Technology, Skill and Long Run Growth

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Abstract

This paper develops a model in which heterogeneous firms invest in technology to increase their profits, and heterogeneous workers invest in human capital to increase their earnings. Production functions are log supermodular in technology and human capital, so the competitive equilibrium features positively assortative matching between firms and workers. Continued investment in technology is profitable only because human capital is growing, and continued investment in human capital is worthwhile only because technology is growing. Both investment technologies have stochastic components, and the balanced growth path has stationary, nondegenerate distributions of technology and human capital, with both growing at a common, constant rate.

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1. OVERVIEW

This paper develops a model in which heterogeneous firms invest in technology to increase their profits, and heterogeneous workers invest in human capital to increase their earnings. Production functions are log supermodular in technology and human capital, so the competitive equilibrium features positively assortative matching between firms and workers. Continued investment in technology is profitable only because human capital is growing, and continued investment in human capital is worthwhile only because technology is growing. Both investment technologies have stochastic components, and the balanced growth path has stationary, nondegenerate distributions of technology and human capital, with both inputs growing at a common, constant rate.

The contribution of the paper is to characterize the interplay between human accumulation and technical change as contributors to long-run growth. On the technology side, there are two types of investment, which can be viewed as process and product innovation. Incumbent firms invest to improve their productivity—process innovation, and they die stochastically. In addition, entering firms invest to obtain technologies for new goods—product innovation, with entry governed by a zero profit condition. On the human capital side, both incumbent and entering workers engage in the same type of investment, to improve their existing skill or obtain initial skill. Population growth is exogenous.

The first main result is to show the existence and uniqueness of a balanced growth path. On the BGP, the (common) growth rate of technology and skill depends mainly on parameters governing the skill distribution and the investment process for skill, while growth in the number of varieties is governed by differences between the parameters governing the two distributions and the two investment processes.

2. RELATED LITERATURE

In almost all of the endogenous growth literature, growth has only one source: either human capital or innovations in technology. The framework analyzed here provides a link between those groups of models.

Among the human capital models, growth can arise from on-the-job learning, as in the LBD models of Arrow (1962), Stokey (1988), Young (1991, 1993), Matsuyama (1992). In others, human capital accumulation competes with production as a use of time, as in Uzawa (1965), Romer (1986), Lucas (1988, 2009), Lucas and Moll (2014), Perla and Tonetti (2014), and others.

In the literature on technology-driven growth, some models emphasize creative destruction, as in Romer (1990), Grossman and Helpman (1991), Aghion and Howitt (1992), Jones (1995), Stokey (1995), Acemoglu (2002), and Klette and Kortum (2004), while in others quality improvements are critical, as in Atkeson and Burstein (2010) and Luttmer (2007).

The model here is also related to the model of technology and wage inequality in Jovanovic (1998) and the model of skill and technology growth in Lloyed-Ellis and Roberts (2002).

The framework here builds on the model of technology growth across firms in Perla and Tonetti (JPE, 2014), adding a similar investment model on the human capital side.

The rest of the paper is organized as follows. Section 3 describes the model of production and pricing, and characterizes the features of an equilibrium, given the number of workers and firms, and the distributions of skill and technology: the allocation of workers to firms, and the associated wages, prices, and outputs. Lemmas 1-3 establish the existence, uniqueness and efficiency of a production equilibrium, as well as some homogeneity properties. Proposition 4 shows that if the technology and skill distributions are Pareto, with locations that are appropriately aligned, then the production equilibrium has a skill-to-technology allocation that is linear, and wage, price, and allocation functions that are isoelastic.

Section 4 looks at dynamics, the investment decisions of incumbent and entering firms, and of current and entering workers. Section 5 provides formal definitions of a competitive equilibrium and a balanced growth path.

Section 6 specializes to the case where technology and skill have Pareto distributions, showing that the isoelastic forms for the wage and profit functions are inherited by the value functions for firms and workers. This fact leads to a tractable model of investment and the evolution of the technology and skill distributions. Section 7 contains results about the existence and uniqueness of BGPs, and how growth rates of skill/technology and variety depend on various features of the model. Section 8 concludes. Various technical derivations and arguments are gathered in the Appendix.

3. PRODUCTION AND PRICES

The single final good is produced by competitive firms using intermediate goods as inputs. Intermediate goods are produced by heterogeneous, monopolistically competitive firms. Each intermediate firm produces a unique product, and all intermediates enter symmetrically into final good production. But intermediate firms differ in their technology level x, which affects their productivity. Let N_p be the number (mass) of intermediate good producers, and let $F(x)$, with with continuous density f, denote the distribution function for technology.

Intermediate good producers use heterogeneous labor, differentiated by its human capital level h, as the only input. Let L_w, be the number (mass) of workers, and let $G(h)$, with continuous density $\widetilde{g}$, denote the distribution function for human capital. This section looks at the the allocation of labor across firms, and wages, prices, output levels, and profits, given N_p, F, L_w, G.

A. Technologies

Although intermediates enter symmetrically into final good production, demands for them differ if their prices differ. Let $p(x)$ denote the price charged by a firm with technology x. The final goods sector takes these prices as given, and each final good producer has the CRS technology

$$y_F = \left[N_p^{1-\nu} \int y(x)^{(\rho-1)/\rho} f(x) dx \right]^{\rho/(\rho-1)}, \tag{1}$$

where $\rho > 1$ is the substitution elasticity and $\nu \in (0, 1/\rho]$ measures diminishing returns to increased variety. Input demands are

$$y^d(x) = N_p^{-\rho\nu} \left(\frac{p(x)}{p_F} \right)^{-\rho} y_F, \qquad \text{all } x,$$

and prices will be normalized by setting the price of the final good to unity,

$$1 = p_F = \left[N_p^{1-\rho\nu} \int p(x)^{1-\rho} f(x) dx \right]^{1/(1-\rho)}. \tag{2}$$

The output of a firm depends on the size and quality of its workforce, as well as its technology. In particular, if a firm with technology x employs ℓ workers with human capital h, then its output is

$$y = \ell \phi(h, x),$$

where $\phi(h, x)$ is the CES function

$$\phi(h, x) \equiv \left[\omega h^{(\eta-1)/\eta} + (1-\omega) x^{(\eta-1)/\eta} \right]^{\eta/(\eta-1)}, \qquad \eta, \omega \in (0, 1). \tag{3}$$

The elasticity of substitution between technology and human capital is assumed to be less than unity, $\eta < 1$. Firms could employ workers with different human capital levels, and in this case their outputs would simply be summed. In equilibrium firms never choose to do so, however, and for simplicity the notation is not introduced.

B. Intermediate goods: price, output, labor

Let $w(h)$ denote the wage function. For a firm with technology x, the cost of producing one unit of output with labor of quality h is $w(h)/\phi(h, x)$. Optimal labor quality h^* minimizes this expression, so h^* satisfies

$$\frac{w'(h^*)}{w(h^*)} = \frac{\phi_h(h^*, x)}{\phi(h^*, x)}. \tag{4}$$

It is straightforward to show that if the (local, necessary) second order condition for cost minimization holds, then $\eta < 1$ implies h^* is strictly increasing. Unit cost

$$c(x) = \frac{w(h^*(x))}{\phi(h^*(x), x)},$$

is strictly decreasing in x,

$$\frac{c'(x)}{c(x)} = -\frac{\phi_x(h^*(x), x)}{\phi(h^*(x), x)} < 0,$$

where (4) implies that the other terms cancel.

As usual, profit maximization by intermediate good producers entails setting a price that is a markup of $\rho/(\rho-1)$ over unit cost. Output is then determined by demand, and labor input by the production function. Hence price, quantity, labor input, and operating profits for the intermediate firm are

$$\begin{aligned} p(x) &= \frac{\rho}{\rho-1}\frac{w(h^*(x))}{\phi(h^*(x),x)}, \\ y(x) &= N_p^{-\rho\nu} p(x)^{-\rho} y_F, \\ \ell(x) &= \frac{y(x)}{\phi(h(x),x)}, \\ \pi(x) &= \frac{1}{\rho} p(x) y(x), \qquad \text{all } x, \end{aligned} \tag{5}$$

where the price normalization requires (2). Firms with higher technology levels x have lower prices, higher sales, and higher profits. They may or may not employ more labor.

Each worker inelastically supplies one unit of labor. The labor market is competitive, and since the production function in (3) is log supermodular, efficiency requires positively assortative matching (Costinot, 2009). Let x_m and h_m denote the lower bounds for the supports of F and G. Then markets clear for all types of labor if

$$h_m = h^*(x_m), \tag{6}$$

$$L_w\left[1 - G(h^*(x))\right] = N_p \int_x^\infty \ell(\xi) f(\xi) d\xi, \qquad \text{all } x \geq x_m. \tag{7}$$

C. Production equilibrium

At any instant, the economy is described by its production parameters, the number of firms and workers, and the distributions of technology and skill.

Definition: A *production environment* $\mathcal{E}_p$ is described by

i. parameters $(\rho, \nu, \omega, \eta)$, with $\rho > 1$, $\nu \in (0, 1/\rho]$, $\omega \in (0,1)$, $\eta \in (0,1)$;

ii. number of producers $N_p > 0$, and workers $L_w > 0$;

iii. distribution functions $F(x)$ with continuous density $f(x)$ and lower bound x_m on its support, and $G(h)$ with continuous density $g(h)$ and lower bound $h_m \geq 0$ on its support.

A production equilibrium consists of price functions and an allocation that satisfy profit maximization and labor market clearing.

Definition: Given a production environment $\mathcal{E}_p$, the prices $w(h), p(x)$, and allocation $h^*(x)$, $y(x)$, $\ell(x)$, $\pi(x)$, y_F, are a *production equilibrium* if (2) and (4)-(7) hold.

The following result is then straightforward.

Lemma 1: For any production environment $\mathcal{E}_p$, an equilibrium exists and it is unique and efficient.

Proof: Use (5) to write labor demand as

$$\ell(x) = N_p^{-\rho\nu} \left(\frac{\rho - 1}{\rho}\right)^{\rho} \frac{\phi(h^*(x), x)^{\rho-1}}{w(h^*(x))^{\rho}} y_F, \qquad \text{all } x \geq x_m, \tag{8}$$

and differentiate (7) to write labor market clearing as

$$L_w \widetilde{g}(h^*(x)) h^{*\prime}(x) = N_p \ell(x) f(x), \qquad \text{all } x \geq x_m, \tag{9}$$

$$L_w = N_p \int_{x_m}^{\infty} \ell(\xi) f(\xi) d\xi. \tag{10}$$

For any fixed y_F, (4) and (9) are a pair of ODEs in $w(h)$ and $h^*(x)$, with $\ell(x)$ given by (8). The price normalization (2) is a boundary condition for w, and (6) is the boundary condition for h^*. Then (10) determines y_F, and (5) determines p, y, π. ■

D. Homogeneity properties

The analysis of BGPs will exploit the fact that production equilibria have certain homogeneity properties. Lemma 2 deals with parallel shifts in the two distribution functions.

LEMMA 2: Fix $\mathcal{E}_p$, and let $\widehat{\mathcal{E}_p}$ be a production environment with the same parameters and N_p, L_w, but with DF's $\hat{F}, \hat{G}$ satisfying

$$\begin{aligned} \hat{F}(X) &= F(X/Q), \qquad \text{all } X, \\ \hat{G}(H) &= G(H/Q), \qquad \text{all } H, \end{aligned}$$

where

$$Q \equiv \mathrm{E}_{\hat{F}}(X).$$

If w, p, h^*, y, ℓ, π, y_F is the production equilibrium for $\mathcal{E}_p$, then the equilibrium for $\widehat{\mathcal{E}_p}$ is

$$\begin{array}{ll} \hat{w}(H) = Qw(H/Q), & \hat{p}(X) = p(X/Q) \\ \hat{h}^*(X) = Qh^*(X/Q), & \hat{y}(X) = Qy(X/Q), \\ \hat{\ell}(X) = \ell(X/Q), & \hat{\pi}(X) = Q\pi(X/Q), \\ \hat{y}_F = Qy_F, & \text{all } X, H. \end{array}$$

Price and employment for any firm depend only on its relative technology $x = X/Q$, while its labor quality, output, and profits are scaled by Q. Wages and final output are also scaled by Q.

Lemma 3 deals with the effects of changes in the number of producers and workers.

LEMMA 3: Fix $\mathcal{E}_p$, and let $\mathcal{E}_p'$ be a production environment with the same parameters and DF's, but with $L_w' = e^{\upsilon}L_w$ workers and $N_p' = e^n N_p$ producers, where $\upsilon, n > 0$. If $[w,\ p,\ h^*,\ y,\ \ell,\ \pi,\ y_F]$ is the production equilibrium for $\mathcal{E}_p$, then $[e^{\Omega n}w$, $e^{\Omega n}p$, h^*, $e^{\upsilon-n}y$, $e^{\upsilon-n}\ell$, $e^{\upsilon+(\Omega-1)n}\pi$, $e^{\upsilon+\Omega n}y_F]$ is the equilibrium for $\mathcal{E}_p'$, where

$$\Omega \equiv \frac{1-\rho\nu}{\rho-1}. \tag{11}$$

An increase in the number of workers leads to a proportionate increase in employment, output and profits at each firm, and in final output, with wages, prices and the allocation of skill to technology unaffected. An increase in the number of firms leads to a proportionate decrease in employment and output at each individual firm. Final output nevertheless rises, with an elasticity of $\Omega \geq 0$. The price of each intermediate and the wage rate for each type of labor also rise, with an elasticity of Ω. Since $\Omega \in [0, 1/(\rho - 1))$, profits per firm—which reflect both the increase in price and decrease in scale—can change in either direction.

E. Pareto distributions

In this section we will show that if the distribution functions F and G are Pareto, with shape parameters that are not too different and location parameters that are appropriately aligned, the production equilibrium has a linear assignment of skill to technology, and wage, price, and profit functions that are isoelastic.

PROPOSITION 4: Let $\mathcal{E}_p$ be a production environment for which F and G are Pareto distributions with parameters (α, x_m) and (γ, h_m). Assume that $\alpha > 1$, $\gamma > 1$, and

$$-1 < \alpha - \gamma < \rho - 1. \tag{12}$$

Define

$$\varepsilon \equiv \frac{1}{\rho}(1 + \alpha - \gamma) \in (0, 1), \tag{13}$$

$$a_h \equiv \left(\frac{1-\varepsilon}{\varepsilon} \frac{1-\omega}{\omega} \right)^{\eta/(\eta-1)}, \tag{14}$$

and in addition, assume

$$h_m = a_h x_m. \tag{15}$$

Then the production equilibrium for $\mathcal{E}_p$ has price and allocation functions

$$h^*(x) = a_h x, \qquad \text{all } x, \tag{16}$$

$$
\begin{aligned}
w(h) &= N_p^{\Omega} w_0 h^{1-\varepsilon}, && \text{all } h, && (17)\\
p(x) &= N_p^{\Omega} p_0 x^{-\varepsilon}, && \text{all } x, && (18)\\
y(x) &= L_w N_p^{-1} A_0 \phi(a_h, 1) x^{\rho\varepsilon}, && \text{all } x,\\
\ell(x) &= L_w N_p^{-1} A_0 x^{\rho\varepsilon - 1}, && \text{all } x,\\
\pi(x) &= L_w N_p^{\Omega-1} \pi_0 x^{\varepsilon(\rho-1)}, && \text{all } x,
\end{aligned}
$$

where

$$
\begin{aligned}
p_0 &\equiv \left(\frac{\alpha}{\gamma - 1 + \varepsilon}\right)^{1/(\rho-1)} x_m^{\varepsilon},\\
w_0 &\equiv \frac{\rho - 1}{\rho} p_0 \phi(a_h, 1) \frac{1}{a_h^{1-\varepsilon}},\\
A_0 &\equiv \frac{\gamma}{\alpha} x_m^{\gamma-\alpha},\\
\pi_0 &\equiv \frac{1}{\rho} p_0 A_0 \phi(a_h, 1),
\end{aligned}
$$

and final output is

$$
y_F = L_w N_p^{\Omega} A_0 \phi(a_h, 1) p_0^{\rho}.
$$

Proof: For the wage function in (17), and the CES function ϕ in (3), optimal labor quality in (4) is as in (16), with a_h as in (14). Then (5) simplifies to (18), where the constants w_0, p_0, A_0, π_0, involve only exogenous parameters. Hence by Lemma 1, it suffices to show that (2), (6), (9) and (10).

Using the fact that F is a Pareto distribution, the prices p in (18) satisfy (2) if

$$
\begin{aligned}
1 &= N_p^{1-\rho\nu} N_p^{\Omega(1-\rho)} p_0^{1-\rho} \alpha x_m^{\alpha} \int_{x_m}^{\infty} x^{(\rho-1)\varepsilon} x^{-\alpha-1} dx\\
&= p_0^{1-\rho} \alpha x_m^{\alpha} \frac{x_m^{(\rho-1)\varepsilon-\alpha}}{\alpha - (\rho-1)\varepsilon}\\
&= p_0^{1-\rho} \frac{\alpha}{\gamma - 1 + \varepsilon} x_m^{(\rho-1)\varepsilon},
\end{aligned}
$$

which holds for p_0 as above. Similarly, $\ell(x)$ in (18) satisfies (10) if

$$\begin{aligned} L_w &= L_w A_0 \alpha x_m^\alpha \int_{x_m}^{\infty} x^{\rho\varepsilon-1} x^{-\alpha-1} dx \\ &= L_w A_0 \frac{\alpha x_m^{\rho\varepsilon-1}}{\alpha+1-\rho\varepsilon} \\ &= L_w A_0 \frac{\alpha}{\gamma} x_m^{\alpha-\gamma} \end{aligned}$$

which holds for A_0 as above.

Clearly (15) implies (6) holds. Then since (10) holds, (9) is satisfied if

$$\frac{\alpha}{\gamma} x_m^{\rho\varepsilon-1} G'(h^*(x)) a_h = x^{\rho\varepsilon-1} f_p(x), \qquad \text{all } x \geq x_m,$$

or using the Pareto densities,

$$\frac{\alpha}{\gamma} x_m^{\rho\varepsilon-1} \gamma \left(a_h x_m\right)^{\gamma} \left(a_h x\right)^{-\gamma-1} a_h = x^{\rho\varepsilon-1} \alpha x_m^{\alpha} x^{-\alpha-1}, \qquad \text{all } x \geq x_m,$$

which holds for ε in (13). The value for y_F follows from (1). ■

The shape parameters α and γ need not be the same, although that is allowed, but (12) puts a restriction on how different they can be. The isoelastic forms for the wage and profit functions, together with the Pareto forms for the skill and technology distributions, will be important showing the existence of a BGP.

4. DYNAMICS

In this section the dynamic aspects of the model are described: investment decisions for incumbent firms and individual workers, the entry decision, the evolution of the number of firms and workers and the DFs for technology and skill, and the consumption/saving decision of households. As in Perla and Tonetti (2014), investment by an incumbent firm or a worker is a zero-one decision, and the cost of investment is the opportunity cost of not producing (for a firm) or working (for an individual). Hence we must distinguish between the total number of firms and the number of producers. Similarly, we must distinguish between the total population and the number of workers. Only producing firms are identified by a technology level, and only working individuals are identified by a skill level, so DFs are defined only for producers and workers.

Time is continuous and the horizon is infinite, $t \geq 0$. Let $N(t), L(t)$ denote the total number of firms and total population at date t; let $N_p(t), L_w(t)$ denote the number of producers and workers; and let $\hat{F}(X,t)$ and $\hat{G}(H,t)$ denote the DFs for technology among producers and skill among workers. Let $W(H,t)$, $H^*(X,t)$, $P(X,t)$, $Y(X,t)$, $\mathfrak{L}(X,t)$, $\Pi(X,t)$, $Y_F(t)$, $t \geq 0$, denote the wage rate, skill allocation, and so on. Let $r(t)$ denote the interest rate at t.

A. Incumbent firms

Firms exit exogenously at a constant rate δ_x, and exit can occur while a firm is producing or investing. While a firm produces, its technology X grows at a constant rate $\mu_x \geq 0$. A firm that chooses to invest stops producing, abandons its current technology and waits to acquire a new one. Call these firms process innovators. The only cost of process innovation is the opportunity cost. A process innovator cannot later reclaim its old technology, so all process innovators at date t are in the same position and all have the same value, call it $V_{f0}(t)$.

Hence at any date t, producing firms with technologies below some threshold $X_m(t)$ become process innovators, while those with technologies above the threshold continue to produce. It follows that at any date t, the value of an incumbent producer with technology X is $V_{f0}(t)$ if $X \leq X_m(t)$, and the irreversibility of investment means that $X_m(t)$ is nondecreasing. For producers with $X > X_m(t)$, their value $V^f(X,t)$ satisfies the Bellman equation[1]

$$[r(t) + \delta_x] V^f(X,t) = \Pi(X,t) + \mu_x X V^f_X(X,t) + V^f_t(X,t).$$

Value matching provides a boundary condition for this ODE, and the optimal choice about when to invest implies that smooth pasting holds, so

$$V^f(X_m(t),t) = V_{f0}(t), \qquad V^f_X(X_m(t),t) = 0, \qquad \text{all } t.$$

Let λ_x denote the hazard rate for success among process innovators, the rate at which they acquire new technologies. Conditional on success, the new technology is drawn from the distribution across current producers, those with $X > X_m(t)$. Hence the value $V_{f0}(t)$ of a process innovator at date t satisfies the Bellman equation

$$[r(t) + \delta_x] V_{f0}(t) = \lambda_x \left\{ \mathrm{E}_{\hat{F}(\cdot,t)} \left[V^f(X,t) \right] - V_{f0}(t) \right\} + V'_{f0}(t).$$

[1] At this stage, it would be easy to assume that the technology X of an incumbent evolves as a geometric Brownian motion. The Bellman equation would then become a (second-order) HJB equation, with the variance appearing as the coefficient on V^f_{XX}. The cross-sectional distribution of technologies among initially identical firms, within each age cohort, would be lognormal, with a growing variance, and the overall distribution would be a mixture of lognormals. When the solution to the model is actually characterized in section 5, however, the argument relies on technologies across incumbents having a Pareto distribution. At that point the mixture of lognormals would be incompatible with the requirement of a Pareto distribution overall, and the variance term would have to be dropped. Therefore, to streamline the notation and analysis, it is not introduced.

B. Entrants

Entering firms—product innovators—choose a hazard rate Λ_e for success. While investing, they pay a flow cost that depends on Λ_e and is scaled by the average profitability of current producers. Conditional on success at date t, their technology is drawn from the distribution of technology among incumbent producers. The firm must then pay a one-time (sunk) cost to start production.

Let $V_e(t)$ denote the value of a product innovator before the sunk cost. $V_e(t)$ satisfies a Bellman equation similar to that of a process innovator, except that the hazard rate is chosen and a direct cost is paid. Hence the Bellman equation for a product innovator is

$$\begin{aligned}[][r(t)+\delta_x]\,V_e(t) \quad &= \quad \max_{\Lambda_e>0}\Big\{-i_e(\Lambda_e)\mathrm{E}_{\hat{F}(\cdot,t)}\left[\Pi(X,t)\right] \\ &\qquad +\Lambda_e\left[\mathrm{E}_{\hat{F}(\cdot,t)}\left[V^f(X,t)\right]-V_e(t)\right]+V_e'(t)\Big\},\end{aligned}$$

where the function i_e is strictly increasing, strictly convex and differentiable, with $\lim_{\Lambda_e\to 0} i_e(\Lambda_e) > 0$, $\lim_{\Lambda_e\to 0} i_E^e(\Lambda_e) = 0$. The positive fixed cost prevents equilibria with an arbitrarily large number of entrants with vanishingly small investment rates. The condition for the optimal choice of research intensity is

$$i_e'(\Lambda_e)\mathrm{E}_{\hat{F}(\cdot,t)}\left[\Pi(X,t)\right] \geq \mathrm{E}_{\hat{F}(\cdot,t)}\left[V^f(X,t)\right]-V_e(t), \qquad \text{w/eq. if } \Lambda_e(t)>0, \quad \text{all } t.$$

The sunk cost for starting production, s_e, is also scaled by the average profitability of current producers. Free entry implies that in equilibrium

$$V_e(t) \leq s_e\mathrm{E}_{\hat{F}(\cdot,t)}\left[\Pi(X,t)\right], \qquad \text{w/eq. if } N_e(t)>0, \quad \text{all } t,$$

where $N_e(t)$ is the number of firms attempting product innovation.

C. Flows between producers and innovators, the DF for technology

At any date, the number of producers grows because of success by innovators of both types, and declines because of exit and decisions to become process innovators. The firms that switch around date t are those with technologies $X(t)$ that are close enough to the threshold $X_m(t)$ so that growth in that threshold overtakes them. Since technologies for producers grow at the rate μ_x, there is a positive level of switching at date t if and only if

$$X_m'(t) - \mu_x X_m(t) > 0. \tag{19}$$

If (19) holds producers are switching at the rate $[X_m'(t) - \mu_x X_m(t)]\,\hat{f}(X_m(t), t)N_p(t)$, while if (19) fails there is no switching. Hence

$$\begin{aligned} N_p'(t) &= \lambda_x N_i(t) + \Lambda_e(t)N_e(t) - \delta_x N_p(t) \\ &\quad - \max\{0,\ X_m'(t) - \mu_x X_m(t)\}\,\hat{f}(X_m(t), t)N_p(t), \qquad \text{all } t. \end{aligned}$$

The mass of process innovators $N_i(t)$ grows because producers switch to investing, while the mass of product innovators $N_e(t)$ grows because new entrants join. Each declines because of exit and success, so

$$\begin{aligned} N_i'(t) &= \max\{0,\ X_m'(t) - \mu_x X_m(t)\}\,\hat{f}(X_m(t), t)N_p(t) - (\delta_x + \lambda_x)\,N_i(t), \\ N_e'(t) &= E(t) - [\delta_x + \Lambda_e(t)]\,N_e(t), \qquad \text{all } t, \end{aligned}$$

where $E(t)$ is the inflow of new entrants—product innovators—at t. As a check, summing the three laws of motion, we find that the total number of firms, $N = N_p + N_i + N_e$, grows because of entry and declines because of exit.

The DF for technology among producers evolves as follows. As noted above, $X_m(t)$ is nondecreasing. Let $\Delta_t > 0$ be a small time increment. For any $t \geq 0$ and any $X \geq X_m(t + \Delta_t)$, the right-tail CDF's at t and $t + \Delta_t$ satisfy

$$\left[1 - \hat{F}(X, t + \Delta_t)\right] N_p(t + \Delta_t)$$

$$\approx \left\{1 - \hat{F}[(1-\mu_x \Delta_t)\, X, t]\right\} [(1-\delta_x \Delta_t)\, N_p(t) + \lambda_x \Delta_t N_i(t) + \Lambda_e(t) \Delta_t N_e(t)].$$

The number of producers at $t + \Delta_t$ with technology greater than X is the share of producers at t with technology greater than $(1-\mu_x \Delta_t)\, X$, multiplied by the sum of the number of producers at t, adjusted for exit, and for the number of successful innovators of both types. Taking a first-order approximation gives

$$\begin{aligned}
&\left[1 - \hat{F}(X,t)\right] \left[N_p(t) + N_p'(t)\Delta_t\right] - \hat{F}_t(X,t)\Delta_t N_p(t) \\
&\qquad \approx \left[1 - \hat{F}(X,t)\right] [(1-\delta_x \Delta_t)\, N_p(t) + \lambda_x \Delta_t N_i(t) + \Lambda_e(t) \Delta_t N_e(t)] \\
&\qquad\quad + \hat{f}(X,t)\mu_x X \Delta_t N_p(t).
\end{aligned}$$

Hence for all $X \geq X_m(t),\ t \geq 0,\ \hat{F}$ satisfies

$$\begin{aligned}
-\hat{F}_t(X,t) &= \hat{f}(X,t)\mu_x X + \left[1 - \hat{F}(X,t)\right] \\
&\quad \times \left[-\frac{N_p'(t)}{N_p(t)} - \delta_x + \lambda_x \frac{N_i(t)}{N_p(t)} + \Lambda_e(t)\frac{N_e(t)}{N_p(t)}\right].
\end{aligned}$$

D. Workers, investment in human capital

Individuals are organized into a continuum of identical, infinitely lived households of total mass one. Each dynastic household comprises a representative cross-section of the population, so each grows in size at the constant rate $\upsilon \geq 0$.

Members of a household pool their earnings, so they face no consumption risk. Each household member inelastically supplies one unit of labor to market activities, work and human capital accumulation, and investment by a worker is chosen to maximize the expected discounted value of his lifetime earnings. Individuals die at the fixed rate $\delta_h > 0$, and are born at a fixed rate $\delta_h + \upsilon$, where $\upsilon \geq 0$. Hence total population at date t is $L(t) = L_0 e^{\upsilon t}$.

As with firms, the investment decision of an individual is a zero-one choice, and the only cost is the opportunity cost of not working. An individual who chooses to invest abandons his old skill and waits to acquire a new one. Since population growth is exogenous, for simplicity we assume that a new entrant into the workforce is in the same position as an older individual who is investing. Hence all investing individuals at date t have the same value, call it $V_{w0}(t)$.

At any date t, all individuals with skill below some threshold $H_m(t)$ stop working and start investing, while those with skill above the threshold continue working. It follows that at any date t, the value of a worker with skill $H \leq H_m(t)$ is $V_{w0}(t)$. As before, irreversibility implies that $H_m(t)$ is nondecreasing.

While a worker produces, his human capital H grows at a constant rate $\mu_h \geq 0$. Hence the value $V^w(H,t)$ for a worker with skill $H > H_m(t)$, satisfies the Bellman equation

$$[r(t) + \delta_h]\, V^w(H,t) = W(H,t) + \mu_h H V_H^w(H,t) + V_t^w(H,t).$$

Value matching and smooth pasting hold at the threshold $H_m(t)$, so

$$V^w\left(H_m(t), t\right) = V_{w0}(t), \qquad V_H^w\left(H_m(t), t\right) = 0, \qquad \text{all } t.$$

Let $\lambda_h > 0$ denote the rate at which individuals who are investing succeed in acquiring new skill. Conditional on success, the individual gets a skill level drawn from the distribution among the currently workforce. Hence the value function for individuals who are investing satisfies the Bellman equation

$$[r(t) + \delta_h]\, V_{w0}(t) = \lambda_h \left\{ \mathrm{E}_{\hat{G}(\cdot,t)}\left[V^w(H,t)\right] - V_{w0}(t) \right\} + V_{w0}'(t).$$

E. Flows between workers and investors, the DF for skill

The number of workers L_w grows because of success by those investing, and declines due to exit and decisions to switch to investing. The latter group consists of those whose human capital falls below the (moving) threshold H_m, despite growth at the rate μ_h. Hence workers are switching at date t if and only if

$$H_m'(t) - \mu_h H_m(t) > 0, \tag{20}$$

and

$$\begin{aligned} L_w'(t) &= \lambda_h L_i(t) - \delta_h L_w(t) \\ &\quad - \max\{0,\ H_m'(t) - \mu_h H_m(t)\}\, \hat{g}[H_m(t), t] L_w(t), \qquad \text{all } t. \end{aligned}$$

The number of investors L_i grows because of entry into the labor force and decisions by current workers to switch to investing, and declines because of success and exit. The arrival rate of entrants is $(\upsilon + \delta_h)[L_i(t) + L_w(t)]$, so

$$\begin{aligned} L_i'(t) &= (\upsilon - \lambda_h) L_i(t) + (\upsilon + \delta_h) L_w(t) \\ &\quad + \max\{0,\ H_m'(t) - \mu_h H_m(t)\}\, \hat{g}[H_m(t), t] L_w(t), \qquad \text{all } t. \end{aligned}$$

As a check, total population $L(t) = L_w(t) + L_i(t)$ grows at the rate υ.

The evolution of $\hat{G}$, the DF for skill, is found using the same argument as for firms. Hence for all $H > H_m(t)$ and $t > 0$, $\hat{G}$ satisfies

$$-\hat{G}_t(H,t) = \hat{g}(H,t)\mu_h H + \left[1 - \hat{G}(H,t)\right]\left[-\frac{L_w'(t)}{L_w(t)} - \delta_h + \lambda_h \frac{L_i(t)}{L_w(t)}\right].$$

F. Consumption

At each date t, the income of the household consists of the wages of its workers plus the profits from its portfolio, net of investment costs. That income is used for consumption and to finance the investment (entry) costs of new firms. All household members share equally in consumption, and the household has the constant-elasticity preferences

$$U = \int_0^\infty L_0 e^{\upsilon t} e^{-\hat{r} t} \frac{1}{1-\theta} c(t)^{1-\theta} dt,$$

where $\hat{r} > 0$ is the rate of pure time preference, $1/\theta > 0$ is the elasticity of intertemporal substitution, and $c(t)$ is per capita consumption.

The household's gross income consists of wages plus profits,

$$Y_F(t) = L_w(t)\mathrm{E}_{\hat{G}(\cdot,t)}\left[W(H,t)\right] + N_p(t)\mathrm{E}_{\hat{F}(\cdot,t)}\left[\Pi(X,t)\right], \qquad \text{all } t.$$

The investment decisions of firms and workers, both incumbents and entrants, maximize, respectively, the expected discounted value of net profits and wages discounted at market interest rates. Hence no further investment decisions are required at the household level. Total investment costs for entrants are

$$I_E(t) = \left[i_e\left(\Lambda_e(t)\right) + s_e\Lambda_e(t)\right] N_e(t)\mathrm{E}_{\hat{F}(\cdot,t)}\left[\Pi(X,t)\right], \qquad \text{all } t,$$

and the household's net income at date t is $Y_F(t) - I_E(t)$.

Given interest rates $r(s)$, $s \geq 0$, define the cumulative interest factor

$$R(t) = \int_0^t r(s)ds, \qquad \text{all } t.$$

The household's consumption/savings decision, given interest rates and its net income stream, is to choose per capita consumption $c(t)$, $t \geq 0$, to maximize utility, subject to a lifetime budget constraint,

$$\int_0^\infty e^{-R(t)}\left\{L_0 e^{\upsilon t}c(t) - \left[Y_F(t) - I_E(t)\right]\right\}dt \leq 0.$$

The condition for an optimum implies that per capita consumption grows at the rate

$$\frac{c'(t)}{c(t)} = \frac{1}{\theta}\left[r(t) - \hat{r}\right], \qquad \text{all } t,$$

with $c(0)$ determined by budget balance.

Final output is used for consumption and for the investment costs of entering firms. Hence market clearing for goods requires

$$Y_F(t) = L_0 e^{\upsilon t}c(t) + I_E(t), \qquad \text{all } t.$$

5. COMPETITIVE EQUILIBRIA, BGP'S

This section provides formal definitions of a competitive equilibrium and a BGP.

We start with the description of a (dynamic) economy, which in addition to the elements characterizing a static economy, includes parameters for preferences, population growth, exit/growth/success rates for firms and worker; and initial levels for the number of investing firms and individuals.

DEFINITION: An *economy* $\mathcal{E}$ is described by

i. parameters $(\rho, \nu, \omega, \eta, \theta, \hat{r}, \upsilon)$, with $\rho > 1$, $\nu \in (0, 1/\rho)$, $\omega \in (0, 1)$, $\eta \in (0, 1)$, $\theta > 0$, $\hat{r} > 0$, $\upsilon \geq 0$;

ii. parameters $\delta_j, \lambda_j > 0, \mu_j > 0, \ \ j = h, x$;

iii. an entry cost function i_e that is strictly increasing and strictly convex, with $i_e(0) = i_e'(0) = 0$, and a sunk cost s_e;

iv. initial conditions $N_{p0}, N_{i0}, N_{e0} > 0$, $L_{w0}, L_{i0} > 0$;

v. initial distribution functions $\hat{F}_0(X)$ with continuous density $\hat{f}_0(X)$ and lower bound X_{m0} on its support, and $\hat{G}_0(H)$ with continuous density $\hat{g}_0(H)$ and lower bound $H_{m0} \geq 0$ on its support.

A. Competitive equilibrium

The definition of a competitive equilibrium is standard.

DEFINITION: A *competitive equilibrium* of an economy $\mathcal{E}$ consists of the following, for all $t \geq 0$:

a. the numbers of producers, process innovators, product innovators, workers, and investing individuals, $N_p(t)$, $N_i(t)$, $N_e(t)$, $L_w(t)$, $L_i(t)$; the inflow rate $E(t)$ to the set of product innovators; and DFs $\hat{F}(X;t), \hat{G}(H;t)$;

b. prices and allocations $W(H;t)$, $P(X,t)$, $H^*(X;t)$, $Y(X,t)$, $\mathfrak{L}(X,t)$, $\Pi(X,t)$, $Y_F(t)$;

c. a value function $V^f(X,t)$ and investment threshold $X_m(t)$ for producers, and a value $V_{f0}(t)$ for process innovators;

d. a value $V_e(t)$ and success rate $\Lambda_e(t)$ for product innovators;

e. a value function $V^w(H;t)$ and investment threshold $H_m(t)$ for workers, and a value $V_{w0}(t)$ for individuals who are investing;

f. income $Y_F(t)$, investment costs $I_E(t)$, per capita consumption $c(t)$ and the interest rate $r(t)$;

such that for all $t \geq 0$,

i. $W, P, H^*, Y, \mathfrak{L}, \Pi, Y_F$, is a production equilibrium, given $N_p, L_w, \hat{F}, \hat{G}$;

ii. the value function V^f and threshold X_m solve the investment problem of producers, given r, Π, V_{f0}; and V_{f0} is consistent with $r, V^f, \hat{F}$;

iii. the value V_e and hazard rate Λ_e solve the problem of entrants, given $r, \Pi, V^f, \hat{F}$; and N_e satisfies the free entry condition, given $V_e, s_e, \Pi, \hat{F}$;

iv. $N_p, N_i, N_e, \hat{F}$ are consistent with X_m, Λ_e, E, and the initial conditions $N_{p0}, N_{i0}, N_{e0}, \hat{F}_0$;

v. the value function V^w and threshold H_m solve the investment problem of workers, given r, W, V_{w0}; and V_{w0} is consistent with $r, V^w, \hat{G}$;

vi. $L_w, L_i, \hat{G}$ are consistent with H_m and the initial conditions $L_{w0}, L_{i0}, \hat{G}_0$;

vii. the investment cost I_E is consistent with $\Lambda_e, N_e, \Pi, \hat{F}$, and c solves the consumption/savings problem of households, given $r, L, Y_F - I_E$; and

viii. the goods market clears.

B. Balanced growth paths

The rest of the analysis focuses on balanced growth paths, competitive equilibria with the property that quantities grow at constant rates, and the normalized distributions of technology and skill are time invariant. In particular, let $Q(t) \equiv E_{\hat{F}(\cdot,t)}(X)$, $t \geq 0$, denote average technology. On a BGP, Q grows at a constant rate g, and the distributions of relative technology $x = X/Q(t)$ and relative human capital

$h = H/Q(t)$ are time invariant.

By assumption, total population L grows at the fixed rate υ. On a BGP, the total number of firms N grows at a constant rate n, and the shares of firms and individuals in each category—$N_p/N, N_i/N, N_e/N$ and $L_w/L, L_i/L$—are constant.

It follows from Lemma 2 that on a BGP the labor allocation in terms of relative technology and relative skill is time invariant. The growth rates for wages, prices, output levels, and so on are then determined from Lemma 3. In particular, average product price grows at rate Ωn, where Ω is as in (11), average output per firm at rate $g + \upsilon - n$, and average employment per firm at rate $\upsilon - n$. The growth rates of aggregate output, g_Y, the average wage, g_w, and the average profit per firm, g_π, are

$$\begin{aligned} g_Y &= g + \Omega n + \upsilon, \\ g_w &= g + \Omega n \qquad\qquad = g_Y - \upsilon, \\ g_\pi &= g + (\Omega - 1)\, n + \upsilon = g_Y - n, \end{aligned} \tag{21}$$

Total consumption grows at rate g_Y, so market clearing for goods requires total investment costs $[i_e(\Lambda_e) + s_e\lambda_e]\, N_e \mathrm{E}_{\hat{F}}[\Pi]$ to grow at rate g_Y as well. The required condition holds if $\Lambda_e = \lambda_e$ is constant.

These observations lead to the following definition.

Definition: A competitive equilibrium for $\mathcal{E}$ is a *balanced growth path* (BGP) if for some $g > 0$ and n, DF's $F(x), G(h)$, production equilibrium $[h^*, w, p, y, \ell, \pi, y_F]$, functions $v_f(x)$, $v_w(h)$, and values $v_{f0}, x_m, v_e, \lambda_e, e_0, v_{w0}, h_m, c_0, i_{E0}$, the CE has the property that for all $t \geq 0$:

a. the number of firms, entrants and individuals satisfy

$$\begin{aligned} N_p(t) &= e^{nt} N_{p0}, \qquad N_i(t) = e^{nt} N_{i0}, \qquad N_e(t) = e^{nt} N_{e0}, \\ E(t) &= e^{nt} E_0, \qquad L_w(t) = e^{\upsilon t} L_{w0}, \qquad L_i(t) = e^{\upsilon t} L_{i0}; \end{aligned}$$

b. average technology satisfies

$$Q(t) \equiv \mathrm{E}_{\hat{F}(\cdot,t)}[X] = e^{gt}Q_0;$$

c. the DF's satisfy

$$\begin{aligned}
\hat{F}(X,t) &= F(X/Q(t)), \qquad \text{all } X, \\
\hat{G}(H,t) &= G(H/Q(t)), \qquad \text{all } H;
\end{aligned}$$

d. the production equilibria satisfy

$$\begin{aligned}
W(H;t) &= e^{g_w t}Q_0 w(H/Q(t)), \qquad \text{all } H; \\
H^*(X;t) &= e^{gt}Q_0 h^*(X/Q(t)), \\
P(X,t) &= e^{\Omega nt}Q_0 p(X/Q(t)), \\
Y(X,t) &= e^{(g+\upsilon-n)t}Q_0 y(X/Q(t)), \\
\mathfrak{L}(X,t) &= e^{(\upsilon-n)t}\ell(X/Q(t)), \\
\Pi(X,t) &= e^{g_\pi t}Q_0 \pi(X/Q(t)), \qquad \text{all } X; \\
Y_F(t) &= e^{g_Y t}Q_0 y_F;
\end{aligned}$$

where $g_Y,\ g_w$ and g_π are as in (21);

e. the value function and policy for producers, and the value of process innovators satisfy

$$\begin{aligned}
V^f(X,t) &= e^{g_\pi t}Q_0 v_f(X/Q(t)), \qquad \text{all } X, \\
X_m(t) &= e^{gt}Q_0 x_m, \\
V_{f0}(t) &= e^{g_\pi t}Q_0 v_{f0};
\end{aligned}$$

f. the value and policy for entrants satisfy

$$\begin{aligned}
V_e(t) &= e^{g_\pi t}Q_0 v_e, \\
\Lambda_e(t) &= \lambda_e;
\end{aligned}$$

g. the value function and policy for workers, and the value of individuals who are investing satisfy

$$\begin{aligned}
V^w(H;t) &= e^{g_w t} Q_0 v_w(H/Q(t)), \qquad \text{all } H, \\
H_m(t) &= e^{gt} Q_0 h_m, \\
V_{w0}(t) &= e^{g_w t} Q_0 v_{w0};
\end{aligned}$$

h. investment costs, per capita consumption and the interest rate satisfy

$$\begin{aligned}
I_E(t) &= e^{g_Y t} Q_0 i_{E0}; \\
c(t) &= e^{g_w t} Q_0 c_0, \\
r(t) &= r \equiv \hat{r} + \theta g_w.
\end{aligned}$$

BGPs arise—if at all—only for initial conditions $(N_{p0}, N_{i0}, N_{e0}, L_{w0}, L_{i0})$ and initial DFs $\hat{F}(X;0), \hat{G}(H;0)$ that satisfy certain restrictions. The rest of the analysis looks at a class of economies for which BGPs exist, and determines the growth rates g, n, on those path.

6. BGPS WITH PARETO DISTRIBUTIONS

In this section we will show that if an economy $\mathcal{E}$ has initial distribution functions $\hat{F}_0$ and $\hat{G}_0$ that are Pareto, with shape and location parameters that satisfy the requirements of Proposition 4, and has initial conditions $(N_{p0}, N_{i0}, N_{e0}, L_{w0}, L_{i0})$ that satisfy certain ratio properties, then it has a competitive equilibrium that is a BGP. We will also show how the growth rates g, n, the required ratios, and various constants are determined.

In the rest of this section we will show how the normalized value functions, decision rules, and so on can be constructed. The key to showing existence of a BGP is the fact that the value functions for producers and workers inherit the isoelastic forms of the profit and wage functions. The arguments are gathered in the Appendix.

A. Production equilibrium

The production equilibria on a BGP can be expressed in terms of the equilibrium for normalized distributions of skill and technology. Suppose the initial distributions $\hat{F}_0$ and $\hat{G}_0$ are Pareto, with parameters (α, X_{m0}) and (γ, H_{m0}). Assume (12) holds, define ε and a_h by (13) and (14), and assume that $H_{m0} = a_h X_{m0}$. Define average technology under the initial distribution

$$Q_0 \equiv \mathrm{E}_{\hat{F}_0}[X] = \frac{\alpha}{\alpha - 1} X_{m0},$$

and use Q_0 to define the normalized DF's

$$\begin{aligned} F(X/Q_0) &\equiv \hat{F}_0(X), \qquad \text{all } X \geq X_m, \\ G(H/Q_0) &\equiv \hat{G}_0(H), \qquad \text{all } H \geq H_m. \end{aligned}$$

The location parameters for F and G are

$$x_m = \frac{X_{m0}}{Q_0} = \frac{\alpha - 1}{\alpha}, \qquad h_m \equiv \frac{H_{m0}}{Q_0}, \tag{22}$$

so the hypotheses of Proposition 4 hold for F, G, and the production equilibrium for the normalized distributions are as in (16)-(18). By construction F has mean $\mathrm{E}_F(x) = 1$.

B. Incumbent firms

For notational convenience, define

$$\zeta \equiv 1 - (\rho - 1)\,\varepsilon. \tag{23}$$

Note that ζ can have either sign, and that $1 - \zeta = \varepsilon\,(\rho - 1)$.

Consider the investment decision and value of a producer. A BGP requires a positive level of process innovation, so (19) must hold,

$$g > \mu_x. \tag{24}$$

As shown in the Appendix, if Π and V^f have the forms required for a BGP, and $\pi(x)$ is as in (18), then the Bellman equation for a producer has the normalized form

$$(r + \delta_x - g_\pi)\,v_f(x) = \pi_1 x^{1-\zeta} + (\mu_x - g)\,x v_f'(x), \qquad x \geq x_m,$$

where

$$\pi_1 \equiv L_{w0} N_{p0}^{\Omega - 1} \pi_0.$$

The solution to this ODE is

$$v_f(x) = B_x \pi_1 x^{1-\zeta} + \left(v_{f0} - B_x \pi_1 x_m^{1-\zeta}\right)(x/x_m)^{R_x}, \qquad x \geq x_m, \tag{25}$$

where

$$\begin{aligned} B_x &= \left[(r + \delta_x - g_\pi) + (g - \mu_x)(1 - \zeta)\right]^{-1}, \\ R_x &= -\frac{r + \delta_x - g_\pi}{g - \mu_x} < 0, \end{aligned} \tag{26}$$

and v_{f0}, the value of a process innovator, is to be determined. The first term in (25) is the value of a producer who operates forever, never investing. The second term represents the additional value from the option to invest in process innovation.

Optimization by producers implies that the investment threshold x_m satisfies the smooth pasting condition, which involves x_m and v_{f0}. Since x_m is determined by (22), this condition determines v_{f0},

$$v_{f0} = B_x \pi_1 x_m^{1-\zeta} \left(1 - \frac{1-\zeta}{R_x} \right). \tag{27}$$

Using this expression in (25) gives

$$v_f(x) = B_x \pi_1 x_m^{1-\zeta} \left[\left(\frac{x}{x_m} \right)^{1-\zeta} - \frac{1-\zeta}{R_x} \left(\frac{x}{x_m} \right)^{R_x} \right], \qquad x \geq x_m,$$

and the Pareto form for F implies

$$\begin{aligned} \mathrm{E}_F \left[x^{1-\zeta} \right] &= x_m^{1-\zeta} \frac{\alpha}{\alpha - 1 + \zeta}, \\ \mathrm{E}_F \left[v_f(x) \right] &= B_x \pi_1 x_m^{1-\zeta} \left(\frac{\alpha}{\alpha - 1 + \zeta} - \frac{1-\zeta}{R_x} \frac{\alpha}{\alpha - R_x} \right). \end{aligned}$$

Similarly, on a BGP the Bellman equation for process innovators has the normalized form

$$(r + \delta_x) v_{f0} = \lambda_x \left\{ \mathrm{E}_F \left[v_f(x) \right] - v_{f0} \right\} + g_\pi v_{f0}.$$

Substituting for the expected value and for v_{f0}, and factoring out $\pi_1 B_x x_m^{1-\zeta}$, gives

$$r + \delta_x - g_\pi = \frac{-R_x (1-\zeta) \lambda_x}{(\alpha - 1 + \zeta)(\alpha - R_x)}. \tag{28}$$

Recall that r, g_π and R_x involve g and n, while all of the other parameters in this expression are exogenous. Hence (28) is one equation relating g and n.

C. Entrants

For product innovators, the optimal choice for the hazard rate λ_e requires

$$i_e'(\lambda_e) \mathrm{E}_F \left[\pi(x) \right] = \mathrm{E}_F \left[v_f(x) \right] - v_e, \tag{29}$$

and the normalized value v_e for a product innovator satisfies

$$(r + \delta_x - g_\pi + \lambda_e) v_e = -i_e(\lambda_e)\mathrm{E}_F[\pi(x)] + \lambda_e \mathrm{E}_F[v_f(x)]. \tag{30}$$

Given $\mathrm{E}_F[\pi(x)]$ and $\mathrm{E}_F[v_f(x)]$, this pair of equations has a unique solution λ_e, v_e, with $\lambda_e > 0$.

On a BGP $N_e > 0$, so the free entry condition requires

$$\begin{aligned} v_e &= s_e \mathrm{E}_F[\pi(x)] \\ &= s_e L_{w0} N_{p0}^{\Omega-1} \pi_0 \mathrm{E}_F\left[x^{1-\zeta}\right], \end{aligned} \tag{31}$$

where the second line uses (18). Given the size of the workforce L_{w0} at date 0, this condition determines the number of producers N_{p0} required for a BGP.

D. Flows of firms, the DF for technology

The entry rate for firms is the sum of the growth and exit rates,

$$E_0 = (n + \delta_x) N_0. \tag{32}$$

The laws of motion for N_p, N_i and N_e then imply that the ratios of process and product innovators to producers are

$$\begin{aligned} \frac{N_i}{N_p} &= \frac{\alpha(g - \mu_x)}{n + \delta_x + \lambda_x}, \\ \frac{N_e}{N_p} &= \frac{n + \delta_x}{\lambda_e}\left[1 + \frac{\alpha(g - \mu_x)}{n + \delta_x + \lambda_x}\right]. \end{aligned} \tag{33}$$

It is easy to check that the DF for technology evolves as required for a BGP.

E. Workers, investment in human capital

The normalized value function and investment rule for workers are determined in the same way as those for producers. A BGP requires positive investment by incumbent workers, so (20) must hold,

$$g > \mu_h. \tag{34}$$

For $w(h)$ as in (17), the Bellman equation for a worker has the normalized form

$$(r + \delta_h - g_w)\, v_w(h) = w_1 h^{1-\varepsilon} + (\mu_h - g)\, h v_w'(h),$$

where

$$w_1 \equiv N_{p0}^{\Omega} w_0.$$

The solution to this ODE is

$$v_w(h) = B_h w_1 h^{1-\varepsilon} + \left(v_{w0} - B_h w_0 h_m^{1-\varepsilon}\right)(h/h_m)^{R_h}, \qquad h \geq h_m, \tag{35}$$

where

$$\begin{aligned} B_h &= \left[(r + \delta_h - g_w) + (g - \mu_h)(1 - \varepsilon)\right]^{-1}, \\ R_h &= -\frac{r + \delta_h - g_w}{g - \mu_h} < 0, \end{aligned} \tag{36}$$

and v_{w0}, the value of an individual who is investing, is to be determined.

Optimization by workers implies the threshold h_m satisfies a smooth pasting condition, which involves h_m and v_{w0}. Since h_m is determined by (22), this condition determines v_{w0},

$$v_{w0} = B_h w_1 h_m^{1-\varepsilon}\left(1 - \frac{1-\varepsilon}{R_h}\right). \tag{37}$$

Using this expression in (35) gives

$$v_w(h) = B_h w_1 h_m^{1-\varepsilon}\left[\left(\frac{h}{h_m}\right)^{1-\varepsilon} - \frac{1-\varepsilon}{R_h}\left(\frac{h}{h_m}\right)^{R_h}\right], \qquad h \geq h_m,$$

and the Pareto form for G implies

$$\mathrm{E}_G\left[v_w(h)\right] = B_h w_1 h_m^{1-\varepsilon}\left[\frac{\gamma}{\gamma - 1 + \varepsilon} - \frac{1-\varepsilon}{R_h}\frac{\gamma}{\gamma - R_h}\right].$$

The normalized Bellman equation for an investing individual is

$$\left(r+\delta_h\right) v_{w0}=\lambda_h\left\{\mathrm{E}_G\left[v_w(h)\right]-v_{w0}\right\}+g_w v_{w0}.$$

Substituting for $\mathrm{E}_G\left[v_w\right]$ and v_{w0}, and factoring out $w_1 B_h h_m^{1-\varepsilon}$, gives

$$r+\delta_h-g_w=\frac{-R_h(1-\varepsilon) \lambda_h}{(\gamma-1+\varepsilon)\left(\gamma-R_h\right)}. \tag{38}$$

This equation, like (28), involves g and n.

F. Flows of individuals, the DF for skill

By assumption population grows at a constant rate υ, and on a BGP the share that is working is constant over time. The law of motion for L_w then implies that the ratio of investors to workers is

$$\frac{L_i}{L_w}=\frac{\lambda_h}{\upsilon+\delta_h+\gamma\left(g-\mu_h\right)}, \tag{39}$$

and it is easy to check that the DF for skill evolves as required for a BGP.

G. Consumption, the interest rate

On a BGP per capita consumption grows at the rate g_w, so the interest rate is

$$r=\hat{r}+\theta g_w. \tag{40}$$

Aggregate income grows at the rate g_Y, so its PDV is finite if and only if $r>g_Y$, or

$$\hat{r}>g_Y-\theta g_w=\upsilon+(1-\theta)(g+\Omega n). \tag{41}$$

Market clearing for goods requires

$$y_{F0}=L_0 c_0+N_{e0}\left[i_e(\lambda_e)+\lambda_e s_e\right] \mathrm{E}_F[\pi(x)], \tag{42}$$

which determines the initial level of per capita consumption, c_0.

7. GROWTH RATES ON THE BGP

In this section we will show how g and n are determined, formally state the main existence result for BGPs, and describe how various parameters affect the growth rates.

The growth rates are determined by (28) and (38), where g_w, g_π the roots R_x and R_h, and the interest rate r involve g, n. Substituting from (21), (26), (36), and (40), these equation are

$$\begin{aligned} g &= \frac{1}{\xi_\alpha}\left[\upsilon + \alpha\mu_x + \frac{1-\zeta}{\alpha-1+\zeta}\lambda_x - \delta_x - \hat{r} - \left[(\theta-1)\,\Omega+1\right]n\right], \\ g &= \frac{1}{\xi_\gamma}\left[\quad \gamma\mu_h + \frac{1-\varepsilon}{\gamma-1+\varepsilon}\lambda_h - \delta_h - \hat{r} - (\theta-1)\,\Omega n\right], \end{aligned} \tag{43}$$

where

$$\xi_\alpha \equiv \alpha - 1 + \theta > 0, \qquad \xi_\gamma \equiv \gamma - 1 + \theta > 0,$$

and the signs follow from the fact that $\alpha, \gamma > 1$.

We then have the following results for BGPs

Proposition 5: Let $\mathcal{E}$ be an economy with initial distributions $\hat{F}_0, \hat{G}_0$ that are Pareto, with shape parameters $\alpha, \gamma > 1$ satisfying (12), and location parameters X_{m0}, H_{m0}. If $\theta < 1$, assume in addition that $\rho > \gamma/(\gamma-1)$. Define ε and a_h by (13) and (14). Then the pair of equations in (43) has a unique solution (g, n). If

(a) g, n satisfy the inequalities in (24), (32), (34), and (41);

(b) $H_{m0}/X_{m0} = a_h$;

(c) the initial ratios N_{i0}/N_{p0} and L_{i0}/L_{w0} satisfy (33) and (39);

(d) the initial ratio $L_{w0}/N_{p0}^{1-\Omega}$ satisfies (31); and

(e) c_0 defined by (42) satisfies $c_0 > 0$,

then $\mathcal{E}$ has a competitive equilibrium that is a BGP.

Proof: For existence and uniqueness of a solution, it suffices to show that the two equations in (43) are not collinear. Here we will prove a slightly stronger result that will be used later, that

$$(\gamma - 1 + \theta)\left[(\theta - 1)\Omega + 1\right] > (\alpha - 1 + \theta)(\theta - 1)\Omega,$$

or

$$\gamma > (\theta - 1)\left[(\alpha - \gamma)\Omega - 1\right]. \tag{44}$$

Since $\nu \in (0, 1/\rho)$ implies $\Omega \in [0, 1/(\rho - 1))$, and (12) implies $\alpha - \gamma \in (-1, \rho - 1)$, it follows that

$$(\alpha - \gamma)\Omega - 1 \in \left(-\frac{\rho}{\rho - 1}, 0\right).$$

If $\theta \geq 1$, the term on the right in (44) is zero or negative. If $\theta < 1$, then by assumption $\gamma > \rho/(\rho - 1)$. In either case (44) holds.

If (a) holds, the growth rates satisfy the required inequalities. The normalized production equilibrium at $t = 0$ is described by Proposition 4, where x_m, h_m are defined by (22), and (b) implies (15) holds. Given g, n, and x_m, (25)-(27) determine v_{f0} and v_f; (29)-(30) determine λ_e and v_e; (32) determines e_0; (35)-(37) determine v_w and v_{w0}, and (40)-(42) determine r and c_0. If (c) holds, the mass of entering firms N_{e0} can immediately adjust so N_{e0}/N_{p0} also satisfies (33). Then (d) and (e) imply the free entry condition holds and consumption is positive, and the solution describes a BGP. ■

To begin, consider two special cases, where $\theta = 1$ and $\Omega = 0$. We will assume throughout that the conditions in (a) hold.

Special case 1: $\theta = 1$.—

Suppose preferences are logarithmic, $\theta = 1$. Then the second equation in (43) simplifies to

$$g = \mu_h + \frac{1}{\gamma}\left[\frac{1 - \varepsilon}{\gamma - 1 + \varepsilon}\lambda_h - \delta_h - \hat{r}\right]. \tag{45}$$

In this case g is a weighted sum of the parameters $\mu_h, \lambda_h, \delta_h$ governing skill accumulation and the rate of time preference $\hat{r}$, and the parameters governing technology accumulation do not enter. Faster human capital growth μ_h among on-the-job workers raises g, as does a higher success rate λ_h for workers who are re-tooling. A higher exit rate δ_h or a higher discount rate $\hat{r}$ reduces g.

The weights depend on the elasticity parameters γ and $1-\varepsilon$. A higher value for $1-\varepsilon$ increases the elasticity of the wage with respect to skill, increasing the returns to investment and increasing g. Recall that ε is increasing in $1/\rho$ and α, and decreasing in γ. Since $1/\rho$ measures the monopoly power of firms, an increase in monopoly power reduces g. An increase in α means the Pareto tail for the technology distribution is thinner, decreasing both the mean and the variance, and reducing g.

The direct effect of an increase in γ is to reduce g. A higher value for γ means the Pareto tail for the skill distribution is thinner, decreasing both the mean and the variance. The indirect effect, through ε, is in the reverse direction, but presumably smaller.

Given g, the first equation in (43) determines n. If the two Pareto distributions have the same shape parameter, if $\alpha=\gamma$, then $\varepsilon=1/\rho$ and $\zeta=\varepsilon$, and that equation simplifies to

$$n=\upsilon+\gamma\left(\mu_x-\mu_h\right)+\frac{\rho-1}{\rho\left(\gamma-1\right)+1}\left(\lambda_x-\lambda_h\right)-\left(\delta_x-\delta_h\right). \tag{46}$$

In this case n is equal to the rate of population growth rate υ, adjusted for weighted differences in the μ_j's, λ_j's and δ_j's, $j=x,h$. An increase in μ_x or λ_x or a decrease in δ_x increases n. Thus, improvements in the investment technology for incumbent firms encourage entry.

If $\alpha\neq\gamma$, then the all the hazard rates have different weights, but qualitatively the pattern is similar.

Special case 2: $\Omega=0$.—

In the limiting case $\nu=1/\rho$, variety is not valued and $\Omega=0$. In this case, too, the growth rate g is determined by the second equation in (43),

$$g = \frac{1}{\xi_\gamma}\left[\gamma\mu_h + \frac{1-\varepsilon}{\gamma-1+\varepsilon}\lambda_h - \delta_h - \hat{r}\right]. \tag{47}$$

The solution in (47) is like the one in (45). If $\alpha = \gamma$, then $\xi_\alpha = \xi_\gamma$, and n is exactly as in (46).

General case.—

The next result extends these comparative statics.

PROPOSITION 6: Let $\mathcal{E}$ be as in Proposition 5, and if $\theta < 1$, assume in addition that $\rho > \max\{\gamma/(\gamma-1), 2\}$. Suppose $\mathcal{E}$ has a competitive equilibrium that is a BGP. Then

a. an increase in μ_h, λ_h or a decrease in δ_h raises g and reduces n;

b. a decrease in $\hat{r}$ raises g, and

—reduces n if $(\theta-1)\Omega \geq 0$, and

—has an ambiguous effect on n if $(\theta-1)\Omega < 0$;

c. an increase in μ_x, λ_x or a decrease in δ_x raises n, and

—raises g if $(\theta-1)\Omega < 0$,

—has no effect on g if $(\theta-1)\Omega = 0$, and

—reduces g if $(\theta-1)\Omega > 0$.

PROOF: First we will show that the equations in (43), plotted in n-g−space, are as shown Figure 1: the line defined by the first equation is downward sloping; the line defined by the second equation has a positive, zero, or negative slope as $(\theta-1)\Omega < 0$, $= 0$, or > 0; and in all case the second line crosses the first from below.

For the first claim, note that $\theta \geq 1$ implies $[(\theta-1)\Omega+1] > 0$. If $\theta < 1$, then $\rho > 2$, so $\Omega < 1/(\rho-1) < 1$ and again $[(\theta-1)\Omega+1] > 0$. The second claim is obvious, and the third follows from (44).

All of the claims follow directly from Figure 1. An increase in μ_h or λ_h, or a

Figure 1a: comparative static (a)

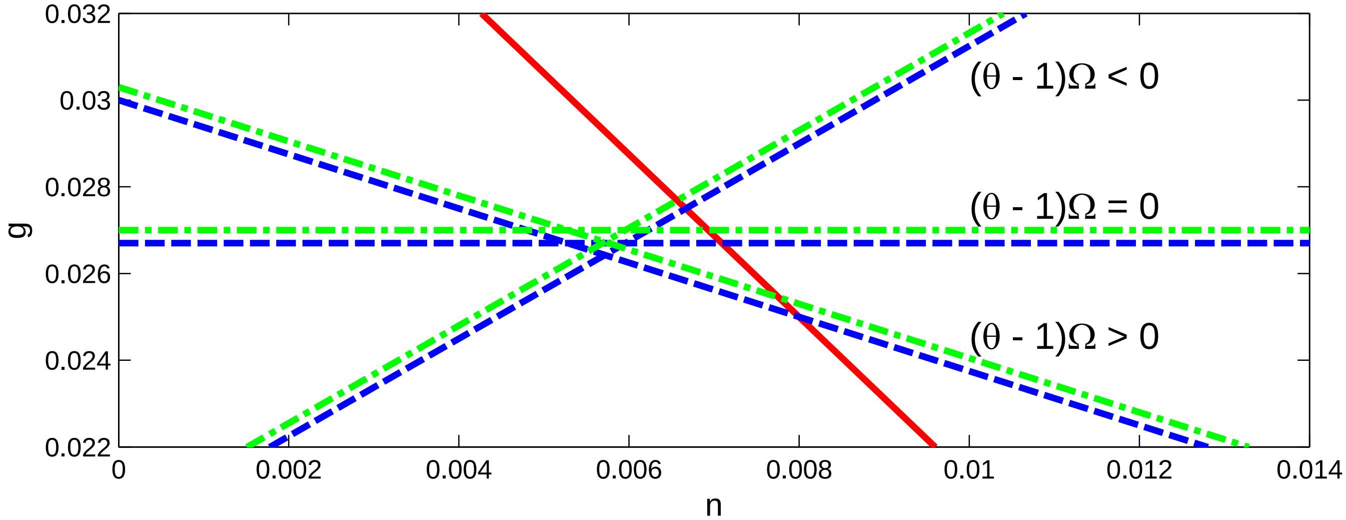


Figure 1b: comparative static (b)

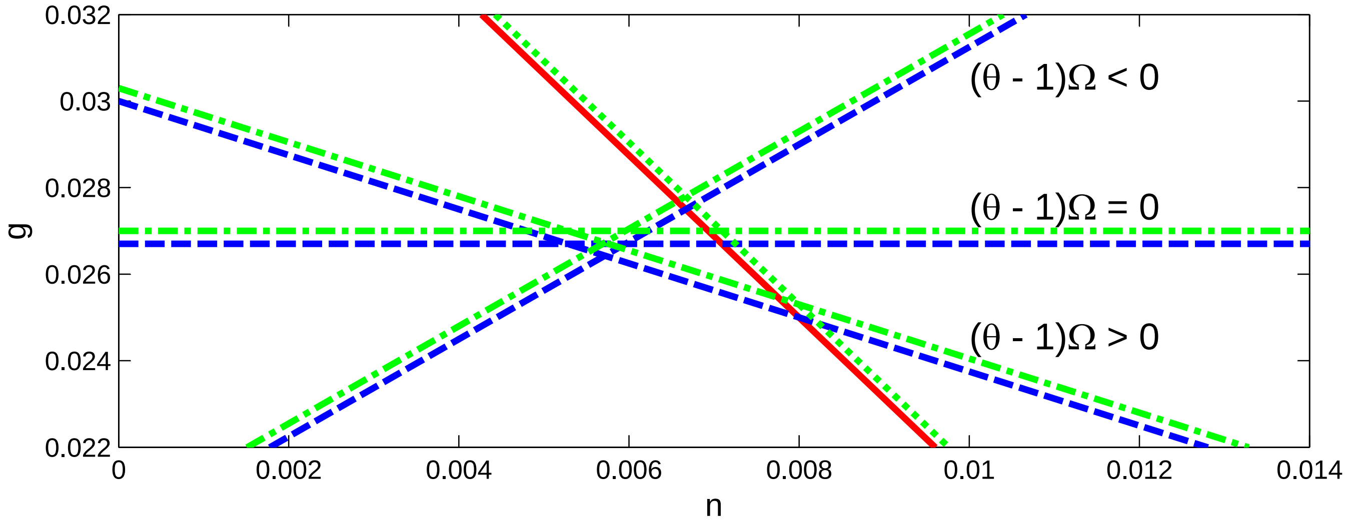
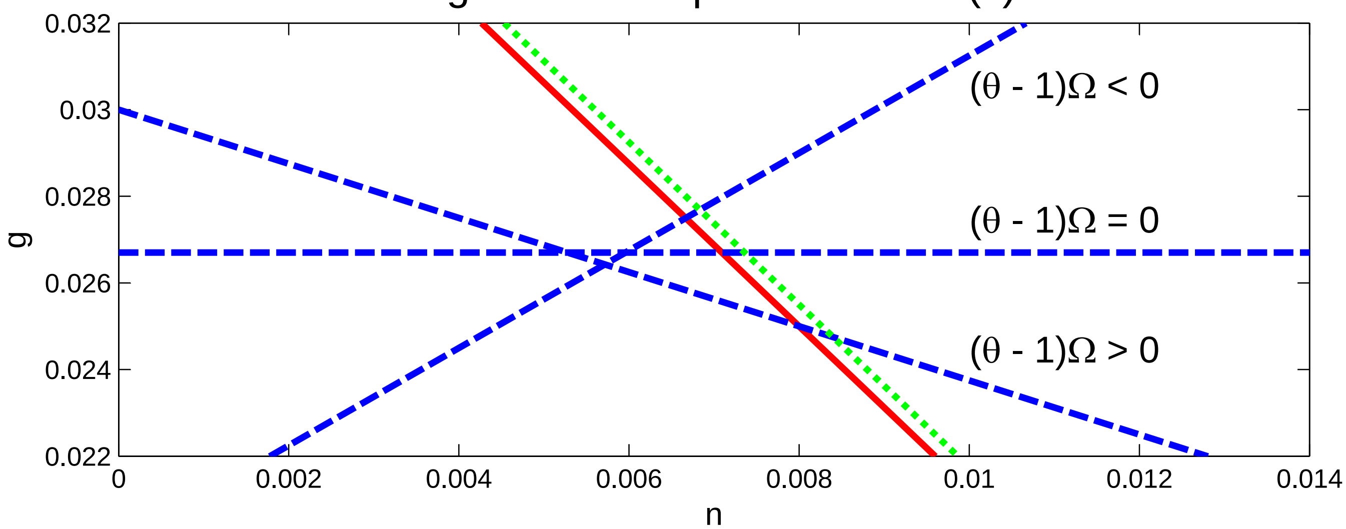


Figure 1c: comparative static (c)



decrease in δ_h, shifts the first line upward, increasing g and decreasing n. A decrease in $\hat{r}$ shifts both lines upward, increasing g. The effect on n depends on the slope of the second line. An increase in μ_x or λ_x, or a decrease in δ_x, shifts the second line upward, increasing n. The effect on g is depends on the slope of the second line. ■

The initial number of workers L_{w0} and number of producers N_{p0} affect initial wage rates and profit levels, but not the growth rates, with the entry condition determining the ratio $N_{p0}^{1-\Omega}/L_{w0}$. Thus, if $\Omega = 0$, an increase in L_{w0} must be matched by a 1-for-1 increase in N_{p0}. If $\Omega > 0$, the number of firms increases less than 1-for-1. In both cases, the effects of an increase in L_{w0} are described by recalling Lemma 3. Profit per firm is unchanged in all cases. If $\Omega = 0$, the average real wage, per capita consumption and output and employment per firm are unchanged, while if $\Omega > 0$ they all increase.

It is interesting that the investment cost function i_e and (endogenous) hazard rate λ_e do not affect g or n. They do affect the level of consumption, as does the sunk cost s_e.

8. CONCLUSION

The contribution of this paper is to develop a model in which human capital accumulation, technological change, and increasing product variety all contribute to long run growth. The main result is to provide conditions for the existence of a balanced growth path, and to show how the rates of growth of skill, technology, variety, average wages, average profits, and per capita consumption depend on various parameters of the model.

On a BGP, skill and technology grow at a common rate. But interestingly, the parameters governing skill accumulation are more important than those governing technological change in determining that rate. The parameters for skill and technology enter more symmetrically—but with opposite signs—in determining growth in product variety. Thus improvements in the parameters for technological change encourage more entry and greater product proliferation, while improvements in the parameters for skill accumulation encourage investment in both skill and technology, but discourage growth in product variety.

In the production function here is log supermodular in its two inputs, technology and skill. Thus, equilibrium there is positively assorative matching betweeen technologies and skills. As a consequence, investments in technology continue to be profitable because human capital continues to grow, and investments in skill continue to be worthwhile because technology continues to grow. In this sense, in the model here, there is no 'race' between human capital and technology: they grow together.

Although the present paper does not study transitional dynamics, the results here suggest that if investment in one input were prevented for some reason, investment in the other input would become less and less worthwhile, and eventually would cease as well.

Other questions for further work:

—Do transition paths look like those for rapidly growing countries, where technology gets "ahead" because of inflows from abroad?

—What types of empirical evidence are useful for assessing the model?

—Investment in the competitive equilibrium is inefficient. What policies would bring it closer to the efficient level?

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APPENDIX: BGPS FOR PARETO DISTRIBUTIONS

A. Incumbent firms

If Π and V^f have the forms required for a BGP, then factoring out $Q_0 e^{g_\pi t}$, the Bellman equation for a producing firm is

$$\begin{aligned}(r+\delta_x)\, v_f(X/Q(t)) \;&=\; \pi(X/Q(t)) + \mu_x \frac{X}{Q(t)} v_f'(X/Q(t)) \\ &\quad + g_\pi v_f(X/Q(t)) - v_f'(X/Q(t)) \frac{X}{Q(t)} \frac{\dot{Q}(t)}{Q(t)},\end{aligned}$$

or

$$(r+\delta_x - g_\pi)\, v_f(x) = \pi(x) + (\mu_x - g)\, x v_f'(x),$$

where $x = X/Q$ and $\dot{Q}/Q = g$. For π as in (18), the normalized HJB equation is as claimed.

It is straightforward to verify that a particular solution this ODE is $v_P(x) = B_x \pi_1 x^{1-\zeta}$, where B_x is as in (26). As usual, $v_P(x)$ is the value of the firm if it never invests, operating with its evolving technology until the exit shock arrives.

In addition, there is a homogeneous solution, of the form $v_H(x) = c_x x^{R_x}$, where R_x in (26) is the root of the characteristic equation. The coefficient c_x is determined by the value matching condition: the value of a firm at the threshold x_m must equal to the value of an investor, $\lim_{x \to x_m} v_f(x) = v_{f0}$. Hence

$$c_x = x_m^{-R_x} \left(v_{f0} - B_x \pi_1 x_m^{1-\zeta} \right),$$

and $v_f(x)$ is as in (25).

Differentiate (25) to get the smooth pasting condition, which represent the optimal choice of the investment threshold x_m by incumbent producers,

$$0 = v_f'(x_m) = (1-\zeta)\, B_x \pi_1 x_m^{-\zeta} + R_x x_m^{-1} \left(v_{f0} - B_x \pi_1 x_m^{1-\zeta} \right).$$

This condition determines v_{f0}, which is in (27). Using v_{f0} in (25) and integrating w.r.t. the density $f(x) = \alpha x_m^{\alpha} x^{-\alpha-1}$ gives $\mathrm{E}_F[v_f]$.

Similarly, if $V_{f0}(t)$ has the form required for a BGP, then the Bellman equation for the normalized value v_{f0} is as claimed. Substituting for the expected value and v_{f0}, and factoring out $\pi_1 B_x x_m^{1-\zeta}$, the Bellman equation requires

$$\begin{aligned} r + \delta_x - g_\pi &= \lambda_x \left\{ \frac{\mathrm{E}_F[v_f(x)]}{v_{f0}} - 1 \right\} \\ &= \frac{\lambda_x R_x}{R_x - 1 + \zeta} \left(\frac{\alpha}{\alpha - 1 + \zeta} - \frac{1-\zeta}{R_x} \frac{\alpha}{\alpha - R_x} + \frac{1-\zeta}{R_x} - 1 \right) \\ &= \frac{\lambda_x R_x}{R_x - 1 + \zeta} \left(\frac{1-\zeta}{\alpha - 1 + \zeta} - \frac{1-\zeta}{\alpha - R_x} \right) \\ &= \frac{\lambda_x R_x (1-\zeta)}{R_x - 1 + \zeta} \frac{-R_x + 1 - \zeta}{(\alpha - 1 + \zeta)(\alpha - R_x)} \\ &= \frac{-R_x (1-\zeta) \lambda_x}{(\alpha - 1 + \zeta)(\alpha - R_x)}, \end{aligned}$$

as in (28)

B. Entrants

If $V_e(t)$ has the form required for a BGP, then the optimal hazard rate λ_e satisfies (29), and v_e is as in (30). To see that (29) has a unique solution, use (30) to write it as

$$\begin{aligned} \frac{\mathrm{E}_F[v_f]}{\mathrm{E}_F[\pi]} &= i_E'(\lambda_e) + \frac{v_e}{\mathrm{E}_F[\pi]} \\ &= i_E'(\lambda_e) + \frac{\lambda_e \mathrm{E}_F[v_f] / \mathrm{E}_F[\pi] - i_e(\lambda_e)}{r + \delta_x - g_\pi + \lambda_e}, \end{aligned}$$

or

$$(r + \delta_x - g_\pi) \frac{\mathrm{E}_F[v_f(x)]}{\mathrm{E}_F[\pi(x)]} = (r + \delta_x - g_\pi + \lambda_e)\, i_e'(\lambda_e) - i_e(\lambda_e),$$

or

$$\frac{\mathrm{E}_F[v_f(x)]}{\mathrm{E}_F[\pi(x)]} = i_e'(\lambda_e) + \frac{\lambda_e i_e'(\lambda_e) - i_e(\lambda_e)}{r + \delta_x - g_\pi},$$

The LHS is positive. The RHS is negative at $\lambda_e = 0$, is strictly increasing in λ_e, and grows without bound as λ_e increases. Hence there is a solution and it is unique.

C. Flows of firms, the DF for technology

On a BGP, $X_m(t)$ grows at the rate g, $N_p(t), N_i(t)$, $N_e(t)$ grow at the rate n, and there is strictly positive process innovation, so (19) holds. Hence the law of motion for N_p requires

$$\begin{aligned} nN_p &= \lambda_x N_i + \lambda_e N_e - \delta_x N_p - (g - \mu_x) \frac{X_m(t)}{Q(t)} f(x_m) N_p \\ &= \lambda_x N_i + \lambda_e N_e - [\delta_x + \alpha (g - \mu_x)] N_p, \end{aligned}$$

where the second line uses the fact that f is a Pareto density with parameters (α, x_m). Hence

$$[n + \delta_x + \alpha (g - \mu_x)] N_p = \lambda_x N_i + \lambda_e N_e.$$

The laws of motion for N_i and N_e require

$$\begin{aligned} (n + \delta_x + \lambda_x) N_i &= \alpha (g - \mu_x) N_p, \\ (n + \delta_x + \lambda_e) N_e &= E = (n + \delta_x) N. \end{aligned}$$

Hence the population shares for firms are

$$\begin{aligned} \frac{N_p}{N} &= \frac{n + \delta_x + \lambda_x}{(n + \delta_x + \lambda_x) + \alpha (g - \mu_x)} \frac{\lambda_e}{n + \delta_x + \lambda_e}, \\ \frac{N_i}{N} &= \frac{\alpha (g - \mu_x)}{(n + \delta_x + \lambda_x) + \alpha (g - \mu_x)} \frac{\lambda_e}{n + \delta_x + \lambda_e}, \\ \frac{N_e}{N} &= \frac{n + \delta_x}{n + \delta_x + \lambda_e}, \end{aligned}$$

and the ratios N_i/N_p and N_e/N_p satisfy (33).

As a check, note that (33) implies

$$\lambda_x \frac{N_i}{N_p} + \lambda_e \frac{N_e}{N_p} - n - \delta_x = \alpha (g - \mu_x).$$

Using this expression in the law of motion for $\hat{F}$, we get

$$-\hat{F}_t(X,t) = \hat{f}(X,t)\mu_x X + \left[1 - \hat{F}(X,t)\right]\alpha\left(g - \mu_x\right), \qquad \text{all } X \geq X_m(t), \quad \text{all } t.$$

If $\hat{F}$ has the form required for a BGP, then

$$\begin{aligned} \hat{f}(X,t) &= f(X/Q(t)) \ / \ Q(t), \\ -\hat{F}_t(X,t) &= f\left(X/Q(t)\right) gX/Q(t), \qquad \text{all } X \geq X_m(t), \quad \text{all } t. \end{aligned}$$

so the required condition is

$$\left(g - \mu_x\right) xf\left(x\right) = \left(g - \mu_x\right)\alpha\left[1 - F\left(x\right)\right], \qquad \text{all } x \geq x_m,$$

which holds since F is a Pareto distribution with parameters $\left(\alpha, x_m\right)$.

D. Workers, investment in human capital

The analysis for workers and individuals investing in human capital is analogous to that for incumbent firms.

E. Flows of individuals, the DF for skill

On a BGP, $H_m(t)$ grows at the rate g; $L_w(t), L_i(t)$ both grow at the rate υ; and ther is positive investment by individuals, so (20) holds. Hence the law of motion for L_w requires

$$\begin{aligned} \upsilon L_w &= \lambda_h L_i - \delta_h L_w - \left(g - \mu_h\right)\frac{H_m(t)}{Q(t)}\widetilde{g}(h_m)L_w \\ &= \lambda_h L_i - \left[\delta_h + \gamma\left(g - \mu_h\right)\right]L_w, \end{aligned}$$

where the second line uses the fact that $\widetilde{g}$ is a Pareto density with parameters $\left(\gamma, h_m\right)$. Hence

$$\lambda_h L_i = \left[\upsilon + \delta_h + \gamma\left(g - \mu_h\right)\right]L_w,$$

the shares of workers and investors in the population are

$$\begin{aligned}\frac{L_w}{L} &= \frac{\lambda_h}{\lambda_h + \upsilon + \delta_h + \gamma\left(g - \mu_h\right)}, \\ \frac{L_i}{L} &= \frac{\upsilon + \delta_h + \gamma\left(g - \mu_h\right)}{\lambda_h + \upsilon + \delta_h + \gamma\left(g - \mu_h\right)},\end{aligned}$$

and the ratio L_i/L_w satisfies (39).

As a check, note that (39) implies

$$-\frac{L_w'(t)}{L_w(t)} - \delta_h + \lambda_h \frac{L_i(t)}{L_w(t)} = \gamma\left(g - \mu_h\right).$$

Using this expression in the law of motion for $\hat{G}$, we find that $\hat{G}$ must satisfy

$$-\hat{G}_t(H,t) = \hat{g}(H,t)\mu_h H + \left[1 - \hat{G}(H,t)\right]\gamma\left(g - \mu_h\right), \qquad \text{all } H \geq H_m(t).$$

If $\hat{G}$ has the form required for a BGP, substituting for $\hat{G}$ and its derivatives gives

$$\left(g - \mu_h\right) h\widetilde{g}(h) = \left(g - \mu_h\right)\gamma\left[1 - G(h)\right], \qquad \text{all } h \geq h_m,$$

which holds since G is a Pareto distribution with parameters (γ, h_m).