Millions and Billions of Martingales: Macroeconomic Uncertainty Prices when Beliefs are Tenuous[1]

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[1] Millions $\sim S_t$, Billions $\sim U_t$

Question from a friend

Why should the agents in your models be like you?

A quote

. . . being based on so flimsy a foundation, a practical theory of the future . . . is subject to sudden and violent changes. *Quarterly Journal of Economics*, Feb. 1937, pp. 214–215.

Barnett, Brock, Hansen

Find it advantageous to explore three components to uncertainty:

- **risk** - uncertainty *within* each model: uncertain outcomes with known probabilities
- **ambiguity** - uncertainty *across* models: unknown weights for alternative possible models
- **misspecification** - uncertainty *about* models: unknown flaws of approximating models

Risks, Returns, Mistakes

- Rational expectations and risk aversion:
  - Higher variances of returns are compensated by higher mean returns
- Representative investor with apparently wrong beliefs:
  - observed average returns depend on both risk aversion and incompletely misunderstood returns distribution
- Effects of *risk aversion* and *distorted beliefs* are confounded
- Belief distortions form a theory of *"stochastic discount factor shocks"*
- Belief distortions make uncertainty prices countercyclical

"The Market's" Markovian baseline model

- $d \log K_t = \left[\widehat{\alpha}_k + \widehat{\beta}_k Z_t + \frac{I_t}{K_t} - \phi\left(\frac{I_t}{K_t}\right) - \frac{|\sigma_k|^2}{2}\right] dt + \sigma_k \cdot dW_t$
- $C_t = \kappa K_t - I_t$
- $dZ_t = \left(\widehat{\alpha}_z - \widehat{\beta}_z Z_t\right) dt + \sigma_z \cdot dW_t$
- stationary distribution for *long-run risk Z* is normal with mean $\bar{z} = \widehat{\alpha}_z / \widehat{\beta}_z$ and variance $|\sigma_z|^2/(2\widehat{\beta}_z)$
- The agent and the econometrician (Lars) share this model

Alternative Structured (i.e., Parametric) Models

- Alternative model indexed by drift distortion S

$$
\begin{aligned}
d\log K_t &= \left[\alpha_k + \beta_k Z_t + \frac{I_t}{K_t} - \phi\left(\frac{I_t}{K_t}\right) - \frac{|\sigma_k|^2}{2}\right] dt + \sigma_k \cdot dW_t^S \\
dZ_t &= (\alpha_z - \beta_z Z_t)\, dt + \sigma_z \cdot dW_t^S
\end{aligned} \tag{1}
$$

- Brownian motions W and W^S are related by

$$
dW_t = S_t dt + dW_t^S \tag{2}
$$

Uncertainty Prices

The equilibrium stochastic discount factor process *Sdf* for our robust representative investor economy is

$$d \log Sdf_t = -\delta dt - .01 \left(\widehat{\alpha}_c + \widehat{\beta}_c Z_t \right) dt - .01 \sigma_c \cdot dW_t + U_t^* \cdot dW_t - \frac{1}{2} |U_t^*|^2 dt$$

minus stochastic discount factor exposure	=	$.01\sigma_c$ **risk price**	$-U_t^*$, **uncertainty price**

$$U_t^* = S_t^* + (U_t^* - S_t^*)$$

Worst-Case Drift Distortions S_t^* and U_t^*

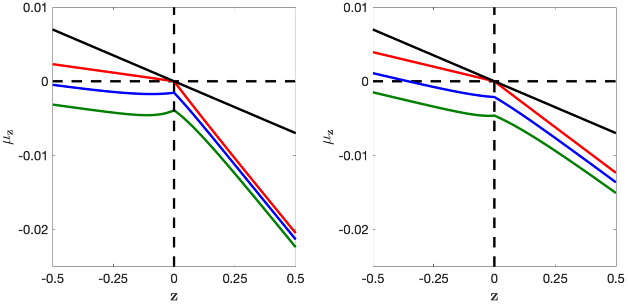

Figure: Worst-case structured model growth rate drifts. Left panel: larger structured entropy ($q_{s,0} = .1$). Right panel: smaller structured entropy ($q_{s,0} = .05$). The penalty parameter θ reset to hit two different targeted values of $q_{u,s}$. **Red**: worst-case structured model; **blue**: $q_{u,s} = .1$; and **green**: $q_{u,s} = .2$.

Worst-Case Drift Distortions S_t^* and U_t^*

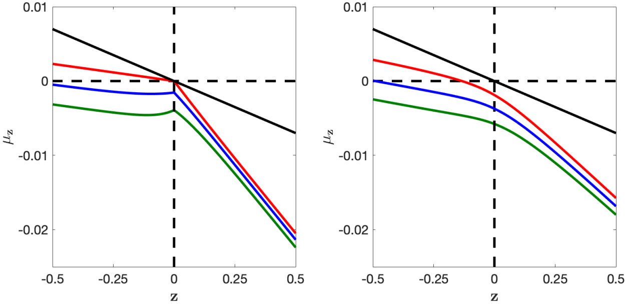

Figure: Distorted growth rate drift for Z. Relative entropy $\mathsf{q}_{s,0} = .1$. Left panel: $\rho_2 = \frac{(.01)}{|\sigma_z|^2}$. Right panel: $\rho_2 = \frac{(.01)}{2|\sigma_z|^2}$. **red**: worst-case structured model; **blue**: $\mathsf{q}_{u,s} = .1$; and **green**: $\mathsf{q}_{u,s} = .2$.

Cons Growth Rate Conditional Densities

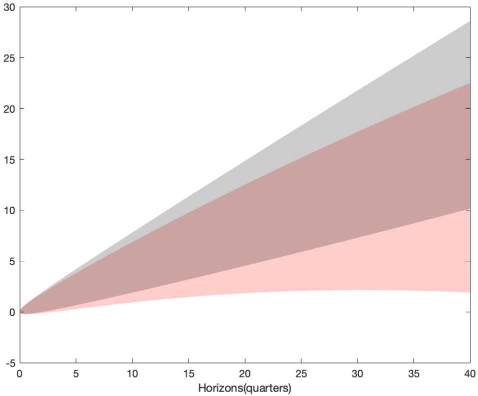

Figure: Distribution of $Y_t - Y_0$ under the baseline model and worst-case model for $q_{s,0} = .1$ and $q_{u,s} = .2$. The gray shaded area depicts the interval between the .1 and .9 deciles for every choice of the horizon under the baseline model. The red shaded area gives the region within the .1 and .9 deciles under the worst-case model.

Calibration via Chernoff entropy

$q_{s,0}$	$q_{u,s}$	$d_{u,s}$	half life u, s	$q_{u,0}$	$d_{u,0}$	half life $u, 0$
.10	.10	.0010	668	.33	.0035	198
.10	.20	.0049	142	.62	.0116	60
.05	.10	.0011	631	.19	.0024	289
.05	.20	.0048	144	.36	.0082	84

Table: Entropies and half lives. $\frac{1}{2}q^2$ measures relative entropy and d measures Chernoff entropy. The subscripts denote the probability models used in performing the computations.

Uncertainty Inside Model

- Baseline *structured* probability model.
- Rectangular set of *structured* probability models.
- Non-rectangular set of *unstructured* probability models constrained by relative entropy

Decision Problem

To construct a set of models, the decision maker:

1) Begins with a Markovian baseline model.
2) Creates from the baseline model a set $\mathcal{M}^o$ of *structured* models by naming a sequence of closed convex sets $\{\Xi_t\}$ and associated drift distortion processes $\{S_t\}$ that satisfy the structured model constraint.
3) Augments $\mathcal{M}^o$ with additional *unstructured* models that violate the structured model constraint but according to discrepancy measure $\Theta(M^U|\mathcal{F}_0)$ are statistically close to models that do satisfy it.

A tension and how we resolve it

- Dynamic consistency
- Admissibility
- Dynamic variational preferences of MMR

Why we want admissibility

- Recommendation by Alan Turing's colleague I.J. Good (1952)
- Sims' "elephant gun" criticism of Hansen

Baseline Model Structure

- Stochastic process $X \doteq \{X_t : t \geq 0\}$ with

$$dX_t = \widehat{\mu}(X_t)dt + \sigma(X_t)dW_t$$

- Plan $\{C_t : t \geq 0\}$ is progressively measurable with respect to filtration $\mathcal{F} = \{\mathcal{F}_t : t \geq 0\}$
- Represent a likelihood ratio by the positive martingale M^U with respect to the baseline Brownian motion specification

$$dM_t^U = M_t^U U_t \cdot dW_t$$

or

$$d \log M_t^U = U_t \cdot dW_t - \frac{1}{2}|U_t|^2 dt,$$

where U is progressively measurable with respect to the filtration $\mathcal{F}$

Likelihood Ratio

- After imposing that $M_0^U = 1$, we can express the solution of M^U's stochastic differential equation as

$$M_t^U = \exp\left(\int_0^t U_\tau \cdot dW_\tau - \frac{1}{2}\int_0^t |U_\tau|^2 d\tau\right)$$

- Associated with U are probabilities defined by the conditional mathematical expectations

$$E^U\left[B_t|\mathcal{F}_0\right] = E\left[M_t^U B_t|\mathcal{F}_0\right]$$

A Set of Martingales

Definition

$\mathcal{M}$ denotes the set of all martingales M^U constructed as stochastic exponentials via representation (4) with a U that satisfies (3) and is progressively measurable with respect to $\mathcal{F} = \{\mathcal{F}_t : t \geq 0\}$.

$$\int_0^t |U_\tau|^2 d\tau < \infty \tag{3}$$

$$M_t^U = \exp\left(\int_0^t U_\tau \cdot dW_\tau - \frac{1}{2}\int_0^t |U_\tau|^2 d\tau\right). \tag{4}$$

A Distorted Model

- Under the baseline model, W has standard Brownian motion, but under U
$$dW_t = U_t dt + dW_t^U,$$
- We can then write
$$d \log M_t^U = U_t \cdot dW_t^U - \frac{1}{2}|U_t|^2 dt.$$
- The distorted model can be expressed as
$$dX_t = \widehat{\mu}(X_t)dt + \sigma(X_t) \cdot U_t dt + \sigma(X_t) dW_t^U.$$

Statistical Discrepancies

- We use a log likelihood ratio $\log M_t^U - \log M_t^S$ with respect to a martingale M_t^S generated by a distortion process S to arrive at

$$E\left[M_t^U\left(\log M_t^U - \log M_t^S\right)|\mathcal{F}_0\right] = \frac{1}{2}E\left(\int_0^t M_\tau^U|U_\tau - S_\tau|^2 d\tau\Big|\mathcal{F}_0\right)$$

- When the limit exists, relative entropy is

$$\begin{aligned}
&\lim_{t\to\infty}\frac{1}{t}E\left[M_t^U\left(\log M_t^U - \log M_t^S\right)\Big|\mathcal{F}_0\right] \\
&= \lim_{t\to\infty}\frac{1}{2t}E\left(\int_0^t M_\tau^U|U_\tau - S_\tau|^2 d\tau\Big|\mathcal{F}_0\right) \\
&= \lim_{\delta\downarrow 0}\frac{\delta}{2}E\left(\int_0^\infty \exp(-\delta\tau)M_\tau^U|U_\tau - S_\tau|^2 d\tau\Big|\mathcal{F}_0\right).
\end{aligned}$$

Statistical Discrepancy from Structured Set

- We define a discrepancy between two martingales M^U and M^S as:

$$\Delta\left(M^U; M^S | \mathcal{F}_0\right) = \frac{\delta}{2}\int_0^\infty \exp(-\delta t) E\left(M_t^U \mid U_t - S_t \mid^2 \middle| \mathcal{F}_0\right) dt.$$

- For a real number $\theta > 0$, define a scaled discrepancy of martingale M^U from a set of martingales $\mathcal{M}^o$ as

$$\Theta(M^U | \mathcal{F}_0) = \theta \inf_{M^S \in \mathcal{M}^o} \Delta\left(M^U; M^S | \mathcal{F}_0\right).$$

- The discrepancy measure $\Theta(M^U|\mathcal{F}_0)$ defines a set of unstructured models that are near the set $\mathcal{M}^o$
- θ measures the decision maker's penalty on her malevolent alter ego for distorting probabilities relative to models in $\mathcal{M}^o$

Hansen's New Way of Constructing a Rectangular Set of Models

- **Secret Weapon:** Construct $\rho(z)$ function from Hansen (2012) and use it to refine relative entropy
- **Lazy Comment:** Decision maker could stop here and use Gilboa-Schmeidler max-min expected utility
- Decision maker doesn't stop here because he fears all of the structured models are misspecified

Log-Likelihood Ratio Process

- For $S_t = \eta(X_t)$, a log-likelihood ratio process is

$$\begin{aligned} L_t &= \int_0^t \eta(X_\tau) \cdot dW_\tau - \frac{1}{2} \int_0^t |\eta(X_\tau)|^2 d\tau \\ &= \int_0^t \eta(X_\tau) \cdot dW_\tau^S + \frac{1}{2} \int_0^t |\eta(X_\tau)|^2 d\tau \end{aligned}$$

- Relative entropy is the limiting average of L_t under M^S probability

Hansen Decomposition of Log-Likelihood Ratio Process

- The process L_t has an additive structure for which there exists the decomposition

$$L_t = \frac{\mathsf{q}^2}{2} t + D_t + \rho(X_0) - \rho(X_t)$$

where

$$D_t = \int_0^t \left[\left(\frac{\partial \rho}{\partial x}(X_\tau) \right)' + \eta(X_\tau) \right] \cdot dW_\tau^S .$$

Log-Likelihood Ratio Long-Horizon Expectation

- Subtracting the time trend and taking date zero conditional expectations under M^S gives

$$
\begin{aligned}
&\lim_{t\to\infty}\left[E\left(M_t^S L_t | X_0 = x\right) - \frac{\mathsf{q}^2}{2} t\right] \\
=&\lim_{t\to\infty} E\left(M_t^S [D_t - \rho(X_t)] \mid X_0 = x\right) + \rho(x) \\
=&\rho(x) - \int \rho dQ,
\end{aligned}
$$

- Q is the limiting stationary distribution under the M^S probability in that

$$
\lim_{t\to\infty} E\left(M_t^S \rho(X_t) | X_0 = x\right) = \int \rho dQ.
$$

Family $\mathcal{M}^o$ of Structured Models

- Create a family of structured probabilities by forming a set of martingales with respect to a baseline probability. So

$$\mathcal{M}^o = \left\{ M^S \in \mathcal{M} \text{ such that } S_t \in \Xi_t \text{ for all } t \geq 0 \right\}$$

- The (undiscounted) entropy for a stochastic process M^S relative to the baseline model is:

$$\varepsilon(M^S) = \lim_{t \to \infty} \frac{1}{2t} \int_0^t E\left(M_\tau^S |S_\tau|^2 \middle| \mathcal{F}_0 \right) d\tau.$$

- ε is the limit as $t \to +\infty$ of a process of mathematical expectations of time series averages

$$\frac{1}{2t} \int_0^t |S_\tau|^2 d\,\tau$$

A Family $\mathcal{M}^o$ of Structured Models

- The infinitesimal generator $\mathcal{A}$ of transitions under the M^S probability is the second-order differential operator:

$$\mathcal{A}^s \rho = \frac{\partial \rho}{\partial z} \cdot (\widehat{\mu} + \sigma s) + \frac{1}{2} \text{trace} \left(\sigma' \frac{\partial^2 \rho}{\partial z \partial z'} \sigma \right)$$

  for $s = \eta(z)$

- Then

$$\mathcal{A}^s \rho = \frac{\mathsf{q}^2}{2} - \frac{|s|^2}{2},$$

  where relative entropy $\varepsilon(M^S) = \frac{\mathsf{q}^2}{2}$ and $\frac{|s|^2}{2}$ measures the magnitude of the corresponding drift distortion.

The function $\rho(z)$

- The function $\rho(z)$ is a long-horizon refinement of relative entropy in the sense that

$$\rho(z) - \int \rho dQ = \lim_{t \to \infty} \frac{1}{2} \int_0^t E\left(M_\tau^S |S_\tau|^2 - \mathrm{q}^2 \mid Z_0 = z \right),$$

  - Q is the stationary distribution for the probability associated with the $S_t = \eta(Z_t)$ model

A Rectangular Family $\mathcal{M}^o$ of Structured Models

- We restrict the S process in terms of an $\mathcal{F}_t$-measurable sequence of convex sets

$$\Xi_t = \left\{ s : \mathcal{A}^s \rho(Z_t) \leq \frac{\mathsf{q}^2}{2} - \frac{|s|^2}{2} \right\}$$

- The boundary of Ξ includes models with the same long-horizon relative entropy $\frac{\mathsf{q}^2}{2}$ and the same refinement $\rho(z) - \int \rho dQ$ of relative entropy
- We could stop here but don't because decision maker wants to investigate *unstructured* models outside set.

Misspecification of structured models

Structured models in terms of S_t appear separately from unstructured models in terms of U_t in statistical discrepancy measures:

- Discrepancy measure

$$\Delta\left(M^U; M^S|\mathcal{F}_0\right) = \frac{\delta}{2}\int_0^\infty \exp(-\delta t) E\left(M_t^U \mid U_t - S_t \mid^2 \Big|\mathcal{F}_0\right) dt$$

- Conditional discrepancy

$$\xi_t(U_t) = \inf_{S_t \in \Xi_t} |U_t - S_t|^2$$

- Scaled integrated discounted discrepancy

$$\Theta\left(M^U|\mathcal{F}_0\right) = \frac{\theta\delta}{2}\int_0^\infty \exp(-\delta t) E\left[M_t^U \xi_t(U_t)\Big|\mathcal{F}_0\right] dt$$

Recursive Representations of Preferences and Decisions

For a consumption plan $\{C_t\}$, the continuation value process $\{V_t\}_{t=0}^{\infty}$ is

$$V_t = \min_{\{U_\tau : t \leq \tau < \infty\}} E\Bigg(\int_0^\infty \exp(-\delta\tau) \left(\frac{M_{t+\tau}^U}{M_t^U} \right) \times \left[\psi(C_{t+\tau}) + \left(\frac{\theta\delta}{2} \right) \xi_{t+\tau}(U_{t+\tau}) \right] d\tau \mid \mathcal{F}_t \Bigg)$$

- ψ is an instantaneous utility function. We will set it to equal log in the following calculations
- $\xi_t(U_t) = \inf_{S_t \in \Xi_t} |U_t - S_t|^2$

Recursive Representations of Preferences and Decisions

The recursive structure of the value function means it can be expressed as

$$V_t = \min_{\{U_\tau : t \le \tau < t+\epsilon\}} \Bigg\{ E\left[\int_0^\epsilon \exp(-\delta\tau) \left(\frac{M^U_{t+\tau}}{M^U_t} \right) \times \right.$$

$$\left. \left[\psi(C_{t+\tau}) + \left(\frac{\theta\delta}{2} \right) \xi_{t+\tau}(U_{t+\tau}) \right] d\tau \mid \mathcal{F}_t \right] +$$

$$\exp(-\delta\epsilon) E\left[\left(\frac{M^U_{t+\epsilon}}{M^U_t} \right) V_{t+\epsilon} \mid \mathcal{F}_t \right] \Bigg\}$$

- View this as an Ito process to write $dV_t = \nu_t dt + \varsigma_t \cdot dW_t$
- A local counterpart to this is

$$0 = \min_{U_t} \left[\psi(C_t) - \frac{\theta\delta}{2} \xi_t(U_t) - \delta V_t + U_t \cdot \varsigma_t + \nu_t \right]$$

Markovian baseline model

- $d \log K_t = \left[\widehat{\alpha}_k + \widehat{\beta}_k Z_t + \frac{I_t}{K_t} - \phi\left(\frac{I_t}{K_t}\right) - \frac{|\sigma_k|^2}{2} \right] dt + \sigma_k \cdot dW_t$
- $C_t = \kappa K_t - I_t$
- $dZ_t = \left(\widehat{\alpha}_z - \widehat{\beta}_z Z_t \right) dt + \sigma_z \cdot dW_t$
- Stationary distribution for *long-run risk Z* is normal with mean $\bar{z} = \widehat{\alpha}_z / \widehat{\beta}_z$ and variance $|\sigma_z|^2/(2\widehat{\beta}_z)$

Structured Parametric Models

- Write state evolution in terms of structured model S

$$d \log K_t = \left[\alpha_k + \beta_k Z_t + \frac{I_t}{K_t} - \phi\left(\frac{I_t}{K_t}\right) - \frac{|\sigma_k|^2}{2}\right] dt + \sigma_k \cdot dW_t^S$$
$$dZ_t = (\alpha_z - \beta_z Z_t)\, dt + \sigma_z \cdot dW_t^S, \tag{5}$$

- Brownian motions W and W^S are related by

$$dW_t = S_t dt + dW_t^S, \tag{6}$$

Structured Parametric Models

- Represent members of a parametric class in terms of our structure with drift distortions S of the form

$$S_t = \eta(Z_t) \equiv \eta_0 + \eta_1(Z_t - \bar{z})$$

- Deduce the following restrictions on η_1:

$$\sigma\eta_1 = \begin{bmatrix} \beta_k - \widehat{\beta}_k \\ \widehat{\beta}_z - \beta_z \end{bmatrix},$$

where

$$\sigma = \begin{bmatrix} (\sigma_k)' \\ (\sigma_z)' \end{bmatrix}.$$

Structured Parametric Models

To compute relative entropy $\frac{\mathsf{q}^2}{2}$ and the function $\rho(z)$, we apply the method of undetermined coefficients to solve the following differential equation:

$$\frac{d\rho}{dz}(z)[-\widehat{\beta}_z(z-\bar{z})+\sigma_z\cdot\eta(z)]+\frac{|\sigma_z|^2}{2}\frac{d^2\rho}{dz^2}(z)-\frac{\mathsf{q}^2}{2}+\frac{|\eta(z)|^2}{2}=0. \quad (7)$$

Under parametric alternatives (5), ρ is quadratic in $z-\bar{z}$:

$$\rho(z)=\rho_1(z-\bar{z})+\frac{1}{2}\rho_2(z-\bar{z})^2.$$

We first compute ρ_1 and ρ_2 by matching coefficients on the terms $(z-\bar{z})$ and $(z-\bar{z})^2$, respectively. Matching constant terms then implies $\frac{\mathsf{q}^2}{2}$.

HJB Equation with Structured Uncertainty Only

If misspecifications of the structured models were not of concern, we would be led to solve the following Hansen-Jacobi-Bellman (HJB) equation:

$$0 = \max_i \min_s \Big\{ \delta \log(\kappa - i) - \delta \widehat{\Psi}(z) + \widehat{\alpha}_k + \widehat{\beta}_k z + i - \phi(i) + \sigma_k \cdot s$$
$$+ [-\widehat{\beta}_z (z - \bar{z}) + \sigma_z \cdot s] \frac{d\widehat{\Psi}}{dz}(z) + \frac{1}{2} |\sigma_z|^2 \frac{d^2 \widehat{\Psi}}{dz^2}(z) \Big\},$$

where i is a potential choice of the investment-capital ratio and s is a potential choice of the structured drift distortion. To assure that $s \in \Xi_t$, we impose that (ρ_1, ρ_2) satisfies:

$$[\rho_1 + \rho_2 (z - \bar{z})] \left[-\widehat{\beta}_z (z - \bar{z}) + \sigma_z \cdot s \right] + \frac{|\sigma_z|^2}{2} \rho_2 - \frac{\mathsf{q}^2}{2} + \frac{s \cdot s}{2} \leq 0$$

The boundary of this set is an ellipsoid.

Structured Parametric Models

1. Fixing $(\rho_1, \rho_2, \mathsf{q})$, we can trace out a one-dimensional family of parametric models having the same relative entropy
2. Given $(\rho_1, \rho_2, \mathsf{q})$, we can first solve equation (7) for η_0 and η_1
3. Matching terms gives three equations in four unknowns that imply a one dimensional curve for η_0 and η_1 that imply nonlinear S_t's as functions of z

In this way, nonlinear structured models are included in the set of structured models near the baseline model as measured by relative entropy. These nonlinear models also have relative entropy $\frac{\mathsf{q}^2}{2}$. We can represent the resulting nonlinear model as a time-varying coefficient model by solving

$$r^*(z) = \sigma \left[\eta_0 + \eta_1(z - \bar{z})\right]$$

An Illustration

1. Suppose that the decision maker sets

$$\eta(z) = \eta_1(z - \bar{z}),$$

2. In this case, $\rho_1 = 0$ and the restriction on (ρ_1, ρ_2) becomes

$$-\frac{\mathsf{q}^2}{2} + \frac{|\sigma_z|^2}{2}\rho_2 = 0$$

   or equivalently,

$$\rho_2 = \frac{\mathsf{q}^2}{|\sigma_z|^2}.$$

3. Notice that restriction on (ρ_1, ρ_2) implies that

$$s = 0$$

   when $z = \bar{z}$. Also given $|\sigma_z|^2$, the value of ρ_2 is determined by q. More generally, q and ρ cannot be specified independently.

Illustration (Continued)

4 Construct the convex set of η_1's that satisfy

$$\frac{1}{2}\eta_1 \cdot \eta_1 + \left(\frac{\mathsf{q}^2}{|\sigma_z|^2}\right)\left(-\widehat{\beta}_z + \sigma_z \cdot \eta_1\right) \leq 0. \tag{8}$$

5 Form the boundary of this convex set of alternative parameter configurations

$$\sigma\eta_1 = \begin{bmatrix} \beta_k - \widehat{\beta}_k \\ \widehat{\beta}_z - \beta_z \end{bmatrix}$$

for (β_k, β_z) associated with alternative choices of η_1

Parameter Countours

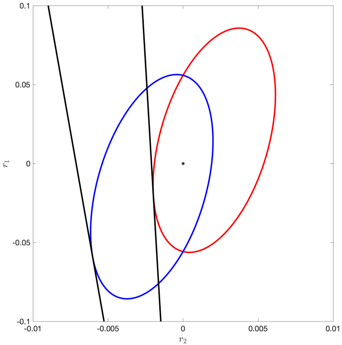

Figure: An illustration for a configuration for $\mathsf{q}_{s,0} = .1$ and $\mathsf{q}_{u,s} = .2$, showing parameter contours for (r_1, r_2) holding relative entropies fixed. The upper right contour depicted in red is for z equal to the .1 quantile of its stationary distribution under the baseline model and the lower left contour is for z at the .9 quantile. The dot depicts the $(r_1, r_2) = (0, 0)$ point corresponding to the baseline model. Tangency points denote worst-case structured models.

Concerns that Structured Models are all Misspecified

The decision maker adds unstructured models via a penalized entropy term.

- The resulting HJB is

$$0 = \max_i \min_{u,s} \Big\{ \delta \log(\kappa - i) - \delta \widehat{\Psi}(z) + \widehat{\alpha}_k + \widehat{\beta}_k z + i - \phi(i) + \sigma_k \cdot u]$$
$$+ [-\widehat{\beta}_z (z - \bar{z}) + \sigma_z \cdot u] \frac{d\widehat{\Psi}}{dz}(z) + \frac{1}{2} |\sigma_z|^2 \frac{d^2 \widehat{\Psi}}{dz^2}(z) + \frac{\theta}{2} |u - s|^2 \Big\}$$

  where s is constrained

- First-order conditions for minimizing with respect to u imply

$$u = s + \sigma' \begin{bmatrix} 1 \\ \frac{d\widehat{\Psi}}{dz}(z) \end{bmatrix}.$$

Misspecified Structured Models

- Substituting this choice of u into the HJB equation leads to

$$
\begin{aligned}
0 = \max_i \min_s \Big\{ & \delta \log(\kappa - i) - \delta \widehat{\Psi}(z) + \widehat{\alpha}_k + \widehat{\beta}_k z + i - \phi(i) + \sigma_k \cdot s \\
& + [-\widehat{\kappa}(z - \bar{z}) + \sigma_z \cdot s] \frac{d\widehat{\Psi}}{dz}(z) + \frac{1}{2} |\sigma_z|^2 \frac{d^2 \widehat{\Psi}}{dz^2}(z) \\
& - \frac{\theta}{2} \begin{bmatrix} 1 & \frac{d\widehat{\Psi}}{dz}(z) \end{bmatrix} \sigma \sigma' \begin{bmatrix} 1 \\ \frac{d\widehat{\Psi}}{dz}(z) \end{bmatrix} \Big\}
\end{aligned}
$$

- Maximization and minimization are both subject to

$$
[\rho_1 + \rho_2(z - \bar{z})] \left[-\widehat{\beta}_z (z - \bar{z}) + \sigma_z \cdot s \right] + \frac{|\sigma_z|^2}{2} \rho_2 - \frac{\mathsf{q}^2}{2} + \frac{s \cdot s}{2} \leq 0.
$$

Structured Models and a Robust Plan

We solve HJB equation

$$0 = \max_i \min_s \left\{ \delta \log(\kappa - i) - \delta \widehat{\Psi}(z) + \widehat{\alpha}_k + \widehat{\beta}_k z + i - \phi(i) + \sigma_k \cdot s \right.$$
$$\left. + [-\widehat{\beta}_z (z - \bar{z}) + \sigma_z \cdot s] \frac{d\widehat{\Psi}}{dz}(z) + \frac{1}{2} |\sigma_z|^2 \frac{d^2 \widehat{\Psi}}{dz^2}(z) \right\}$$

for three different configurations of structured models We compute a solution by first focusing on a specification in which $\rho_1 = 0$ and ρ_2 satisfies:

$$\rho_2 = \frac{\mathsf{q}^2}{|\sigma_z|^2}$$

where here we use q as a synonym for $\mathsf{q}_{s,0}$. When η is restricted to be $\eta_1(z - \bar{z})$, a given value of q imposes a restriction on η_1 and implicitly on (β_c, β_k)

Structured Models Parameter Contour

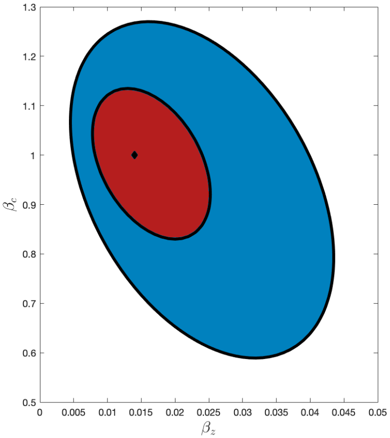

Figure: Parameter contours for (β_c, β_k) holding relative entropy $q_{s,0}$ fixed. The outer curve depicts $q_{s,0} = .1$ and the inner curve $q_{s,0} = .05$. The small diamond depicts the baseline model.

Expanded Caption

Figure 4 also reported iso-entropy contours when z is at the .1 and .9 quantile of the stationary distribution under the baseline model. The larger value of z results in a downward shift of the contour relative to the smaller value of z. The points of tangency in Figure 4 are the worst-case structured models. A tangency point occurs at a lower drift distortion for the .9 quantile than for the .1 quantile.