Informational Braess' Paradox: The Effect of Information on Traffic Congestion

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Abstract

To systematically study the implications of additional information about routes provided to certain users (e.g., via GPS-based route guidance systems), we introduce a new class of congestion games in which users have differing information sets about the available edges and can only use routes consisting of edges in their information set. After defining the notion of Information Constrained Wardrop Equilibrium (ICWE) for this class of congestion games and studying its basic properties, we turn to our main focus: whether additional information can be harmful (in the sense of generating greater equilibrium costs/delays). We formulate this question in the form of Informational Braess' Paradox (IBP), which extends the classic Braess' Paradox in traffic equilibria, and asks whether users receiving additional information can become worse off. We provide a comprehensive answer to this question showing that in any network in the series of linearly independent (SLI) class, which is a strict subset of series-parallel network, IBP cannot occur, and in any network that is not in the SLI class, there exists a configuration of edge-specific cost functions for which IBP will occur. In the process, we establish several properties of the SLI class of networks, which include the characterization of the complement of the SLI class in terms of embedding a specific set of networks, and also an algorithm which determines whether a graph is SLI in linear time. We further prove that the worst-case inefficiency performance of ICWE is no worse than the standard Wardrop equilibrium.

Contents

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1 Introduction

The advent of GPS-based route guidance systems, such as Waze or Google maps, promises a better traffic experience to its users, as it can inform them about routes that they were not aware of or help them choose dynamically between routes depending on recent levels of congestion. Though other drivers might plausibly suffer increased congestion as the routes they were using become more congested due to this reallocation of traffic, or certain residents may experience elevated noise levels in their side streets, it is generally presumed that the users of these systems (and perhaps society as a whole) will benefit. In this paper, we present a framework for systematically analyzing how changes in the information sets of users in a traffic network (e.g., due to route guidance systems) impact the traffic equilibrium, and show the conditions under which even those with access to additional information may suffer greater congestion.

Our formal model is a version of the well-known congestion games, augmented with multiple types of users (drivers), each with a different information set about the available edges in the network. These different information sets represent the differing knowledge of drivers about the road network, which may result from their past experiences, from inputs from their social network, or from the different route guidance systems they might rely on. A user can only utilize a route (path between origin and destination) consisting of edges belonging to her information set. Each edge is endowed with a latency/cost function representing costs due to congestion. We generalize the classic notion of *Wardrop equilibrium* (Wardrop (1952); Beckmann et al. (1956) and Schmeidler (1973)), where each (non-atomic) user takes the level of congestion on all edges as given and chooses a route with minimum cost (defined as the summation of costs of edges on the route). Our notion of *Information Constrained Wardrop Equilibrium (ICWE)*, also imposes the same equilibrium condition as Wardrop equilibrium, but only for routes that are contained in the information set of each type of user.

After establishing the existence and essential uniqueness of ICWE and characterizing its main properties for networks with a single origin-destination pair (an assumption we impose for simplicity and later relax), we turn to our key question of whether expanding the information sets of some group of users can make them worse off — in the sense of increasing the level of congestion they suffer in equilibrium. For this purpose, we define the notion of *Informational Braess' Paradox (IBP)*, designating the possibility that users with expanded information sets experience greater equilibrium cost. We then provide a tight characterization of when IBP is and is not possible in a traffic network.

Our main result is that IBP does not occur if and only if the network is *series of linearly independent (SLI)*. More specifically, this result means that in an SLI network, IBP can never occur, ensuring that users with expanded information sets always benefit from their additional information. Conversely, if the network is not SLI, then there exists a configuration of latency/cost functions for edges for which IBP will occur. To understand this result, let us consider what the relevant class of networks comprises. The set of SLI networks is a subset of *series-parallel* networks, which are those for which two routes never pass through any edge in opposite directions. An SLI network is obtained by joining together a collection of *linearly independent (LI)* networks in series. LI networks are those in which each route includes at least one edge that is not part of any other route. The key step in establishing our main result is the following. In an LI network with two subsets of routes A and B, if we decrease the flow on routes in A and increase the flow on routes in B and the cost of all routes in A increases, then there exists a route in B whose cost must have increased as well. This argument is used to establish the "if" part of our main result. The "only if" part is proved by showing that every non-SLI network embeds one of the collection of networks, and we demonstrate constructively that each one of these networks generates IBP (for some configuration of costs).

We should also note that, since SLI is a restrictive class of networks, and few real-world networks would fall into this class, we take this characterization to imply that IBP is difficult to rule out in practice, and thus the new, highly-anticipated route guidance technologies may make traffic problems worse.

Since the class of SLI networks plays a central role in our analysis, a natural question is whether identifying SLI networks is straightforward. We answer this question by showing that whether a given network is SLI or not can be determined in linear time. This result is based on the algorithms for identifying series-parallel networks proposed by Valdes et al. (1979); Schoenmakers (1995), and Eppstein (1992).

If, rather than considering a general change of information sets, we specialize the problem so that only one user type does not have complete information about the available set of routes and the change in question is to bring all users complete information, then we show that an IBP is possible if and only if the network is not *series-parallel*. It is intuitive that this class of networks is less restrictive than SLI, since we are now considering a specific change in information sets (thus making IBP less likely to occur).

Our main focus is on traffic networks with a single origin-destination pair for which we provide a full characterization of network topologies for occurrence of IBP. In Section 6.4, we consider multiple origin-destination pairs and use our characterization to provide a sufficient condition on the network topology under which IBP does not occur.

Our notion of IBP closely relates to the classic Braess' Paradox (BP), introduced in Braess (1968) and further studied in Murchland (1970) and Arnott and Small (1994), which considers whether an additional edge in the network can increase equilibrium cost. When BP occurs in a network, IBP with a single information type also occurs (since IBP with a single information type can be shown to be identical to BP). Various aspects of BP and congestion games in general is studied in Murchland (1970); Steinberg and Zangwill (1983); Dafermos and Nagurney (1984); Patriksson (1994); Meir and Parkes (2014); Nikolova and Stier-Moses (2014); Chen et al. (2015), and Feldman and Friedler (2015). Our characterization of ICWE and IBP clarifies that our notion is different and, at least mathematically, more general. This can be seen readily from a comparison of our results to the most closely related papers to ours in the literature, Milchtaich (2005, 2006). The characterizations in Milchtaich (2006) imply that BP can be ruled out in series-parallel networks. Since IBP is a generalization of BP, it should occur in a wider class of networks, and this is indeed what our result shows- indeed SLI is a strict subset of series-parallel networks. This result also indicates that IBP is a considerably more pervasive phenomenon than BP. Notably, the mathematical argument for our key theorem is different from Milchtaich (2006) due to the key difficulty relative to BP that not all users have access to the same set of edges, and thus changes in traffic that benefit some groups of users might naturally harm others by increasing the congestion on the routes that they were previously utilizing.

Issues related to Braess' Paradox arise not only in the context of models of traffic, but in various models of communication, pricing and choice over congested goods, and electrical circuits. See e.g., Orda et al. (1993), Korilis et al. (1997), Kelly et al. (1998), and Low and Lapsley (1999) for communication networks; the classic works by Pigou (1920) and Samuelson (1952) as well as more recent works by Johari and Tsitsiklis (2003), Acemoglu and Ozdaglar (2007); Ashlagi et al. (2009) and Perakis (2004) for related economic problems; Frank (1981), Cohen and Horowitz (1991), and Cohen and Jeffries (1997) for mechanical systems electrical circuits; and Rosenthal (1973) and Vetta (2002) for general game-theoretic approaches. This observation also implies that the results we present here are relevant beyond traffic networks, in fact to any resource allocation problem over a network subject to congestion considerations. As pointed out in Newell (1980) and Sheffi (1985), the Braess' paradox and related inefficiencies are a clear and present challenge to traffic engineers, who often try to restrict travel choices to improve congestion (e.g., via systems such as ramp metering on freeway entrances). Issues related to the effects of providing information in other contexts are studied in Maheswaran and Başar (2003); Sanghavi and Hajek (2004); Yang and Hajek (2005); Harel et al. (2014); Rogers et al. (2015), and Liu et al. (2016).

Because our analysis also presents "price of anarchy" type results, i.e., bounds on the overall level of inefficiency that can occur in an ICWE, our paper is related to previous work on the price of anarchy in congestion and related games started by seminal works of Koutsoupias and Papadimitriou (1999) and Roughgarden and Tardos (2002) and followed by Correa et al. (2004, 2005), and Friedman (2004), as well as more generally to the analysis of equilibrium and inefficiency in the variants of this class of games, including Milchtaich (2004a,b), Acemoglu et al. (2007); Mavronicolas et al. (2007); Nisan et al. (2007); Arnott and Small (1994); Lin et al. (2004); Meir and Parkes (2015), and Anshelevich et al. (2008). Here, our result is that the presence of users with different information sets does not change the worst-case inefficiency traffic equilibrium as characterized, for example, in

Roughgarden and Tardos (2002).

The rest of the paper is organized as follows. In Section 2, we introduce our model of traffic equilibrium with users that are heterogeneous in terms of the information about routes/edges they have access to, and then define the notion of Information Constrained Wardrop Equilibrium for this setting. In Section 3 we prove the existence and essential uniqueness of Information Constrained Wardrop Equilibrium. Before moving to our main focus, in Section 4 we review some graph-theoretic notions about series-parallel and linearly independent networks, and then introduce the class of series of linearly independent networks and prove some basic properties of this class of networks, which are then used in the rest of our analysis. Section 5 defines our notion of Informational Braess' Paradox. Section 6 contains our main result, showing that Informational Braess' Paradox occurs "if and only if" the network is not in the class of series of linearly independent networks. Section 7 characterizes the worst-case inefficiency of Information Constrained Wardrop Equilibrium, and finally, Section 8 concludes. All the omitted proofs are included in the Appendix.

2 Model

We first describe the environment and then introduce our notion of Information Constrained Wardrop Equilibrium.

2.1 Environment

We consider an undirected multigraph $G = (V, \mathcal{E})$ with vertex set V and edge set $\mathcal{E} \subseteq V \times V$. We allow multiple edges between two distinct vertices (hence our use of the term multigraph), but we assume G has no self-loops. We use the terms node and vertex interchangeably. Each edge $e \in \mathcal{E}$ joins two (distinct) vertices u and v, referred to as the *end vertices* of e, and is denoted by $e = (u, v)$. An edge e and a vertex v are said to be *incident* to each other if v is an end vertex of e. A *path* $p \in G$ of length n ($n \geq 0$) is an alternating sequence of edges $e_1 \dots e_n$ in $\mathcal{E}$ where e_i and e_{i+1} share the same end vertex. If an edge e appears on a path p, we write $e \in p$. The first and last vertices of a path p are called the initial and terminal vertices of p, respectively. If q is a path of the form $e_{n+1} \dots e_m$, with the initial vertex the same as the terminal vertex of p but all the other vertices and edges of q do not belong to p, then $e_1 \dots e_n e_{n+1} \dots e_m$ is also a path, denoted by $p + q$. For a path p and two nodes v and u on it, we denote the section of path between u and v by p_{uv}.

Throughout the paper, we focus on an undirected multigraph $G = (V, \mathcal{E})$ together with an ordered pair of distinct vertices, called terminals, an origin O and a destination D, referred to as a *network*. A subnetwork of G is defined as $(V', \mathcal{E}')$, where $V' \subseteq V$ and $\mathcal{E}' = \mathcal{E} \cap (V' \times V')$. We assume that each vertex and edge belong to at least one path between the initial vertex O and the terminal vertex D. This assumption is without loss of generality because the vertices and edges that do not belong to any path from O to D are irrelevant for the purpose of sending traffic from O to D. Any path r with O as the initial vertex and D as the terminal vertex will be called a *route*. The set of all routes in a network is denoted by $\mathcal{R}$.

We suppose there are $K \geq 1$ types of users (we use the terms users and players interchangeably) and use the shorthand notation $[K] = \{1, \dots, K\}$ to denote the set of users. Each type $i \in [K]$ has total *traffic demand* $s_i \in \mathbb{R}^+$, and we denote the vector of traffic demands by $s_{1:K} = (s_1, \dots, s_K)$. For each type i, we use $\mathcal{E}_i \subseteq \mathcal{E}$ to denote the set of edges that type i knows and $\mathcal{R}_i$ to denote the routes formed by edges in $\mathcal{E}_i$ (assumed non-empty). We refer to $\mathcal{E}_i$ and $\mathcal{R}_i$ as the edges and routes in type i's information set. We use $\mathcal{E}_{1:K} = (\mathcal{E}_1, \dots, \mathcal{E}_K)$ to denote the information sets of all types.

We use $f^{(i)} = (f_r^{(i)} \ : \ r \in \mathcal{R}_i)$ to denote the flow vector of type i, where for all $r \in \mathcal{R}_i$, $f_r^{(i)} \geq 0$ represents the amount of traffic (flow) that type i sends on route r. We use $f^{(1:K)} = (f^{(1)}, \dots, f^{(K)})$ to denote the flow vector of all types. Each edge of the network has a cost (latency) function $c_e : \mathbb{R}^+ \to \mathbb{R}^+$ which is continuous, nonnegative, and nondecreasing. We denote the set of all cost functions by

$$\mathbf{c} = \{c_e \ : \ e \in \mathcal{E}\}.$$

For instance, if all the cost functions are affine functions, then for any $e \in \mathcal{E}$, we would have $c_e(x) = a_e x + b_e$, for some $a_e, b_e \in \mathbb{R}^+$. We refer to $(G, \mathcal{E}_{1:K}, s_{1:K}, \mathbf{c})$ as a *traffic network with multiple information types*. A feasible flow is a flow vector $f^{(1:K)} = (f^{(1)}, \dots, f^{(K)})$ such that for all $i \in [K]$, $f^{(i)}$ is a flow vector of type i, i.e., $f^{(i)} : \mathcal{R}_i \to \mathbb{R}^+$ and $\sum_{r \in \mathcal{R}_i} f_r^{(i)} = s_i$. We denote the total flow on each route r by f_r, i.e., $f_r = \sum_{i=1}^K f_r^{(i)}$.

2.2 Information Constrained Wardrop Equilibrium

The cost of a route r with respect to a flow $(f^{(1)}, \dots, f^{(K)})$ is the sum of the cost of the edges that belong to this route, i.e., $c_r(f^{(1:K)}) = \sum_{e \in r} c_e(f_e)$, where f_e denotes the amount of traffic that passes through edge e, i.e., $f_e = \sum_{r \in \mathcal{R} \ : \ e \in r} f_r$.

We assume flows get allocated at equilibrium according to a "constrained" version of Wardrop's principle: flows of each type of users are routed along routes in their information set with minimal (and hence equal) cost. We next formalize this equilibrium notion.

Definition 1 (Information Constrained Wardrop Equilibrium (ICWE)) A feasible flow $f^{(1:K)} = (f^{(1)}, \dots, f^{(K)})$ is an Information Constrained Wardrop Equilibrium (ICWE) if for every $i \in [K]$ and every pair $r, \tilde{r} \in \mathcal{R}_i$ with $f_r^{(i)} > 0$, we have

$$c_r(f^{(1:K)}) \leq c_{\tilde{r}}(f^{(1:K)}). \tag{2.1}$$

This implies that all the routes in $\mathcal{R}_i$ with positive flow from type i have the same cost, which is smaller or equal to the cost of any other route in $\mathcal{R}_i$. *The equilibrium cost of type* i, denoted by $c^{(i)}$, is then given by the cost of any route in $\mathcal{R}_i$ with positive flow from type i. Note that the Wardrop Equilibrium (WE) is a special case of this definition for a traffic network with a single information type, i.e., $K = 1$.

We next provide an example that illustrates this definition and how it differs from the classic Wardrop Equilibrium.

Example 1 Consider the network $G = (V, \mathcal{E})$ given in Figure 1 with $s_1 = s$, $s_2 = 1 - s$, and the cost functions specified in Figure 1. There are 5 different routes from origin to destination, which we denote by $r_1 = e_1e_3e_4$, $r_2 = e_1e_3e_5$, $r_3 = e_2e_3e_4$, $r_4 = e_2e_3e_5$, and $r_5 = e_6e_7$. We let $\mathcal{E}_1 = \mathcal{E}$ and $\mathcal{E}_2 = \{e_6, e_7\}$, which results in $\mathcal{R}_1 = \{r_1, r_2, r_3, r_4, r_5\}$ and $\mathcal{R}_2 = \{r_5\}$, respectively.

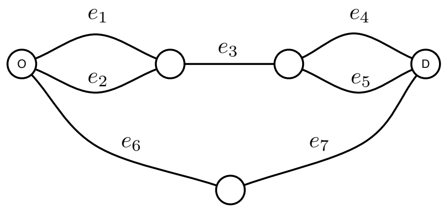


Figure 1: Example of a network with edge cost functions given by $c_{e_1}(x) = c_{e_4}(x) = c_{e_6}(x) = x$ and $c_{e_2}(x) = c_{e_5}(x) = c_{e_7}(x) = 1 + ax$ and $c_{e_3} = ax$ for some $a > 0$.

- If $s \leq \frac{2+a}{3+2a}$, ICWE is $f^{(2)}_{r_5} = 1 - s$ and $f^{(1)}_{r_1} = s$. The equilibrium cost of type 1 is $c^{(1)} = c_{r_1}(f^{(1:2)}) = s + as + s = s(a+2)$. The equilibrium cost of type 2 is $c^{(2)} = c_{r_5}(f^{(1:2)}) = (1-s)+(1+a(1-s)) = (1-s)(1+a)+1$. Hence, the equilibrium cost of type 1 and type 2 users need not be the same.
- If $s > \frac{2+a}{3+2a}$, ICWE is $f^{(1)}_{r_1} = \frac{2+a}{3+2a}$, $f^{(1)}_{r_5} = s - \frac{2+a}{3+2a} > 0$ and $f^{(2)}_{r_5} = 1 - s$, which give $c^{(1)} = c^{(2)} = \frac{(2+a)^2}{3+2a}$. This illustrates that when different types use a common path in an equilibrium, their costs are the same.

3 Existence of Information Constrained Wardrop Equilibrium

In this section, we show that given a traffic network with multiple information types $(G, \mathcal{E}_{1:K}, s_{1:K}, \mathbf{c})$, an ICWE always exists and it is "essentially" unique, i.e., for each type, equilibrium cost is the same for all equilibria. Our proof for existence and essential uniqueness of ICWE relies on the following characterization, which is a straightforward extension of the well-known optimization characterization of Wardrop Equilibrium (see Beckmann et al. (1956) and Smith (1979)).

Proposition 1 *A flow $f^{(1:K)}$ is an ICWE if and only if it is a solution of the following optimization problem:*

$$\begin{aligned} &\min \sum_{e \in \mathcal{E}} \int_0^{f_e} c_e(z)dz \\ &f_e = \sum_{i=1}^{K} \sum_{r \in \mathcal{R}_i \,:\, e \in r} f^{(i)}_r, \\ &\sum_{r \in \mathcal{R}_i} f^{(i)}_r = s_i, \ \text{and } f^{(i)}_r \geq 0 \text{ for all } r \in \mathcal{R}_i. \end{aligned} \tag{3.1}$$

We call $\sum_{e \in \mathcal{E}} \int_0^{f_e} c_e(z)dz$ the potential function and denote it by Φ.

Using the characterization of ICWE as the minimizer of a potential function, we can now show the existence and essential uniqueness.

Theorem 1 (Existence and Uniqueness of ICWE) *Let $(G, \mathcal{E}_{1:K}, s_{1:K}, \mathbf{c})$ be a traffic network with multiple information types.*

- *There exists an ICWE flow* $f^{(1:K)} = (f^{(1)}, \ldots, f^{(K)})$.
- *The ICWE is essentially unique in the sense that if* $f^{(1:K)}$ *and* $\tilde{f}^{(1:K)}$ *are both ICWE flows, then* $c_e(f_e) = c_e(\tilde{f}_e)$ *for every edge* $e \in \mathcal{E}$.

Remark 1 As shown in Milchtaich (2005); Gairing et al. (2006), and Mavronicolas et al. (2007) the essential uniqueness of equilibrium does not hold for multiple type congestion games where different types of users have different cost functions for the same edge. This class of congestion games is also referred to as *player-specific congestion games.* Several conditions on the edge cost functions and network topology have been proposed to guarantee the existence of an essentially unique equilibrium (see Konishi et al. (1997); Voorneveld et al. (1999); Milchtaich (2005); Mavronicolas et al. (2007); Georgiou et al. (2009), and Gairing and Klimm (2013)). In particular, Milchtaich (2005) provides sufficient and necessary conditions on the network topology under which an essentially unique equilibrium exists . Mavronicolas et al. (2007) show that when the edge costs are affine functions and differ by a player-specific additive constant, an essentially unique equilibrium exists. Georgiou et al. (2009) show that for any player-specific congestion game a necessary and sufficient condition for the existence of an essentially unique equilibrium is that for every pair of types, the corresponding cost functions are affine transformations of each other. Our model is a special case of a player-specific congestion game in which the cost of an edge e for a type i user is $c_e(\cdot)$ if $e \in \mathcal{E}_i$ and ∞, otherwise. Therefore, the results of Mavronicolas et al. (2007) and Georgiou et al. (2009) can directly be used to establish the existence of an essentially unique equilibrium in our model. For completeness, we provide an alternative proof of Theorem 1 in the Appendix 9.1 based on the classical results of Beckmann et al. (1956); Schmeidler (1973); Smith (1979), and Milchtaich (2000).

Theorem 1 assumes that the cost functions are non-decreasing. If we strengthen this assumption to strictly increasing cost functions, the results of Roughgarden and Tardos (2002), Mavronicolas et al. (2007), and Georgiou et al. (2009) show that the essential uniqueness result can be strengthened. In this case, the total flow on any edge at any equilibrium would be the same.

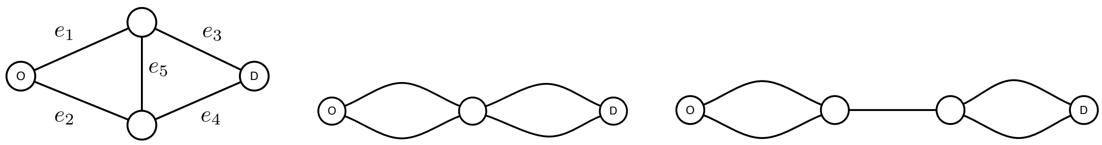
4 Some Graph-Theoretic Notions

In this section, we first present two classes of networks namely series-parallel and linearly independent networks which we use in our characterization of IBP. In preparation for our main graph-theoretic results, we also present equivalent characterizations of these networks and delineate the relations among them. Finally, we define a new class of networks termed series of linearly independent and show a characterization of it in terms of embedding of a few basic networks.

Definition 2 (Series-Parallel Network (SP)) A (two-terminal) network is called series-parallel if two routes never pass through an edge in opposite directions. Equivalently, as was shown by Riordan and Shannon (1942), a network is series-parallel if and only if

(i) it comprises a single edge between O and D, or

(ii) it is constructed by connecting two series-parallel networks in series, i.e., by joining the destination of one series-parallel network with the origin of the other one, or



(a) (b) (c)

Figure 2: Networks that cannot be embedded in SP and LI networks: Network (a) is not embedded in SP networks, Networks (a),(b), and (c) are not embedded in LI networks.

(iii) it is constructed by connecting two series-parallel networks in parallel, i.e., by joining the origins and destinations of two series-parallel networks.

As an example, the networks shown in Figure 2b and Figure 2c are series-parallel networks, while the network shown in Figure 2a is not series-parallel. The reason is that two routes $e_1e_5e_4$ and $e_2e_5e_3$ pass through the edge e_5 in opposite directions.

An important subclass of series-parallel networks are *linearly independent* networks.

Definition 3 (Linearly Independent Network (LI)) A (two terminal) network is called linearly independent if each route has at least one edge that does not belong to any other route. Equivalently, as was shown by Holzman and Law-yone (2003), a network is linearly independent if and only if

(i) it comprises of a single edge between O and D, or

(ii) it is constructed by connecting a linearly independent network in series with a single edge network, or

(iii) it is constructed by connecting two linearly independent networks in parallel.

This class is termed linearly independent because of an algebraic characterization of the routes when viewed as vectors in the edge space. In particular, for any $r \in \mathcal{R}$, let $\mathbf{v}_r \in \mathbb{F}_2^{|\mathcal{E}|}$ be $\mathbf{v}_r = (v_r^1, \ldots, v_r^{|\mathcal{E}|})$, where $v_r^i = 1$ if $e_i \in r$ and 0, otherwise. A network G is LI if and only if the set of vectors $\{\mathbf{v}_r \ : \ r \in \mathcal{R}\}$ is linearly independent (see Milchtaich (2006) and Diestel (2000)).

As Definitions 2 and 3 make it clear, the class of linearly independent networks is a subset of the class of series-parallel networks. An alternate characterization of linearly independent and series-parallel networks is based on the "graph embedding" notion, shown by Duffin (1965) and Milchtaich (2006), respectively. We next define a graph embedding and then present these characterizations which will be used later in our analysis.

Definition 4 (Embedding) A network H is embedded in the network G if we can start from H and construct G by applying the following steps in any order:

(i) Divide an edge, i.e., replace an edge with two edges with a single common end node.

(ii) Add an edge between two nodes.

(iii) Extend origin or destination by one edge.

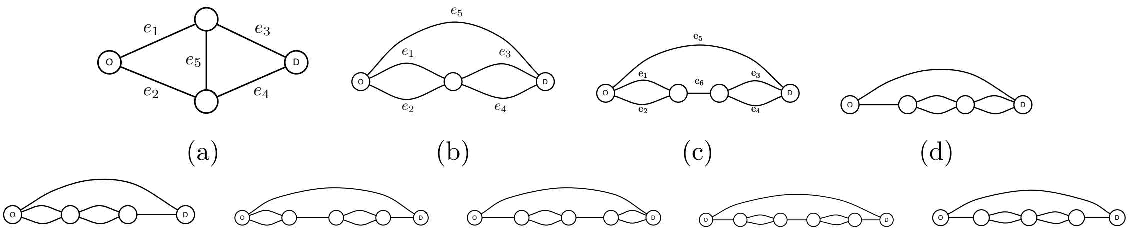


Figure 3: Networks that cannot be embedded in SLI networks.

Proposition 2 *(a) [Milchtaich (2006)] A network G is LI if and only if none of the networks shown in Figure 2 are embedded in it. Furthermore, a network G is LI if and only if for every pair of routes r and r' and every vertex $v \neq O, D$ common to both routes, either the path r_{Ov} is equal to r'_{Ov}, or r_{vD} is equal to r'_{vD}.*

(b) [Duffin (1965)] A network G is SP if and only if the network shown in Figure 2a is not embedded in it.

This proposition shows that series-parallel networks are those in which the network shown in Figure 2a, which is referred to as *Wheatstone network* (see Braess (1968)) is not embedded. LI networks, in addition, also exclude embeddings of series-parallel networks that have routes that "cross" as indicated in Figure 2b and Figure 2c.

We now introduce a new class of networks, which we refer to as *series of linearly independent networks* (SLI).

Definition 5 (Series of Linearly Independent Network (SLI)) A (two-terminal) network G is called series of linearly independent (SLI) if and only if

(i) it comprises of a single linearly independent network, or

(ii) it is constructed by connecting two SLI networks in series.

A biconnected LI network is called an *LI block*, where a graph is biconnected if it is connected and after removing any node and its incident edges the graph remains connected (see Bondy and Murty (1976, Chapter 3)). Equivalently, a network G is SLI if and only if it is constructed by connecting several LI blocks in series (see Appendix 9.2.1 for a formal proof). We refer to each of these blocks as an LI block of SLI network G.

We next provide a new characterization of SLI networks in terms of a graph embedding using the characterizations for SP and LI networks presented in Proposition 2.

Theorem 2 (Characterization of SLI) *A network G is SLI if and only if none of the networks shown in Figure 3 are embedded in it.*

The class of SLI networks is a subset of series-parallel networks and a superset of linearly independent networks. It plays an important role in our characterization of networks that exhibit IBP. Valdes et al. (1979) provided an algorithm to determine whether a given network is SP in $O(|\mathcal{E}| + |V|)$ steps based on a tree decomposition of SP networks. This leads to the question whether one can find a linear time algorithm (i.e., linear in the numbers of vertices and edges) to recognize an SLI network. We next use the results of Valdes et al. (1979) to show that we can recognize whether a given network is SLI in linear time.

Proposition 3 *There exists an algorithm that can determine whether a given network G is SLI in $O(|\mathcal{E}| + |V|)$.*

5 Informational Braess' Paradox

We first present the classical Braess' Paradox (BP) which is defined for a traffic network with single type of users with $\mathcal{E}_1 = \mathcal{E}$, denoted by $(G, \mathcal{E}_1, s_1, \mathbf{c})$.

Definition 6 (Braess' Paradox (BP)) Consider a traffic network with single information type $(G, \mathcal{E}_1, s_1, \mathbf{c})$. BP occurs if there exists another set of cost functions $\hat{\mathbf{c}}$ with $\hat{c}_e(x) \leq c_e(x)$ for all $e \in \mathcal{E}$ and $x \in \mathbb{R}^+$, such that the equilibrium cost of $(G, \mathcal{E}_1, s_1, \hat{\mathbf{c}})$ is strictly larger than the equilibrium cost of $(G, \mathcal{E}_1, s_1, \mathbf{c})$.

BP refers to an unexpected increase in equilibrium cost in response to a decrease in edge costs. We next discuss the Informational Braess' Paradox (IBP), which arises when providing more information to a subset of users in a traffic network increases those users' costs.

Definition 7 (Informational Braess' Paradox (IBP)) Consider a traffic network with multiple information types $(G, \mathcal{E}_{1:K}, s_{1:K}, \mathbf{c})$. IBP occurs if there exist expanded information sets $\tilde{\mathcal{E}}_{1:K}$ with $\mathcal{E}_1 \subset \tilde{\mathcal{E}}_1 \subseteq \mathcal{E}$ and $\tilde{\mathcal{E}}_i = \mathcal{E}_i$, for $i = 2, \ldots, K$, such that the equilibrium cost of type 1 in $(G, \tilde{\mathcal{E}}_{1:K}, s_{1:K}, \mathbf{c})$ is strictly larger than the equilibrium cost of type 1 in $(G, \mathcal{E}_{1:K}, s_{1:K}, \mathbf{c})$. We denote the equilibrium cost of type $i \in [K]$ before and after the expansion of information sets by $c^{(i)}$ and $\tilde{c}^{(i)}$, respectively.

The choice of type 1 users in this definition is without loss of generality, i.e., we assume that the information set of only one type expands and the information sets of the rest of the types remain the same. In comparing IBP to BP, first note that BP occurs in a network if and only if a special case of BP occurs in which we decrease the cost of one of the edges from infinity to its actual cost, i.e., equilibrium cost increases by adding a new edge to the network. The "if "part holds by definition and the "only if "part holds because the special case of BP occurs in Wheatstone network (as presented in Example 2(a)) and Wheatstone network is embedded in any network that features BP as shown by Milchtaich (2005). In light of this, it follows that the occurrence of IBP is a generalization of that of BP since addition of a new edge to the network can be viewed as expansion of the information set of a type to include that edge in a traffic network with single information type.

The next example shows that IBP occurs in all networks shown in Figure 3, i.e., all the basic networks that are embedded in non-SLI networks.

Example 2 In this example we will show that for all networks shown in Figure 3, there exists an assignment of cost functions along with information sets for which IBP occurs.

(a) IBP occurs for Wheatstone network shown in Figure 3a. This follows from the occurrence of BP on Wheatstone network as shown in Braess (1968). We will provide the example for the sake of completeness in Appendix 9.3.1.

(b) Consider the network shown in Figure 3b with cost functions given by $c_{e_1}(x) = \frac{1}{2}x$, $c_{e_2}(x) = x + \frac{3}{4}$, $c_{e_3}(x) = \frac{4}{3}x$, $c_{e_4}(x) = 2$ and $c_{e_5}(x) = x$. The information sets are

$\mathcal{E}_2 = \{e_1, e_4, e_5\}$, $\mathcal{E}_1 = \{e_2, e_3, e_5\}$, and $\tilde{\mathcal{E}}_1 = \{e_1, e_2, e_3, e_5\}$. For $s_1 = \frac{13}{4}$ and $s_2 = 1$, the equilibrium flows are

$$f^{(2)}_{e_1e_4} = 1, f^{(2)}_{e_5} = 0, f^{(1)}_{e_2e_3} = \frac{3}{4}, f^{(1)}_{e_5} = \frac{10}{4},$$
$$\tilde{f}^{(2)}_{e_1e_4} = 0, \tilde{f}^{(2)}_{e_5} = 1, \tilde{f}^{(1)}_{e_2e_3} = 0, \tilde{f}^{(1)}_{e_1e_3} = \frac{6}{4}, \tilde{f}^{(1)}_{e_5} = \frac{7}{4}.$$

The resulting equilibrium costs are $c^{(1)} = c^{(2)} = \frac{10}{4}$ and $\tilde{c}^{(1)} = \tilde{c}^{(2)} = \frac{11}{4}$. Since $\tilde{c}^{(1)} > c^{(1)}$, IBP occurs in this network. The main intuition for this example is as follows. After adding e_1 to $\mathcal{E}_1$, type 1 users will no longer use e_2e_3 and instead redirect part of their flow over e_1e_3. This in turn will increase the cost of e_1e_4 for type 2 users, and induce them to redirect all their flow from e_1e_4 to e_5. In balancing the cost of e_1e_3 and e_5 for type 1 users, their equilibrium cost goes up.

(c) Finally, for the networks shown in Figures 3c, 3d, 3e, 3f, 3g, 3h, and 3i, IBP occurs if we use the same setting as part (b) and include extra edges in all information sets with zero cost.

Remark 2 In Appendix 9.3.2, we show that Example 2(b) is not degenerate and provide an infinite set of (affine) cost functions for which IBP occurs in this network. Similar to Example 2(c), this argument extends to show that there are infinitely many cost functions for which IBP occurs in networks shown in Figures 3c,..., 3i. Finally, for the network shown in Figure 3a, there are infinitely many cost functions for which BP occurs when edge e_5 is added, hence IBP occurs as well (see e.g. Steinberg and Zangwill (1983)).

In a seminal paper, Milchtaich (2006) provided necessary and sufficient conditions on the network topology under which BP occurs. In particular, Milchtaich (2006) showed that for a given traffic network with single information type $(G, \mathcal{E}_1, s_1, \mathbf{c})$, BP does not occur if and only if G is SP. That is, if G is SP, then for any assignment of cost functions $\mathbf{c}$ and traffic demand, BP does not occur, and if G is not SP, then there exists an assignment of cost functions $\mathbf{c}$ for which BP occurs.

We next investigate conditions on the topology of the network under which IBP occurs. Similar to the characterization provided by Milchtaich (2006), we will identify classes of networks for which IBP does not occur regardless of the cost functions of the edges. Since, as already noted, IBP is a strict generalization of BP, we will see that IBP can occur in a broader class of networks, underscoring the problem mentioned in the introduction that IBP is likely to be a more pervasive problem.

6 Characterization of Informational Braess' Paradox

In this section, after establishing the key lemmas which underpin the rest of our analysis, we provide our main characterization of IBP. In Subsection 6.3 we provide a characterization for IBP for a more restricted type of change in information sets. We conclude this section with a discussion of extensions of our results to multiple origin-destination pairs.

6.1 Three Key Lemmas

The following lemmas identify properties of the traffic network consisting of heterogeneous users over an LI network.

Lemma 1 *(a) Given an LI network G , let $f^{(1:K)}$ and $\tilde{f}^{(1:K)}$ be two arbitrary non-identical feasible flows for two traffic networks $(G, \mathcal{E}_{1:K}, s_{1:K}, \mathbf{c})$ and $(G, \mathcal{E}_{1:K}, \tilde{s}_{1:K}, \mathbf{c})$, respectively. If $\sum_{i=1}^{K} s_i \geq \sum_{i=1}^{K} \tilde{s}_i$, then there exists a route r such that $\sum_{i=1}^{K} f_r^{(i)} > \sum_{i=1}^{K} \tilde{f}_r^{(i)}$ and $f_e \geq \tilde{f}_e$, for all $e \in r$.*

(b) Given an LI network G, let $\mathrm{c}^{(i)}$ and $\tilde{\mathrm{c}}^{(i)}$ denote the equilibrium cost of type $i \in [K]$ users in traffic networks $(G, \mathcal{E}_{1:K}, s_{1:K}, \mathbf{c})$ and $(G, \tilde{\mathcal{E}}_{1:K}, s_{1:K}, \mathbf{c})$, respectively. If $\mathcal{E}_1 \subseteq \tilde{\mathcal{E}}_1$ and $\tilde{\mathcal{E}}_i = \mathcal{E}_i$, for $i = 2, \ldots, K$, then there exists some $i \in [K]$ such that $\tilde{\mathrm{c}}^{(i)} \leq \mathrm{c}^{(i)}$.

This lemma directly follows from of Milchtaich (2006, Lemma 5 and Theorem 3). The first part of the this lemma shows that in an LI network, if the total traffic increases, then there exists at least one route whose flow strictly increases, and the flow on each of its edges does not decrease. The second part shows that in an LI network, if we expand the information set of type 1 users, then the equilibrium cost of at least one of the types does not increase. Note that this result is not sufficient for establishing that IBP does not occur over LI networks because what we need to establish is that it is the equilibrium cost of type 1 users that does not increase. For completeness, in Appendix 9.4.1, we show how this lemma follows from the results of Milchtaich (2006).

The next lemma shows a property of equilibrium flows and equilibrium costs in a network which is the result of attaching two networks in series. We use the following definition to state the lemma. Suppose $f^{(1:K)}$ is a feasible flow for $(G, \mathcal{E}_{1:K}, \mathbf{s}_{1:K}, \mathbf{c})$ where G is the result of attaching G_1 and G_2 in series. We denote the attaching point of G_1 and G_2 by D_1. The *restriction* of $f^{(1:K)}$ to G_1 (similarly to G_2) is defined as $\bar{f}^{(1:K)} = (\bar{f}^{(1)}, \ldots, \bar{f}^{(K)})$ where the flow of type i on any route $\bar{r}$ in G_1 is the summation of the flows of type i on all routes of G which contain $\bar{r}$. Formally, for any $i \in [K]$ we have $\bar{f}^{(i)}(\bar{r}) = \sum_{r \in \bar{\mathcal{R}}_i(\bar{r})} f^{(i)}(r)$, where $\bar{\mathcal{R}}_i(\bar{r}) = \{r \in \mathcal{R}_i \ : \ r_{OD_1} = \bar{r}\}$. Note that $\bar{f}^{(1:K)}$ is a feasible flow on G_1.

Lemma 2 *(a) If G is the result of attaching two networks G_1 and G_2 in series, then the restriction of an equilibrium flow for G to each of G_1 and G_2 is an equilibrium flow.*

(b) If G is the result of attaching two networks G_1 and G_2 in series, then the equilibrium cost of any type on G is the summation of the equilibrium costs of that type on G_1 and G_2.

The third lemma shows our key lemma that we will use in the proof of Theorem 3. Intuitively, this lemma states that in an LI network, if we decrease the traffic on one subset of routes $\mathcal{R}_A$ of the network and reroute it through the rest of the routes in the network, denoted by $\mathcal{R}_B = \mathcal{R} \setminus \mathcal{R}_A$, then the maximum cost improvement over all the routes in $\mathcal{R}_A$ cannot be smaller than the minimum cost improvement over all the routes in $\mathcal{R}_B$. This result will enable us to establish that in an LI or SLI network, the reallocation of traffic due to one class of users obtaining more information cannot harm that group.

Lemma 3 *Given an LI network G, we let $\mathcal{R}_A, \mathcal{R}_B \neq \emptyset$ denote a partition of routes $\mathcal{R}$, i.e., $\mathcal{R}_B = \mathcal{R} \setminus \mathcal{R}_A$. We let $f^{(1:K)}$ and $\tilde{f}^{(1:K)}$ be two feasible flows for traffic networks $(G, \mathcal{E}_{1:K}, s_{1:K}, \mathbf{c})$ and $(G, \mathcal{E}_{1:K}, \tilde{s}_{1:K}, \mathbf{c})$, respectively. For these two flows, we let the traffic over $\mathcal{R}_A$ and $\mathcal{R}_B$ be $s_A = \sum_{r \in \mathcal{R}_A} \sum_{i=1}^{K} f_r^{(i)}$, $\tilde{s}_A = \sum_{r \in \mathcal{R}_A} \sum_{i=1}^{K} \tilde{f}_r^{(i)}$, $s_B =$*

$\sum_{r\in\mathcal{R}_B}\sum_{i=1}^{K} f_r^{(i)}$, *and* $\tilde{s}_B = \sum_{r\in\mathcal{R}_B}\sum_{i=1}^{K} \tilde{f}_r^{(i)}$. *If* $\tilde{s}_A \le s_A$ *and* $\tilde{s}_B \ge s_B$, *then we have*

$$\max_{r\in\mathcal{R}_A}\{c_r - \tilde{c}_r\} \ge \min_{r\in\mathcal{R}_B}\{c_r - \tilde{c}_r\},$$

where for any route r, c_r *and* $\tilde{c}_r$ *denote the cost of this route with flows* $f^{(1:K)}$ *and* $\tilde{f}^{(1:K)}$, *respectively.*

Before proving this lemma for a general LI network, let us show it for the special case where G consists of parallel edges from O to D. In this case $\mathcal{R}_A$ and $\mathcal{R}_B$ are two disjoint sets of edges from O to D. Since $\tilde{s}_A \le s_A$, there exists an edge e_A in $\mathcal{R}_A$ such that $\tilde{f}_{e_A} \le f_{e_A}$. Similarly, since $\tilde{s}_B \ge s_B$, there exists $e_B \in \mathcal{R}_B$ such that $\tilde{f}_{e_B} \ge f_{e_B}$. Since the cost functions are nondecreasing, we have

$$\max_{r\in\mathcal{R}_A}\{c_r - \tilde{c}_r\} \ge c_{e_A}(f_{e_A}) - c_{e_A}(\tilde{f}_{e_A}) \ge 0 \ge c_{e_B}(f_{e_B}) - c_{e_B}(\tilde{f}_{e_B}) \ge \min_{r\in\mathcal{R}_B}\{c_r - \tilde{c}_r\},$$

which is the desired result. The proof for the general case is by induction on the number of edges and is included next.

Proof We first note a consequence of Proposition 2:

Claim 1: If a network G is LI then for any vertex such as v, either there exists a unique path from origin to v or there exists a unique path from v to destination. This claim follows since if the contrary holds, then there exists two paths from O to v denoted by r_{Ov} and r'_{Ov} and two paths from v to D denoted by r_{vD} and r'_{vD}. Therefore, the two routes $r = r_{Ov} + r_{vD}$ and $r' = r'_{Ov} + r'_{vD}$ have a common vertex v, but $r_{Ov} \ne r'_{Ov}$ and $r_{vD} \ne r'_{vD}$ which contradicts the statement of part (a) of Proposition 2.

We now prove Lemma 3 using induction on the number of edges. For a single edge it evidently holds. For a general LI network, we have the following cases:

- There exists $r \in \mathcal{R}_A$ such that $c_r \ge \tilde{c}_r$ and $r' \in \mathcal{R}_B$ such that $c_{r'} \le \tilde{c}_{r'}$. This leads to

  $$\max_{r\in\mathcal{R}_A}\{c_r - \tilde{c}_r\} \ge c_r - \tilde{c}_r \ge 0 \ge c_{r'} - \tilde{c}_{r'} \ge \min_{r\in\mathcal{R}_B}\{c_r - \tilde{c}_r\},$$

  which concludes the proof in this case.

- For any $r \in \mathcal{R}_A$, we have $c_r < \tilde{c}_r$. We break the proof into three steps.

  Step 1: There exists a route $r \in \mathcal{R}_A$ and an edge $e \in r$ with the following properties: (i) The flow on r from $\tilde{s}_A$ is less than or equal to the flow on r from s_A. (ii) The flow on e from $\tilde{s}_B$ is larger than the flow on e from s_B and the flow on e from $\tilde{s}_A$ is less than or equal to the flow on e from s_A.

  This step follows from invoking part (a) of Lemma 1. Since $\tilde{s}_A \le s_A$, using part (a) of Lemma 1 there exists a route $r \in \mathcal{R}_A$ such that the flow on each edge of r from $\tilde{s}_A$ is less than or equal to the flow from s_A. However, we know that the overall cost of any $r \in \mathcal{R}_A$ has gone up, i.e., $\tilde{c}_r > c_r$. This implies that there exists an edge $e \in r$ such that the flow from $\tilde{s}_B$ on e is more than the flow from s_B on e.

  Step 2: Let $\mathcal{R}_e$ denote the set of routes using edge $e = (u_e, v_e)$ as defined in Step 1. $\mathcal{R}_e$ has the following properties: (i) There exists a vertex $D' \in G$ such that all routes $r \in \mathcal{R}_e$ have a common path from O to v_e and a common path from D' to D. (ii) There exists a subnetwork G' with origin $O' = v_e$ and destination D' such that for the restricted parts of $\mathcal{R}_A$ and $\mathcal{R}_B$ over G', denoted by $\mathcal{R}'_A$ and $\mathcal{R}'_B$, if we let s'_A, $\tilde{s}'_A$, s'_B, and $\tilde{s}'_B$ to denote the corresponding traffic on $\mathcal{R}'_A$ and $\mathcal{R}'_B$, then we have $\tilde{s}'_A \leq s'_A$ and $\tilde{s}'_B \geq s'_B$.

Using Claim 1, for an edge $e = (u_e, v_e)$ either there is a unique path from O to v_e, or there is a unique path from v_e to D; we assume without loss of generality it is the former case. We let D' to be the first node on route r such that all routes in $\mathcal{R}_e$ coincide from D' to D. Therefore, all routes $r \in \mathcal{R}_e$ have a common path from O to v_e and a common path from D' to D, showing the first property.
We next show that the subnetwork consisting of all routes from v_e to D', denoted by G', satisfies the second property. To see this, note that the flows on G' are only the ones that are passing through edge e. From Step 1, we know that the flow on e from $\tilde{s}_B$ is larger than the flow on e from s_B and the flow on e from $\tilde{s}_A$ is less than or equal to the flow on e from s_A, showing the second property.

Step 3: Using steps 1 and 2, and induction hypothesis for G', we will show that

$$\max_{r\in\mathcal{R}_A}\{c_r - \tilde{c}_r\} \geq \min_{r\in\mathcal{R}_B}\{c_r - \tilde{c}_r\}.$$

First note that $\mathcal{R}_e \cap \mathcal{R}_A \neq \emptyset$ since $r \in \mathcal{R}_A$ and $e \in r$. Furthermore, $\mathcal{R}_e \cap \mathcal{R}_B \neq \emptyset$, since as explained in Step 1, the flow on e from $\tilde{s}_B$ is strictly positive. This in turn shows that $\mathcal{R}'_A$ and $\mathcal{R}'_B$ are nonempty. Using Step 2, all the conditions of Lemma 3 hold for subnetwork G'. Therefore, we can use the induction hypothesis for LI network G' to obtain

$$\max_{r\in\mathcal{R}'_A}\{c_r - \tilde{c}_r\} \geq \min_{r\in\mathcal{R}'_B}\{c_r - \tilde{c}_r\}.$$

Using Step 2, for all the routes in $\mathcal{R}_e$ the costs of going from O to O', denoted by $c_{O'\to O}$ are the same. Similarly, the costs of all routes in $\mathcal{R}_e$ going from D' to D, denoted by $c_{D'\to D}$ are the same. Therefore, we have

$$\max_{r\in\mathcal{R}_A}\{c_r - \tilde{c}_r\} \geq \max_{r\in\mathcal{R}_A\cap\mathcal{R}_e}\{c_r - \tilde{c}_r\} = (c_{O\to O'} - \tilde{c}_{O\to O'}) + \max_{r\in\mathcal{R}'_A}\{c_r - \tilde{c}_r\} + (c_{D'\to D} - \tilde{c}_{D'\to D})$$
$$\geq (c_{O\to O'} - \tilde{c}_{O\to O'}) + \min_{r\in\mathcal{R}'_B}\{c_r - \tilde{c}_r\} + (c_{D'\to D} - \tilde{c}_{D'\to D}) = \min_{r\in\mathcal{R}_B\cap\mathcal{R}_e}\{c_r - \tilde{c}_r\} \geq \min_{r\in\mathcal{R}_B}\{c_r - \tilde{c}_r\},$$

which concludes the proof in this case.

- For any $r \in \mathcal{R}_B$, we have $c_r > \tilde{c}_r$. The proof of this case is similar to the previous case. We state the three steps without repeating the reasoning for each of them.

  Step 1: There exists a route $r \in \mathcal{R}_B$ and an edge $e \in r$ with the following properties: (i) The flow on r from $\tilde{s}_B$ is larger than or equal to the flow on r from s_B. (ii) The flow on e from $\tilde{s}_A$ is smaller than the flow on e from s_A and the flow on e from $\tilde{s}_B$ is larger than or equal to the flow on e from s_B.

  Step 2: Let $\mathcal{R}_e$ denote the set of routes using edge $e = (u_e, v_e)$ as defined in Step 1. We have the following properties: (i) There exists a vertex $D' \in G$ such that all routes $r \in \mathcal{R}_e$ have a common path from O to v_e and a common path from D' to D. (ii) There exists a subnetwork G' with origin $O' = v_e$ and destination D' such that for the restricted parts of $\mathcal{R}_A$ and $\mathcal{R}_B$ over G', denoted by $\mathcal{R}'_A$ and $\mathcal{R}'_B$, if we let s'_A, $\tilde{s}'_A$, s'_B, and $\tilde{s}'_B$ to denote the corresponding traffic on $\mathcal{R}'_A$ and $\mathcal{R}'_B$, then we have $\tilde{s}'_A \leq s'_A$ and $\tilde{s}'_B \geq s'_B$.

Step 3: Again, using steps 1 and 2, and induction hypothesis for G', we have

$$\max_{r \in \mathcal{R}_A} \{c_r - \tilde{c}_r\} \geq \min_{r \in \mathcal{R}_B} \{c_r - \tilde{c}_r\},$$

which completes the proof.

■

6.2 Characterization of Informational Braess' Paradox

We next present our main result, which states that IBP does not occur if and only if the network is SLI. The idea of this result, as already discussed in the Introduction, is the following. To show the "if" part, we note using Lemma 2 that it suffices to show IBP does not occur in LI networks. Consider an expansion of the information set of type 1 and the new equilibrium flows. If the equilibrium cost of type 1 increases, then ICWE definition implies the following:

- Consider all types with increased equilibrium costs (including type 1). All routes used by these types (in the equilibrium before information expansion) have higher costs in the new equilibrium.
- Consider all types with decreased equilibrium costs. All routes used by these types (in the equilibrium after information expansion) have lower costs in the new equilibrium.

Using these two claims, it follows that the total flow sent over the routes with higher costs is lower, and the total flow sent over the routes with lower costs is higher (see Figure 4). Since the network is LI, Lemma 3 leads to a contradiction. The "only if" part holds because any non-SLI network embeds one of the networks shown in Figure 3, and an IBP can be constructed for each of them (Example 2) which then extends to an IBP for the non-SLI network.

Theorem 3 (Characterization of IBP) *IBP does not occur if and only if G is SLI. More specifically, we have the following.*

(a) If G is SLI, for any traffic network $(G, \mathcal{E}_{1:K}, s_{1:K}, \mathbf{c})$ with arbitrary assignment of cost functions $\mathbf{c}$, K, traffic demands $s_{1:K}$, and information sets $\mathcal{E}_{1:K}$, IBP does not occur.

(b) If G is not SLI, there exists an assignment of cost functions $\mathbf{c}$, K, traffic demands $s_{1:K}$, and information sets $\mathcal{E}_{1:K}$ in which IBP occurs.

Proof [Proof of part (a):] To reach a contradiction, suppose that $\tilde{c}^{(1)} > c^{(1)}$. By Definition 5, G is obtained from attaching several LI blocks in series, denoted by $G_1, \ldots, G_N$ for some $N \geq 1$. Using part (b) of Lemma 2, we have $\tilde{c}^{(1)} = \sum_{t=1}^{N} \tilde{c}_t^{(1)} > \sum_{t=1}^{N} c_t^{(1)} = c^{(1)}$, where $c_t^{(1)}$ denotes the equilibrium cost of type 1 users in G_t. Therefore, there exists one LI block such as j for which $\tilde{c}_j^{(1)} > c_j^{(1)}$. Also, using part(a) of Lemma 2, the restriction of equilibrium flows $f^{(1:K)}$ and $\tilde{f}^{(1:K)}$ to G_j creates an equilibrium flow for this LI block. Therefore, IBP occurs in LI block G_j. In the rest of the proof of part (a), we will assume IBP occurs in an LI block (and hence LI network) and reach a contradiction. We let $f^{(1:K)}$

and $\tilde{f}^{(1:K)}$ be the equilibrium flows before and after the information set expansion. Also, for any route $r \in \mathcal{R}$, we let c_r and $\tilde{c}_r$ to denote the cost of route r with flows $f^{(1:K)}$ and $\tilde{f}^{(1:K)}$, respectively.
We partition the set $[K]$ into groups A and B as follows

$$A = \{i \in [k] \ : \ \tilde{c}^{(i)} > c^{(i)}\},$$

and

$$B = \{i \in [k] \ : \ \tilde{c}^{(i)} \leq c^{(i)}\},$$

i.e., set A denotes all types with higher equilibrium cost in the game with higher information, and set B denotes the rest of the types.
We also partition the routes of the network into two subsets $\mathcal{R}_A$ and $\mathcal{R}_B$, where

$$\mathcal{R}_A = \{r \in \mathcal{R} \ : \ \tilde{c}_r > c_r\},$$

and

$$\mathcal{R}_B = \{r \in \mathcal{R} \ : \ \tilde{c}_r \leq c_r\},$$

i.e., $\mathcal{R}_A$ denotes all routes that have higher costs in the game with higher information, and $\mathcal{R}_B$ denotes the rest of the routes. We show the following claims:
Claim 1: For any type $i \in A$ and any route $r \in \mathcal{R}_B$, we have $f_r^{(i)} = 0$, i.e., for a given type i, if the equilibrium cost increases in the game with higher information, then the cost of all routes that type i was using (with strictly positive flow) also increases. This follows since if $r \notin \mathcal{R}_i$, then $f_r^{(i)} = 0$. Otherwise, $r \in \mathcal{R}_i$ which implies $r \in \tilde{\mathcal{R}}_i$ as well, where $\tilde{\mathcal{R}}_i$ denotes the set of available routes to type i in the expanded information set. Assuming $i \in A$ and $r \in \mathcal{R}_B$, we have

$$c_r \geq \tilde{c}_r \geq \tilde{c}^{(i)} > c^{(i)},$$

where the first inequality follows from the definition of the set $\mathcal{R}_B$. The second inequality follows from the definition of ICWE. The third inequality follows from the definition of set A. The overall inequality and the definition of ICWE show that $f_r^{(i)} = 0$.
Claim 2: For any type $i \in B$ and any route $r \in \mathcal{R}_A$, we have $\tilde{f}_r^{(i)} = 0$, i.e., for a given route, if the cost of the route in the equilibrium increases in the game with higher information, then the equilibrium costs of all types that are using this route in the equilibrium of the higher information game also increases. This follows since if $r \notin \tilde{\mathcal{R}}_i$, then $\tilde{f}_r^{(i)} = 0$. Otherwise, $r \in \tilde{\mathcal{R}}_i$ which implies $r \in \mathcal{R}_i$, because $1 \notin B$ and information set of all other types are fixed. Assuming $i \in B$ and $r \in \mathcal{R}_A$, we have

$$\tilde{c}_r > c_r \geq c^{(i)} \geq \tilde{c}^{(i)},$$

where the first inequality follows from the definition of the set $\mathcal{R}_A$. The second inequality follows from the definition of ICWE. The third inequality follows from the definition of set B. The overall inequality and the definition of ICWE show that $\tilde{f}_r^{(i)} = 0$.
Claim 3: Letting $s_A = \sum_{r \in \mathcal{R}_A} \sum_{i=1}^{K} f_r^{(i)}$, $\tilde{s}_A = \sum_{r \in \mathcal{R}_A} \sum_{i=1}^{K} \tilde{f}_r^{(i)}$, $s_B = \sum_{r \in \mathcal{R}_B} \sum_{i=1}^{K} f_r^{(i)}$, and $\tilde{s}_B = \sum_{r \in \mathcal{R}_B} \sum_{i=1}^{K} \tilde{f}_r^{(i)}$, we have $\tilde{s}_A \leq s_A$ and $\tilde{s}_B \geq s_B$.
This follows from Claims 1 and 2. The traffic on the routes in $\mathcal{R}_A$ from $f^{(1:K)}$ is s_A which is the entire traffic demand s_i for all $i \in A$ (Claim 1) and possibly some portion of the traffic demand s_j for $j \in B$. On the other hand, the traffic on the routes in $\mathcal{R}_A$ from $\tilde{f}^{(1:K)}$ is $\tilde{s}_A$, which contains only some portion of the traffic demand s_i for $i \in A$. Claim

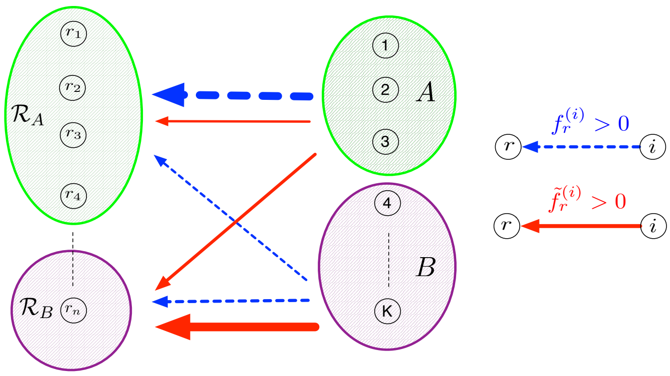


Figure 4: Proof of Theorem 3: set A (B) represents types with higher (lower) equilibrium costs and set $\mathcal{R}_A$ ($\mathcal{R}_B$) represents routes with higher (lower) costs. There is no dashed (blue) arrow from A to $\mathcal{R}_B$ which illustrates Claim 1 and there is no solid arrow (red) from B to $\mathcal{R}_A$ which illustrates Claim 2.

2 implies that for all $j \in B$ the traffic demand $\tilde{s}_j$ is only sent on the routes in $\mathcal{R}_B$. This shows that $\tilde{s}_A \leq s_A$ which in turn leads to $\tilde{s}_B \geq s_B$ (see Figure 4 for an illustration of the partitioning and the flows).

Part (b) of Lemma 1 shows that there exists type i for which $\tilde{c}^{(i)} \leq c^{(i)}$, which in turn shows that set B is nonempty. Also by the contradiction assumption $1 \in A$, which implies that both A and B are nonempty. Using Claim 1, if A is nonempty, then $\mathcal{R}_A \neq \emptyset$ as the flow $f^{(1:K)}$ of the types in A can only go to routes in $\mathcal{R}_A$. Also using Claim 2, since B is nonempty, we have $\mathcal{R}_B \neq \emptyset$ as the flow $\tilde{f}^{(1:K)}$ of the types in B can only go to routes in $\mathcal{R}_B$. Therefore, we have partitioned the routes of the network into two nonempty sets $\mathcal{R}_A$ and $\mathcal{R}_B$ such that $\tilde{c}_r > c_r$ for all $r \in \mathcal{R}_A$ and $\tilde{c}_r \leq c_r$ for all $r \in \mathcal{R}_B$. In other words, we have $\max_{r\in\mathcal{R}_A}\{c_r - \tilde{c}_r\} < 0$ and $\min_{r\in\mathcal{R}_B}\{c_r - \tilde{c}_r\} \geq 0$. We now have all the pieces to use Lemma 3 which yields to

$$0 > \max_{r\in\mathcal{R}_A}\{c_r - \tilde{c}_r\} \geq \min_{r\in\mathcal{R}_B}\{c_r - \tilde{c}_r\} \geq 0,$$

which is a contradiction, completing the proof of part (a). ■

Proof [Proof of part (b):] The proof of this part follows from Theorem 2. Let G be a non-SLI network. Using Theorem 2, one of the networks shown in Figure 3 must be embedded in G. Using Example 2, for all networks shown in Figure 3, there exists an assignment of cost functions and information sets for which IBP occurs.

To construct an example for G we start from the cost functions for which the embedded network features IBP (as shown in Example 2) and then following the steps of embedding, given in Definition 4, we will update the information sets as well as the cost functions in a way that IBP occurs in the final network G. The updates of information sets and cost functions are as follows.

(i) If the step of embedding is to divide an edge, we assign half of the original edge cost to each of the new edges and update the information set by adding both newly created edges to the same information set as of the original edge. This guarantees that the equilibrium flow of the network after dividing an edge is the same as the one before.

(ii) If the step of embedding is to add an edge, then we include that edge in none of the information sets (or equivalently assign cost infinity to it). This guarantees that the new edge is never used in any equilibrium.

(iii) If the step of embedding is to extend origin or destination, we let the cost of the new edge to be $c(x) = x$ The argument of and update all of the information sets by adding this edge to them. Since this edge will be used by all types and the flow on it will not change, this embedding does not affect the equilibrium flow.

This construction establishes that since IBP is present in the initial network, i.e., one of the networks shown in Figure 3, it will be present in the network G as well. This completes the proof of part (b). ■

Recall that in Remark 2 we showed for each of the networks shown in Figure 3 there exist infinitely many cost functions for which IBP occurs. This shows that if IBP occurs in a network, then it occurs for infinitely many cost functions. Because, if IBP occurs in a network G, Theorem 3 part (a) implies G is not SLI and Theorem 2 shows that one of the basic networks shown in Figure 3 is embedded in G. Finally, by construction of the proof of Theorem 3 part (b), the cost function configuration of the basic network can be extended to network G, showing that IBP occurs for infinitely many cost functions.

6.3 IBP with Restricted Information Sets

In this subsection, we show that restricting focus to networks with a much more specific information structure — whereby only one type does not know all the edges and the change in question informs this type of all edges — allows us to establish that IBP does not occur in a larger set of networks. Interestingly, in this case, IBP does not occur in exactly the same set of networks on which BP does not occur, series-parallel networks, though the two concepts continue to be very different even under this more specific information structure. The similarity is that after the change, as in the classic Wardrop Equilibrium setting studied for BP, there is no more heterogeneity among users. We first define IBP with restricted information sets and then state the characterization of network topology which leads to it.

Definition 8 (IBP with Restricted Information Sets) Consider a traffic network with multiple information types $(G, \mathcal{E}_{1:K}, s_{1:K}, \mathbf{c})$. IBP with restricted information sets occurs if there exist expanded information sets $\tilde{\mathcal{E}}_{1:K}$ with $\mathcal{E}_1 \subset \tilde{\mathcal{E}}_1 = \mathcal{E}$, and $\mathcal{E}_i = \tilde{\mathcal{E}}_i = \mathcal{E}$ for $i = 2, \dots, K$, such that the equilibrium cost of type 1 in $(G, \tilde{\mathcal{E}}_{1:K}, s_{1:K}, \mathbf{c})$ is strictly larger than the equilibrium cost of type 1 in $(G, \mathcal{E}_{1:K}, s_{1:K}, \mathbf{c})$. We denote the equilibrium cost of type $i \in [K]$ before and after expansion of information by $c^{(i)}$ and $\tilde{c}^{(i)}$, respectively.

Theorem 4 *IBP with restricted information sets does not occur if and only if the network G is SP. More specifically, we have the following.*

(a) If G is SP, for any network with multiple information sets $(G, \mathcal{E}_{1:K}, s_{1:K}, \mathbf{c})$ with arbitrary assignment of cost functions $\mathbf{c}$, K, traffic demands $s_{1:K}$, and information set $\mathcal{E}_1$, IBP with the restricted information sets does not occur.

(b) If G is not SP, there exists an assignment of cost functions $\mathbf{c}$, K, traffic demands $s_{1:K}$, and information set $\mathcal{E}_1$ in which IBP with restricted information sets occurs.

6.4 Extension to Multiple Origin-Destination Pairs

In this subsection, we consider networks with multiple information types and multiple origin-destination pairs as defined next.

Definition 9 Consider a graph $G = (V, \mathcal{E})$ containing m origin-destination pairs denoted by (O_i, D_i), $i \in [m]$. For any $i \in [m]$, there are K_i types of users, each with information set $\mathcal{E}_{i,j} \subseteq \mathcal{E}$, for $j \in [K_i]$. We refer to (i, j) as the type of a user where $i \in [m]$ denotes the origin-destination pair of this type and $j \in [K_i]$ represents its information set. The traffic network with multiple information types and multiple origin-destination pairs is denoted by $\left(G, \{\mathcal{E}_{i,1:K_i}\}_{i=1}^m, \{s_{i,1:K_i}\}_{i=1}^m, \mathbf{c}\right)$. We let $\mathcal{R}_{i,j}$ to denote the set of routes available to a user of type (i, j) (i.e., routes formed by edges in $\mathcal{E}_{i,j}$). A feasible flow is a flow vector $f = (f^{(1,1:K_1)}, \ldots, f^{(m,1:K_m)})$ such that $f^{(i,1:K_i)}$ is a feasible flow for origin-destination pair (O_i, D_i).

We denote the total flow on an edge e by f_e where $f_e = \sum_{i=1}^m \sum_{j=1}^{K_i} \sum_{r \in \mathcal{R}_{i,j} \,:\, e \in r} f_r^{(i,j)}$. Note that since G is an undirected graph, the total flow on each edge is the sum of the flows sent through that edge in either direction (see Lin et al. (2011) and Holzman and Monderer (2015)). The cost of a route r is defined as $c_r(f) = \sum_{e \in r} c_e(f_e)$. ICWE in this case is defined naturally as follows:

A feasible flow $f = (f^{(1,1:K_1)}, \ldots, f^{(m,1:K_m)})$ is an Information Constrained Wardrop Equilibrium (ICWE) if for every $i \in [m]$ and $j \in [K_i]$ and every pair $r, \tilde{r} \in \mathcal{R}_{i,j}$ with $f_r^{(i,j)} > 0$, we have

$$c_r(f) \leq c_{\tilde{r}}(f). \tag{6.1}$$

This implies that all routes of type (i, j) with positive flow have the same cost, which is smaller than or equal to the cost of any other route in $\mathcal{R}_{i,j}$. The equilibrium cost of type (i, j), denoted by $c^{(i,j)}$, is then given by the cost of any route in $\mathcal{R}_{i,j}$ with positive flow from type (i, j).

The existence of ICWE in this setting follows from an identical argument to that of Theorem 1. Finally, the definition of IBP for this extended setting is as follows.

Definition 10 (IBP with Multiple Origin-destination Pairs) Consider a traffic network with multiple information types and multiple origin-destination pairs $\left(G, \{\mathcal{E}_{i,1:K_i}\}_{i=1}^m, \{s_{i,1:K_i}\}_{i=1}^m, \mathbf{c}\right)$. IBP occurs if there exists an expanded information set $\left\{\tilde{\mathcal{E}}_{i,1:K_i}\right\}_{i=1}^m$ with $\mathcal{E}_{1,1} \subset \tilde{\mathcal{E}}_{1,1}$ and $\tilde{\mathcal{E}}_{i,j} = \mathcal{E}_{i,j}$, for all $(i, j) \neq (1, 1)$, $i \in [m]$, $j \in [K_i]$, such that the equilibrium cost of type $(1, 1)$ in $\left(G, \left\{\tilde{\mathcal{E}}_{i,1:K_i}\right\}_{i=1}^m, \{s_{i,1:K_i}\}_{i=1}^m, \mathbf{c}\right)$ is strictly larger than the equilibrium cost of type $(1, 1)$ in $\left(G, \{\mathcal{E}_{i,1:K_i}\}_{i=1}^m, \{s_{i,1:K_i}\}_{i=1}^m, \mathbf{c}\right)$.

Note that the choice of type $(1, 1)$ for information expansion is arbitrary and without loss of generality. We next establish a sufficient condition on the network topology under which IBP with multiple origin-destination pairs does not arise. We will use the following definitions from Chen et al. (2016).

Definition 11

- For any origin-destination pair (O_i, D_i), the *relevant network* i denoted by $G_i = (V_i, \mathcal{E}_i)$ consists of all edges and nodes of G that belong to at least one route from O_i to D_i in G.

- For an SLI network G_i, each LI block has two terminal nodes, an origin and a destination, such that the origin is the first node and the destination is the last node in the block visited on any route in G_i. For two SLI networks G_i and G_j, a *coincident LI block* is a common LI block of G_i and G_j with the same set of terminals, allowing origin of one to be the destination of other.

Based on this definition, we next provide a sufficient condition for excluding IBP.

Proposition 4 *Let G be a graph with $m \geq 1$ origin-destination pairs. For any $i \in [m]$, let $G_i = (V_i, \mathcal{E}_i)$ be the relevant network for origin-destination pair (O_i, D_i). IBP does not occur if the following two conditions hold.*

(a) For any $i \in [m]$, the network G_i is SLI.

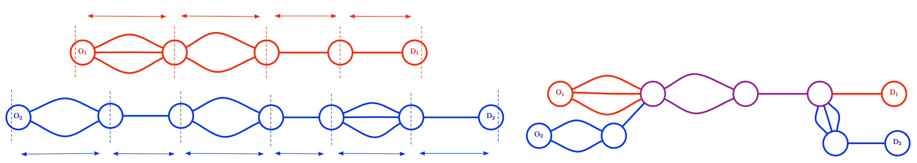
(b) For any $i, i' \in [m]$ either $\mathcal{E}_i \cap \mathcal{E}_{i'} = \emptyset$ or $\mathcal{E}_i \cap \mathcal{E}_{i'}$ consists of all coincident blocks of G_i and $G_{i'}$.

Proof We let f and $\tilde{f}$ denote the equilibrium flows before and after the expansion of information set of type $(1,1)$. To reach a contradiction suppose $\tilde{c}^{(1,1)} > c^{(1,1)}$. Using Lemma 2 part (b) and condition (a) of the proposition, the equilibrium cost of type $(1,1)$ users is the sum of equilibrium cost of the LI blocks of G_1. Since $\tilde{c}^{(1,1)} > c^{(1,1)}$, there exists an LI block of G_1 for which the equilibrium cost after expanding information set of type $(1,1)$ increases. We denote this LI block by G^* and its corresponding origin and destination by O^* and D^*. Using condition (b) for any $i \neq 1$ we have one of the following two cases: (i) G_i does not have any common edge with G^* and therefore none of the route flows of (O_i, D_i) go through any edge of G^* (ii) O^* and D^* belong to all routes of G_i and therefore all route flows of (O_i, D_i) go through G^*. We let C be the set of indices of such origin-destination pairs, i.e., $C = \{i \in [m] \ : \ O^*, D^* \in r, \forall r \in G_i\}$. We next define a traffic network with single origin-destination pair (O^*, D^*) over G^* for which IBP has occurred. The types of users are $\left(\bigcup_{i \in C}\{(i,j) \ : \ j \in [K_i]\}\right) \bigcup \{(1,j) \ : \ j \in [K_1]\}$ with their corresponding traffic demands. Note that for all $i \in C$, even though our definition of coincident LI block allows the route flows of (O_i, D_i) to go from $O*$ to D^* in either direction, without loss of generality, we can assume that route flows go from O^* to D^*. This is because the cost of any edge is a function of the sum of the flows that passes through that edge in either direction and reversing the flows does not change the equilibrium flows on edges. Using Lemma 2 part (a) the restriction of equilibrium flows f and $\tilde{f}$ to G^* are equilibrium flows for the congestion game with multiple information types and single origin-destination pair defined on G^*. Note that G^* is an LI network and the equilibrium cost of type $(1,1)$ users after expanding their information set has gone up, which is a contradiction using Theorem 3. ■

Figure 5a shows two SLI networks with their corresponding LI blocks and Figure 5b shows a graph with two origin-destination pairs which satisfies our sufficient condition.

The next example shows that the conditions of Proposition 4 are not necessary for non-occurrence of IBP.

Example 3 Consider the network G shown in Figure 6. The common LI block of relevant networks G_1 and G_2 is G itself which is not a coincident LI block as the sets of terminals of this block for G_1 and G_2 are different. Therefore, this network does not satisfy the conditions of Proposition 4. However, in Appendix 9.4.4 we show that for any set of edge cost functions, IBP does not occur in this network.



(a) (b)

Figure 5: (a) Two SLI networks with their corresponding LI blocks. (b) A graph with two OD pairs for which IBP does not occur.

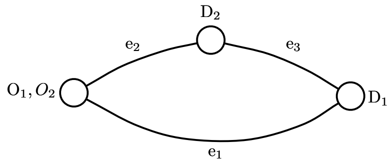


Figure 6: Example 3: IBP does not occur on this network.

In concluding this subsection we should note that BP with multiple origin destination pairs has been studied in Epstein et al. (2009); Lin et al. (2011); Fujishige et al. (2015); Holzman and Monderer (2015), and Chen et al. (2016). In particular, recently Chen et al. (2016) provides a full characterization of network topologies for which BP occurs with multiple origin-destination pairs. BP as defined in Chen et al. (2016) occurs if adding an edge (decreasing cost of an edge) increases the equilibrium cost of the users of one of the origin-destination pairs, even if that edge is never used by the users of that origin-destination pair. With this definition it is possible to have a network for which IBP does not occur while BP occurs. For instance, BP occurs in the network considered in Example 3 (see Chen et al. (2016)) while we showed IBP does not occur in this network.

7 Efficiency of Information Constrained Wardrop Equilibrium

In this section, we provide bounds on the inefficiency of ICWE. We show that the worst-case inefficiency remains the same as the standard Wardrop Equilibrium, even though our notion of ICWE is considerably more general than Wardrop Equilibrium since it allows for a rich amount of heterogeneity among users.

We start by defining the social optimum defined as the feasible flow vector that minimizes the total cost over all edges. We focus on aggregate efficiency loss defined as the ratio of total cost experienced by all users at social optimum and ICWE. We provide tight bounds on this measure of efficiency loss which are realized for different classes of cost functions. We also consider type-specific efficiency loss defined as the ratio of total cost experienced by type i users at social optimum and ICWE. We show that the bounds in this case are different from the ones in the standard Wardrop Equilibrium.

Given a traffic network with multiple information types $(G, \mathcal{E}_{1:K}, s_{1:K}, \mathbf{c})$, we define the social optimum, denoted by $f_{\text{so}}^{(1:K)} = (f_{\text{so}}^{(1)}, \ldots, f_{\text{so}}^{(K)})$ (or simply f_{so}), as the optimal solution of the following optimization problem:

$$\begin{aligned}
&\min \sum_{e\in\mathcal{E}} f_e\ c_e(f_e),\\
&f_e = \sum_{i=1}^{K} \sum_{r\in\mathcal{R}_i\ :\ e\in r} f_r^{(i)},\\
&\sum_{r\in\mathcal{R}_i} f_r^{(i)} = s_i,\ \text{and}\ f_r^{(i)} \geq 0 \text{ for all } r\in\mathcal{R}_i \text{ and } i.
\end{aligned} \tag{7.1}$$

This optimization problem minimizes the total cost over all edges incurred by all users of all types. Under the assumption that each cost function is continuous, it follows that the optimal solution of problem (7.1) and hence a social optimum always exists. We denote the total cost of a feasible flow $f^{(1:K)}$ by

$$C(f^{(1:K)}) \triangleq \sum_{e\in\mathcal{E}} f_e c_e(f_e).$$

Similarly, for a feasible flow $f^{(1:K)}$, we define the total cost incurred by type i users as

$$C^{(i)}(f^{(1:K)}) \triangleq \sum_{e\in\mathcal{E}} f_e^{(i)} c_e(f_e).$$

Consequently, we define the socially optimal cost of type i as $C_{\text{so}}^{(i)} = C^{(i)}(f_{\text{so}}^{(1:K)})$ for $i\in[K]$ and the overall cost (over all types) of social optimum as $C_{\text{so}} = C(f_{\text{so}}^{(1:K)})$. Similarly, we define equilibrium cost of type i as $C_{\text{cwe}}^{(i)} = C^{(i)}(f_{\text{cwe}}^{(1:K)})$ for $i\in[K]$ and the overall cost (over all types) of ICWE as $C_{\text{cwe}} = C(f_{\text{cwe}}^{(1:K)})$, where $f_{\text{cwe}}^{(1:K)}$ (or simply f_{cwe}) denotes an ICWE. Note that $C^{(i)}(f_{\text{cwe}}^{(1:K)})$ is different from equilibrium cost of type i denoted by $c^{(i)}$, as the latter notion is the cost per unit of flow and the former is the aggregate cost. The relation between these two is simply $C_{\text{cwe}}^{(i)} = s_i\ c^{(i)}$, $i\in[K]$.

The following result from Roughgarden and Tardos (2002) and Correa et al. (2005) presents bounds on the efficiency loss of Wardrop Equilibrium, which provides bounds on the efficiency loss of ICWE in a traffic network with single information type with $\mathcal{E}_1 = \mathcal{E}$ denoted by $(G, \mathcal{E}_1, s_1, \mathbf{c})$.

Proposition 5 (Roughgarden and Tardos (2002)) *Consider a traffic network with a single information type $(G, \mathcal{E}_1, s_1, \mathbf{c})$. Let f_{we} be a Wardrop Equilibrium and f_{so} be a social optimum. Then, we have*

(a) $\inf_{(G,\mathcal{E}_1,s_1,\mathbf{c}):\ c_e \text{ convex}} \frac{C_{\text{so}}}{C_{\text{we}}} = 0.$

(b) Suppose $c_e(x)$ is an affine function for all $e\in\mathcal{E}$. Then, we have $\frac{C_{\text{so}}}{C_{\text{we}}} \geq \frac{3}{4}$, and this bound is tight.

(c) Let $\mathcal{C}$ be a class of latency functions and let $\beta(\mathcal{C}) = \sup_{c\in\mathcal{C},\ x\geq 0} \beta(c,x)$, where

$$\beta(c,x) = \max_{z\geq 0} \frac{z\ (c(x) - c(z))}{x\ c(x)}.$$

Then, we have $\frac{C_{\text{so}}}{C_{\text{we}}} \geq 1 - \beta(\mathcal{C})$, and the bound is tight.

Our next result shows that Proposition 5 holds exactly for ICWE, indicating that within the class of heterogeneous, information-constrained traffic equilibria we consider, the worst-case scenario occurs for networks with homogeneous users.

Proposition 6 *Consider a traffic network with multiple information types* $(G, \mathcal{E}_{1:K}, s_{1:K}, \mathbf{c})$*. Let* f_{cwe} *be an ICWE and* f_{so} *be a social optimum. Then, we have*

(a) $\inf_{(G,\mathcal{E}_{1:K},s_{1:K},\mathbf{c}):\ c_e \text{ convex}} \frac{C_{\text{so}}}{C_{\text{cwe}}} = 0.$

(b) Suppose $c_e(x)$ *is an affine function for all* $e \in \mathcal{E}$*. Then, we have* $\frac{C_{\text{so}}}{C_{\text{cwe}}} \geq \frac{3}{4}$*, and this bound is tight.*

(c) Let $\mathcal{C}$ *be a class of latency functions and let* $\beta(\mathcal{C}) = \sup_{c\in\mathcal{C},x\geq 0} \beta(c, x)$*, where*

$$\beta(c, x) = \max_{z\geq 0} \frac{z\ (c(x) - c(z))}{x\ c(x)}.$$

Then, we have $\frac{C_{\text{so}}}{C_{\text{cwe}}} \geq 1 - \beta(\mathcal{C})$*, and the bound is tight.*

Proof We first show that for any type i, and any feasible flow $f^{(i)}$ for this type, we have

$$\sum_{e\in\mathcal{E}} c_e(f_{e,\text{cwe}})({f_{e,\text{cwe}}}^{(i)} - f_e^{(i)}) \leq 0. \tag{7.2}$$

The reason is that in ICWE each type uses only the routes with the minimal costs. Therefore, for any type i and any feasible flow $f^{(i)}$ for type i, we have

$$\sum_{r\in\mathcal{R}_i} c_r(f_{\text{cwe}}^{(1:K)}) f_{r,\text{cwe}}^{(i)} \leq \sum_{r\in\mathcal{R}_i} c_r(f_{\text{cwe}}^{(1:K)}) f_r^{(i)}.$$

This leads to

$$\begin{aligned}
0 &\geq \sum_{r\in\mathcal{R}_i} c_r(f_{\text{cwe}}^{(1:K)})(f_{r,\text{cwe}}^{(i)} - f_r^{(i)}) = \sum_{r\in\mathcal{R}_i} \left(\sum_{e:e\in r} c_e(f_{e,\text{cwe}}) \right) (f_{r,\text{cwe}}^{(i)} - f_r^{(i)}) \\
&= \sum_{e\in\mathcal{E}} c_e(f_{e,\text{cwe}}) \sum_{r\in\mathcal{R}_i:\ e\in r} (f_{r,\text{cwe}}^{(i)} - f_r^{(i)}) = \sum_{e\in\mathcal{E}} c_e(f_{e,\text{cwe}})(f_{e,\text{cwe}}^{(i)} - f_e^{(i)}),
\end{aligned}$$

which is the desired inequality, showing Equation (7.2). We next proceed with the proof.
Part (a): this holds because a traffic network with one type is a special case of traffic network with multiple information types and part (a) of Proposition 5 shows the infimum is zero.
Part (b): using Equation (7.2) for $f^{(i)} = f_{\text{so}}^{(i)}$ for any $i \in [K]$, and taking summation over all types $i \in [K]$, we obtain

$$\begin{aligned}
C_{\text{cwe}} = \sum_{e\in\mathcal{E}} f_{e,\text{cwe}} c_e(f_{e,\text{cwe}}) &= \sum_{i=1}^{K} \sum_{e\in\mathcal{E}} c_e(f_{e,\text{cwe}}) f_{e,\text{cwe}}^{(i)} \leq \sum_{i=1}^{K} \sum_{e\in\mathcal{E}} c_e(f_{e,\text{cwe}}) f_{e,\text{so}}^{(i)} \\
&= \sum_{e\in\mathcal{E}} c_e(f_{e,\text{cwe}}) \sum_{i=1}^{K} f_{e,\text{so}}^{(i)} = \sum_{e\in\mathcal{E}} f_{e,\text{so}} c_e(f_{e,\text{cwe}}) \\
&= \sum_{e\in\mathcal{E}} f_{e,\text{so}} c_e(f_{e,\text{so}}) + \sum_{e\in\mathcal{E}} f_{e,\text{so}} \left(c_e(f_{e,\text{cwe}}) - c_e(f_{e,\text{so}})\right) \\
&\leq \sum_{e\in\mathcal{E}} f_{e,\text{so}} c_e(f_{e,\text{so}}) + \frac{1}{4} \sum_{e\in\mathcal{E}} f_{e,\text{cwe}} c_e(f_{e,\text{cwe}}),
\end{aligned}$$

where the last inequality comes from the fact that with $c_e(x) = a_e x + b_e$ for $b_e, a_e \geq 0$, we have

$$f_{e,\text{so}}\left(c_e(f_{e,\text{cwe}}) - c_e(f_{e,\text{so}})\right) = a_e f_{e,\text{so}}(f_{e,\text{cwe}} - f_{e,\text{so}}) \leq \frac{1}{4} f_{e,\text{cwe}}^2 a_e \leq \frac{1}{4} f_{e,\text{cwe}} c_e(f_{e,\text{cwe}}).$$

The proof of tightness follows from part (b) of Proposition 5 as a traffic network with one type is a special case of a traffic network with multiple information types.
Part (c): using the same argument as in part (b), we obtain

$$\begin{aligned} C_{\text{cwe}} = \sum_{e\in\mathcal{E}} f_{e,\text{cwe}} c_e(f_{e,\text{cwe}}) &\leq \sum_{e\in\mathcal{E}} f_{e,\text{so}} c_e(f_{e,\text{so}}) + \sum_{e\in\mathcal{E}} f_{e,\text{so}}\left(c_e(f_{e,\text{cwe}}) - c_e(f_{e,\text{so}})\right) \\ &\leq \sum_{e\in\mathcal{E}} f_{e,\text{so}} c_e(f_{e,\text{so}}) + \beta(\mathcal{C}) \sum_{e\in\mathcal{E}} f_{e,\text{cwe}} c_e(f_{e,\text{cwe}}), \end{aligned}$$

where the last inequality comes from the fact that

$$f_{e,\text{so}}\left(c_e(f_{e,\text{cwe}}) - c_e(f_{e,\text{so}})\right) \leq \beta(c_e, f_{e,\text{cwe}}) f_{e,\text{cwe}} c_e(f_{e,\text{cwe}}) \leq \beta(\mathcal{C}) f_{e,\text{cwe}} c_e(f_{e,\text{cwe}}).$$

The proof of the tightness follows from part (c) of Proposition 5. ■

In concluding this section, we should note that in this environment with heterogeneous users, there are alternatives to our formulation of the social optimum problem, which considers the "utilitarian" social optimum, summing over the costs of all groups. An alternative would be to consider a weighted sum or focus on the class of users suffering the greatest costs. We next illustrate that if we focus on type-specific costs, even with affine cost functions, some groups of users may have worse than 3/4 performance relative to the social optimum.

Example 4 Consider the network shown in Figure 7 with $\mathcal{E}_1 = \{e_1\}$, $\mathcal{E}_2 = \{e_1, e_2\}$. The ICWE is $f^{(1)}_{e_1,\text{cwe}} = s_1$ and $f^{(2)}_{e_1,\text{cwe}} = \frac{1}{a} - s_1$, $f^{(2)}_{e_2,\text{cwe}} = s_2 - \frac{1}{a} + s_1$. The equilibrium costs are $C^{(1)}_{\text{cwe}} = s_1$ and $C^{(2)}_{\text{cwe}} = s_2$. The social optimum is $f^{(1)}_{e_1,\text{so}} = s_1$ and $f^{(2)}_{e_1,\text{so}} = \frac{1}{2a} - s_1$, $f^{(2)}_{e_2,\text{so}} = s_2 - \frac{1}{2a} + s_1$. The corresponding costs are $C^{(1)}_{\text{so}} = \frac{s_1}{2}$ and $C^{(2)}_{\text{so}} = \frac{-1}{4a} + s_2 + \frac{s_1}{2}$ (assuming $\frac{1}{2a} \geq s_1$ and $s_2 \geq \frac{1}{a} - s_1$). Therefore, we have

$$\frac{C^{(1)}_{\text{so}}}{C^{(1)}_{\text{cwe}}} = \frac{1}{2}, \quad \frac{C^{(2)}_{\text{so}}}{C^{(2)}_{\text{cwe}}} = \frac{\frac{-1}{4a} + s_2 + \frac{s_1}{2}}{s_2}, \quad \text{and} \quad \frac{C^{(1)}_{\text{so}} + C^{(2)}_{\text{so}}}{C^{(1)}_{\text{cwe}} + C^{(2)}_{\text{cwe}}} = \frac{s_1 + s_2 - \frac{1}{4a}}{s_1 + s_2}.$$

We next show that the ratio of the aggregate costs is greater than or equal to 3/4. We have $s_1 + s_2 \geq \frac{1}{a}$ which leads to

$$\frac{C_{\text{so}}}{C_{\text{cwe}}} = \frac{s_1 + s_2 - \frac{1}{4a}}{s_1 + s_2} = 1 - \frac{1}{4a}\frac{1}{s_1 + s_2} \geq 1 - \frac{1}{4a} a = \frac{3}{4}.$$

However, the type-specific efficiency loss can be smaller than $\frac{3}{4}$ as we have $\frac{C^{(1)}_{\text{so}}}{C^{(1)}_{\text{cwe}}} < \frac{3}{4}$.

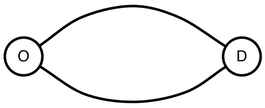


Figure 7: Example 4, type-specific efficiency loss versus aggregate efficiency loss.

8 Concluding Remarks

GPS-based route guidance systems, such as Waze or Google maps, are rapidly spreading among drivers because of their promise of reduced delays as they inform their users about routes that they were not aware of or help them choose dynamically between routes depending on recent levels of congestion. Nevertheless, there is no systematic analysis of the implications for traffic equilibria of additional information provided to subsets of users. In this paper, we systematically studied this question. We first extended the class of standard congestion games used for analysis of traffic equilibria to a setting where users are heterogeneous because of their different information sets about available routes. In particular, each user's information set contains information about a subset of the edges in the entire road network, and drivers can only utilize routes consisting of edges that are in their information sets. We defined the notion of Information Constrained Wardrop Equilibrium (ICWE), an extension of the classic Wardrop Equilibrium notion, and established the existence and essential uniqueness of ICWE.

We then turned to our main focus, which we formulate in the form of Informational Braess' Paradox (IBP). IBP asks that whether users receiving additional information can become worse off. Our main result is a comprehensive answer to this question. We showed that in any network in the Series of Linearly Independent (SLI) class, which is a strict subset of series-parallel network, IBP cannot occur, and in any network that is not in the SLI class, there exists a configuration of edge-specific cost functions for which IBP will occur. The SLI class is comprised of networks that join linearly independent networks in series, and linearly independent networks are those for which every path between origin and destination contains at least one edge that is not in any other such path. This is the property that enables us to prove that IBP cannot occur in any SLI network. We also showed that any network that is not in the SLI class necessarily embeds at least one of a specific set of basic networks, and then used this property to show that IBP will occur for some cost configurations in any non-SLI network. We further proved that whether a given network is SLI can be determined in linear time. Finally, we also established that the worst-case inefficiency performance of ICWE is no worse than the standard Wardrop Equilibrium with one type of users.

There are several natural research directions which are opened up by our study. These include:

- Our analysis focused on the effect of additional information on the set of users receiving the information. For what classes of networks is additional information very harmful for other users? This question is important from the viewpoint of fairness and other social objectives. We may like that users utilizing route guidance systems are experiencing lower delays, but not if this comes at the cost of significantly longer delays for others.

- How "likely" are the cost function configurations that will cause IBP to occur in non-SLI networks. This question is important for determining, ex-ante before knowing the exact traffic flows, whether additional information for some sets of users, coming for example from route guidance systems, might be harmful.

- Is there an "optimal information" configuration for users of a traffic network? Specifically, one could consider the following question: given the traffic demands of K types, $s_1, \ldots, s_K$, find the information sets $\mathcal{E}_1, \ldots, \mathcal{E}_K$ that generate the minimum overall cost for all types in an ICWE. This question is related to Roughgarden (2001, 2006) who investigate the question of finding the subnetwork of the initial network that leads to optimal equilibrium cost with one type of user.

- We established a sufficient condition under which IBP does not occur on a traffic network with multiple origin-destination pairs. One natural question is to find sufficient and necessary condition for this problem.

- Finally, our study poses an obvious empirical question, complementary to similar studies for the Braess' Paradox: are there real-world settings where we can detect IBP?

Acknowledgment

We would like to thank anonymous reviewers for their useful comments and suggestions.

9 Appendix

9.1 Proofs from Section 3

9.1.1 Proof of Proposition 1

Since for any $e \in \mathcal{E}$ the function $c_e(\cdot)$ is nondecreasing, $\int_0^{f_e} c_e(z)dz$ as a function of $f_r^{(i)}$ is convex and continuously differentiable.
Claim 1: If $f^{(1:K)}$ is an optimal solution of (3.1), then it is an ICWE.
Since the objective function is convex and the constraints are affine functions, regularity conditions holds and KKT conditions are satisfied, i.e., there exists $\mu_{i,r} \leq 0$ and λ_i such that for all $i \in [K]$ and $r \in \mathcal{R}_i$ we have

$$\frac{\partial}{\partial f_r^{(i)}} \left(\sum_{e\in\mathcal{E}} \int_0^{f_e} c_e(z)dz - \sum_{i=1}^{K} \lambda_i \left(\sum_{r\in\mathcal{R}_i} f_r^{(i)} - s_i \right) + \sum_{r,i} \mu_{r,i} f_r^{(i)} \right) = 0, \tag{9.1}$$

where $\mu_{r,i} = 0$ for $f_r^{(i)} > 0$ Bertsekas (1999, Chapter 3). We show that the flow $f^{(1:K)}$ is an ICWE with the equilibrium cost of type i being λ_i. First, note that $f^{(1:K)}$ is a feasible flow by the constraints of (3.1). Second, we can rewrite (9.1) as

$$\sum_{e\in\mathcal{E}} \frac{\partial f_e}{\partial f_r^{(i)}} c_e(f_e) = \sum_{e\in\mathcal{E}\,:\,e\in r} c_e(f_e) = \begin{cases} = \lambda_i & \text{if } f_r^{(i)} > 0, \\ \geq \lambda_i & \text{if } f_r^{(i)} = 0, \end{cases} \tag{9.2}$$

where we used $\mu_{r,i} = 0$ for $f_r^{(i)} > 0$ in the first case and $\mu_{r,i} \leq 0$ for $f_r^{(i)} = 0$ in the second case. This is exactly the definition of ICWE which completes the proof of Claim 1.

Claim 2: If $f^{(1:K)}$ is an ICWE, then it is an optimal solution of (3.1).
We let the equilibrium cost of type i users be λ_i which leads to the following relation

$$\sum_{e\in\mathcal{E}\,:\,e\in r} c_e(f_e) = \begin{cases} = \lambda_i & \text{if } f_r^{(i)} > 0, \\ \geq \lambda_i & \text{if } f_r^{(i)} = 0. \end{cases} \tag{9.3}$$

For all $i \in [K]$ and $r \in \mathcal{R}_i$, if $f_r^{(i)} > 0$, then we define $\mu_{i,r} = 0$ and if $f_r^{(i)} = 0$, then we define $\mu_{i,r} = \lambda_i - \sum_{e\in\mathcal{E}\,:\,e\in r} c_e(f_e)$. First, note that $\mu_{i,r} \leq 0$ and if $f_r^{(i)} > 0$, then $\mu_{i,r} = 0$. Second, note that

$$\frac{\partial}{\partial f_r^{(i)}}\left(\sum_{e\in\mathcal{E}}\int_0^{f_e} c_e(z)dz - \sum_{i=1}^{K}\lambda_i\left(\sum_{r\in\mathcal{R}_i} f_r^{(i)} - s_i\right) + \sum_{r,i}\mu_{r,i} f_r^{(i)}\right) = 0. \tag{9.4}$$

Therefore, the flow $f^{(1:K)}$ together with λ_i and $\mu_{i,r}$ satisfy the KKT conditions. Since the objective function of (3.1) is convex and the constraints are affine functions, KKT conditions are sufficient for optimality Bertsekas (1999, Chapter 3), proving the claim.

9.1.2 Proof of Theorem 1

The set of feasible flows $f^{(1:K)}$ is a compact subset of $K|\mathcal{R}|$-dimensional Euclidean space. Since edge cost functions are continuous, the potential function is also continuous. Weierstrass extreme value theorem establishes that optimization problem (3.1) attains its minimum which by Proposition 1 is an ICWE.
We next, show that in two different equilibria $f^{(1:K)}$ and $\tilde{f}^{(1:K)}$, the equilibrium cost for each type is the same. By Proposition 1, both $f^{(1:K)}$ and $\tilde{f}^{(1:K)}$ are optimal solutions of (3.1). Since $\Phi(\cdot)$ is a convex function, we have

$$\Phi\left(\alpha f^{(1:K)} + (1-\alpha)\tilde{f}^{(1:K)}\right) \leq \alpha\Phi\left(f^{(1:K)}\right) + (1-\alpha)\Phi\left(\tilde{f}^{(1:K)}\right),$$

for any $\alpha \in [0,1]$. Since $\Phi\left(f^{(1:K)}\right)$ and $\Phi\left(\tilde{f}^{(1:K)}\right)$ are both equal to optimal value of (3.1), and for each e, the function $\int_0^{f_e} c_e(z)dz$ is convex (its derivative with respect to f_e is $c_e(f_e)$ which is non-decreasing), the functions $\int_0^{f_e} c_e(z)dz$ for any $e \in \mathcal{E}$ must be linear between values of f_e and $\tilde{f}_e$. This shows that all cost functions c_e are constant between f_e and $\tilde{f}_e$ and in particular the equilibrium costs are the same.

9.2 Proofs of Section 4

9.2.1 Proof of Equivalence in Definition 5

We first show that each LI network G is the result of attaching several LI blocks in series. This follows by induction on the number of edges. Using Definition 3, G is either the result of attaching two LI networks in parallel or the result of attaching an LI network and a single edge in series. If G is the result of attaching two LI networks in parallel, then G is biconnected and so is an LI block. If G is the result of attaching an LI network G_1 with a single edge, then the single edge is an LI block and by induction hypothesis G_1 is series of several LI blocks. Therefore, G is the result of attaching several LI blocks in series.

We next show that the following two definitions are equivalent.

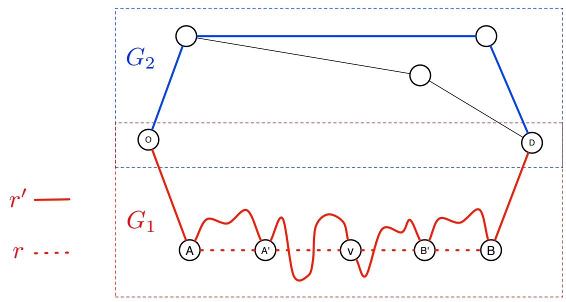


Figure 8: Proof of Theorem 2: G_1 is not LI and G_2 has at least one route from O to D.

- An SLI network is either a single LI network or the connection of two SLI networks in series. We let SET1 to denote the set of such networks.
- An SLI network consists of attaching several LI blocks in series. We let SET2 to denote the set of such networks.

We show that SET1=SET2 by induction on the number of edges, i.e., we suppose that for any network with number of edges less than or equal to m these two sets are equal and then show that for networks with $m+1$ edges the two sets are equal as well (note that the base of this induction for $m=1$ corresponds to a single edge which evidently holds).

- If a network G belongs to SET1, then either it is a single LI network or is the result of attaching two SLI networks in series. In the former case, it belongs to SET2 as we have shown each LI network is the result of attaching several LI blocks. In the latter case, by induction hypothesis both SLI subnetworks are the series of several LI blocks and so is their attachment in series. This shows SET1 $\subseteq$ SET2.
- If a network G belongs to SET2, then either it is a single LI block or is the result of attaching several LI blocks in series. In the former case, by definition it belongs to SET1. In the latter case, we let G_1 to denote the LI block that contains origin and the series of the rest of LI blocks by G_2. G_1 is SLI by definition as it is a single LI block and G_2 is SLI by induction hypothesis. Therefore, the series attachment of G_1 and G_2 belongs to SET1. This shows SET2 $\subseteq$ SET1, completing the proof.

9.2.2 Proof of Theorem 2

We first show that if a network G belongs to the class SLI, then none of the networks shown in Figure 3 is embedded in it. First note that since all networks in the class SLI are series-parallel, using part (b) of Proposition 2 implies that the Wheatstone network shown in Figure 3a is not embedded in it. The SLI network G consists of several LI blocks that are attached in series. Using part (a) of Proposition 2 none of the networks shown in Figure 3 (i.e., networks shown in Figs. 3b, 3c, 3d, 3e, 3f, 3g, 3h, and 3i) can be embedded in one of the LI blocks. We next show that they cannot be embedded in the series of two LI blocks as well. We let G_1 and G_2 be two LI blocks that are attached in series where the resulting network from this attachment is H. Also, we let the node c be the attaching node of these two networks. We will show that the network shown in Figure 3b can not be embedded in H (a similar argument shows that the rest of the networks shown in Figure 3 cannot be embedded in it). In order to reach to a contradiction, we suppose

the contrary, i.e., H is obtained from the network shown in Figure 3b by applying the embedding procedure described in Definition 4. We define the corresponding routes to e_5, e_1e_4, and e_2e_3 in H by r_3, r_1 and r_2. Formally, we start from $r_3 = e_5$, $r_1 = e_1e_4$, and $r_2 = e_2e_3$ in the network shown in Figure 3b and at each step of the embedding procedure whenever we divide an edge on r_i ($i = 1, 2, 3$) we will update r_i by adding that edge and whenever we extend origin or destination we will add the new edge to all r_is. Given this construction, in the network H we have three routes r_3, r_1, and r_2, where r_1 and r_2 have a common node and they do not have any common node (except O and D) with r_3. This is a contradiction as all routes in H must have node c in common. This completes the proof of the first part.
We next show that if none of networks shown in Figure 3 is embedded in G, then G belongs to the class SLI. Proposition 2(b) implies that since Figure 3a is not embedded in G, it is series-parallel. We next show that given a series-parallel network G, if G is not SLI then we can find an embedding of one of networks shown in Figures 3b, $\cdots$,3i in it. The proof is by induction on the number of edges of G. Following Definition 2, consider the last building step of the network G. If the last step, is attaching two networks G_1 and G_2 in series, then assuming that G is not SLI, we conclude that either G_1 or G_2 (or both) is not SLI. Therefore, by induction hypothesis, we can find an embedding of one of the networks shown in Figures 3b,$\cdots$, 3i in either G_1 or G_2, which in turn shown it is embedded in G. If the last step, is attaching two networks G_1 and G_2 in parallel, then it must be the case that that either G_1 or G_2 is not LI. Because otherwise the parallel attachment of two LI networks is LI (Definition 3) and hence SLI, which contradicts the fact that G is not SLI. Without loss of generality, we let the network that is not LI to be G_1. Therefore, part (a) of Proposition 2 shows that there exist two routes r and r' and a vertex v common to both routes such that both sections r_{Ov} and r'_{Ov} as well as r_{vD} and r'_{vD} are not equal (note that $v \notin \{O, D\}$ because otherwise if $v = O$, then $r_{Ov} = r'_{Ov}$ as both are the single node O).

We let A be the last vertex before which the two routes r and r' become the same (this vertex can be O itself). Since v is the common vertex of theses two routes and $r_{Ov} \neq r'_{Ov}$ such a vertex exists. Because v is a common vertex of r and r' the two routes r and r' have a common vertex between A and v. We let A' to be the first such vertex (it can be v itself). Similarly, we define B as the first vertex after which r and r' become the same (B can be D itself) and B' as the last vertex after v for which r and r' coincide (B' can be v itself). Given these definitions for the nodes v, A, A', B, and B', we know that $r_{AA'}$ (the path between A and A' on r) and $r'_{AA'}$ (the path between A and A' on r') do not have any vertex in common and similarly $r_{BB'}$ and $r'_{BB'}$ do not have any vertex in common. The definition of the nodes A, A', B, B' is illustrated in Figure 8. Next, we show that one of the networks shown in Figures 3b,..., 3i is embedded in G. We have the following cases:

- $A = O$, $B = D$, $A' = v$, and $B' = v$: in this case, the network shown in Figure 3b is embedded in G. This is because there are two disjoint paths from O to v and from v to D and there is at least one path from O to D in G_2. Since any other edge and vertex of the network belongs to a path that connects O to D, we can construct the graph G by starting from the network shown in Figure 3b and applying the embedding procedure.

- $A = O$, $B = D$, and $A' \neq v$ or $B' \neq v$: in this case, the network shown in Figure 3c is embedded in G. This is because there is at least one path from O to D in G_2

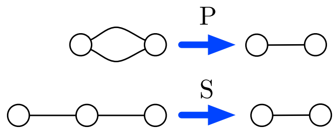


Figure 9: Two operations that turns a series-parallel network into a single edge.

and the network shown in Figure 3c is embedded in G_1. To see this, note that the edges e_1 and e_2 are embedded in the section of the routes r and r' between O and A', and the edges e_3 and e_4 are embedded in the the section of the routes r and r' between B' and D. Also, note that the single edge e_6 is embedded in the network between A' and B' (single edge is embedded in any network).

- $A \neq O$, $B = D$, and $A' = v$ and $B' = v$: the network shown in Figure 3d is embedded in G.
- $A = O$, $B \neq D$, and $A' = v$ and $B' = v$: the network shown in Figure 3e is embedded in G.
- $A = O$, $B \neq D$, and $A' \neq v$ or $B' \neq v$: the network shown in Figure 3f is embedded in G.
- $A \neq O$, $B = D$, and $A' \neq v$ or $B' \neq v$: the network shown in Figure 3g is embedded in G.
- $A \neq O$, $B \neq D$, and $A' \neq v$ or $B' \neq v$: the network shown in figure 3h is embedded in G.
- $A \neq O$, $B \neq D$, and $A' = v$ and $B' = v$: the network shown in Figure 3i is embedded in G.

This completes the proof.

9.2.3 Proof of Proposition 3

We use the following results and definitions in this proof.

Proposition 7 (Valdes et al. (1979)) *A network is series-parallel if following the steps S and P shown in Figure 9 in any order, turns the network into a single edge connecting origin to destination. Moreover, if a network is series-parallel, then in linear time $O(|\mathcal{E}| + |V|)$ we can obtain a binary tree decomposition (shown in Figure 10) which indicates a sequence of S and P that turns G into a single edge.*

We now proceed with the proof of Proposition 3. Using Proposition 7, we first verify whether G is series-parallel or not, which can be done in linear time. If G is not series-parallel, then it is not SLI as well. If G is series-parallel, then a binary tree decomposition can be obtained in linear time (again using Proposition 7). Note that the binary tree decomposition is not unique and the following argument works with any binary tree decomposition. In this tree the edges of G are represented by the leaves of the tree. We label the incident edges to origin by O and the incident edges to destination by D (an edge might be labeled both O and D). Since G is SP, by definition it is the result of attaching

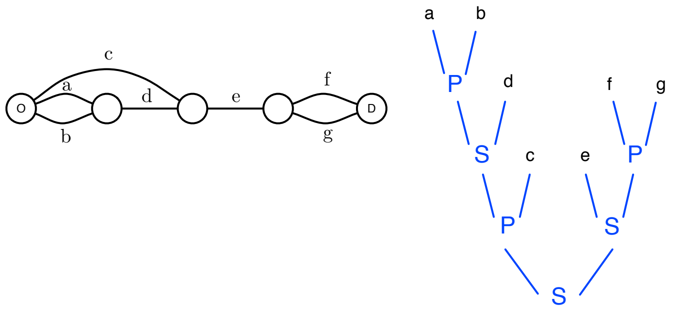


Figure 10: Binary tree decomposition of a series-parallel network.

two SP networks in series or parallel. If it is the result of attaching two SP networks in series, then there exist a node of the tree labeled S, referred to as the root of the tree, such that on one of the subtrees starting from that node we only have O labeled leaves and on the other subtree we only have D labeled leaves (this can be done in linear time by traversing the tree). If G is the result of attaching two SP networks in parallel, then there exists a node of the tree labeled P, again referred to as the root of the tree, such that on both subtrees starting from it we have both O and D labeled leaves.

We next show by induction on the size of tree that whether the binary tree represents an SLI network can be verified in linear time. If the root of the tree is S, then we have series of two networks. By induction hypothesis in linear time we can verify whether each of these subtrees represent and SLI network, which in turn determines whether G is SLI. If the root of the tree is P, we need to check whether each subtree represents an LI network. We next show this can be done in linear time which concludes the proof.

Claim: Given the binary tree decomposition, we can verify whether the underlying network is LI in linear time.

We show this claim by induction on the size of the tree as well. Starting from the root of the tree, if the root has label P, then by induction hypothesis, for each of the subtrees denoted by T_1 and T_2, we can verify whether the underlying network is LI in $O(V_{T_1})$ and $O(V_{T_2})$, respectively. The underlying network is LI if and only if both of these subtrees represent an LI network. Therefore, in $O(V)$ it can be verified whether the underlying network is LI. If the root is labeled S, then the underlying network is LI if and only if one of the subtrees is only labeled S, and the other subtree is LI. Using any traversing algorithm (breadth first search, or depth first search), one can visit all nodes in both subtrees in linear time, verifying if it only has S labels. Furthermore, by induction we can verify whether each subtree represents an LI network. Therefore, in linear time, we can verify whether the network is LI, completing the proof.

9.3 Proofs of Section 5

9.3.1 Expansion of Example 2

We provide the example for part (a) of Example 2. Let $K = 1$, $c_{e_1}(x) = x, c_{e_2}(x) = 1$, $c_{e_3}(x) = 1, c_{e_4}(x) = x, c_{e_5}(x) = 0$ and $s_1 = 1$. Also, we let the information sets be $\mathcal{E}_1 = \{e_1, e_2, e_3, e_4\}$ and $\tilde{\mathcal{E}}_1 = \{e_1, e_2, e_3, e_4, e_5\}$. In equilibrium, we have $f^{(1)}_{e_1e_3} = f^{(1)}_{e_2e_4} = \frac{1}{2}$ with $c^{(1)} = \frac{3}{2}$ and $\tilde{f}^{(1)}_{e_1e_3} = \tilde{f}^{(1)}_{e_2e_4} = 0$, $\tilde{f}^{(1)}_{e_1e_5e_4} = 1$ with $\tilde{c}^{(1)} = 2$. Therefore, after expanding the information set of type 1 users their equilibrium cost has increased from $\frac{3}{2}$ to 2.

9.3.2 Proof of the Claim of Remark 2

We will show that there are infinitely many cost functions for the network shown in Figure 3b for which IBP occurs. In particular, we show the following claim.

Claim: For any $a_1, a_3, a_5 > 0$ such that $a_1+a_3 > a_5$, there exist non-negative $b_1, b_2, b_3, b_4, b_5, a_2, s_1$, and s_2 such that with cost functions $c_{e_i}(x) = a_i x + b_i$, $1 \leq i \leq 5$, IBP occurs in the network shown in Figure 3b. In particular, we show the following cost function parameters along with $\mathcal{E}_2 = \{e_1, e_4, e_5\}$, $\mathcal{E}_1 = \{e_2, e_3, e_5\}$, and $\tilde{\mathcal{E}_1} = \{e_1, e_2, e_3, e_5\}$ leads to IBP.

$$\begin{aligned}
&a_4 = b_1 = b_3 = b_5 = 0,\\
&b_2 = a_1 y = a_1 \frac{a_5(s_1+s_2)}{a_1+a_3+a_5},\\
&b_4 = \frac{a_5 a_3 (s_1+s_2)}{a_1+a_3+a_5},\\
&\frac{s_1}{s_1+s_2} \in \left(\frac{a_1+a_3}{a_1+a_3+a_5}, \min\left\{ \frac{(a_3+a_5)(a_3a_5+a_1^2+a_1a_3+a_1a_5) - a_1 a_5^2}{(a_1+a_3+a_5)\,(a_3a_5+a_3a_1+a_1a_5)}, 1 \right\} \right),\\
&a_2 = \frac{a_5^2 \left(\frac{s_1}{s_1+s_2} (a_1+a_3+a_5) - a_1 \right)}{a_5\,(a_1+a_3+a_5)\,\frac{s_1}{s_1+s_2} - a_1\,(a_1+a_3+a_5)\,\frac{s_2}{s_1+s_2} - a_3 a_5} - a_3 - a_5.
\end{aligned}$$

Proof: we let $a_4 = b_1 = b_3 = b_5 = 0$ and then find a_2, b_2, b_4, s_1, and s_2 for which IBP occurs with $\mathcal{E}_2 = \{e_1, e_4, e_5\}$, $\mathcal{E}_1 = \{e_2, e_3, e_5\}$, and $\tilde{\mathcal{E}_1} = \{e_1, e_2, e_3, e_5\}$. We will find the a_2, b_2, b_4, s_1, and s_2 parameters such that before expanding the information set, the equilibrium flow is $f^{(2)}_{e_5} = 0$, $f^{(2)}_{e_1e_4} = s_2$, and $f^{(1)}_{e_5} = s_1 - x$, $f^{(1)}_{e_2e_3} = x$. We will further impose the constraint that the cost of route e_5 for type 2 users is equal to the cost of route e_1e_4. For this to hold it is sufficient and necessary to have $a_5(s_1 - x) = a_2 x + b_2 + a_3 x$, which leads to

$$x = \frac{a_5 s_1 - b_2}{a_2 + a_3 + a_5} \in [0, s_1]. \tag{9.5}$$

We also have $a_1 s_2 + b_4 = a_5(s_1 - x)$, which leads to

$$a_1 s_2 + b_4 = a_5 \left(s_1 - \frac{a_5 s_1 - b_2}{a_2 + a_3 + a_5} \right) \tag{9.6}$$

We will also choose a_2, b_2, b_4, s_1, and s_2 parameters such that after expanding the information set, the equilibrium flow becomes $\tilde{f}^{(2)}_{e_5} = s_2$, $\tilde{f}^{(2)}_{e_1e_4} = 0$, $\tilde{f}^{(1)}_{e_5} = s_1 - y$, $\tilde{f}^{(1)}_{e_2e_3} = 0$, and $\tilde{f}^{(1)}_{e_1e_3} = y$. We will further impose the constraint that the cost of all available routes for each type of users are equal. For this to hold it is sufficient and necessary to have $a_5(s_1 + s_2 - y) = a_1 y + a_3 y$, which leads to

$$y = \frac{a_5(s_2 + s_1)}{a_1 + a_3 + a_5} \in (0, s_1). \tag{9.7}$$

We also have $a_1 y + a_3 y = b_2 + a_3 y$, which after substituting y from (9.7) leads to

$$b_2 = a_1 y = a_1 \frac{a_5(s_2 + s_1)}{a_1 + a_3 + a_5}. \tag{9.8}$$

Also, for type 2 users we have $a_1 y + b_4 = a_5(s_2 + s_1 - y)$, which after substituting y from (9.7) leads to

$$b_4 = \frac{a_5 a_3 (s_2 + s_1)}{a_1 + a_3 + a_5}. \tag{9.9}$$

Therefore, Equations (9.9) and (9.8) determine b_2 and b_4 as a function of other parameters. In what follows we will show how to choose non-negative s_1, s_2, and a_2 such that Equations (9.5), (9.6), and (9.7) hold as well. After some rearrangements, we can see that the constraints imposed by Equations (9.5) and (9.7) are equivalent to

$$\frac{s_1}{s_2 + s_1} \geq \frac{\max\{a_5, a_1\}}{a_1 + a_3 + a_5}. \tag{9.10}$$

Furthermore, IBP occurs if we have $a_1 y + b_4 > a_1 s_2 + b_4$, which leads to $\frac{s_2}{s_2+s_1} < \frac{a_5}{a_1+a_3+a_5}$ or equivalently

$$\frac{s_1}{s_2 + s_1} > \frac{a_1 + a_3}{a_1 + a_3 + a_5}. \tag{9.11}$$

Using $a_1 + a_3 > a_5$, Equations (9.10) and (9.11) become equivalent to

$$\frac{s_1}{s_2 + s_1} > \frac{a_1 + a_3}{a_1 + a_3 + a_5}. \tag{9.12}$$

Using Equation (9.6), we can find a_2 as follows

$$a_2 = \frac{a_5^2 \left(\left(\frac{s_1}{s_2+s_1}(a_1 + a_3 + a_5) \right) - a_1 \right)}{a_5 \left(\frac{s_1}{s_2+s_1}(a_1 + a_3 + a_5) \right) - a_5 a_3 + a_1 \left(\frac{s_1}{s_2+s_1}(a_1 + a_3 + a_5) \right) - a_1(a_1 + a_3 + a_5)} - a_3 - a_5, \tag{9.13}$$

with the condition that the right-hand side of Equation (9.13) is non-negative. From (9.12) the non-negativity of a_2 becomes equivalent to

$$\frac{s_1}{s_1 + s_2}(a_1 + a_3 + a_5) \leq \frac{(a_3 + a_5)(a_3 a_5 + a_1^2 + a_1 a_3 + a_1 a_5) - a_1 a_5^2}{(a_3 + a_5)(a_5 + a_1) - a_5^2}. \tag{9.14}$$

Choosing $\frac{s_1}{s_1+s_2}(a_1 + a_3 + a_5)$ which satisfies both Equations (9.14) and (9.12) is feasible if we have

$$a_1 + a_3 < \frac{(a_3 + a_5)(a_3 a_5 + a_1^2 + a_1 a_3 + a_1 a_5) - a_1 a_5^2}{(a_3 + a_5)(a_5 + a_1) - a_5^2},$$

which after simplification becomes equivalent to $a_3 a_5^2 > 0$, and therefore holds. Hence, by choosing $\frac{s_1}{s_1+s_2}(a_1 + a_3 + a_5)$ such that

$$\begin{aligned} &\frac{s_1}{s_1 + s_2}(a_1 + a_3 + a_5) \\ &\in \left(a_1 + a_3, \min \left\{ \frac{(a_3 + a_5)(a_3 a_5 + a_1^2 + a_1 a_3 + a_1 a_5) - a_1 a_5^2}{(a_3 + a_5)(a_5 + a_1) - a_5^2}, a_1 + a_3 + a_5 \right\} \right), \end{aligned}$$

all the conditions are satisfied and IBP occurs in this network for infinitely many cost functions.

9.4 Proofs of Section 6

9.4.1 Proof of Lemma 1

Given the feasible flow $f^{(1:K)}$ for $(G, \mathcal{E}_{1:K}, s_{1:K}, \mathbf{c})$ we construct a feasible flow f with load $\sum_{i=1}^{K} s_i$ for a single type of users by letting $f_r = \sum_{i=1}^{K} f_r^{(i)}$. Using this constructions, from two feasible flows $f^{(1:K)}$ and $\tilde{f}^{(1:K)}$ we obtain two feasible flows f and $\tilde{f}$ for a single type congestion game such that the load of f is larger than or equal to the traffic demand of $\tilde{f}$. Therefore, part (a) follows from Milchtaich (2006, Lemma 5).

We next show part (b). Since part (a) holds for any two feasible flows, we can apply it for the equilibrium flows $f^{(1:K)}$ and $\tilde{f}^{(1:K)}$ over the traffic networks $(G, \mathcal{E}_{1:K}, s_{1:K}, \mathbf{c})$ and $(G, \tilde{\mathcal{E}}_{1:K}, s_{1:K}, \mathbf{c})$, respectively (we can view $f^{(1:K)}$ as a feasible flow over the traffic network $(G, \tilde{\mathcal{E}}_{1:K}, s_{1:K}, \mathbf{c})$ as well). It follows that there exists a route r such that $\sum_{i=1}^{K} f_r^{(i)} > \sum_{i=1}^{K} \tilde{f}_r^{(i)}$ and $f_e \geq \tilde{f}_e$ for all $e \in r$. From the first inequality it follows that $\sum_{i=1}^{K} f_r^{(i)} > 0$ which shows at least one of the types, say type i, put a positive traffic on route r. Note that i can be any element of $[K]$ (it can also be 1 as the flow $f^{(1:K)}$ is a feasible flow for the traffic network $(G, \mathcal{E}_{1:K}, s_{1:K}, \mathbf{c})$). We obtain

$$c^{(i)} = c_r \geq \tilde{c}_r \geq \tilde{c}^{(i)},$$

where the first equality follows from $f_r^{(i)} > 0$. The first inequality follows from $f_e \geq \tilde{f}_e$ for all $e \in r$. The second inequality follows from the definition of ICWE and the fact that if type i users can use route r in $(G, \mathcal{E}_{1:K}, s_{1:K}, \mathbf{c})$, then they can use it in $(G, \tilde{\mathcal{E}}_{1:K}, s_{1:K}, \mathbf{c})$ as well since the information sets are not smaller in the second game. This completes the proof.

9.4.2 Proof of Lemma 2

Part (a): Suppose $f^{(1:K)}$ is an equilibrium flow on G. We next show that the restriction of $f^{(1:K)}$ to G_1 creates an equilibrium for G_1. Consider type i users and let r_1 be a route in G_1 such that $f_{r_1}^{(i)} > 0$ and let r_1' be another route in G_1 which belongs to the information set of type i users. The route r_1 is part of a route r in G for which $f_r^{(i)} > 0$. We let r_2 be the restriction of r to G_2 (so that $r = r_1 + r_2$). Since $f^{(1:K)}$ is an equilibrium of G, we have $c_r = c_{r_1} + c_{r_2} \leq c_{r_1'} + c_{r_2} = c_{r_1'+r_2}$ which leads to $c_{r_1} \leq c_{r_1'}$, showing that the restriction of $f^{(1:K)}$ to G_1 is an equilibrium. Similarly, the restriction to G_2 is an equilibrium.
Part (b): We consider an equilibrium $f^{(1:K)}$ for G and then using part (a) we consider the equilibria of G_1 and G_2 obtained by restriction of $f^{(1:K)}$ to G_1 and G_2. For a type i and route r such that $f_r^{(i)} > 0$, we have $c^{(i)} = c_r = c_{r_1} + c_{r_2}$, where r_1 and r_2 are the restriction of r to G_1 and G_2, respectively. Since $f_{r_1}^{(i)} > 0$ and $f_{r_2}^{(i)} > 0$ we have $c_{r_1} = c_1^{(i)}$ and $c_{r_2} = c_2^{(i)}$, which leads to $c^{(i)} = c_1^{(i)} + c_2^{(i)}$.

9.4.3 Proof of Theorem 4

We first show two lemmas that we will use in the proof. The first lemma directly follows from the results of Milchtaich (2006) for single type congestion game.

Lemma 4 *Consider a traffic network with multiple information types* $(G, \mathcal{E}_{1:K}, s_{1:K}, \mathbf{c})$*. Let* $f^{(1:K)}$ *and* $\tilde{f}^{(1:K)}$ *be two (arbitrary) non-identical feasible flows such that* $\sum_{i=1}^{K} s_i \geq$

$\sum_{i=1}^{K} \tilde{s}_i$. *If G is series-parallel, there exists a route r such that $f_e \geq \tilde{f}_e$ and $f_e > 0$ for all $e \in r$.*

Proof Similar to the proof of Lemma 1, given a feasible flow $f^{(1:K)}$ for $(G, \mathcal{E}_{1:K}, s_{1:K}, \mathbf{c})$ we define a feasible flow f with traffic demand $\sum_{i=1}^{K} s_i$ for a congestion game with a single information type. Therefore, this lemma follows from Milchtaich (2006, Lemma 2). ∎

Lemma 5 *Consider a traffic network with multiple information types $(G, \mathcal{E}_{1:K}, s_{1:K}, \mathbf{c})$ with equilibrium costs $c^{(i)}$, $i = 1, \dots, K$. Consider an ICWE with flow $(f^{(1)}, \dots, f^{(K)})$ and let r be a route for which $f_e > 0$ for any $e \in r$. We have that*

$$c_r \in \left[\min\{c^{(1)}, \dots, c^{(K)}\}, \max\{c^{(1)}, \dots, c^{(K)}\}\right].$$

Proof We will prove this lemma by induction on the number of edges of G. Since G is series-parallel, it is either the result of attaching two series-parallel networks in series or attaching two series-parallel networks in parallel. We consider both cases in the following. If G is the result of attaching two series-parallel networks, G_A and G_B in series, then using Lemma 2 an ICWE for the overall network is obtained by concatenating an ICWE for G_A with an ICWE for G_B. Moreover, we have $c^{(i)} = c_A^{(i)} + c_B^{(i)}$ for $i = 1, \dots, K$. By induction on the number of edges for the part of r in the network G_A denoted by r_A, we have

$$c_{r_A} \in \left[\min\{c_A^{(1)}, \dots, c_A^{(K)}\}, \max\{c_A^{(1)}, \dots, c_A^{(K)}\}\right].$$

We similarly have

$$c_{r_B} \in \left[\min\{c_B^{(1)}, \dots, c_B^{(K)}\}, \max\{c_B^{(1)}, \dots, c_B^{(K)}\}\right].$$

By adding the previous two relations, we obtain

$$c_r \in \left[\min\{c^{(1)}, \dots, c^{(K)}\}, \max\{c^{(1)}, \dots, c^{(K)}\}\right].$$

Now suppose that G is the result of attaching G_A and G_B in parallel and suppose $r \in G_A$. Let $T = \{i \in [K] \ : \ f_A^{(i)} > 0\}$. For all $i \in T$ since $f_A^{(i)} > 0$, we have $c^{(i)} = c_r^{(i)}$. Therefore, by induction hypothesis we have

$$c_r \in \left[\min_{i \in T}\{c^{(i)}\}, \max_{i \in T}\{c^{(i)}\}\right].$$

Since we have $\min_{i\in[K]}\{c^{(i)}\} \leq \min_{i\in T}\{c^{(i)}\}$ and $\max_{i\in T}\{c^{(i)}\} \leq \max_{i\in[K]}\{c^{(i)}\}$, we obtain

$$c_r \in \left[\min_{i\in[K]}\{c^{(i)}\}, \max_{i\in[K]}\{c^{(i)}\}\right],$$

which concludes the proof of lemma. ∎

Proof of part (a) of Theorem 4: After expanding information set of type 1 users to $\mathcal{E}$, we obtain $\tilde{c}^{(i)} = \tilde{c}^{(1)}$ for all $i \in [K]$. Using Lemma 4, there exists a route r such that $f_e \geq \tilde{f}_e$ and $f_e > 0$ for any $e \in r$. We have

$$c_r \geq \tilde{c}_r \geq \tilde{c}^{(i)} = \tilde{c}^{(1)}, \quad \forall\, i \in [K],$$

where the first inequality follows form $f_e \geq \tilde{f}_e$, the second inequality follows from the definition of ICWE, and the equality follows from $\tilde{\mathcal{E}}_i = \mathcal{E}$ for all $i = 1, \dots, K$. Since $\mathcal{E}_1 \subseteq \mathcal{E}$, we have $c^{(i)} = c^{(j)} \leq c^{(1)}$ for all $i, j = 2, \dots, K$. Using Lemma 5, this leads to

$$c_r \in [c^{(i)}, c^{(1)}], \qquad \text{for } i = 2, \dots, K.$$

Combining the previous two relations leads to $\tilde{c}^{(1)} \leq c^{(1)}$.

Proof of part (b) of Theorem 4: The proof is similar to the proof of part (b) of Theorem 3. In Example 2 we have provided an example showing that IBP with restricted information sets can occur over Wheatstone network shown in Figure 3a.

Suppose that a network G is not series-parallel. Using Proposition 2, G can be constructed from Wheatstone network shown in Figure 3a by following the steps of embedding. To construct an example for G, we start from the cost functions for which the embedded network features IBP with restricted information sets and then following the steps of embedding we will update the information sets as well as the cost functions in a way that IBP occurs in the final network which is G. The updates are identical to those described in the proof of part (b) of Theorem 3 and establishes that if IBP with restricted information sets is present in the initial network (i.e., the Whetstone network shown in Figure 3a), it will be present in network G as well. This completes the proof of part (b).

9.4.4 Omitted Proof of Example 3

First, note that after expansion of information, without loss of generality, each type of users (i, j), $i = 1, 2$ have at least two routes from O_i to D_i. Because, otherwise, if a type with traffic demand s has only one route r, we can consider an equivalent game in which we update the cost of all edges on r from $c_e(x)$ to $c_e(x + s)$. Also, note that due to symmetry we can only consider the information expansion of one of the types of the form $(1, j)$. Therefore, without loss of generality we assume that there exists one type from O_2 to D_2 with information about all edges of the network and there exist either one or two types from O_1 to D_1. Below, we examine all possible cases and show that IBP does not occur:

(1) There exist two types $\{(1, 1), (2, 1)\}$ such that $\mathcal{R}_{2,1} = \{e_1e_3, e_2\}$, $\mathcal{R}_{1,1} = \{e_1\}$, and $\tilde{\mathcal{R}}_{1,1} = \{e_1, e_2e_3\}$: If type $(1, 1)$ does not use route e_2e_3 after information expansion, then equilibrium remains the same. Now suppose, type $(1, 1)$ uses route e_2e_3 (i.e., $\tilde{f}_{e_1}^{(1,1)} < f_{e_1}^{(1,1)} = s_{1,1}$). If $\tilde{f}_{e_1} \leq f_{e_1}$, then we have

$$\tilde{c}^{(1,1)} \leq c_{e_1}(\tilde{f}_{e_1}) \leq c_{e_1}(f_{e_1}) = c^{(1,1)},$$

which shows IBP does not occur. Now suppose $\tilde{f}_{e_1} > f_{e_1}$, which in turn shows $\tilde{f}_{e_2} < f_{e_2}$ as $\tilde{f}_{e_1} + \tilde{f}_{e_2} = f_{e_1} + f_{e_2} = s_{1,1} + s_{2,1}$. We have

$$\tilde{f}_{e_3} = \tilde{f}_{e_2e_3}^{(1,1)} + \tilde{f}_{e_1e_3}^{(2,1)} > f_{e_2e_3}^{(1,1)} + f_{e_1e_3}^{(2,1)} = f_{e_3},$$

where we used $\tilde{f}_{e_2e_3}^{(1,1)} > f_{e_2e_3}^{(1,1)} = 0$ and $\tilde{f}_{e_1e_3}^{(2,1)} \geq f_{e_1e_3}^{(2,1)}$, which holds because $\tilde{f}_{e_1e_3}^{(2,1)} = \tilde{f}_{e_1} - \tilde{f}_{e_1}^{(1,1)} > f_{e_1} - s_{1,1} = f_{e_1} - f_{e_1}^{(1,1)} = f_{e_1e_3}^{(2,1)}$. Therefore, we have

$$c_{e_1}(\tilde{f}_{e_1}) + c_{e_3}(\tilde{f}_{e_3}) \leq c_{e_2}(\tilde{f}_{e_2}) \leq c_{e_2}(f_{e_2}) \leq c_{e_1}(f_{e_1}) + c_{e_3}(f_{e_3}), \tag{9.15}$$

where the first inequality holds because $\tilde{f}^{(2,1)}_{e_1e_3} > f^{(2,1)}_{e_1e_3} \geq 0$, the second inequality holds because $\tilde{f}_{e_2} < f_{e_2}$, and the third inequality holds because $f^{(2,1)}_{e_2} = s_{2,1} - f^{(2,1)}_{e_1e_3} > s_{2,1} - \tilde{f}^{(2,1)}_{e_1e_3} = \tilde{f}^{(2,1)}_{e_2} \geq 0$. Inequality (9.15) together with $c_{e_1}(\tilde{f}_{e_1}) \geq c_{e_1}(f_{e_1})$ and $c_{e_3}(\tilde{f}_{e_3}) \geq c_{e_3}(f_{e_3})$ shows that the cost of all three edges before and after information expansion are the same, leading to the same equilibrium cost for all types. Therefore, IBP does not occur in this case.

(2) There exist two types $\{(1,1),(2,1)\}$ such that $\mathcal{R}_{2,1} = \{e_1e_3, e_2\}$, $\mathcal{R}_{1,1} = \{e_2e_3\}$, and $\tilde{\mathcal{R}}_{1,1} = \{e_1, e_2e_3\}$: If type $(1,1)$ does not use e_1 after information expansion, then equilibrium remains the same. Now suppose type $(1,1)$ uses route e_1. We show IBP does not occur in this case by considering all possibilities as follows:

- $\tilde{f}_{e_1} \leq f_{e_1}$: Since $f_{e_1} + f_{e_2} = s_{1,1} + s_{2,1} = \tilde{f}_{e_1} + \tilde{f}_{e_2}$, we have $\tilde{f}_{e_2} \geq f_{e_2}$. We also have $\tilde{f}_{e_3} < f_{e_3}$, because

$$\tilde{f}_{e_3} = \tilde{f}^{(1,1)}_{e_2e_3} + \tilde{f}^{(2,1)}_{e_1e_3} < f^{(1,1)}_{e_2e_3} + f^{(2,1)}_{e_1e_3} = f_{e_3},$$

  where we used $\tilde{f}^{(1,1)}_{e_2e_3} < f^{(1,1)}_{e_2e_3}$ as type $(1,1)$ is using e_1 after information expansion and $\tilde{f}^{(2,1)}_{e_1e_3} < f^{(2,1)}_{e_1e_3}$ as $\tilde{f}^{(2,1)}_{e_1e_3} = \tilde{f}_{e_1} - \tilde{f}^{(1,1)}_{e_1} < f_{e_1} = f^{(2,1)}_{e_1e_3}$. The inequality $\tilde{f}^{(2,1)}_{e_1e_3} < f^{(2,1)}_{e_1e_3}$ implies $\tilde{f}^{(2,1)}_{e_2} > f^{(2,1)}_{e_2} \geq 0$. Therefore, we have

$$c_{e_2}(\tilde{f}_{e_2}) \leq c_{e_1}(\tilde{f}_{e_1}) + c_{e_3}(\tilde{f}_{e_3}) \leq c_{e_1}(f_{e_1}) + c_{e_3}(f_{e_3}) \leq c_{e_2}(f_{e_2}), \tag{9.16}$$

  where the first inequality follows from $\tilde{f}^{(2,1)}_{e_2} > 0$, the second inequality follows from $\tilde{f}_{e_1} \leq f_{e_1}$ and $\tilde{f}_{e_3} \leq f_{e_3}$, and the third inequality follows from $f^{(2,1)}_{e_1e_3} > \tilde{f}^{(2,1)}_{e_1e_3} \geq 0$. Inequality (9.16) leads to

$$c^{(1,1)} = c_{e_2}(\tilde{f}_{e_2}) + c_{e_3}(\tilde{f}_{e_3}) \leq c_{e_2}(f_{e_2}) + c_{e_3}(f_{e_3}) \leq \tilde{c}^{(1,1)},$$

  showing IBP does not occur.

- $\tilde{f}_{e_1} > f_{e_1}$: Since $f_{e_1} + f_{e_2} = s_{1,1} + s_{2,1} = \tilde{f}_{e_1} + \tilde{f}_{e_2}$, we have $\tilde{f}_{e_2} < f_{e_2}$. If $\tilde{f}_{e_3} \leq f_{e_3}$, then we have

$$c^{(1,1)} = c_{e_2}(\tilde{f}_{e_2}) + c_{e_3}(\tilde{f}_{e_3}) \leq c_{e_2}(f_{e_2}) + c_{e_3}(f_{e_3}) \leq \tilde{c}^{(1,1)},$$

  showing IBP does not occur. Otherwise, we have $\tilde{f}_{e_3} > f_{e_3}$. First note that if $\tilde{f}_{e_3} > f_{e_3}$, then $\tilde{f}^{(2,1)}_{e_1e_3} > f^{(2,1)}_{e_1e_3}$. This inequality holds because

$$\tilde{f}^{(2,1)}_{e_1e_3} = \tilde{f}_{e_3} - \tilde{f}^{(1,1)}_{e_2e_3} > f_{e_3} - f^{(1,1)}_{e_2e_3} = f^{(2,1)}_{e_1e_3},$$

  where we used $\tilde{f}_{e_3} > f_{e_3}$ and $\tilde{f}^{(1,1)}_{e_2e_3} < f^{(1,1)}_{e_2e_3}$ as $(1,1)$ uses e_1 after the expansion of information (i.e., $f^{(1,1)}_{e_2e_3} = s_{1,1}$ and $\tilde{f}^{(1,1)}_{e_2e_3} < s_{1,1}$). Therefore, we have

$$c_{e_1}(\tilde{f}_{e_1}) + c_{e_3}(\tilde{f}_{e_3}) \leq c_{e_2}(\tilde{f}_{e_2}) \leq c_{e_2}(f_{e_2}) \leq c_{e_1}(f_{e_1}) + c_{e_3}(f_{e_3}), \tag{9.17}$$

  where the first inequality holds because $\tilde{f}^{(2,1)}_{e_1e_3} > f^{(2,1)}_{e_1e_3} \geq 0$, the second inequality holds because $\tilde{f}_{e_2} < f_{e_2}$, and the third inequality holds because $f^{(2,1)}_{e_2} = s_{2,1} - f^{(2,1)}_{e_1e_3} > s_{2,1} - \tilde{f}^{(2,1)}_{e_1e_3} = \tilde{f}^{(2,1)}_{e_2} \geq 0$. Inequality (9.17) together with $\tilde{f}_{e_1} > f_{e_1}$

and $\tilde{f}_{e_3} > f_{e_3}$ leads to $c_{e_1}(f_{e_1}) = c_{e_1}(\tilde{f}_{e_1})$, $c_{e_2}(f_{e_2}) = c_{e_2}(\tilde{f}_{e_2})$, and $c_{e_3}(f_{e_3}) = c_{e_3}(\tilde{f}_{e_3})$. Therefore, we have

$$\tilde{c}^{(1,1)} \leq c_{e_2}(\tilde{f}_{e_2}) + c_{e_3}(\tilde{f}_{e_3}) = c_{e_2}(f_{e_2}) + c_{e_3}(f_{e_3}) = c^{(1,1)},$$

showing IBP does not occur.

(3) There exist three types $\{(1,1),(1,2),(2,1)\}$ such that $\mathcal{R}_{2,1} = \{e_1e_3, e_2\}$, $\mathcal{R}_{1,2} = \{e_1, e_2e_3\}$, $\mathcal{R}_{1,1} = \{e_1\}$, and $\tilde{\mathcal{R}}_{1,1} = \{e_1, e_2e_3\}$: This case is similar to the first case. If type $(1,1)$ does not use e_2e_3 after the expansion of information, then equilibrium remains the same. Now suppose, type $(1,1)$ uses route e_2e_3 (i.e., $\tilde{f}_{e_1}^{(1,1)} < f_{e_1}^{(1,1)} = s_{1,1}$). If $\tilde{f}_{e_1} \leq f_{e_1}$, then we have

$$\tilde{c}^{(1,1)} \leq c_{e_1}(\tilde{f}_{e_1}) \leq c_{e_1}(f_{e_1}) = c^{(1,1)},$$

which shows IBP does not occur. Now suppose $\tilde{f}_{e_1} > f_{e_1}$, which in turn shows $\tilde{f}_{e_2} < f_{e_2}$ as $\tilde{f}_{e_1} + \tilde{f}_{e_2} = f_{e_1} + f_{e_2} = s_{1,1} + s_{1,2} + s_{2,1}$. We consider the following two cases:

– $\tilde{f}_{e_1}^{(1,1)} + \tilde{f}_{e_1}^{(1,2)} < f_{e_1}^{(1,1)} + f_{e_1}^{(1,2)}$ (note that $f_{e_1}^{(1,1)} = s_{1,1}$): we have

$$\tilde{f}_{e_1e_3}^{(2,1)} = \tilde{f}_{e_1} - \left(\tilde{f}_{e_1}^{(1,1)} + \tilde{f}_{e_1}^{(1,2)}\right) > f_{e_1} - \left(f_{e_1}^{(1,1)} + f_{e_1}^{(1,2)}\right) = f_{e_1e_3}^{(2,1)},$$
$$\tilde{f}_{e_2e_3}^{(1,1)} + \tilde{f}_{e_2e_3}^{(1,2)} = s_{1,1} + s_{1,2} - \left(\tilde{f}_{e_1}^{(1,1)} + \tilde{f}_{e_1}^{(1,2)}\right) > s_{1,1} + s_{1,2} - \left(f_{e_1}^{(1,1)} + f_{e_1}^{(1,2)}\right) = f_{e_2e_3}^{(1,2)}.$$

These two inequalities lead to

$$\tilde{f}_{e_3} = \tilde{f}_{e_1e_3}^{(2,1)} + \tilde{f}_{e_2e_3}^{(1,1)} + \tilde{f}_{e_2e_3}^{(1,2)} > f_{e_1e_3}^{(2,1)} + f_{e_2e_3}^{(1,2)} = f_{e_3}.$$

Therefore, we have

$$c_{e_1}(\tilde{f}_{e_1}) + c_{e_3}(\tilde{f}_{e_3}) \leq c_{e_2}(\tilde{f}_{e_2}) \leq c_{e_2}(f_{e_2}) \leq c_{e_1}(f_{e_1}) + c_{e_3}(f_{e_3}), \tag{9.18}$$

where the first inequality holds because $\tilde{f}_{e_1e_3}^{(2,1)} > f_{e_1e_3}^{(2,1)} \geq 0$, the second inequality holds because $\tilde{f}_{e_2} < f_{e_2}$, and the third inequality holds because $f_{e_2}^{(2,1)} = s_{2,1} - f_{e_1e_3}^{(2,1)} > s_{2,1} - \tilde{f}_{e_1e_3}^{(2,1)} = \tilde{f}_{e_2}^{(2,1)} \geq 0$. Inequality (9.18) together with $\tilde{f}_{e_1} > f_{e_1}$ and $\tilde{f}_{e_3} > f_{e_3}$ leads to $c_{e_1}(f_{e_1}) = c_{e_1}(\tilde{f}_{e_1})$, $c_{e_2}(f_{e_2}) = c_{e_2}(\tilde{f}_{e_2})$, and $c_{e_3}(f_{e_3}) = c_{e_3}(\tilde{f}_{e_3})$. Therefore, the cost of all three edges before and after information expansion are the same, leading to the same equilibrium cost for all types. Therefore, IBP does not occur in this case.

– $\tilde{f}_{e_1}^{(1,1)} + \tilde{f}_{e_1}^{(1,2)} \geq f_{e_1}^{(1,1)} + f_{e_1}^{(1,2)}$: We have

$$\begin{aligned} \tilde{f}_{e_1}^{(1,2)} &= \left(\tilde{f}_{e_1}^{(1,1)} + \tilde{f}_{e_1}^{(1,2)}\right) - \tilde{f}_{e_1}^{(1,1)} \geq \left(f_{e_1}^{(1,1)} + f_{e_1}^{(1,2)}\right) - \tilde{f}_{e_1}^{(1,1)} \\ &> \left(f_{e_1}^{(1,1)} + f_{e_1}^{(1,2)}\right) - f_{e_1}^{(1,1)} = f_{e_1}^{(1,2)}, \end{aligned} \tag{9.19}$$

$$\begin{aligned} \tilde{f}_{e_2e_3}^{(1,1)} + \tilde{f}_{e_2e_3}^{(1,2)} &= s_{1,1} + s_{1,2} - \left(\tilde{f}_{e_1}^{(1,1)} + \tilde{f}_{e_1}^{(1,2)}\right) \\ &\leq s_{1,1} + s_{1,2} - \left(f_{e_1}^{(1,1)} + f_{e_1}^{(1,2)}\right) = f_{e_2e_3}^{(1,2)}. \end{aligned} \tag{9.20}$$

If $\tilde{f}^{(2,1)}_{e_1e_3} \leq f^{(2,1)}_{e_1e_3}$, then Inequality (9.20) leads to

$$\tilde{f}_{e_3} = \tilde{f}^{(1,1)}_{e_2e_3} + \tilde{f}^{(1,2)}_{e_2e_3} + \tilde{f}^{(2,1)}_{e_1e_3} \leq f^{(1,2)}_{e_2e_3} + f^{(2,1)}_{e_1e_3} = f_{e_3}.$$

Therefore, we obtain

$$c_{e_1}(\tilde{f}_{e_1}) = c_{e_2}(\tilde{f}_{e_2}) + c_{e_3}(\tilde{f}_{e_3}) \leq c_{e_2}(f_{e_2}) + c_{e_3}(f_{e_3}) \leq c_{e_1}(f_{e_1}), \tag{9.21}$$

where the first equality holds because using Inequality (9.19) we obtain $\tilde{f}^{(1,2)}_{e_1} > f^{(1,2)}_{e_1} \geq 0$ and $\tilde{f}^{(1,1)}_{e_2e_3} > 0$, the first inequality holds because $\tilde{f}_{e_2} < f_{e_2}$ and $\tilde{f}_{e_3} < f_{e_3}$, and the second inequality holds because using Inequality (9.19) we obtain $f^{(1,2)}_{e_2e_3} = s_{1,2} - f^{(1,2)}_{e_1} > s_{1,2} - \tilde{f}^{(1,2)}_{e_1} \geq 0$. Inequality (9.21) leads to

$$\tilde{c}^{(1,1)} \leq c_{e_1}(\tilde{f}_{e_1}) \leq c_{e_1}(f_{e_1}) = c^{(1,1)},$$

showing IBP does not occur in this case.
Now suppose $\tilde{f}^{(2,1)}_{e_1e_3} > f^{(2,1)}_{e_1e_3}$ which leads to

$$c_{e_1}(\tilde{f}_{e_1}) \leq c_{e_1}(\tilde{f}_{e_1}) + c_{e_3}(\tilde{f}_{e_3}) \leq c_{e_2}(\tilde{f}_{e_2}) \leq c_{e_2}(f_{e_2}) \leq c_{e_2}(f_{e_2}) + c_{e_3}(f_{e_3}) \leq c_{e_1}(f_{e_1}), \tag{9.22}$$

where the second inequality holds because $\tilde{f}^{(2,1)}_{e_1e_3} > f^{(2,1)}_{e_1e_3} \geq 0$, the third inequality holds because $\tilde{f}_{e_2} < f_{e_2}$, and the last inequality holds because using Inequality (9.19) we obtain $f^{(1,2)}_{e_2e_3} = s_{1,2} - f^{(1,2)}_{e_1} > s_{1,2} - \tilde{f}^{(1,2)}_{e_1} \geq 0$. Inequality (9.22) leads to

$$\tilde{c}^{(1,1)} \leq c_{e_1}(\tilde{f}_{e_1}) \leq c_{e_1}(f_{e_1}) = c^{(1,1)},$$

showing IBP does not occur in this case.

(4) There exist three types $\{(1,1),(1,2),(2,1)\}$ such that $\mathcal{R}_{2,1} = \{e_1e_3, e_2\}$, $\mathcal{R}_{1,2} = \{e_1, e_2e_3\}$, $\mathcal{R}_{1,1} = \{e_2e_3\}$, and $\tilde{\mathcal{R}}_{1,1} = \{e_1, e_2e_3\}$: This case is similar to the second case.

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