TERM STRUCTURE OF RISK PRICES AND EXPOSURES IN NONLINEAR MODELS

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The manner in which risk operates upon time preference will differ, among other things, according to the particular periods in the future to which the risk applies.

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There are several alternative ways in which one may approach the impulse problem .... One way which I believe is particularly fruitful and promising is to study what would become of the solution of a determinate dynamic system if it were exposed to a stream of erratic shocks that constantly upsets the continuous evolution, and by so doing introduces into the system the energy necessary to maintain the swings.

Ragnar Frisch (1933)

Idea

- quantify cash-flow exposures to shocks over alternative investment horizons;
- and the associated compensations to investors holding these cash flows

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Shock elasticities: refinement of the analysis of asset pricing models

- shock-exposure elasticity – sensitivity of **expected payoffs** to a shock
  - risk quantification
- shock-price elasticity – sensitivity of **expected returns** to shock exposure
  - risk pricing

Dynamic value decomposition

- expected returns and payoffs $\implies$ shock elasticities

Dynamic value decomposition

- expected returns and payoffs $\Longrightarrow$ shock elasticities

Alternative payoff horizons

- term structure of risk

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Alternative payoff horizons

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Shock elasticities

- an alternative to log-linearization methods in asset pricing
- asset pricing counterparts to **nonlinear impulse response functions**.
- framework explicitly takes into account:
  - nonlinearity (stochastic volatility, regime shifts, ...)
  - growth and discounting (compounding over time, permanent components)

Empirical evidence

- growing literature on term structure of risky returns
- new evidence to help us understand differences between existing models and how to distinguish them

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Theoretical modeling

- models make assumptions about the temporal composition of risk
  - transitory vs permanent components
  - substantial role of the permanent component for pricing
- methods for formal description of these components

Shock elasticities

- One-period pricing;
- Elasticities over multiple periods;
- Martingale components in stochastic discount factors

Applications

- Exponential-quadratic framework solvable in closed form;
  - Second-order perturbations of DSGE models (Dynare implementation);
  - Model of intangible capital [Ai, Croce and Li (2012)]
  - Model with financial constraints [Gertler and Kiyotaki (2011)]
- Continuous-time Brownian information framework
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- Suppose X is first-order Markov, and $W \sim N(0, I)$ iid

$$X_{t+1} = \psi(X_t, W_{t+1})$$

- A **multiplicative functional** M is a conditionally Gaussian model

$$\log M_{t+1} - \log M_t = \kappa(X_t, W_{t+1})$$

  - Captures growth and decay.

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- Examples of multiplicative functionals
  - stochastic discount factor S
  - macroeconomic growth functional G (consumption, priced cash flows, ...)

- Given state X_0, suppose that

$$\begin{aligned} \log G_1 &= \beta_g(X_0) + \alpha_g(X_0) \cdot W_1 \\ \log S_1 &= \beta_s(X_0) + \alpha_s(X_0) \cdot W_1 \end{aligned}$$

- One-period return

$$R_1 = \frac{G_1}{E\left[S_1 G_1 \mid X_0\right]}$$

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- Logarithm of the expected return

$$\begin{aligned} \log E\left[R_1 \mid X_0 = x\right] &= \log E\left[G_1 \mid X_0 = x\right] - \log E\left[S_1 G_1 \mid X_0 = x\right] = \\ &= -\beta_s(x) - \frac{1}{2}\left|\alpha_s(x)\right|^2 - \alpha_g(x) \cdot \alpha_s(x) \end{aligned}$$

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- $-\alpha_s(x)$ is the risk price vector for exposure to components of W_1.
- Asset pricing puzzle: Modeled versions of $|\alpha_s|$ are too small for ‘plausible’ values of risk aversion.

- Consider a parameterized family of **perturbations**

$$\log H_1(\mathrm{r}) = -\frac{1}{2}\mathrm{r}^2 \left|\eta(X_0)\right|^2 + \mathrm{r}\eta(X_0) \cdot W_1$$

  - **exposure direction** η, scaling r, and $E[H_1(\mathrm{r}) \mid X_0] = 1$

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- Form $G_1 H_1(\mathrm{r})$:

$$\log G_1 + \log H_1(\mathrm{r}) = \beta_g(X_0) - \frac{1}{2}\mathrm{r}^2 |\eta(X_0)|^2 + \underbrace{[\alpha_g(X_0) + \mathrm{r}\eta(X_0)]}_{\text{new shock exposure}} \cdot W_1$$

- Parameterized family of payoffs to be priced.

Compute expected returns

$$\log E\left[G_1 H_1(\mathrm{r}) \mid X_0 = x\right] - \log E\left[S_1 G_1 H_1(\mathrm{r}) \mid X_0 = x\right]$$

Compute expected returns and differentiate

$$\frac{d}{dr} \log E\left[G_1 H_1(\mathrm{r}) \mid X_0 = x\right] - \log E\left[S_1 G_1 H_1(\mathrm{r}) \mid X_0 = x\right] \Bigg|_{\mathrm{r}=0}$$

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Shock elasticities

1. shock-exposure elasticity

$$\varepsilon_g(x) = \frac{d}{dr} \log E\left[G_1 H_1(\mathrm{r}) \mid X_0 = x\right]\Big|_{\mathrm{r}=0} = \alpha_g(x) \cdot \eta(x)$$

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2. shock-cost elasticity

$$\varepsilon_v(x) = \frac{d}{dr} \log E\left[S_1 G_1 H_1(\mathrm{r}) \mid X_0 = x\right]\Big|_{\mathrm{r}=0} = \left[\alpha_s(x) + \alpha_g(x)\right] \cdot \eta(x)$$

Compute expected returns and differentiate

$$\frac{d}{dr} \log E\left[G_1 H_1(r) \mid X_0 = x\right] - \log E\left[S_1 G_1 H_1(r) \mid X_0 = x\right] \Big|_{r=0}$$

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3. shock-price elasticity

$$\varepsilon_p(x) = \varepsilon_g(x) - \varepsilon_v(x) = -\alpha_s(x) \cdot \eta(x)$$

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- Time-t payoff G_t (zero-coupon equity), stochastic discount factor S_t
- Perturb payoff **in period 1**: $G_t H_1(\mathrm{r})$
- Expected return

$$\frac{1}{t}\log E\left[G_t H_1(\mathrm{r}) \mid X_0 = x\right] - \frac{1}{t}\log E\left[S_t G_t H_1(\mathrm{r}) \mid X_0 = x\right]$$

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- Form shock elasticities:
  1. Shock-**exposure** elasticity

$$\varepsilon_g(x,t) = \left.\frac{d}{d\mathrm{r}}\log E\left[G_t H_1(\mathrm{r}) \mid X_0 = x\right]\right|_{\mathrm{r}=0}$$

  2. Shock-**value** elasticity

$$\varepsilon_v(x,t) = \left.\frac{d}{d\mathrm{r}}\log E\left[S_t G_t H_1(\mathrm{r}) \mid X_0 = x\right]\right|_{\mathrm{r}=0}$$

  3. Shock-**price** elasticity

$$\varepsilon_p(x,t) = \varepsilon_g(x,t) - \varepsilon_v(x,t)$$

Let M be a multiplicative functional (either G or SG):

$$\varepsilon(x,t) = \eta(x) \cdot \frac{E\left[M_t W_1 | X_0 = x\right]}{E\left[M_t | X_0 = x\right]}$$

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Observations

- Log-linear framework:
  - $\varepsilon(x,t)$ recovers the impulse response function for $\log M$ in response to a shock $\eta(x) \cdot W_1$
  - shock-**exposure** elasticity $\varepsilon_g(x,t)$ reflects the IRF for $\log G$
  - shock-**price** elasticity $\varepsilon_p(x,t)$ reflects the IRF for $-\log S$

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- With stochastic volatility and other sources of nonlinearity, $\varepsilon_p(x,t)$ will also depend on the choice of G

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What do long-term pricing implications depend on?

- martingale components in the stochastic discount factor and cash flows

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Multiplicative structure of M allows the extraction of a martingale component

- Hansen and Scheinkman (2009), Alvarez and Jermann (2005)
- long-horizon elasticity limits driven by permament components of M

- We want to pin down long-horizon limits (role of permanent shocks).
- Consider the eigenvalue problem (M_0 normalized to one)

$$E\left[M_1 e(X_1) \mid X_0 = x\right] = \exp(\eta)\, e(x)$$

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We obtain a martingale

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- $\widetilde{M}$ is a martingale
- $\exp(\eta t)$ is a deterministic drift
- $e(X_t)$ is a stationary (eigen)function of the Markov state X_t

- Martingale factorization

$$M_t = \exp\left(\eta t\right) \widetilde{M}_t \frac{e\left(X_0\right)}{e\left(X_t\right)}$$

- Thus (using $\hat{e}\left(x\right) = 1/e\left(x\right)$)

$$\varepsilon(x,t) = \eta(x) \cdot \frac{E\left[M_t W_1 | X_0 = x\right]}{E\left[M_t | X_0 = x\right]} = \eta(x) \cdot \frac{\widetilde{E}\left[\hat{e}\left(X_t\right) W_1 | X_0 = x\right]}{\widetilde{E}\left[\hat{e}\left(X_t\right) | X_0 = x\right]}$$

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- Long-horizon limit

$$\varepsilon(x,\infty) = \eta(x) \cdot \widetilde{E}\left[W_1 \mid X_0 = x\right]$$

Short-horizon limit $\varepsilon(x, 1)$

- corresponds to the local exposure and price of risk in the continuous-time limit

Long-horizon limit $\varepsilon(x, \infty)$

- pricing determined by the martingale components in cash flows and SDFs

Shock elasticities fill the rest

- term structure of risk

Martingale component in the SDF generated by

- permanent shocks to investors' consumption
- nonseparabilities in preferences (Epstein–Zin)
- financial constraints

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How big is it?

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Result

$$L_t\left(\frac{\widetilde{S}_{t+1}}{\widetilde{S}_t}\right) \geq E_t \log R_{t,t+1} - E_t \log R^{\infty}_{t,t+1}$$

- $R_{t,t+1}$ — any one-period return
- $R^{\infty}_{t,t+1}$ — one-period return on the long-horizon bond
- $L_t\left(Z_{t+1}\right) = \log E_t\left[Z_{t+1}\right] - E_t\left[\log Z_{t+1}\right]$ — entropy of random variable Z_{t+1}

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Triangular state vector system:

$$\begin{aligned} X_{1,t+1} &= \Delta_{10} + \Delta_{11} X_{1,t} + \Lambda_{10} W_{t+1} \\ X_{2,t+1} &= \Delta_{20} + \Delta_{21} X_{1,t} + \Delta_{22} X_{2,t} + \Delta_{23} \left(X_{1,t} \otimes X_{1,t} \right) + \\ &\quad + \Lambda_{20} W_{t+1} + \Lambda_{21} \left(X_{1,t} \otimes W_{t+1} \right) + \Lambda_{23} \left(W_{t+1} \otimes W_{t+1} \right) \end{aligned}$$

- Stable dynamics if Δ_{11} and Δ_{22} have stable eigenvalues.
- Structure allows for stochastic volatility through $X_{1,t} \otimes W_{t+1}$

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A class of quadratic functions for increments of multiplicative functionals

$$\begin{aligned} \log M_{t+1} - \log M_t &= \Gamma_0 + \Gamma_1 X_t + \Gamma_2 \left(X_{1,t} \otimes X_{1,t} \right) + \\ &\quad + \Theta_0 W_{t+1} + \Theta_1 \left(X_{1,t} \otimes W_{t+1} \right) + \Theta_2 \left(W_{t+1} \otimes W_{t+1} \right) \end{aligned}$$

- Use to model stochastic discount factors and growth functionals.

The framework allows for quasi-analytical formulas for conditional expectations of multiplicative functionals and for elasticities.

- Start with

$$\log f(x) = \phi + \Phi x + \frac{1}{2}(x_1)'\Psi(x_1)$$

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$$\log f(x) = \phi + \Phi x + \frac{1}{2}(x_1)'\Psi(x_1)$$
- Then
$$\begin{aligned} \log E\left[\left(\frac{M_{t+1}}{M_t}\right) f(X_{t+1}) | X_t = x\right] &= \phi^* + \Phi^* x + \frac{1}{2}(x_1)'\Psi^*(x_1) = \\ &= \log f^*(x) \end{aligned}$$
- Multiperiod conditional expectations calculated by iterating on the coefficient mapping.
- Series expansion (perturbation) framework as a special case.

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Value premium

- High book-to-market (B/M) firms (value stocks) have high expected returns relative to low B/M firms (growth stocks) [Fama and French (1992, 1995)]
- Does this reflect intangible risk exposure? [Hansen, Heaton and Li (2005)].

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Use a model by Ai, Croce, Li (2012)

- distinction between physical and intangible capital
- growth firms have more intangible capital that is a source of new investment projects
- new vintages of capital are less exposed to aggregate risk than old vintages
- growth firms are less risky, and thus earn lower expected returns

Aggregate output in the economy

$$Y_t = (K_t)^{\lambda} (A_t N_t)^{1-\lambda} = C_t + I_t + I_t^* \qquad N_t = 1$$

Two types of capital

- physical capital

$$K_{t+1} = (1-\delta) K_t + B_{t+1} G(I_t, K_t^*)$$

- intangible capital

$$K_{t+1}^* = [K_t^* - G(I_t, K_t^*)] (1-\delta^*) + H(I_t^*, K_t)$$

- production of new physical capital

$$G(I_t, K_t^*) = \left[\phi (I_t)^{1-\psi} + (1-\phi)(K_t^*)^{1-\psi}\right]^{\frac{1}{1-\psi}}$$

Representative household with recursive preferences

Technology modeled as permanent (growth rate) shocks

$$\log A_{t+1} - \log A_t = \Theta + Z_t + \Gamma W_{t+1}^1$$

where the 'long-run risk' component Z evolves as:

$$Z_{t+1} = \Psi Z_t + \Lambda W_{t+1}^2$$

Shock to **newly installed capital** is modeled as :

$$\log B_{t+1} = -\frac{1-\lambda}{\lambda}\left(Z_t + \Gamma W_{t+1}^1\right)$$

Neutral technology growth rate and the investment specific shocks are perfectly negatively correlated.

The common component to $\log A_{t+1} - \log A_t$ and $\log B_{t+1}$ is $Z_t + \Gamma W^1_{t+1}$ where the 'long-run risk' component Z evolves as:

$$Z_{t+1} = \Psi Z_t + \Lambda W^2_{t+1}$$

Two shocks

- immediate shock: ΓW^1_{t+1} (rescaled to have a unit sd)
- long-run risk shock: ΛW^2_{t+1} (rescaled to have a unit sd)

Impact on $\log B_{t+1}$ temporary but impact on $\log A_{t+1}$ is permanent.

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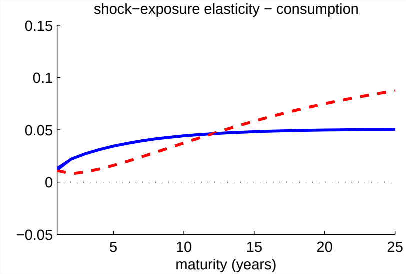
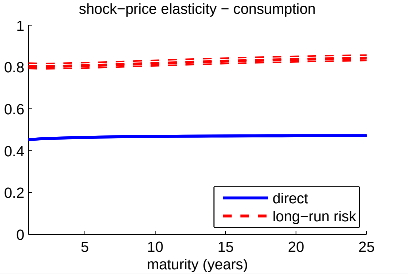
Questions

- What are the sources of risk premia? And what generates the heterogeneity?
  - Which shocks get priced (direct, long-run risk)?
  - At what horizons?
  - Through which channels (neutral, investment-specific)?

Shock-exposure elasticity

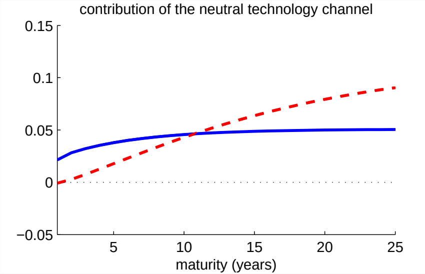


Shock-price elasticity



- shock-exposure elasticities gradually build (capital accumulation
- shock-price elasticities are close to flat (recursive preferences)

Shock-exposure elasticity



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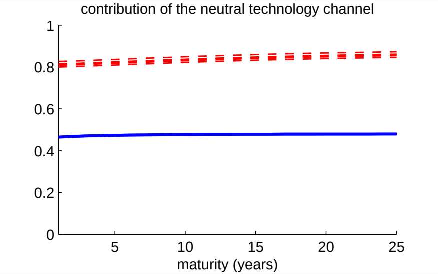

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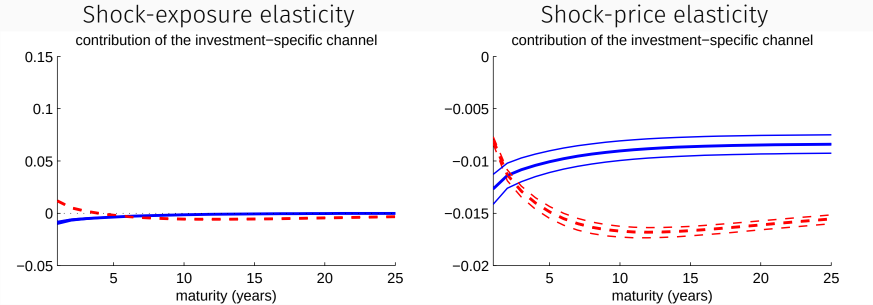
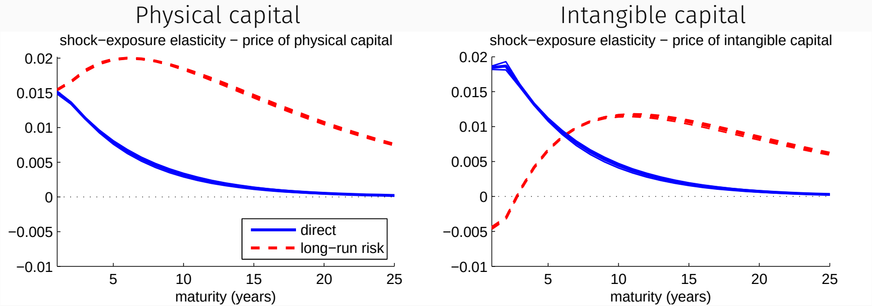
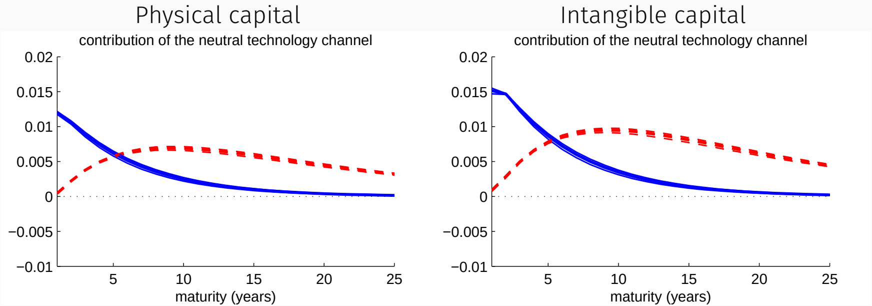


- shock-exposure elasticities gradually build (capital accumulation
- shock-price elasticities are close to flat (recursive preferences)
- **neutral technology growth rate shock** drives the pricing results
- **shock to newly installed capital** has only small transitory impact on consumption dynamics $\Longrightarrow$ minimal impact on pricing

SHOCK ELASTICITIES FOR CAPITAL VALUATION



- dramatic difference in the responses to the long-run risk shock
- which channel contributes?



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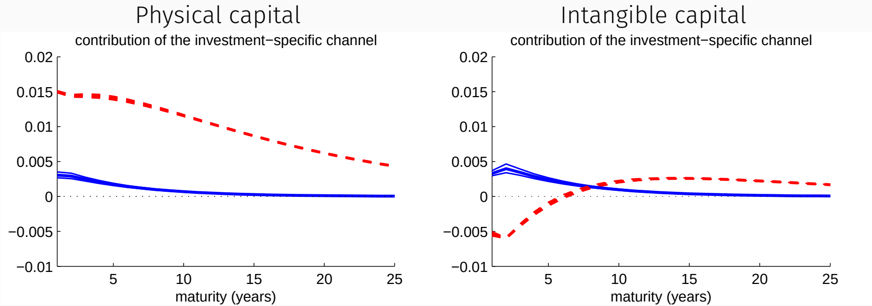

- dramatic difference in the responses to the long-run risk shock
- which channel contributes?
- responses propagated through the neutral technology channel are similar
- differences caused by the exposure to the long-run risk shock **propagated through the investment-specific channel**

Shock elasticities

- One-period pricing;
- Elasticities over multiple periods;
- Martingale components in stochastic discount factors

Applications

- Exponential-quadratic framework solvable in closed form;
  - Second-order perturbations of DSGE models (Dynare implementation);
  - Model of intangible capital [Ai, Croce and Li (2012)]
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EXAMPLE: MODELS WITH FINANCIAL CONSTRAINTS (GERTLER AND KIYOTAKI (2011))

A large class of models with frictions in the financial sector

- Households cannot invest in productive capital directly.
- Financing friction in banks casts a wedge between MRS and prices.
- Asset valuation depends on cash flows and ability to relax constraints.

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Gertler and Kiyotaki (2011)

- Households $\Longleftrightarrow$ financial sector $\Longleftrightarrow$ firms
- Occasionally binding constraint.
- Here: Series expansion around the deterministic steady state.
  - fits the exponential-quadratic framework.

- preferences

$$U_t\left(C, L\right) \doteq E\left[\sum_{j=0}^{\infty}\left(\log\left(C_{t+j} - hC_{t+j-1}\right) - \frac{\chi}{1+\varepsilon}\left(L_{t+j}\right)^{1+\varepsilon}\right) \mid \mathcal{F}_t\right].$$

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- stochastic discount factor

$$\frac{S_{t+1}^h}{S_t^h} = \beta\frac{U_{c,t+1}}{U_{c,t}}$$

where U_c is the marginal utility of consumption

$$U_{c,t} = \frac{1}{C_t - hC_{t-1}} - h\beta E\left[\frac{1}{C_{t+1} - hC_t} \mid \mathcal{F}_t\right].$$

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- priced asset: deposit contract

$$1 = E\left[\frac{S^h_{t+1}}{S^h_t}R^d_{t+1} \mid \mathcal{F}_t\right].$$

- objective

$$V(k_t, b_t, d_t) = \max_{(k_{t+1}, b_{t+1}, d_{t+1})} E\left[\frac{S^h_{t+1}}{S^h_t}\left[(1-\sigma)\, n_{t+1} + \sigma V(k_{t+1}, b_{t+1}, d_{t+1})\right] \mid \mathcal{F}_t\right].$$

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- balance-sheet constraint

$$Q_t k_t = n_t + d_t + b_t$$

- accumulation of net worth

$$n_{t+1} = R^k_{t+1} Q_t k_t - R^b_{t+1} b_t - R^d_{t+1} d_t$$

- financial constraint

$$V(k_t, b_t, d_t) \geq \theta^k Q_t k_t - \theta^b b_t$$

- Neutral productivity shock

$$Y_t = A_t K_t^{\alpha} L_t^{1-\alpha}$$

- Capital-quality shock

$$K_{t+1} = \psi_{t+1} \left[I_t + (1-\delta) K_t \right]$$

- Both modeled as stationary
  - but capital-quality shocks gain a lot of persistence through capital accumulation

- Return on capital

$$R_{t+1}^{k} = \psi_{t+1} \frac{Z_{t+1} + (1-\delta) Q_{t+1}}{Q_t}$$

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- Revenue stream from a unit of capital produced at time t:

$$G_{t+j} = Z_{t+j}\,(1-\delta)^{j-1} \prod_{m=1}^{j} \psi_{t+m}, \qquad j \geq 1,$$

  - corresponds to a multiplicative functional

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- Profit cash flow of the whole firm

$$G_t = \psi_t Z_t k_t$$

- Net worth paid to households

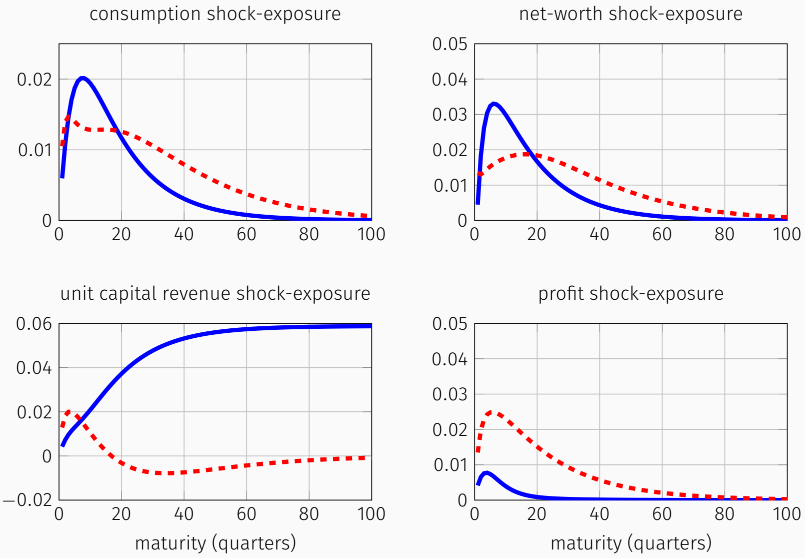
$$G_t = (1-\sigma)\, n_t$$

Euler equation

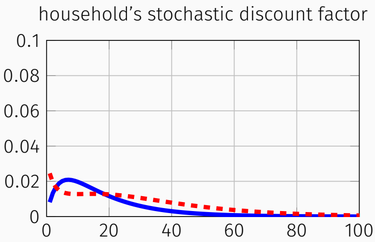
$$Q_t = E\left[\underbrace{\frac{S^h_{t+1}}{S^h_t}\left(1-\sigma+\sigma\mu_{t+1}\right)\frac{1+\lambda_t}{\mu_t+\lambda_t\theta^k}}_{\text{'stochastic discount factor'}}\underbrace{\psi_{t+1}\left(Z_{t+1}+(1-\delta)\,Q_{t+1}\right)}_{\text{payoff in period } t+1} \mid \mathcal{F}_t\right]$$

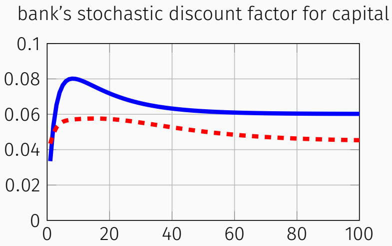
-
  - λ_{t+1} Lagrange multiplier on the financial constraint
  - μ_{t+1} Lagrange multiplier on the balance-sheet constraint
  - frictionless economy: $\lambda_{t+1}=0$, $\mu_{t+1}=1$
- 'stochastic discount factor' is asset-specific (θ^k)

SHOCK-EXPOSURE ELASTICITIES



SHOCK-PRICE ELASTICITIES





$$\frac{S_{t+1}(\theta^k)}{S_t\left(\theta^k\right)} \doteq \frac{S^h_{t+1}}{S^h_t}\left(1-\sigma+\sigma\mu_{t+1}\right)\frac{1+\lambda_t}{\mu_t+\lambda_t\theta^k}$$

- shock-price elasticities for $S(\theta^k)$ are much larger ...
- ... and **do not decay to zero.**
- **state dependence** (!!!)

Shock elasticities

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- Elasticities over multiple periods;
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Environment

- State dynamics
$$dX_t = \mu(X_t)\,dt + \sigma(X_t)\,dW_t$$
- Multiplicative processes (SDFs, cash flows)
$$d\log M_t = \beta(X_t)\,dt + \alpha(X_t)\cdot dW_t$$

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Shock elasticities

- Perturbation over a 'short time interval' dW_0
- Related to the Malliavian derivative

$$\varepsilon(x,t) = \nu(x) \cdot \left[\underbrace{\sigma(x)' \left(\frac{\partial}{\partial x} \log E\left[\frac{M_t}{M_0} \mid X_0 = x \right] \right)}_{\text{state propagation}} + \underbrace{\alpha(x)}_{\text{initial impact}} \right]$$

Central object in the analysis is the computation of

$$\phi_t(x) \doteq E\left[\frac{M_t}{M_0}\phi_0(X_t) \mid X_0 = x\right].$$

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Differential equation for $\phi_t(x)$

$$\frac{\partial}{\partial t}\phi_t = \left(\beta + \frac{1}{2}|\alpha|^2\right)\phi_t + (\mu + \sigma\alpha)\cdot\frac{\partial \phi_t}{\partial x} + \frac{1}{2}\mathrm{tr}\left[\sigma\sigma'\frac{\partial^2 \phi_t}{\partial x \partial x'}\right]$$

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Boundary conditions

- terminal: $\phi_0(x) \equiv 1$
- lateral: determined by the economics of the model

Solution

- finite differences, perturbation methods, MC simulations, ...

The Perron–Frobenius problem

$$E\left[M_t e\left(X_t\right) \mid X_0 = x\right] = \exp\left(\eta t\right) e\left(x\right)$$

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$$\lim_{t \to 0} \frac{1}{t}\left[E\left[M_t e\left(X_t\right) \mid X_0 = x\right] - \exp\left(\eta\right) e\left(x\right)\right] = 0.$$

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Resulting PDE

$$\eta e = \left(\beta + \frac{1}{2}\left|\alpha\right|^2\right) e + \left(\mu + \sigma\alpha\right) \cdot \frac{\partial e}{\partial x} + \frac{1}{2}\mathrm{tr}\left[\sigma\sigma' \frac{\partial^2 e}{\partial x \partial x'}\right]$$

- Sturm–Liouville problem: finding $(\eta, e) \implies \phi_t\left(x\right) = \exp\left(\eta t\right) e\left(x\right)$

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Martingale factorization

$$M_t = \exp\left(\eta t\right) \widetilde{M}_t \frac{e\left(X_0\right)}{e\left(X_t\right)}$$

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Two types of agents: households and experts

- Households less efficient in managing capital
- Lending capital to experts is only possible using risk-free bond contracts
    - financial constraints on efficient allocation of capital

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Preferences

- Both agents linear preferences. Experts

$$E\left[\int_0^\infty e^{-\rho t} dC_t^h \mid \mathcal{F}_0\right]$$

- Households more patient than experts: $r < \rho$.
- Marginal value of experts' wealth $\Theta_t \geq 1$ due to the financial constraint.

- wealth share of experts:

$$X_t = \frac{N_t}{Q_t K_t}$$

- aggregate capital

$$d \log K_t = \beta_k (X_t)\, dt + \bar{\alpha}_k dW_t$$

  - drift depends on wealth distribution, since the two agents have different capital accumulation rates

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  - drift depends on wealth distribution, since the two agents have different capital accumulation rates

State $X_t \in (0, \bar{x}]$

- X_t large — households invest all capital with firms
- X_t small — limited intermediation
- $X_t = \bar{x}$ — marginal value of experts' wealth declines to one, experts consume
  - reflecting boundary

Evolution of the state

$$dX_t = \mu(X_t)\,dt + \sigma(X_t)\,dW_t - X_t d\zeta_t$$

- $d\zeta_t$ — consumption of experts at $X_t = \bar{x}$.

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Stochastic discount factor of experts

$$\frac{S_t}{S_0} = e^{-\rho t}\frac{\theta(X_t)}{\theta(X_0)}$$

- $\Theta_t = \theta(X_t)$ — marginal utility of experts' wealth
  - not equal to one due to limited risk-sharing

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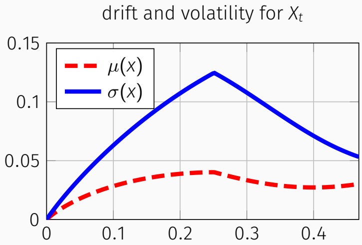
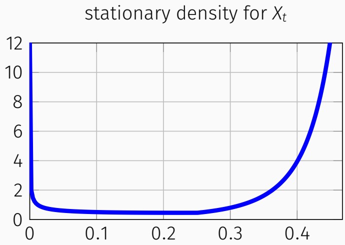
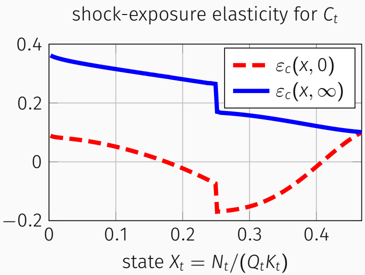
Cash flow: household consumption

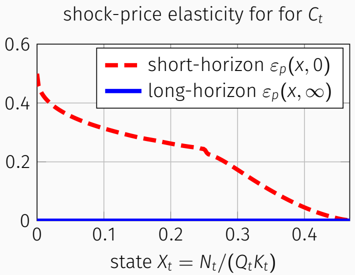
$$d\log C_t = \beta_c(X_t)\,dt + \alpha_c(X_t)\,dW_t$$



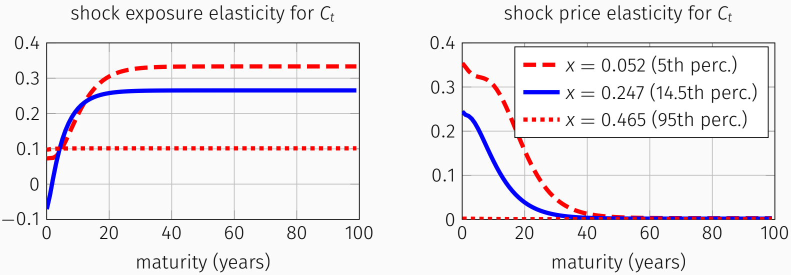


- Wealth in hands of experts valuable $\implies$ accumulation, $\mu(x) > 0$
- Experts consume when $\theta(X_t)$ reaches one $\implies$ reflecting right boundary
- Stationary density has two peaks, 'normal times' and 'financial crises'

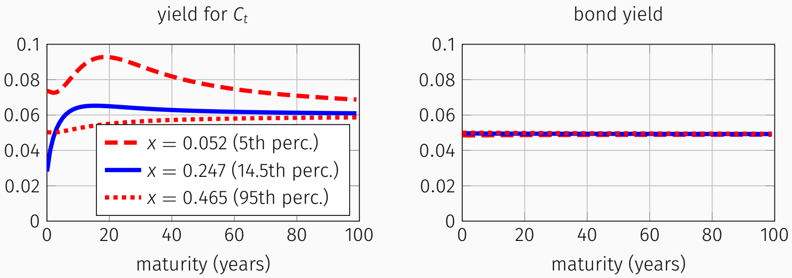




- Short-horizon consumption response small, even countercyclical.
- Shock-price elasticity is large during ‘financial crises’, zero in the long run.



- shock exposures generally increasing, shock prices decrasing



- Shock-exposure and shock-price elasticities explain the shape of yield differences between risky and risk-free cash flows
  - hump shaped in risky region
  - negative for short horizons in the center of the distribution

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Let Z_t be an n-state continuous time Markov chain

- realization a coordinate vector
- intensity matrix A

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Multiplicative functional

$$\log M_t = \sum_{0 \le u \le t} (Z_{u-})' \kappa Z_u + \int_0^t (Z_{u-})' \beta du + \int_0^t (Z_{u-})' \alpha dW_t$$

- β — elements are conditional (on Z_{u-}) growth rates
- α — rows are vectors of conditional exposures to dW_t
- κ — jump risk matrix, zeros on diagonal

Jump-risk perturbation

$$\log H_t(\mathrm{r}) = \sum_{0 \le u \le t} (Z_{u-})' (\mathrm{r}\kappa_d) Z_u + \int_0^t (Z_{u-})' \beta_h(\mathrm{r})\, du$$

- κ_d — perturbation direction (jump combination) that we are interested in

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- κ_d — perturbation direction (jump combination) that we are interested in

Shock elasticity function

$$\varepsilon(t) = \mathrm{dvec}\left\{\kappa_d \left(\hat{A} - A\right)'\right\} + \mathrm{dvec}\left\{[\Xi(t) \times \kappa_d]\, \hat{A}'\right\}$$

- $\hat{A}$ — intensity matrix distorted by the martingale component of M
- $\Xi(t)$ traces out the transitory response

- Consumption modeled as multiplicative functional

$$\log C_t = \int_0^t (Z_{u-})' \beta_c du + \int_0^t (Z_{u-})' \alpha_c dW_t$$

- Agents endowed with Epstein–Zin preferences with $IES = 1$

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- Continuation value

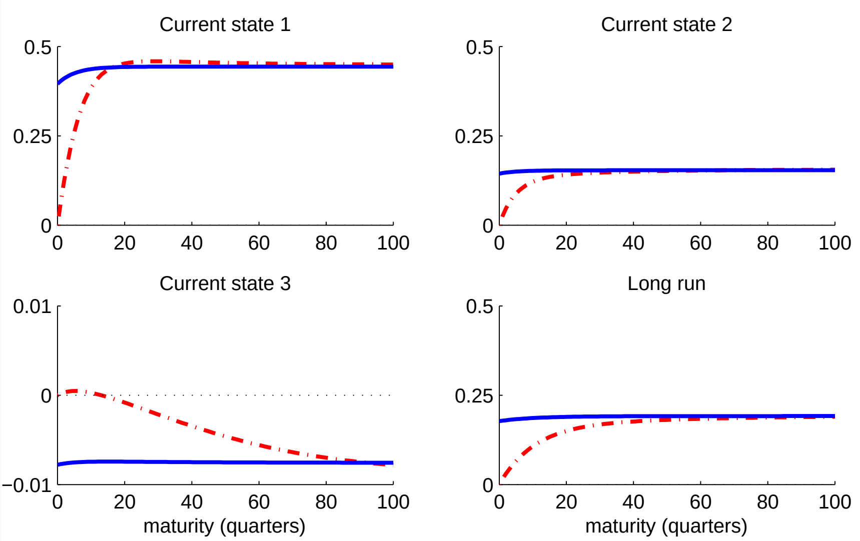
$$\log V_t = v \cdot Z_t + \log C_t$$

- Even though C_t does not jump at regime changes, V_t does:

$$\kappa_v = \mathbf{1}_n v' - v \mathbf{1}_n'$$

- Compare shock-price elasticities for CRRA and EZ stochastic discount factors

EXAMPLE: BONOMO AND GARCIA (1996)



- κ_d — normalized jump component of the martingale from $\log C$

Substantial information in the term structure of risk premia.

- yields on dividend strips and other maturity specific assets
  - empirical work by Koijen, van Binsbergen and others
  - many models produce counterfactual results
- holding period returns on long-term assets
  - entropy and variance bounds on the martingale component

Shock elasticities

- risk exposures and associated compensation
- term structure of risk

Theoretical framework

- continuous time
  - Borovička, Hansen, Hendricks, Scheinkman (2011); Borovička, Hansen (2016)
- discrete time
  - Borovička, Hansen (2014); Nakamura, Sergeyev, Steinsson (2016)
- non-Gaussian framework
  - Borovička, Hansen, Hendricks, Scheinkman (2011); Zviadadze (2016)
- nonlinear impulse response functions
  - Gallant, Rossi and Tauchen (1993); Koop, Pesaran and Potter (1996); Gourieroux and Jasiak (2005); Borovička, Hansen, Scheinkman (2014)

Long-run pricing

- martingale decomposition
  - Hansen, Scheinkman (2009, 2012); Borovička, Hansen, Scheinkman (2016); Qin and Linetsky (2014a,b)
- role of the martingale component
  - Alvarez and Jermann (2005); Qin, Linetsky and Nie (2016); Bakshi, Chabi-Yo and Gao (2016); Christensen (2014); ...

Applications

- currency returns: Zviadadze (2016)
- long-run risk: Nakamura, Sergeyev, Steinsson (2016)

Empirical evidence

- downward sloping term structure of risk
  - van Binsbergen, Brandt and Koijen (2012); van Binsbergen, Hueskes, Koijen and Vrugt (2013); Ai, Croce, Diercks and Li (2013); Belo, Collin-Dufresne and Goldstein (2015); Hasler and Marfè (2015), Lopez, Lopez-Salido and Vazquez-Grande (2015), van Binsbergen and Koijen (2016)