Computation in Heterogeneous-Agent Models with Financial Frictions
For more information, see

- Survey “Money, Macro and Finance: A Continuous-Time Approach” with Markus Brunnermeier
- The Princeton Initiative, initiative.princeton.edu
Approach

• Heterogeneous agents
  – agents specialize in specific economic activities (e.g. intermediation, production in a particular sector)
  – level of these activities stems from risk taking (i.e. solution to optimal consumption and portfolio choice)
  – hence, wealth distribution matters for aggregate production / efficiency of asset allocation

• Examples: Basak-Cuoco, He-Krishnamurthy, Brunnermeier-Sannikov, Di Tella

• Solution method based on value function iteration (in continuous time) – solve for value functions of each agent class, and determine risk taking / asset allocation from value functions
Goals

• Build a model capable of generating crises
  – a regime different from normal, with high endogenous risk, asset missallocation

• Understanding the resilience of the financial system
  – frequency of crises, level of endogenous risk, speed of recovery
  – role of asset liquidity (market, technological), leverage, asset price level, financial innovations

• How does the system respond to various policies? How do policies affect spillovers/welfare?
  – policies often have unintended consequences, the model finds some of those
Some interesting phenomena

- Dynamics near vs. away from steady state
- Endogenous tail risk
- Endogenous risk and asset correlation
- “Volatility paradox”
- Innovation / better risk sharing might lead to instability
Basic Model: Technology

experts

Output \((a - i_t) k_t\)

Investment \(i_t\) creates new capital at rate \(\Phi(i_t) k_t\)

\[dk_t = (\Phi(i_t) - \delta) k_t \, dt + \sigma k_t \, dZ_t\]

less productive households

Output \((a - i_t) k_t\)

Investment \(i_t\) creates new capital at rate \(\Phi(i_t) k_t\)

\[dk_t = (\Phi(i_t) - \delta) k_t \, dt + \sigma k_t \, dZ_t\]
## Basic Model: Preferences

<table>
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<th></th>
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<th>less productive households</th>
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<td>$\rho &gt; r$</td>
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$a \leq a$
### Basic Model: Financial Frictions

#### Experts

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#### Less Productive Households

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<td>same</td>
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## Basic Model: Asset Markets

### Output

\[
(a - \eta_t) k_t
\]

### Investment

Investment \( \eta_t \) creates new capital at rate \( \Phi(\eta_t) k_t \)

\[
dk_t = (\Phi(\eta_t) - \delta) k_t \, dt + \sigma k_t \, dZ_t
\]

### CRRA Utility, Discount Rate \( \rho \)

CRRA utility, discount rate \( \rho \)

### Market Liquidity

Liquid markets for capital \( k_t \) with endogenous price per unit \( q_t \)

\[
dq_t/q_t = \mu_t^q \, dt + \sigma_t^q \, dZ_t
\]

### Experts

less productive households

\[ a \leq a \]

### Differences

Market liquidity

\[ \rho > r \]

### Constraints

CRRA utility, discount rate \( \rho \) may issue debt + equity

must retain at least \( \chi \) of risk + solvency constraint

same
Variations

• Basak-Cuoco: $a = -\infty$ (households cannot hold capital)
  $\chi = 1$ (experts cannot issue equity)
  utility is logarithmic, $\rho = r$

• He-Krishnamurthy: $a = -\infty$ (households cannot hold capital)
  $\rho = r$, but newborn households get labor income $lK_t$
  same return on inside and outside equity of experts
  (so rationing occurs)

• Here I assume that return on inside and outside equity is different
Equilibrium Definition

• Equilibrium is a map

histories of shocks \{Z_s, s \leq t\} \rightarrow \text{prices } q_t, r_t^F, \text{ allocations}

(of capital } \psi_t, \text{ expert equity } \chi_t \leq \chi, \text{ risk-free asset, consumption)

s.t.
• experts, HH solve optimal consumption/portfolio choice (capital, risk-free asset, equity) problems (Merton)
• markets clear
Equilibrium Characterization

- Equilibrium is a map

histories of shocks \( \{Z_s, s \leq t\} \)

prices \( q_t, r_t^F \), allocations
(of capital \( \psi_t \), expert equity \( \chi_t \leq \chi \), risk-free asset, consumption)

wealth distribution:
fraction \( \eta_t = N_t/(q_t K_t) \in (0, 1) \) owned by experts

\[ \eta_t = \frac{N_t}{q_t K_t} \in (0, 1) \] owned by experts
Equilibrium in time

Markov equilibrium should be a fixed point of this, we can iterate (from a reasonable terminal condition) to find it

value functions

price $q_t$, allocation (portfolios) law of motion of $\eta$

value functions (of $\eta$)

$t - \varepsilon$ $t$ time
Back to our model...

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Liquid markets for capital $k_t$ with endogenous price per unit $q_t$

$$dq_t/q_t = \mu_t^q dt + \sigma_t^q dZ_t$$
Notation

- Risk, $dZ_t$, required risk premia $\varsigma_t$ (experts) $\geq \varsigma_t$ (households)
- Allocation: experts hold $\psi_t \leq 1$ of capital, retain $\chi_t \geq \chi$ equity
- Balance sheets
Log utility

Risk / risk premia

Ito ⇒

\[ \xi_t = \sigma_t^N = \frac{\chi \psi_t}{\eta_t} (\sigma + \sigma_t^q) \]

if \( \psi_t < 1 \), subtracting

\[ \frac{a - a}{q_t} = (\sigma + \sigma_t^q)^2 \chi \left( \frac{\chi \psi_t}{\eta_t} - \frac{1 - \chi \psi_t}{1 - \eta_t} \right) \]

market clearing

\[ \frac{a \psi + a(1 - \psi) - \rho(q)}{q} = \rho \eta + r(1 - \eta) \]

Experts

\[ \xi_t = \sigma_t^N = \frac{1 - \chi \psi_t}{1 - \eta_t} (\sigma + \sigma_t^q) \]

Households

Capital pricing

\[ \mathbb{E}[dr_t^k] / dt - r_t^F = \underbrace{(\sigma + \sigma_t^q)(\chi \xi_t + (1 - \chi)\xi_t)}_{\text{risk of capital}} \]

\[ \mathbb{E}[dr_t^k] / dt - r_t^F \leq \underbrace{(\sigma + \sigma_t^q)\xi_t}_{\text{risk of capital}} \]

= if \( \psi_t < \):
Log utility

Risk / risk premia

\[ \xi_t = \sigma_t^N = \frac{\chi \psi_t}{\eta_t} (\sigma + \sigma_t^q) \]

Ito \[\Rightarrow\]

\[ \eta \sigma_t^q = \eta_t (1 - \eta_t) (\sigma_t^N - \sigma_t^N) = (\sigma + \sigma_t^q) (\chi \psi_t - \eta_t) \]

capital pricing

\[ E[dr^k_t] / dt - r_t^F = \left( \sigma + \sigma_t^q \right) \left( \chi \xi_t + (1 - \chi) \xi_t \right) \]

if \( \psi < 1 \), subtracting

\[ \frac{a-a}{q_t} = (\sigma + \sigma_t^q)^2 \chi \left( \frac{\chi \psi_t}{\eta_t} - \frac{1 - \chi \psi_t}{1 - \eta_t} \right) \]

market clearing

\[ \frac{a\psi + a(1 - \psi) - \iota(q)}{q} = \rho \eta + r(1 - \eta) \]

\[ \eta, q(\eta) \rightarrow \psi \rightarrow \sigma^q \rightarrow q'(\eta) \]
Example: $\gamma = 1$ (log utility), $r = 5\%$, $\rho = 6\%$, $a = 11\%$, $\bar{a} = 7\%$, $\delta = 5\%$, $\sigma = 10\%$, equity issuance (up to 50%), $\Phi(\iota) = ((1 - 2\kappa\iota)^{1/2} - 1)/\kappa$ with $\kappa = 2$.
CRRA utility

Volatility of wealth

Ito $\Rightarrow$

Rel. capital

Pricing

$\eta \sigma^q_t = \eta_t (1 - \eta_t)(\sigma^N_t - \sigma^N_t) = (\sigma + \sigma^q_t)(\chi \psi_t - \eta_t)$

$mkt$

$\xi_t \neq \sigma^N_t = \frac{\frac{\chi \psi_t}{\eta_t} (\sigma + \sigma^q_t)}{1 - \eta_t}$

$clearing$

$\eta \sigma^q_t = \eta_t (1 - \eta_t)(\sigma^N_t - \sigma^N_t) = (\sigma + \sigma^q_t)(\chi \psi_t - \eta_t)$

$\frac{a - a}{q_t} = (\sigma + \sigma^q_t)\chi(\xi_t - \xi_t)$

$households$

$q(\eta)\sigma^q_t = q'(\eta)(\sigma + \sigma^q_t)\chi \psi_t - \eta_t)$

$\frac{a \psi + a(1 - \psi) - \iota(q)}{K_t} = \frac{C_t + C_t}{K_t}$
CRRA utility

Volatility of wealth

Ito ⇒

\[ \eta \sigma_t^N = \eta_t (1 - \eta_t) (\sigma_t^N - \sigma_t^N) = (\sigma + \sigma_t^q) (\chi \psi_t - \eta_t) \]

Rel. capital

Pricing

Value functions

marginal utility / consumption

-risk prem = volatility

experts

households

\[ \xi_t = \sigma_t^\gamma - \sigma_t^\eta - \sigma_t^q - \gamma \sigma \]

\[ \xi_t = \sigma_t^\gamma + \eta \sigma_t^\eta / (1 - \eta) - \sigma_t^q - \gamma \sigma \]
CRRA utility

Volatility of wealth

Ito ⇒ \[ \eta \sigma_t^\eta = \eta_t (1 - \eta_t) (\sigma_t^N - \sigma_t^N) = (\sigma + \sigma_t^q) (\chi \psi_t - \eta_t) \]

Rel cap

Pricing

Value functions

marginal utility / consumption

risk prem = -volatility

\[ \xi_t \neq \sigma_t^N = \frac{\chi \psi_t}{\eta_t} (\sigma + \sigma_t^q) \]

experts

\[ \xi_t \neq \sigma_t^N = \frac{1 - \chi \psi_t}{1 - \eta_t} (\sigma + \sigma_t^q) \]

households

\[ q(\eta) \sigma_t^q = q'(\eta) (\sigma + \sigma_t^q) (\chi \psi_t - \eta_t) \]

\[ a \psi + g(1 - \psi) - u'(q) = \frac{C_t + C_t}{K_t} \]

\[ v_i K_t^{1 - \gamma} = \frac{v_i}{(\eta_t q_i)^{1 - \gamma}} N_t^{1 - \gamma} \]

\[ v_i K_t^{1 - \gamma} = \frac{v_i K_t^{1 - \gamma}}{1 - \gamma} \]

\[ C_t^{-\gamma} = \frac{v_i}{(\eta_t q_i)^{1 - \gamma}} N_t^{-\gamma} = \frac{v_i}{\eta_t q_i} K_t^{-\gamma} \]

\[ C_t^{-\gamma} = \frac{v_i}{(1 - \eta_t) q_t} K_t^{-\gamma} \]
Determine prices / allocations

value functions (of $\eta$)

price $q_t$, allocation (portfolios)

law of motion of $\eta$

$t - \varepsilon$  \hspace{1cm}  $t$  \hspace{1cm}  time
CRRA utility

Volatility of wealth

Ito ⇒

\[ \eta \sigma^\eta_t = \eta_t (1 - \eta_t) (\sigma_t^N - \sigma_t^N) = (\sigma + \sigma_q^q) (\chi \psi_t - \eta_t) \]

Rel \( \Rightarrow \)

\[ a - a \]

Pricing

Value functions

marginal utility / consumption

risk prem = -volatility

\[ \eta, q(\eta) \rightarrow C/K, \frac{C}{K} \rightarrow \psi \rightarrow \sigma_q^q \rightarrow q'(\eta) \]
Law of motion of $\eta$

(value functions (of $\eta$))

price $q_t$, allocation (portfolios)

law of motion of $\eta$

$t - \varepsilon$  $t$  time
Law of motion of $\eta$

### Net Worth

**Experts**

$$\frac{dN_t}{N_t} = r_t^F dt + \frac{\chi_i \psi_i}{\eta_t} (\sigma + \sigma_i^q) (\xi_i dt + dZ_t) - \frac{C_t}{N_t} dt$$

**Households**

$$\frac{dN_t}{N_t} = r_t^F dt + \frac{1 - \chi_i \psi_i}{1 - \eta_t} (\sigma + \sigma_i^q) (\xi_i dt + dZ_t) - \frac{C_t}{N_t} dt$$

### Ito

$$\frac{d\eta_t}{\eta_t(1 - \eta_t)} = (\mu_t^N - \mu_t^N) dt + (\sigma_t^N - \sigma_t^N) (dZ_t - (\eta_t \sigma_t^N + (1 - \eta_t) \sigma_t^N) dt)$$

$$\left( \frac{\chi_i \psi_i}{\eta_t} \xi_i - \frac{1 - \chi_i \psi_i}{1 - \eta_t} \xi_i \right) (\sigma + \sigma_i^q) \left( \frac{C_t}{N_t} + \frac{C_t}{N_t} \right) dt + \left( \frac{\chi_i \psi_i}{\eta_t} - \frac{1 - \chi_i \psi_i}{1 - \eta_t} \right) (\sigma + \sigma_i^q) (dZ_t - (\sigma + \sigma_i^q) dt)$$
Value function iteration

Markov equilibrium should be a fixed point of this, we can iterate (from a reasonable terminal condition) to find it.
Value function iteration

Value functions

Have drifts

Ito ⇒

$$\rho \frac{v_t K_t^{1-\gamma}}{1-\gamma} - C_t^{1-\gamma} =$$

$$\frac{\mu_t^v + (1-\gamma) \left( (\Phi(t) - \delta - \frac{\gamma}{2} \sigma^2 + \sigma \sigma_t^v) \right)}{1-\gamma}$$

Hence, we can calculate

$$\mu_t^v$$ and $$\mu_t^v$$

Ito ⇒

$$\mu_t^v v(\eta, t) = \mu_t^v \eta v_\eta(\eta, t) + \frac{(\sigma_t^v \eta)^2}{2} v_{\eta \eta}(\eta, t) + v_t(\eta, t)$$

$$\mu_t^v v(\eta, t) = \mu_t^v \eta v_\eta(\eta, t) + \frac{(\sigma_t^v \eta)^2}{2} v_{\eta \eta}(\eta, t) + v_t(\eta, t).$$
Advantages of the “Iterative method”

1) Easy to extend to multiple interdependent functions (prices of other assets, more state variables)
2) It resembles familiar value function iteration
3) Boundary conditions are easy to accommodate
Solving the PDE

\[ \mu_t^x v(\eta, t) = \mu_t^\eta \eta v_\eta(\eta, t) + \frac{(\sigma_t^\eta \eta)^2}{2} v_{\eta\eta}(\eta, t) + v_t(\eta, t) \]

\[ \mu_t^\eta v(\eta, t) = \mu_t^\eta \eta v_\eta(\eta, t) + \frac{(\sigma_t^\eta \eta)^2}{2} v_{\eta\eta}(\eta, t) + v_t(\eta, t). \]

- Choose any reasonable terminal conditions for very large T, and solve backwards (until convergence)
- Fixed grid over η, on which 1st and 2nd derivatives are evaluated using finite differences
- Explicit scheme: solve as a system of 1st-order ODEs in time, one equation for each grid point η
- Which method to use to solve the ODEs? The simplest Euler method is adequate, since we are looking for a fixed point
  \[ v(\eta, t - \Delta t) = v(\eta, t) - v_t(\eta, t) \Delta t \]
- Very important: evaluate the 1st η-derivative (left or right) according to the sign of µ^n (this is crucial for numerical stability)
Implicit Method

\[ \mu^\eta_t v(\eta, t) = \mu^\eta_t v_\eta(\eta, t) + \frac{(\sigma^\eta_t \eta)^2}{2} v_{\eta\eta}(\eta, t) + v_t(\eta, t) \]

• Given \( v(\eta, t) \), find \( v(\eta, t - \Delta t) \) that solves this equation, but with \( \eta \)-derivatives evaluated at \( t - \Delta t \), not \( t \).
• This involves solving a linear equation for \( v(\bullet, t - \Delta t) \)
Remarks

• Again: it is very important to use the first derivative of $\theta$ in the correct direction

• With explicit method, stability requires that time step must be $O(d\eta^2)$ – amount of time it takes to reach the next grid point with volatility. This makes the explicit method slow for large number of grid points (time $O(N^3)$)

• Implicit method can work a lot faster, $O(N)$
Example: γ = 2, r = 5%, ρ = 6%, a = 11%, a̅ = 3%, δ = 5%, σ = 10%, χ = 0.5, \( \Phi(ι) = \log(κι + 1)/κ \) with κ = 10
Example: $\gamma = 2$, $r = 5\%$, $\rho = 6\%$, $a = 11\%$, $\bar{a} = 3\%$, $\delta = 5\%$, $\sigma = 10\%$, $5\%$, $1\%$, $\chi = 0.5$, $\Phi(i) = \log(\kappa i + 1)/\kappa$ with $\kappa = 10$
Example: $\gamma = 2$, $r = 5\%$, $\rho = 6\%$, $a = 11\%$, $\underline{a} = 3\%$, $\delta = 5\%$, $\sigma = 10\%$, $\chi = 0.5$, $0.2$, $0.1$, $\Phi(\iota) = \log(\kappa \iota + 1)/\kappa$ with $\kappa = 10$
Example: $\gamma = 2$, $r = 5\%$, $\rho = 6\%$, $a = 11\%$, $\bar{a} = 3\%$, $-3\%$, $-9\%$, $\delta = 5\%$, $\sigma = 10\%$, $\chi = 0.5$, $\Phi(i) = \log(\kappa i + 1)/\kappa$ with $\kappa = 10$
Example: \( \gamma = 2, 5, .5, r = 5\%, \rho = 6\%, a = 11\%, \alpha = 3\%, \delta = 5\%, \sigma = 10\%, \chi = 0.5, \Phi(i) = \log(\kappa i + 1)/\kappa \) with \( \kappa = 10 \)
Conclusion

- Continuous time offers a powerful methodology to analyze heterogeneous-agent models with financial frictions
- Models are capable of generating crises
  - can measure fraction of time spent in crises, expected time to crisis from steady state, speed of recovery
- Endogenous risk-taking leads to paradoxes
  - surprisingly, dynamics not too sensitive to exogenous risk $\sigma$
  - endogenous risk persists even as $\sigma \to 0$
- Computation
  - Method based on value function iteration (for each type of agents) has the advantage that it can work in a similar fashion in many models of this class