) implies \( X_1 + \cdots + X_n \) dominates \( Y_1 + \cdots + Y_n \) for large \( n \). By Proposition 3,
\[ \mathbb{P}[X_1 + \cdots + X_n \geq na] \geq \mathbb{P}[Y_1 + \cdots + Y_n \geq na] \]
Thus, \( \rho \) holds. Theorem 1 holds. The pair still holds for every such that \( L_Y(t) > L_X(t) \) for \( t \geq 0 \). Thus, applying Proposition 3 to the pair \(-Y, -X\), we obtain
\[
P[-Y_1 - \cdots - Y_n \geq nb] \geq P[-X_1 - \cdots - X_n \geq nb]
\]
for every \( b \leq \mathbb{E}[-X] \) and \( n \geq N \). Setting \( a = -b \), this is equivalent to
\[
P[X_1 + \cdots + X_n > na] \geq P[Y_1 + \cdots + Y_n > na]
\]
for every \( a \leq \mathbb{E}[X] \). Thus the inequality holds for all \( a \) when \( n \) is sufficiently large, and Theorem 1 holds.

To prove Proposition 3, let \( b \) be a positive number so that \( X \) and \( Y \) are supported on \([-b/2, b/2]\). Without loss of generality we assume \( X \) and \( Y \) are non-degenerate.\(^{18}\) Moreover, since \( L_X(t) > L_Y(t) \) for all \( t \geq 0 \), letting \( t \to \infty \) yields \( \max[X] \geq \max[Y] \). Since they are unequal by assumption, we in fact have \( \max[X] > \max[Y] \).

Denote by \( F^{*n}(a) \) the c.d.f. of the sum of \( n \) i.i.d. copies of \( X \). We need to show \( 1 - F^{*n}(na) \geq 1 - G^{*n}(na) \) for \( a \geq \mathbb{E}[Y] \) and \( n \) large. We divide the proof into cases.

**Case 1**: \( a > \max[Y] \). In this case \( G^{*n}(na) = 1 \), and so trivially \( 1 - F^{*n}(na) \geq 1 - G^{*n}(na) \) for any \( n \).

**Case 2**: \( \mathbb{E}[X] \leq a \leq \max[Y] \). Assume, without loss of generality, that \( \max[Y] > \mathbb{E}[X] \). Let \( A = [\mathbb{E}[X], \max[Y]] \) and consider \( C_A, n_A \) as defined in Corollary 1, applied to the random variable \( X \). When \( a \in A \) we have \( e^{-at}M_X(t) > e^{-at}M_Y(t) \) for every \( t > 0 \).

Since for \( a > \mathbb{E}[X] \) we have \( t_X(a) > 0 \), this implies
\[
\rho_X(a) = M_{X-a}(t_X(a)) > M_{Y-a}(t_X(a)) \geq \rho_Y(a).
\]
But even if \( a = \mathbb{E}[X] \), it still holds that \( \rho_X(a) = 1 = M_{Y-a}(0) > \rho_Y(a) \) since \( t_Y(a) > 0 \). Thus \( \rho_X(a) > \rho_Y(a) \) whenever \( a \in A \).

Now, Corollary 1 implies that for all \( a \in A \) and \( n \geq n_A \),
\[
1 - F^{*n}(an) \geq C_A \cdot \frac{\rho_X(a)^n}{\sqrt{n}}, \quad (29)
\]
while Lemma 3 implies
\[
1 - G^{*n}(an) \leq \rho_Y(a)^n. \quad (30)
\]

\(^{18}\) Otherwise, we can find non-degenerate random variables \( \tilde{X} \) and \( \tilde{Y} \) with distributions close to \( X \) and \( Y \), such that \( X \) dominates \( \tilde{X} \) and \( \tilde{Y} \) dominates \( Y \) in first-order stochastic dominance, and that \( L_{\tilde{X}}(t) > L_{\tilde{Y}}(t) \) still holds for every \( t \geq 0 \). The result of Proposition 3 for the pair \( \tilde{X}, \tilde{Y} \) implies the corresponding result for the pair \( X, Y \).
As $\rho_X$ and $\rho_Y$ are continuous functions and $\rho_X(a) > \rho_Y(a)$ on $A$, the ratio $\rho_X/\rho_Y$ is bounded below by $1 + \varepsilon$ for some $\varepsilon > 0$.\footnote{For $\mathbb{E}[X] \leq a \leq \max\{Y\}$, and $\rho_Y(a) = 0$ if and only if $a = \max\{Y\}$ and the distribution of $Y$ has an atom at $\max\{Y\}$. On the other hand, $\rho_X$ is strictly positive on this interval.}

Hence, for any $n$ such that

$$C_A > \frac{\sqrt{n}}{(1 + \varepsilon)n} \quad \text{and} \quad n \geq n_A$$

it follows from (29) and (30) that $1 - F^{*\eta}(an) > 1 - G^{*\eta}(an)$ for all $a \in A$.

**Case 3:** $\mathbb{E}[Y] \leq a \leq \mathbb{E}[X]$. By the Berry-Esseen Theorem there exist constants $k_X$ and $k_Y$ such that for all $a$,

$$\left| F^{*\eta}(na) - \Phi \left( \sqrt{n} \cdot \frac{a - \mathbb{E}[X]}{\sigma_X} \right) \right| \leq \frac{k_X}{\sqrt{n}} \quad (31)$$

$$\left| G^{*\eta}(na) - \Phi \left( \sqrt{n} \cdot \frac{a - \mathbb{E}[Y]}{\sigma_Y} \right) \right| \leq \frac{k_Y}{\sqrt{n}}$$

where $\Phi$ denotes the cdf of a standard Gaussian distribution. Fix $a_0 = \frac{1}{2}(\mathbb{E}[X] + \mathbb{E}[Y])$. Since $a_0 > \mathbb{E}[Y]$ there exists an $N$ such that $n \geq N$ implies

$$G^{*\eta}(na_0) \geq \Phi \left( \sqrt{n} \cdot \frac{a_0 - \mathbb{E}[Y]}{\sigma_Y} \right) - \frac{k_Y}{\sqrt{n}} > 0.99 - \frac{k_Y}{\sqrt{n}} \geq \frac{1}{2} + \frac{k_X}{\sqrt{n}} \geq F^{*\eta}(n \cdot \mathbb{E}[X]).$$

where the first and the last inequalities follow directly from (31). Similarly, there exists $N'$ such that $n \geq N'$ implies

$$F^{*\eta}(na_0) \leq \Phi \left( \sqrt{n} \cdot \frac{a_0 - \mathbb{E}[X]}{\sigma_Y} \right) + \frac{k_X}{\sqrt{n}} < 0.01 + \frac{k_X}{\sqrt{n}} \leq \frac{1}{2} - \frac{k_Y}{\sqrt{n}} \leq G^{*\eta}(n \cdot \mathbb{E}[Y]).$$

Hence for $n \geq \max\{N, N'\}$, if $a_0 \leq a \leq \mathbb{E}[X]$, then

$$G^{*\eta}(na) \geq G^{*\eta}(na_0) > F^{*\eta}(n \cdot \mathbb{E}[X]) \geq F^{*\eta}(na).$$

Conversely, if $\mathbb{E}[Y] \leq a \leq a_0$ then

$$F^{*\eta}(na) \leq F^{*\eta}(na_0) < G^{*\eta}(n \cdot \mathbb{E}[Y]) \leq G^{*\eta}(na).$$

Therefore $1 - F^{*\eta}(na) > 1 - G^{*\eta}(na)$ holds for all $a$ in this range. Proposition 3 follows.

**C Preliminaries for Comparison of Experiments**

We collect here some useful facts regarding the distributions of log-likelihood ratios induced by an experiment. Let $P = (\Omega, P_0, P_1)$ be an experiment and let

$$\Pi = \frac{dP_1/dP_0}{1 + dP_1/dP_0}$$
be the random variable corresponding to the posterior probability that $\theta$ equals 1. For every $A \subseteq [0, 1]$ we have

$$
\pi_1(A) = \int_{\Pi \in A} dP_1 = \int_{\Pi \in A} \frac{dP_1}{dP_0} dP_0 = \int_{\Pi \in A} \frac{\Pi}{1 - \Pi} dP_0
$$

Thus

$$
\frac{d\pi_1}{d\pi_0}(p) = \frac{p}{1 - p}.
$$

(32)

Recall that $\pi = \frac{1}{2} \pi_0 + \frac{1}{2} \pi_1$, so

$$
\frac{d\pi}{d\pi_1}(p) = \frac{1}{2p}
$$

(33)

We also observe that for every function $\phi$ that is integrable with respect to $F_1$, defined as in (10),

$$
\int_\mathbb{R} \phi(u) dF_1(u) = \int_\mathbb{R} \phi(-v)e^{-v} dF_0(v).
$$

(34)

This implies that the moment generating function of $F_1$

$$
M_{F_1}(u) = \int_{-\infty}^{\infty} e^{tu} dF_1(u)
$$

satisfies

$$
M_{F_1}(t) = M_{F_0}(-t - 1)
$$

(35)

Hence, in particular, $M_{F_1}(-1) = 1$.

C.1 Proof of Lemma 1

Given an exponential distribution with support $\mathbb{R}_+$ and cdf $H(x) = 1 - e^{-x}$ for all $x \geq 0$, $\tilde{F}$ and $\tilde{G}$ can be written as

$$
\tilde{F}(a) = \int_0^\infty F_1(a + u) dH(u) = \int_0^\infty F_1(a + u) e^{-u} du
$$

and similarly

$$
\tilde{G}(a) = \int_0^\infty G_1(a + u) e^{-u} du.
$$

Consider the first part of the lemma. Suppose $a \geq b$, then by assumption $F_1(a + u) \leq G_1(a + u)$ for all $u \geq 0$, which implies $\tilde{F}(a) \leq \tilde{G}(a)$.

For the second part of the lemma, we will establish the following identities:

$$
\tilde{F}(a) = \int_{-a}^{\infty} F_0(v) e^{-v} dv \quad \text{and} \quad \tilde{G}(a) = \int_{-a}^{\infty} G_0(v) e^{-v} dv.
$$

(36)

Given this, the result would follow easily: If $F_0(a) \leq G_0(a)$ for all $a \geq b$, then the above implies $\tilde{F}(-a) \leq \tilde{G}(-a)$ for all $a \geq b$. 32
We now show how (36) follows from (34). We first observe that by taking \( \phi(u) = 1_{(a, \infty)}(u) \cdot e^{-u} \), (34) implies
\[
\tilde{F}(a) = F_1(a) + e^a \int_{(a, \infty)} e^{-u} dF_1(u) = F_1(a) + e^a F_0((-a)_{-})
\]  
(37)
where \( F_0((-a)_{-}) \) denotes the left limit of \( F_0 \) evaluated at \(-a\). Moreover, taking \( \phi \) to be the indicator function of \(( -\infty, a ] \) implies
\[
F_1(a) = \int_{-a}^{\infty} e^{-v} dF_0(v).
\]
Integration by parts leads to
\[
F_1(a) = \int_{-a}^{\infty} e^{-v} dF_0(v) = -e^a F_0((-a)_{-}) - \int_{-a}^{\infty} F_0(v) e^{-v} dv
\]
Hence by (37), we obtain
\[
\tilde{F}(a) = \int_{-a}^{\infty} F_0(v) e^{-v} dv
\]
as desired.

\section{Proof of Theorem 2}
Throughout the proof, we use the notation introduced in §3.6 and §3.7 and further discussed in §C, as well as the notation related to large deviation estimates introduced in §A.

We first show that (i) implies (ii). As discussed in the main text, the comparison of Rényi divergences between two experiments is independent of the number of repetitions. Thus it suffices to show that the Rényi order refines the Blackwell order.

For \( t > 1 \), the function \( v_1(p) = 2p^t (1-p)^{1-t} \) is strictly convex. Thus it is the indirect utility function induced by some decision problem. Moreover, it is straightforward to check that
\[
\int_0^1 v_1(p) d\pi(p) = \exp((t-1)\mathcal{R}_P^1(t)),
\]
which is a monotone transformation of the Rényi divergence. Thus, experiment \( P \) yields higher expected payoff in this decision problem (with indirect utility \( v \)) than \( Q \) only if \( \mathcal{R}_P^1(t) > \mathcal{R}_Q^1(t) \).

For \( t \in (0, 1) \), we consider the indirect utility function \( v_2(p) = -2p^t (1-p)^{1-t} \), which is now convex due to the negative sign. Observe similarly that
\[
\int_0^1 v_2(p) d\pi(p) = -\exp((t-1)\mathcal{R}_P^1(t))
\]
is again a monotone transformation of the Rényi divergence. So \( P \) yields higher payoff in this decision problem only if \( \mathcal{R}_P^1(t) > \mathcal{R}_Q^1(t) \).
For $t = 1$, we consider the indirect utility function $v_3(p) = 2p \log \left( \frac{p}{1-p} \right)$, which is strictly convex. Since
\[
\int_0^1 v_3(p) \, d\pi(p) = R^1_P(1),
\]
$P$ yields higher payoff only if $R^1_P(1) > R^1_Q(1)$.

Summarizing, the above family of decision problems show that $P$ Blackwell-dominates $Q$ only if $R^0_P(t) > R^0_Q(t)$ for all $t > 0$. Since the two states are symmetric, we also have $R^0_P(t) > R^0_Q(t)$ for all $t > 0$. This shows $P$ dominates $Q$ in the Rényi order.

We now show that (ii) implies (i). The assumptions that $R^0_P(1) > R^0_Q(1)$ and that $R^1_P(1) > R^1_Q(1)$ are, in terms of the notation introduced in (11), is equivalent to
\[
\mathbb{E}[G_0] < \mathbb{E}[F_0] \quad \text{and} \quad \mathbb{E}[G_1] < \mathbb{E}[F_1],
\]
where, with slight abuse of notation, given a cdf $H$ we denote by $\mathbb{E}[H]$ the expectation of a random variable with distribution $H$.

Let $X, X_1, \ldots, X_n$ be i.i.d. and distributed according to $F_1$ and let $Y, Y_1, \ldots, Y_n$ be i.i.d. and distributed according to $G_1$. By (17) and (18), the assumption that $R^0_P(t) > R^0_Q(t)$ for all positive $t \neq 1$ is equivalent to having $M_X(t) > M_Y(t)$ for $t > 0$ and $t < -1$, and $M_X(t) < M_Y(t)$ for $t \in (-1, 0)$. In particular, $L_X(t) > L_Y(t)$ for all $t > -1$.

By Proposition 2, it suffices to show that for $n$ large,
\[
X_1 + \cdots + X_n - E \geq Y_1 + \cdots + Y_n - E.
\]
where $E$ is an independent, positive, exponential random variable with density $e^{-x}$. That is, we need to show for $n$ large and all $a \in \mathbb{R}$,
\[
\mathbb{P}[X_1 + \cdots + X_n - E \geq na] \geq \mathbb{P}[Y_1 + \cdots + Y_n - E \geq na] \quad (38)
\]
We consider a number of cases.

**Case 1: $a \geq \mathbb{E}[G_1]$.** The random variables $X$ and $Y$ satisfy the conditions of Proposition 3. Thus, for every $n$ large enough and every $a$ in this range it holds that
\[
\mathbb{P}[X_1 + \cdots + X_n \geq na] \geq \mathbb{P}[Y_1 + \cdots + Y_n \geq na]
\]
Hence $F_1^{\infty}(na) \leq G_1^{\infty}(na)$, and so the first statement of Lemma 1 applied to $F_1^{\infty}, G_1^{\infty}$ implies
\[
F^{\infty}(na) \leq G^{\infty}(na),
\]
which implies (38).
**Case 2:** \( a \leq -\mathbb{E}[G_0] \). Here we repeat the argument of the previous case, but applied to \( F_0 \) and \( G_0 \), instead of \( F_1 \) and \( G_1 \). The hypothesis that \( M_{F_1}(t) < M_{G_1}(t) \) for all \( t < -1 \) is equivalent, by (35), to \( M_{F_1}(t) > M_{G_0}(t) \) for all \( t > 0 \). Moreover \( \mathbb{E}[F_0] > \mathbb{E}[G_0] \), and so the same conditions that applied in the previous case apply here. Thus there exists \( N \) such that for \( n \geq N \) it holds that

\[
F_0^{*n}(na) \leq G_0^{*n}(na),
\]

for every \( a \geq \mathbb{E}[G_0] \). Hence the second statement of Lemma 1 implies

\[
\overline{F}^{*n}(na) \leq \overline{G}^{*n}(na)
\]

for all \( a \leq -\mathbb{E}[G_0] \).

**Case 3:** \( -\mathbb{E}[G_0] \leq a \leq \mathbb{E}[G_1] \). Here we will still show (as in case 1) that

\[
\mathbb{P}[X_1 + \cdots + X_n \geq na] \geq \mathbb{P}[Y_1 + \cdots + Y_n \geq na],
\]

which would imply the result via Lemma 1.

Recall that \( t_Y(a) \) satisfies \( K_Y'(t_Y(a)) = a \). Observe that \( K_Y'(0) = \mathbb{E}[G_1] \geq a \) and

\[
K_Y'(-1) = K_{G_1}'(-1) = -K_{G_0}'(0) = -\mathbb{E}[G_0] \leq a.
\]

Thus by convexity of \( K_Y \), we have \( t_Y(a) \in [-1, 0] \). Since \( \mathbb{E}[F_1] > \mathbb{E}[G_1] \) and \( \mathbb{E}[F_0] > \mathbb{E}[G_0] \), it follows that \( t_X(a) \in (-1, 0) \).

Denote \( A = [-\mathbb{E}[G_0], \mathbb{E}[G_1]] \). By (24), we have for all \( n \geq \left( \frac{106}{\min_{a \in A} \sigma_Y(a)} \right)^2 \) and \( a \in A \),

\[
\mathbb{P}[X_1 + \cdots + X_n \geq na] \geq 1 - \rho_X(a)^n
\]

and

\[
\mathbb{P}[Y_1 + \cdots + Y_n \geq na] \leq 1 - \frac{C(a)}{\sqrt{n}} \rho_Y(a)^n,
\]

where

\[
C(a) = \frac{e^{10t_Y(a)b} \cdot b}{\sigma_Y(a)}
\]

is strictly positive when \( a \in A \).

We now argue that \( \rho_X(a) < \rho_Y(a) \) for \( a \) in this range. Indeed, since \( M_Y(t) > M_X(t) \) for \( t \in (0, 1) \), and since (as is true for any distribution of a log-likelihood ratio) \( M_X(0) = M_X(-1) = M_Y(0) = M_Y(-1) = 1 \), we have

\[
\rho_Y(a) = e^{-at_Y(a)} \cdot M_Y(t_Y(a)) \geq e^{-at_Y(a)} \cdot M_X(t_Y(a)) \geq e^{-at_X(a)} \cdot M_X(t_X(a)) = \rho_X(a).
\]

But the first inequality holds equal only if \( t_Y(a) \in \{-1, 0\} \), in which case the second inequality must be strict, because \( t_X(a) = \arg \min_t e^{-at} \cdot M_X(t) \) is strictly between \(-1\) and \( 0 \).
Therefore $\rho_X(a) < \rho_Y(a)$ for $a \in A$. By continuity,

$$\gamma := \max_{a \in A} \frac{\rho_X(a)}{\rho_Y(a)}$$

is strictly below 1. We therefore conclude that

$$\mathbb{P}[X_1 + \cdots + X_n \geq na] \geq \mathbb{P}[Y_1 + \cdots + Y_n \geq na]$$

for every $n$ large enough to satisfy

$$\gamma^n < \frac{\min_{a \in A} C(a)}{\sqrt{n}}.$$

This completes the proof.

E Proof of Proposition 1

It is easily checked that the condition $R_P^1(1/2) > R_Q^1(1/2)$ reduces to

$$\sqrt{\alpha(1-\alpha)} > \sqrt{\beta(1/2 - \beta)} + \frac{1}{4}. \quad (39)$$

Since the experiments form a generic pair, by Theorem 2, we just need to check dominance in the Rényi order. Equivalently, we need to show

$$(1/2 - \beta)^r \beta^{1-r} + (1/2 - \beta)^{1-r} \beta^r + \frac{1}{2} < (1-\alpha)^r \alpha^{1-r} + (1-\alpha)^{1-r} \alpha^r, \quad \forall 0 < r < 1; \quad (40)$$

$$(1/2 - \beta)^r \beta^{1-r} + (1/2 - \beta)^{1-r} \beta^r + \frac{1}{2} > (1-\alpha)^r \alpha^{1-r} + (1-\alpha)^{1-r} \alpha^r, \quad \forall r < 0 \text{ or } r > 1; \quad (41)$$

$$\beta \cdot \ln(\frac{\beta}{1/2 - \beta}) + (1/2 - \beta) \cdot \ln(\frac{1-\beta}{\beta}) > \alpha \cdot \ln(\frac{\alpha}{1-\alpha}) + (1-\alpha) \cdot \ln(\frac{1-\alpha}{\alpha}). \quad (42)$$

To prove these, it suffices to consider the $\alpha$ that makes (39) hold with equality.\(^{20}\) We will show that the above inequalities hold for this particular $\alpha$, except that (40) holds equal at $r = \frac{1}{2}$. Let us define the following function

$$\Delta(r) := (1/2 - \beta)^r \beta^{1-r} + (1/2 - \beta)^{1-r} \beta^r + \frac{1}{2} - (1-\alpha)^r \alpha^{1-r} - (1-\alpha)^{1-r} \alpha^r.$$

When (39) holds with equality, we have $\Delta(0) = \Delta(\frac{1}{2}) = \Delta(1) = 0$. Thus $\Delta$ has roots at 0, 1 as well as a double-root at $\frac{1}{2}$. But since $\Delta$ is a weighted sum of 4 exponential functions plus a constant, it has at most 4 roots (counting multiplicity).\(^{21}\) Hence these

\(^{20}\)It is clear that inequalities are easier to satisfy when $\alpha$ increases in the range $[0, \frac{1}{2}]$.

\(^{21}\)This follows from Rolle's theorem and an induction argument.
are the only roots, and we deduce that the function $\Delta$ has constant sign on each of the intervals $(-\infty, 0), (0, \frac{1}{2}), (\frac{1}{2}, 1), (1, \infty)$.

Now observe that since $2\beta < \alpha \leq \frac{1}{2}$, it holds that $\frac{1/2 - \beta}{\beta} > \frac{1 - \alpha}{\alpha} > 1$. It is then easy to check that $\Delta(r) \to \infty$ as $r \to \infty$. Thus $\Delta(r)$ is strictly positive for $r \in (1, \infty)$. As $\Delta(1) = 0$, its derivative is weakly positive. But recall that we have enumerated the 4 roots of $\Delta$. So $\Delta$ cannot have a double-root at $r = 1$, and it follows that $\Delta'(1)$ is strictly positive. Hence (42) holds.

Note that $\Delta'(1) > 0$ and $\Delta(1) = 0$ also implies $\Delta(1 - \varepsilon) < 0$. Thus $\Delta$ is negative on $(\frac{1}{2}, 1)$. A symmetric argument shows that $\Delta$ is positive on $(-\infty, 0)$ and negative on $(0, \frac{1}{2})$. Hence (40) and (41) both hold, completing the proof.

**F Necessity of Genericity Assumption**

Here we present examples to show that Theorem 1 and Theorem 2 do not hold without the genericity assumption.

**Gambles.** The following is an example where $L_X(t) > L_Y(t)$ for all $t \in \mathbb{R}$, but $X$ does not dominate $Y$ in the aggregate because $\max[X] = \max[Y]$. Fix any $q \in (0, 1)$, and consider

$$X = \begin{cases} 10, \text{ w.p. } q \\ 2, \text{ w.p. } \frac{1-q}{2} \\ 0, \text{ w.p. } \frac{1-q}{2} \end{cases}; \quad Y = \begin{cases} 10, \text{ w.p. } q \\ 1, \text{ w.p. } \frac{2(1-q)}{3} \\ -1, \text{ w.p. } \frac{1-q}{3} \end{cases}$$

Let $\hat{X}$ be the random variable that takes values 2 and 0 with equal probabilities; note that $\hat{X}$ is distributed as $X$, conditional on $X \neq 10$. Similarly define $\hat{Y}$ to take value 1 w.p. $2/3$ and value $-1$ w.p. $1/2$. It is easy to check that $\hat{X}_1 + \hat{X}_2$ first-order stochastically dominates $\hat{Y}_1 + \hat{Y}_2$. As a result, $L_{\hat{X}}(t) > L_{\hat{Y}}(t)$ for all $t$. Since $M_X(t) = q \cdot e^{10t} + (1 - q) \cdot M_{\hat{X}}(t)$, we conclude that $L_X(t) > L_Y(t)$ for all $t$.

Nonetheless, we now show that $X$ does not dominate $Y$ in the aggregate. For each $n$, consider $P[X_1 + \cdots + X_n \geq 10n - 9]$. In other for this to happen, either every $X_i$ takes value 10, or all but one $X_i$ equals 10 and the remaining one equals 2. Thus

$$P[X_1 + \cdots + X_n \geq 10n - 9] = q^n + nq^{n-1} \cdot \frac{1-q}{2}.$$

Similarly we have

$$P[Y_1 + \cdots + Y_n \geq 10n - 9] = q^n + nq^{n-1} \cdot \frac{2(1-q)}{3}.$$
Since the latter probability is larger, $X_1 + \cdots + X_n$ does not first-order dominate $Y_1 + \cdots + Y_n$.\textsuperscript{22}

**Nonexistence of a Generator.** The above example shows that aggregate stochastic dominance does not admit a generator. Suppose, by way of contradiction, that $V$ was a family of functions $\phi : \mathbb{R} \to \mathbb{R}$ with the property that for all bounded random variables $X$ and $Y$, $X$ first-order dominates $Y$ in the aggregate if and only if $\mathbb{E} [\phi(X)] \geq \mathbb{E} [\phi(Y)]$ for all $\phi \in V$.

Let $X, \hat{X}, Y$ and $\hat{Y}$ be defined as in the previous paragraph. Since $\hat{X}$ dominates $\hat{Y}$ in the aggregate, then it must hold that $\mathbb{E} [\phi(\hat{X})] \geq \mathbb{E} [\phi(\hat{Y})]$ for all $\phi \in V$. Hence,

$$\mathbb{E} [\phi(X)] = (1 - q)\mathbb{E} [\phi(\hat{X})] + q\phi(10) = (1 - q)\mathbb{E} [\phi(\hat{Y})] + q\phi(10) \geq \mathbb{E} [\phi(Y)],$$

implying that $X$ dominates $Y$, a contradiction.

**Experiments.** Consider the experiments $P$ and $Q$ described in §3.5. Fix $\alpha = \frac{1}{4}$ and $\beta = \frac{1}{16}$, which satisfy (39). Then by Proposition 1, $P$ dominates $Q$ in large samples.

But similar to the preceding example, we will perturb these two experiments by adding another signal realization (to each experiment) which strongly indicates the true state is 1. The perturbed conditional probabilities are given below:

<table>
<thead>
<tr>
<th>$\theta$</th>
<th>$x_0$</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$\hat{Q}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$\varepsilon$</td>
<td>$\frac{1}{16}$</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{7}{16} - \varepsilon$</td>
<td>$y_0$</td>
</tr>
<tr>
<td>1</td>
<td>$100\varepsilon$</td>
<td>$\frac{7}{16}$</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{16} - 100\varepsilon$</td>
<td></td>
</tr>
<tr>
<td>$\hat{P}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\hat{Q}^\oplus n$</td>
</tr>
</tbody>
</table>

If $\varepsilon$ is a small positive number, then by continuity $\hat{P}$ still dominates $\hat{Q}$ in the Rényi order. Nonetheless, we show below that $\hat{P}^\oplus n$ does not Blackwell-dominate $\hat{Q}^\oplus n$ for any $n$ and $\varepsilon > 0$.

To do this, let $\overline{p} := \frac{100^{n-1}}{100^n+1}$ be a threshold belief. We will show that a decision maker whose indirect utility function is $(p-\overline{p})^+$ strictly prefers $\hat{Q}^\oplus n$ to $\hat{P}^\oplus n$. Indeed, it suffices to focus on posterior beliefs $p > \overline{p}$; that is, the likelihood-ratio should exceed $100^{n-1}$. Under $\hat{Q}^\oplus n$, this can only happen if every signal realization is $y_0$, or all but one signal is $y_0$ and the remaining one is $y_1$. Thus, in the range $p > \overline{p}$, the posterior belief has the following distribution under $\hat{Q}^\oplus n$:

$$p = \begin{cases} \frac{100^n}{100^n+1} & \text{w.p. } \frac{1}{2}(100^n + 1)\varepsilon^n \\ \frac{3 \cdot 100^n - 1}{3 \cdot 100^n + 1} & \text{w.p. } \frac{1}{2}(3 \cdot 100^n - 1 + 1)\varepsilon^{n-1} \end{cases}$$

\textsuperscript{22}Related, a slightly modification of this example shows that even if $X_1 + \cdots + X_n \geq 1, Y_1 + \cdots + Y_n$ for all large $n$, this does not imply that $X_1 + \cdots + X_n \geq 1, X_1 + \cdots + X_{n-1} + Y_n$ for all large $n$. Indeed, suppose that $Y = 9$ (instead of 10) w.p. $q$ in this example, then Theorem 1 applies and shows that $X$ dominates $Y$ in the aggregate, but $\mathbb{P} [X_1 + \cdots + X_n \geq 10n - 9] < \mathbb{P} [X_1 + \cdots + X_{n-1} + Y_n \geq 10n - 9]$.\textsuperscript{22}
Similarly, under $\tilde{P}^{\otimes n}$ the relevant posterior distribution is

$$p = \begin{cases} 
\frac{100^n}{100^n + 1} & \text{w.p. } \frac{1}{2}(100^n + 1)\varepsilon^n \\
\frac{7 \cdot 100^n}{7 \cdot 100^n + 1} & \text{w.p. } \frac{n}{32}(7 \cdot 100^n + 1)\varepsilon^{n-1}
\end{cases}$$

Recall that the indirect utility function is $(p - \bar{p})^+$. So $\tilde{Q}^{\otimes n}$ yields higher expected payoff than $\tilde{P}^{\otimes n}$ if and only if

$$\frac{n}{8}(3 \cdot 100^{n-1} + 1)\varepsilon^{n-1} \left(\frac{3 \cdot 100^{n-1}}{3 \cdot 100^{n-1} + 1} - \bar{p}\right) > \frac{n}{32}(7 \cdot 100^{n-1} + 1)\varepsilon^{n-1} \left(\frac{7 \cdot 100^{n-1}}{7 \cdot 100^{n-1} + 1} - \bar{p}\right).$$

That is,

$$4(3 \cdot 100^{n-1} + 1)\varepsilon^{n-1} \left(\frac{3 \cdot 100^{n-1}}{3 \cdot 100^{n-1} + 1} - \frac{100^{n-1}}{100^{n-1} + 1}\right) > (7 \cdot 100^{n-1} + 1)\varepsilon^{n-1} \left(\frac{7 \cdot 100^{n-1}}{7 \cdot 100^{n-1} + 1} - \frac{100^{n-1}}{100^{n-1} + 1}\right).$$

The LHS is computed to be $\frac{8 \cdot 100^{n-1}}{100^{n-1} + 1}$, while the RHS is $\frac{6 \cdot 100^{n-1}}{100^{n-1} + 1}$. Hence the above inequality holds, and it follows that $\tilde{P}^{\otimes n}$ does not Blackwell dominate $\tilde{Q}^{\otimes n}$.

**G Proof of Theorem 3**

If $X$ second-order dominates $Y$ in the aggregate, then, by considering risk-averse CARA utility functions, we obtain that $L_X(t) > L_Y(t)$ for all $t < 0$; the strict inequality is because these utility functions are strictly concave. By continuity we thus have $L_X(0) \geq L_Y(0)$, which implies $E[X] \geq E[Y]$. Since by assumption they are unequal, we in fact have $E[X] > E[Y]$. Hence (ii) implies (i).

To show (i) implies (ii), suppose $L_X(t) > L_Y(t)$ for all $t \leq 0$. As in the proof of Theorem 1, We assume without loss of generality that $X$ and $Y$ are non-degenerate, and denote by $F^{*n}$ (resp. $G^{*n}$) the c.d.f. of the sum of $n$ i.i.d. copies of $X$ (resp. $Y$). Furthermore, by shifting $X$ and $Y$ by a constant, we can assume $E[X] = \mu$ and $E[Y] = -\mu$ for some positive number $\mu$.

To prove second-order stochastic dominance, we need to show that for $n$ large enough and for every $x \in \mathbb{R}$ it holds that

$$\int_{-\infty}^{x} G^{*n}(t) - F^{*n}(t) \, dt \geq 0$$

(43)

We again consider a few cases.

**Case 1**: $x \leq 0$. In this case Proposition 3 applied to the random variables $-Y$ and $-X$ implies $G^{*n}(x) \geq F^{*n}(x)$ for all $x \leq n\mu$. Hence (43) holds too.
Case 2: $x \geq 0$. Note that, as can be shown by integration by parts,
\[
\int_{-\infty}^{\infty} G^{*n}(t) - F^{*n}(t) \, dt = n\mathbb{E} [X] - n\mathbb{E} [Y] = 2n\mu
\]
Hence
\[
\int_{-\infty}^{x} G^{*n}(t) - F^{*n}(t) \, dt = 2n\mu - \int_{x}^{\infty} G^{*n}(t) - F^{*n}(t) \, dt
\]
\[
= 2n\mu - \int_{x}^{\infty} (1 - F^{*n}(t)) - (1 - G^{*n}(t)) \, dt
\]
\[
\geq 2n\mu - \int_{0}^{\infty} 1 - F^{*n}(t) \, dt \quad \text{(44)}
\]
Now, again using integration by parts we have that
\[
\int_{0}^{\infty} 1 - F^{*n}(t) \, dt = n\mu + \int_{-\infty}^{0} F^{*n}(t) \, dt
\]
\[
= n\mu + \int_{0}^{\min [X]} F^{*n}(t) \, dt \leq n\mu + n \cdot |\min [X]| \cdot F^{*n}(0).
\]
By the Chernoff bound (i.e., (22) in Lemma 3), $F^{*n}(0) \leq \rho_{-X}(0)^{n}$. Since $\rho_{-X}(0) < 1$, for $n$ large enough we have that the above is at most $\frac{3}{2} n\mu$. Applying this estimate to (44) yields
\[
\int_{-\infty}^{x} G^{*n}(t) - F^{*n}(t) \, dt \geq \frac{1}{2} n\mu
\]
for all $x \geq 0$, and our proof is complete.

H Proof of Theorem 4

We first show (ii) implies (i). Observe that by assumption, $\mathbb{E} [X_{1} + \cdots + X_{n}] = \mathbb{E} [Y_{1} + \cdots + Y_{n}]$ for each $n$. Thus $X_{1} + \cdots + X_{n}$ second-order stochastically dominates $Y_{1} + \cdots + Y_{n}$ if and only if the latter is a mean-preserving spread of the former. Thus, for every strictly convex function $\phi : \mathbb{R} \to \mathbb{R}$ (not necessarily increasing), it holds that
\[
\mathbb{E} [\phi (X_{1} + \cdots + X_{n})] < \mathbb{E} [\phi (Y_{1} + \cdots + Y_{n})].
\]
Choosing $\phi(x) = x^{2}$ and using $\mathbb{E} [X_{1} + \cdots + X_{n}] = \mathbb{E} [Y_{1} + \cdots + Y_{n}]$, we deduce that $\text{Var}(X_{1} + \cdots + X_{n}) < \text{Var}(Y_{1} + \cdots + Y_{n})$, and so $\text{Var}(X) < \text{Var}(Y)$. Moreover, choosing $\phi(x) = e^{tx}$ implies that $M_{X}(t) < M_{Y}(t)$ for all $t \neq 0$. This is equivalent to $L_{X}(t) < L_{Y}(t)$ for all $t > 0$ and $L_{X}(t) > L_{Y}(t)$ for all $t < 0$, as we desire to show.

Below we prove that (i) implies (ii). Since $L_{X}(t) < L_{Y}(t)$ for all $t > 0$, taking $t \to \infty$ yields $\max [X] \leq \max [Y]$ by continuity. But since $X$ and $Y$ are generic, we in fact have $\max [X] < \max [Y]$. Similarly we have $\min [X] > \min [Y]$. We also assume without loss of generality that $\mathbb{E} [X] = \mathbb{E} [Y] = 0$. Thus $X$ and $Y$ are bounded, zero mean random variables satisfying the following conditions:
(i). \( \min[X] > \min[Y] \).
(ii). \( \max[X] < \max[Y] \).
(iii). \( \text{Var}(X) < \text{Var}(Y) \).
(iv). \( M_X(t) < M_Y(t) \) for all \( t \neq 0 \).

As in the proof of Theorem 1, denote by \( F^{*n} \) (resp. \( G^{*n} \)) the c.d.f. of the sum of \( n \) i.i.d. copies of \( X \) (resp. \( Y \)). Let \( b > 0 \) be a number such that \( X \) and \( Y \) are supported on \([−b/2, b/2] \).

To prove (ii) we need to show that for \( n \) large enough and for every \( x \in \mathbb{R} \) it holds that

\[
W(x) := \int_{-\infty}^{x} G^{*n}(y) - F^{*n}(y) \, dy \geq 0. \tag{45}
\]

Since \( E[X] = E[Y] \), integration by parts shows that \( W(x) = 0 \) for \( x \) sufficiently large. Thus we also have \( W(x) = \int_{x}^{\infty} F^{*n}(y) - G^{*n}(y) \, dy \). The above inequality reduces to

\[
\int_{x}^{\infty} F^{*n}(t) - G^{*n}(t) \, dt \geq 0. \tag{46}
\]

We will show that (46) holds for all \( x \geq 0 \). The case of \( x \leq 0 \) follows by applying the same argument to \(-X\) and \(-Y\), which also satisfy the above four conditions.

As before, we write \( x = na \) and consider a few cases.

**Case 1:** \( \max[X] < a \). In this range \( F^{*n} = 1 \), and hence \( F^{*n} \geq G^{*n} \) point-wise.

**Case 2:** \( \varepsilon \leq a \leq \max[X] \), with \( \varepsilon > 0 \) chosen in case 3 below. Note that \( L_Y(t) > L_X(t) \) for all \( t > 0 \), so the random variables \( Y \) and \( X \) almost satisfy the assumptions of Proposition 3, except that \( L_Y(0) = L_X(0) \) (which equals their common expected value). However, since we have \( a \geq \varepsilon \), we can follow the analysis in the proof of Proposition 3 and deduce that \( \rho_Y(a) > \rho_X(a) \) in this range. The result of Proposition 3 thus gives

\[
1 - G^{*n}(na) \geq 1 - F^{*n}(na)
\]

for all \( n \) large enough (depending on \( \varepsilon \)) and \( a \geq \varepsilon \). The integral in (46) is thus positive in this range.

**Case 3:** \( \sqrt{\frac{1}{2} \text{Var}(X) \frac{\log n}{n}} \leq a \leq \varepsilon \). Define \( r_X(a) = \log \rho_X(a) \) (and \( r_Y \) analogously). It follows from Lemma 4 that

\[
1 - F^{*n}(na) \leq \exp(n \cdot r_X(a)) \cdot \frac{1 + \sqrt{2\pi t_X(a)b}}{\sqrt{2\pi\sigma_X(a)t_X(a)\sqrt{n}}},
\]

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and that

\[ 1 - G^*(na) \geq \exp \left( n \cdot r_Y(a) \right) \frac{1 - 2\sqrt{2\pi} t_Y(a) b}{2\sqrt{2\pi \sigma_Y(a) t_Y(a) n}} , \]

provided that

\[ n \geq [\sigma_Y(a) t_Y(a)]^{-2} . \]

By Lemma 5 below \( r'_X(0) = -t_X(0) = 0 \) and 
\[ r''_X(0) = -t'_X(0) = -\frac{1}{K''_X(0)} = -\frac{1}{\text{Var}(X)} . \]

Hence by Taylor expansion, we can write

\[ r_X(a) = r_X(0) + r'_X(0) + \frac{1}{2} r''_X(0) a^2 + O(a^3) = -\frac{1 + O(\varepsilon)}{2 \text{Var}(X)} \]

for \( 0 \leq a \leq \varepsilon \); similarly for \( r_Y(a) \).

Note that \( t_X(0) = 0 \), so \( t_X(a) = O(\varepsilon) \). Also, \( \sigma_X(0) = \text{Std}(X) \) (the standard deviation of \( X \)), which implies \( \sigma_X(a) = (1 + O(\varepsilon)) \text{Std}(X) \). Moreover, \( t'_X(0) = \frac{1}{\text{Var}(X)} > t'_Y(0) \), so \( t_X(a) > t_Y(a) \) for \( 0 \leq a \leq \varepsilon \) whenever \( \varepsilon \) is sufficiently small. Plugging all of these estimates into the above inequality for \( F^*n \), we have

\[ 1 - F^*n(na) \leq \exp \left( -\frac{D(\varepsilon) a^2 n}{2 \text{Var}(X)} \right) \frac{1}{2\sqrt{2\pi \text{Var}(X) n} \cdot t_Y(a) D(\varepsilon)} , \quad (47) \]

where \( D(\varepsilon) < 1 \) is a shorthand for \( 1 - O(\varepsilon) \), which approaches 1 as \( \varepsilon \to 0 \).

Similarly,

\[ 1 - G^*(na) \geq \exp \left( -\frac{a^2 n}{2D(\varepsilon) \text{Var}(Y)} \right) \frac{D(\varepsilon)}{2\sqrt{2\pi \text{Var}(Y) n} \cdot t_Y(a)} , \quad (48) \]

provided that

\[ n \geq [\sigma_Y(a) t_Y(a)]^{-2} . \quad (49) \]

Considering the ratio between (47) and (48), we obtain

\[ \frac{1 - G^*(na)}{1 - F^*n(na)} \geq \exp \left( \frac{1}{2} \left( \frac{D(\varepsilon)}{\text{Var}(X)} - \frac{1}{D(\varepsilon) \text{Var}(Y)} \right) a^2 n \right) \frac{\text{Std}(X)}{2 \text{Std}(Y)} . \]

Denote

\[ V_{XY} = \frac{1}{4} \left( \frac{1}{\text{Var}(X)} - \frac{1}{\text{Var}(Y)} \right) , \quad (50) \]

which is positive since \( \text{Var}(X) < \text{Var}(Y) \). We now choose \( \varepsilon \) small enough so that

\[ \frac{1}{2} \left( \frac{D(\varepsilon)}{\text{Var}(X)} - \frac{1}{D(\varepsilon) \text{Var}(Y)} \right) > V_{XY} > 0 . \]
For this $\varepsilon$, it thus holds that

$$\frac{1 - G^{*n}(na)}{1 - F^{*n}(na)} \geq \exp\left(\frac{V_{XY} a^2 n}{2 \text{Std}(Y)}\right).$$

Since we are considering the case that $a^2 \geq \frac{1}{2} \text{Var}(X) \log \frac{n}{n}$ we have that

$$\frac{1 - G^{*n}(na)}{1 - F^{*n}(na)} \geq \exp\left(\frac{1}{2} V_{XY} \text{Var}(X) \log n\right) \frac{\text{Std}(X)}{2 \text{Std}(Y)}.$$

This is larger than one for $n$ large enough. So we still have $F^{*n}(na) \geq G^{*n}(na)$ point-wise.

Lastly, we need to verify the condition (49). As we noted above, $\sigma_Y(0) = \text{Std}(Y)$, $t_Y(0) = 0$ and $t_Y'(0) = 1 / \text{Var}(Y)$. So for $\varepsilon$ small enough and all $a$ such that $\frac{1}{2} \text{Var}(X) \log \frac{n}{n} \leq a^2 \leq \varepsilon^2$ we have $\sigma_Y(a) = (1 + O(\varepsilon)) \text{Std}(Y)$ and $t_Y(a) = (1 + O(\varepsilon))a / \text{Var}(Y)$. Hence

$$\left[\sigma_Y(a) t_Y(a)\right]^{-2} \leq \frac{2 \text{Var}(Y)}{a^2} \leq \frac{4 \text{Var}(Y)}{\text{Var}(X) \log n},$$

and so condition (49) will hold for all $n$ sufficiently large.

**Case 4:** $\sqrt{\frac{1}{n}} \leq a \leq \sqrt{\frac{1}{2} \text{Var}(X) \log \frac{n}{n}}$. By the Berry-Esseen Theorem

$$F^{*n}(na) \geq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a \sqrt{n}/\text{Std}(X)} e^{-x^2/2} \, dx - \frac{k}{\sqrt{n}}$$

and

$$G^{*n}(na) \leq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a \sqrt{n}/\text{Std}(Y)} e^{-x^2/2} \, dx + \frac{k}{\sqrt{n}},$$

where $k$ is a constant depending only on the distribution of $X$ and $Y$. Hence

$$F^{*n}(na) - G^{*n}(na) \geq \frac{1}{\sqrt{2\pi}} \int_{a \sqrt{n}/\text{Std}(X)}^{a \sqrt{n}/\text{Std}(Y)} e^{-x^2/2} \, dx - \frac{2k}{\sqrt{n}}.$$

Since $e^{-x^2/2}$ is decreasing in this range we can lower bound the integrand by its right limit, yielding

$$F^{*n}(na) - G^{*n}(na) \geq \left(\frac{1}{\text{Std}(X)} - \frac{1}{\text{Std}(Y)}\right) a \sqrt{n} \cdot e^{-a^2 n/(2 \text{Var}(X))} - \frac{2k}{\sqrt{n}}. \quad (51)$$

Applying the assumption $\frac{1}{n} \leq a^2 \leq \frac{1}{2} \text{Var}(X) \log \frac{n}{n}$ yields

$$F^{*n}(na) - G^{*n}(na) \geq \left(\frac{1}{\text{Std}(X)} - \frac{1}{\text{Std}(Y)}\right) n^{-1/4} - 2k \cdot n^{-1/2}, \quad (52)$$

which is again positive for all $n$ large enough.
Case 5: $0 \leq a \leq \sqrt{\frac{1}{n}}$. Recall that we defined

$$W(x) = \int_x^{\infty} F^*(y) - G^*(y) \, dy.$$ 

From cases 1-4, we have shown that for $y \geq \sqrt{n}$, $F^*(y) - G^*(y) \geq 0$ point-wise. Moreover, from (52) we in fact have

$$F^*(y) - G^*(y) \geq c n^{-1/4}$$

for $n$ large enough and $\sqrt{n} \leq y \leq \sqrt{\frac{1}{2} \text{Var}(X)n \log n}$, where $c$ is a positive constant independent of $n$. Integrating this estimate over the range of $y$ to which it applies, we deduce

$$W(\sqrt{n}) \geq \int_{\sqrt{n}}^{\sqrt{\frac{1}{2} \text{Var}(X)n \log n}} F^*(y) - G^*(y) \, dy = cn^{1/4}.$$ 

On the other hand, for $y \in [0, \sqrt{n}]$, it follows from 51 that

$$F^*(x) - G^*(x) \geq -\frac{2k}{\sqrt{n}}.$$ 

So for any $x \in [0, \sqrt{n}]$,

$$W(x) = \int_x^{\infty} F^*(y) - G^*(y) \, dt$$

$$= \int_{\sqrt{n}}^{x} F^*(y) - G^*(y) \, dy + W(\sqrt{n})$$

$$\geq -\frac{2k}{\sqrt{n}} \sqrt{n} + W(\sqrt{n})$$

$$= -2k + cn^{1/4}$$

which is positive for $n$ large enough. This completes the proof that $W(x) \geq 0$ for all $x \geq 0$, and the theorem follows.

H.1 Additional Lemma

**Lemma 5.** Let $X$ be a bounded, zero mean random variable, and define $r_X(a) = \log \rho_X(a)$. Then $r_X'(0) = -t_X(0) = 0$ and $r_X''(0) = -t_X'(0) = -1/\text{Var}(X)$.

**Proof.** We suppress the subscript $X$ in this proof. Observe that $r(a) = \inf_t K(t) - at$. So by the envelope theorem, $r'(a) = -t(a)$. Since $t(\mathbb{E}[X]) = 0$, we deduce $r'(0) = -t(0) = 0$.

Moreover, we have $r''(a) = -t'(a)$. Now recall that $t(a)$ satisfies $K'(t(a)) = a$, and so $t'(a) = \frac{1}{K''(t(a))}$. But from $K(t) = \log \mathbb{E}[e^{tX}]$ it is easy to deduce $K''(0) = \text{Var}(X)$. Hence $r_X''(0) = -t_X'(0) = -1/\text{Var}(X)$ as desired. \qed

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I Omitted Lemma in the Proof of Proposition 4

To prove (iii) implies (i), we establish the following lemma:

**Lemma 6.** Suppose $X \geq_k Y$ for some $k \geq 1$. Then for each $n$ and i.i.d. replicas $X_1, \ldots, X_n$ of $X$ and $Y_1, \ldots, Y_n$ of $Y$, it holds that

$$X_1 + \cdots + X_n \geq_k Y_1 + \cdots + Y_n.$$ 

**Proof.** We first show that if $X \geq_k Y$, then $X + Z \geq_k Y + Z$ whenever $Z$ is independent of both $X$ and $Y$. Indeed, by definition we need to show $E[u(X + Z)] \geq E[u(Y + Z)]$ for any $u$ whose first $k$ derivatives have alternating signs. The assumption that $X \geq_k Y$ shows $E[u(X + Z)] \geq E[u(Y + Z)]$ for every realization $z$, since $u(\cdot + z)$ also has $k$ derivatives that alternate signs. Integrating over $z$ then yields the claim. Repeatedly applying this result, we obtain

$$X_1 + \cdots + X_n \geq_k X_1 + \cdots + X_{n-1} + Y_n \geq_k X_1 + \cdots + X_{n-2} + Y_{n-1} + Y_n \geq_k \cdots \geq_k Y_1 + \cdots + Y_n.$$ 

This proves the lemma.

J Proof of Proposition 5

That (i) implies (ii) is immediate: whenever $X$ dominates $Y$ in the aggregate, $L_X(t) \geq L_Y(t)$ for all $t \leq 0$. Thus $E[-e^{-rX}] \geq E[-e^{-rY}]$ for all $r \geq 0$, and $E[u(X)] \geq E[u(Y)]$.

It remains to show the negation of (i) implies the negation of (ii). Suppose we are given a utility function $u \not\in C$. Without loss we can assume $u$ is increasing and concave; otherwise we can find $X \geq_2 Y$ such that $E[u(X)] < E[u(Y)]$, which is stronger than the negation of (ii). In particular, we will assume $u$ is continuous.

Equip the space of functions $\mathbb{R}^R$ with the product topology (i.e. topology of pointwise convergence), and let $C \subset \mathbb{R}^R$ be the set of MRA utility functions. It is immediate that $C$ is a convex cone. It is also closed, because the set of functions that are Laplace Transforms of non-negative measures—that is, the set of functions of the form $\int e^{-rx} d\mu(r)$—is closed under pointwise convergence (Curtiss, 1942, Theorem 2).

Since $\mathbb{R}^R$ is locally convex, and since $C$ is a closed convex cone, it follows from the Hahn-Banach Separation Theorem that there is a continuous linear functional $\lambda: \mathbb{R}^R \to \mathbb{R}$ and a constant $\delta > 0$ such that for all $v \in C$ it holds that

$$\lambda(u) < -\delta < 0 \leq \lambda(v).$$

As a continuous linear functional on $\mathbb{R}^R$ with the product topology, $\lambda$ is of the form

$$\lambda(v) = \sum_{i=1}^n \alpha_i v(x_i) - \beta_i v(y_i).$$

Recall that the usual stochastic order implies the aggregate stochastic order.
for some \( n \geq 1 \), non-negative \((\alpha_i)\), non-negative \((\beta_i)\), and real numbers \((x_i)\) and \((y_i)\).\(^{24}\) Equivalently, there are finitely supported random variables \( X \) and \( Y \), and constants \( A, B \geq 0 \), such that

\[
\lambda(v) = A \cdot E[v(X)] - B \cdot E[v(Y)].
\]

Since \( \mathcal{C} \) contains all constant functions, we conclude that \( A = B \). Without loss of generality, we can therefore assume that \( A = B = 1 \).

Now recall \( \lambda(u) < -\delta \), and so

\[
E[u(X)] < E[u(Y)] - \delta.
\]

Also, \( \lambda(v) \geq 0 \) for all \( v \in \mathcal{C} \), and so for any such \( v \)

\[
E[v(X)] \geq E[v(Y)].
\]

This implies \( L_X(t) \geq L_Y(t) \) for all \( t \leq 0 \).

To complete the proof, let \( \varepsilon > 0 \) and define \( Z = Y - \varepsilon \). Since \( u \) is continuous, and using (53), we can choose \( \varepsilon \) small enough to satisfy \( E[u(X)] < E[u(Z)] \). On the other hand, \( X \) and \( Z \) form a generic pair and \( L_X(t) \geq L_Y(t) > L_Z(t) \) for all \( t \leq 0 \). Thus by Theorem 3, \( X \) second-order dominates \( Z \) in the aggregate. This leads to the negation of (ii).

### K Connection to Other Stochastic Orders

Aumann and Serrano (2008) and Foster and Hart (2009) propose two criteria for measuring the riskiness of a gamble. They focus on random variables \( X \) with \( E[X] > 0 \) and \( P[X < 0] > 0 \). The Aumann-Serrano riskiness index is the unique positive number \( R_{AS}(X) \) such that

\[
E\left[e^{-\frac{X}{R_{AS}(X)}}\right] = 1.
\]

On the other hand, the Foster-Hart measure of riskiness is the unique positive number \( R_{FH}(X) \) such that

\[
E\left[\log\left(1 + \frac{X}{R_{FH}(X)}\right)\right] = 0.
\]

Hart (2011) recognizes that these indices induce two complete orderings over gambles that refine second-order stochastic dominance. That is, we can define \( X \) to dominate \( Y \) if and only if \( R_{AS}(X) \leq R_{AS}(Y) \) (or \( R_{FH}(X) \leq R_{FH}(Y) \), respectively). He provides behavioral characterizations of these orders, which are called “uniform-wealth dominance” and “uniform-utility dominance.”

In what follows, we show that if \( L_X(t) \geq L_Y(t) \) for all \( t \leq 0 \), then \( X \) is less risky than \( Y \) according to both Aumann-Serrano and Foster-Hart. This, together with one direction \(^{24}\)In other words, \( \lambda(v) \) is a fixed linear combination of the value of the function \( v \) at certain fixed points.
of Theorem 3, then proves that the two uniform dominance orders in Hart (2011) both refine our second-order aggregate dominance order.

To show $R_{AS}(X) \leq R_{AS}(Y)$, let $a$ denote $\frac{1}{R_{AS}(X)}$ and $b$ denote $\frac{1}{R_{AS}(Y)}$. By definition we have $M_X(-a) = 1$. But since $L_X(-a) \geq L_Y(-a)$ by assumption, we obtain $M_Y(-a) \geq M_X(-a) = 1$. From $M_Y(-a) \geq 1$, $M_Y(0) = M_Y(-b) = 1$, and the strict convexity of $M_Y(t)$, we can conclude that $b \leq a$. Thus

$$R_{AS}(X) = \frac{1}{a} \leq \frac{1}{b} = R_{AS}(Y).$$

To show $R_{FH}(X) \leq R_{FH}(Y)$, we similarly denote $c = \frac{1}{R_{FH}(X)}$ and $d = \frac{1}{R_{FH}(Y)}$. By definition,

$$\mathbb{E} [\log(1 + cX)] = 0 = \mathbb{E} [\log(1 + dY)].$$

Consider the utility function $u(x) = \log(1 + dx)$. Observe that for $x > -\frac{1}{d}$, $u(x)$ has derivatives that alternate signs. By Bernstein’s theorem, $u(x)$ can be written as a mixture of linear functions and exponential functions $\{-e^{-tx} \}_{0 \leq t \leq \infty}$. Since by assumption $M_X(t) \leq M_Y(t)$ for all $t \leq 0$, we deduce that $\mathbb{E} [u(X)] \geq \mathbb{E} [u(Y)]$.\(^{25}\) In other words,

$$\mathbb{E} [\log(1 + dX)] \geq \mathbb{E} [\log(1 + dY)] = 0.$$

Now observe that the function $g(\lambda) = \mathbb{E} [\log(1 + \lambda X)]$ is strictly concave in $\lambda$, and $g(0) = g(c) = 0 \leq g(d)$. Hence $d \leq c$. It follows that

$$R_{FH}(X) = \frac{1}{c} \leq \frac{1}{d} = R_{FH}(Y).$$

\(^{25}\)Note that $L_X(t) \geq L_Y(t)$ for $t \to -\infty$ implies $\min[X] \geq \min[Y]$. Thus whenever $\mathbb{E} [\log(1 + dY)]$ is defined, so is $\mathbb{E} [\log(1 + dX)]$. 

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References


