TERM STRUCTURE OF RISK PRICES AND EXPOSURES IN NONLINEAR MODELS

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The manner in which risk operates upon time preference will differ, among other things, according to the particular periods in the future to which the risk applies.

Irving Fisher (1930)
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There are several alternative ways in which one may approach the impulse problem .... One way which I believe is particularly fruitful and promising is to study what would become of the solution of a determinate dynamic system if it were exposed to a stream of erratic shocks that constantly upsets the continuous evolution, and by so doing introduces into the system the energy necessary to maintain the swings.

Ragnar Frisch (1933)
Idea

- quantify *cash-flow exposures* to shocks over alternative investment horizons;
- and the associated *compensations to investors* holding these cash flows
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- quantify cash-flow exposures to shocks over alternative investment horizons;
- and the associated compensations to investors holding these cash flows

**Shock elasticities**: refinement of the analysis of asset pricing models

- **shock-exposure elasticity** – sensitivity of expected payoffs to a shock
  - risk quantification
- **shock-price elasticity** – sensitivity of expected returns to shock exposure
  - risk pricing
Dynamic value decomposition

- expected returns and payoffs $\implies$ shock elasticities
Dynamic value decomposition

- expected returns and payoffs $\rightarrow$ shock elasticities

Alternative payoff horizons

- term structure of risk
Dynamic value decomposition

- expected returns and payoffs $\rightarrow$ shock elasticities

Alternative payoff horizons

- term structure of risk

Shock elasticities

- an alternative to log-linearization methods in asset pricing
- asset pricing counterparts to nonlinear impulse response functions.
- framework explicitly takes into account:
  - nonlinearity (stochastic volatility, regime shifts, ...)
  - growth and discounting (compounding over time, permanent components)
Empirical evidence

- growing literature on term structure of risky returns
- new evidence to help us understand differences between existing models and how to distinguish them
WHY IS THIS IMPORTANT / INTERESTING?

Empirical evidence

- growing literature on term structure of risky returns
- new evidence to help us understand differences between existing models and how to distinguish them

Theoretical modeling

- models make assumptions about the temporal composition of risk
  - transitory vs permanent components
  - substantial role of the permanent component for pricing
- methods for formal description of these components
Shock elasticities

- One-period pricing;
- Elasticities over multiple periods;
- Martingale components in stochastic discount factors

Applications

- Exponential-quadratic framework solvable in closed form;
  - Second-order perturbations of DSGE models (Dynare implementation);
  - Model of intangible capital [Ai, Croce and Li (2012)]
  - Model with financial constraints [Gertler and Kiyotaki (2011)]
- Continuous-time Brownian information framework
  - Models with financial constraints [He and Krishnamurthy (2013), Brunnermeier and Sannikov (2014)]
- Regime shift risk
  - Endowment economy with jump risk [Bonomo and Garcia (1996)]
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· Suppose $X$ is first-order Markov, and $W \sim N(0, I)$ iid

$$X_{t+1} = \psi(X_t, W_{t+1})$$

· A **multiplicative functional** $M$ is a conditionally Gaussian model

$$\log M_{t+1} - \log M_t = \kappa(X_t, W_{t+1})$$

· Captures growth and decay.
Suppose $X$ is first-order Markov, and $W \sim N(0, I)$ iid

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$$\log M_{t+1} - \log M_t = \kappa(X_t, W_{t+1})$$

  - Captures growth and decay.

- Examples of multiplicative functionals
  - stochastic discount factor $S$
  - macroeconomic growth functional $G$ (consumption, priced cash flows, ...)
· Given state $X_0$, suppose that

$$\log G_1 = \beta_g (X_0) + \alpha_g (X_0) \cdot W_1$$
$$\log S_1 = \beta_s (X_0) + \alpha_s (X_0) \cdot W_1$$

· One-period return

$$R_1 = \frac{G_1}{E [S_1 G_1 \mid X_0]}$$
· Given state $X_0$, suppose that

$$
\log G_1 = \beta_g (X_0) + \alpha_g (X_0) \cdot W_1
$$

$$
\log S_1 = \beta_s (X_0) + \alpha_s (X_0) \cdot W_1
$$

· One-period return

$$
R_1 = \frac{G_1}{E [S_1 G_1 \mid X_0]}
$$

· Logarithm of the expected return

$$
\log E [R_1 \mid X_0 = x] = \log E [G_1 \mid X_0 = x] - \log E [S_1 G_1 \mid X_0 = x] =
$$

$$
= -\beta_s (x) - \frac{1}{2} |\alpha_s (x)|^2 - \alpha_g (x) \cdot \alpha_s (x)
$$
• Given state $X_0$, suppose that

\[
\log G_1 = \beta_g (X_0) + \alpha_g (X_0) \cdot W_1 \\
\log S_1 = \beta_s (X_0) + \alpha_s (X_0) \cdot W_1
\]

• One-period return

\[
R_1 = \frac{G_1}{E[S_1G_1 | X_0]}
\]

• Logarithm of the expected return

\[
\log E[R_1 | X_0 = x] = \log E[G_1 | X_0 = x] - \log E[S_1G_1 | X_0 = x] = \\
= -\beta_s (x) - \frac{1}{2} |\alpha_s (x)|^2 - \alpha_g (x) \cdot \alpha_s (x)
\]

• $-\alpha_s (x)$ is the risk price vector for exposure to components of $W_1$.

• Asset pricing puzzle: Modeled versions of $|\alpha_s|$ are too small for ‘plausible’ values of risk aversion.
AN ALTERNATIVE APPROACH: PERTURBATIONS

- Consider a parameterized family of perturbations

\[
\log H_1 (r) = -\frac{1}{2} r^2 |\eta (X_0)|^2 + r\eta (X_0) \cdot W_1
\]

- exposure direction \( \eta \), scaling \( r \), and \( E [H_1 (r) \mid X_0] = 1 \)
AN ALTERNATIVE APPROACH: PERTURBATIONS

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- exposure direction \( \eta \), scaling \( r \), and \( E[H_1 (r) \mid X_0] = 1 \)

- Form \( G_1 H_1 (r) \):

\[
\log G_1 + \log H_1 (r) = \beta_g (X_0) - \frac{1}{2} r^2 |\eta(X_0)|^2 + \left[ \alpha_g (X_0) + r\eta(X_0) \right] \cdot W_1
\]

new shock exposure

- Parameterized family of payoffs to be priced.
Compute expected returns

\[
\log E [G_1H_1 (r) \mid X_0 = x] - \log E [S_1G_1H_1 (r) \mid X_0 = x]
\]
AN ALTERNATIVE APPROACH: PERTURBATIONS

Compute expected returns and differentiate

\[
\frac{d}{dr} \log E [G_1H_1 (r) \mid X_0 = x] - \log E [S_1G_1H_1 (r) \mid X_0 = x] \bigg|_{r=0}
\]
Compute expected returns and differentiate

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\frac{d}{dr} \log E [G_1 H_1 (r) \mid X_0 = x] - \log E [S_1 G_1 H_1 (r) \mid X_0 = x]
\]

\[= \left. \frac{d}{dr} \log E [G_1 H_1 (r) \mid X_0 = x] \right|_{r=0}
\]

Shock elasticities

1. shock-exposure elasticity

\[
\varepsilon_g (x) = \left. \frac{d}{dr} \log E [G_1 H_1 (r) \mid X_0 = x] \right|_{r=0} = \alpha_g (x) \cdot \eta (x)
\]
Compute expected returns and differentiate

\[ \frac{d}{dr} \log E [G_1 H_1 (r) | X_0 = x] - \log E [S_1 G_1 H_1 (r) | X_0 = x] \bigg|_{r=0} \]

Shock elasticities

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2. shock-cost elasticity

\[ \varepsilon_v (x) = \frac{d}{dr} \log E [S_1 G_1 H_1 (r) | X_0 = x] \bigg|_{r=0} = [\alpha_s (x) + \alpha_g (x)] \cdot \eta (x) \]
Compute expected returns and differentiate
\[
\frac{d}{dr} \log E [G_1 H_1 (r) \mid X_0 = x] - \log E [S_1 G_1 H_1 (r) \mid X_0 = x] \bigg|_{r=0}
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Shock elasticities

1. shock-exposure elasticity
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\varepsilon_v (x) = \frac{d}{dr} \log E [S_1 G_1 H_1 (r) \mid X_0 = x] \bigg|_{r=0} = [\alpha_s (x) + \alpha_g (x)] \cdot \eta (x)
\]

3. shock-price elasticity
\[
\varepsilon_p (x) = \varepsilon_g (x) - \varepsilon_v (x) = -\alpha_s (x) \cdot \eta (x)
\]
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- Regime shift risk
  - Endowment economy with jump risk [Bonomo and Garcia (1996)]
· Time-\(t\) payoff \(G_t\) (zero-coupon equity), stochastic discount factor \(S_t\)
· Perturb payoff \textbf{in period 1}: \(G_t H_1(r)\)
· Expected return

\[
\frac{1}{t} \log E [G_t H_1(r) \mid X_0 = x] - \frac{1}{t} \log E [S_t G_t H_1(r) \mid X_0 = x]
\]
EXTENDING THE INVESTMENT HORIZON

- Time-$t$ payoff $G_t$ (zero-coupon equity), stochastic discount factor $S_t$
- Perturb payoff **in period 1**: $G_t H_1(r)$
- Expected return

$$\frac{1}{t} \log E \left[ G_t H_1(r) \mid X_0 = x \right] - \frac{1}{t} \log E \left[ S_t G_t H_1(r) \mid X_0 = x \right]$$

- Form shock elasticities:
  1. **Shock-exposure** elasticity
     \[ \varepsilon_g(x, t) = \frac{d}{dr} \log E \left[ G_t H_1(r) \mid X_0 = x \right] \bigg|_{r=0} \]
  2. **Shock-value** elasticity
     \[ \varepsilon_v(x, t) = \frac{d}{dr} \log E \left[ S_t G_t H_1(r) \mid X_0 = x \right] \bigg|_{r=0} \]
  3. **Shock-price** elasticity
     \[ \varepsilon_p(x, t) = \varepsilon_g(x, t) - \varepsilon_v(x, t) \]
Let $M$ be a multiplicative functional (either $G$ or $SG$):

\[ \varepsilon(x, t) = \eta(x) \cdot \frac{E[M_t W_1 | X_0 = x]}{E[M_t | X_0 = x]} \]
Let $M$ be a multiplicative functional (either $G$ or $SG$):

$$
\varepsilon(x, t) = \eta(x) \cdot \frac{E[M_t W_1 | X_0 = x]}{E[M_t | X_0 = x]}
$$

Observations

- Log-linear framework:
  - $\varepsilon(x, t)$ recovers the impulse response function for $\log M$ in response to a shock $\eta(x) \cdot W_1$
  - **shock-exposure elasticity** $\varepsilon_g(x, t)$ reflects the IRF for $\log G$
  - **shock-price elasticity** $\varepsilon_p(x, t)$ reflects the IRF for $-\log S$
Let $M$ be a multiplicative functional (either $G$ or $SG$):

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\varepsilon(x, t) = \eta(x) \cdot \frac{E[M_t W_1 | X_0 = x]}{E[M_t | X_0 = x]}
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Observations

· Log-linear framework:
  
  · $\varepsilon(x, t)$ recovers the impulse response function for $\log M$ in response to a shock $\eta(x) \cdot W_1$
  
  · shock-exposure elasticity $\varepsilon_g(x, t)$ reflects the IRF for $\log G$
  
  · shock-price elasticity $\varepsilon_p(x, t)$ reflects the IRF for $- \log S$

· With stochastic volatility and other sources of nonlinearity, $\varepsilon_p(x, t)$ will also depend on the choice of $G$
Shock elasticities

- One-period pricing;
- Elasticities over multiple periods;
- Martingale components in stochastic discount factors

Applications

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  - Second-order perturbations of DSGE models (Dynare implementation);
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What do long-term pricing implications depend on?

- martingale components in the stochastic discount factor and cash flows
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- martingale components in the stochastic discount factor and cash flows

Multiplicative structure of $M$ allows the extraction of a martingale component

- Hansen and Scheinkman (2009), Alvarez and Jermann (2005)
- long-horizon elasticity limits driven by permanent components of $M$
· We want to pin down long-horizon limits (role of permanent shocks).
· Consider the eigenvalue problem ($M_0$ normalized to one)

$$E [M_1 e(X_1) \mid X_0 = x] = \exp (\eta) e(x)$$
We want to pin down long-horizon limits (role of permanent shocks).

Consider the eigenvalue problem ($M_0$ normalized to one)

$$E \left[ M_1 e(X_1) \mid X_0 = x \right] = \exp(\eta) e(x)$$

We obtain a martingale

$$\tilde{M}_t = M_t \exp(-\eta t) \frac{e(X_t)}{e(X_0)}$$
· We want to pin down long-horizon limits (role of permanent shocks).
· Consider the eigenvalue problem ($M_0$ normalized to one)

\[
E [M_1 e(X_1) \mid X_0 = x] = \exp (\eta) e(x)
\]

We obtain a martingale

\[
\tilde{M}_t = M_t \exp (-\eta t) \frac{e(X_t)}{e(X_0)} \quad \Longrightarrow \quad M_t = \exp (\eta t) \tilde{M}_t \frac{e(X_0)}{e(X_t)}
\]
• We want to pin down long-horizon limits (role of permanent shocks).
• Consider the eigenvalue problem \((M_0 \text{ normalized to one})\)

\[
E [M_1 e(X_1) \mid X_0 = x] = \exp (\eta) e(x)
\]

We obtain a martingale

\[
\tilde{M}_t = M_t \exp (-\eta t) \frac{e(X_t)}{e(X_0)} \quad \implies \quad M_t = \exp (\eta t) \tilde{M}_t \frac{e(X_0)}{e(X_t)}
\]

• \(\tilde{M}\) is a martingale
• \(\exp (\eta t)\) is a deterministic drift
• \(e(X_t)\) is a stationary (eigen)function of the Markov state \(X_t\)
\[ M_t = \exp(\eta t) \frac{\tilde{M}_t e(X_0)}{e(X_t)} \]

\[ \varepsilon(x, t) = \eta(x) \cdot \frac{E[M_t W_1|X_0 = x]}{E[M_t|X_0 = x]} = \eta(x) \cdot \frac{\tilde{E}[\hat{e}(X_t) W_1|X_0 = x]}{\tilde{E}[\hat{e}(X_t)|X_0 = x]} \]
• **Martingale factorization**

\[ M_t = \exp(\eta t) \tilde{M}_t \frac{e(X_0)}{e(X_t)} \]

• Thus (using \( \hat{e}(x) = 1/e(x) \))

\[ \varepsilon(x, t) = \eta(x) \cdot \frac{E[M_t W_1 | X_0 = x]}{E[M_t | X_0 = x]} = \eta(x) \cdot \frac{\tilde{E}[\hat{e}(X_t) W_1 | X_0 = x]}{\tilde{E}[\hat{e}(X_t) | X_0 = x]} \]

• **Long-horizon limit**

\[ \varepsilon(x, \infty) = \eta(x) \cdot \tilde{E}[W_1 | X_0 = x] \]
Short-horizon limit \( \varepsilon (x, 1) \)

- corresponds to the \textit{local exposure and price of risk} in the continuous-time limit

Long-horizon limit \( \varepsilon (x, \infty) \)

- pricing determined by the martingale components in cash flows and SDFs

Shock elasticities fill the rest

- \textit{term structure of risk}
Martingale component in the SDF generated by

- permanent shocks to investors’ consumption
- nonseparabilities in preferences (Epstein–Zin)
- financial constraints
Martingale component in the SDF generated by

- permanent shocks to investors’ consumption
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How big is it?

- Alvarez and Jermann (2005) — asset prices impose bounds on the magnitude of $\tilde{S}$
THE ROLE OF THE MARTINGALE COMPONENT

Martingale component in the SDF generated by

- permanent shocks to investors’ consumption
- nonseparabilities in preferences (Epstein–Zin)
- financial constraints

How big is it?

- Alvarez and Jermann (2005) — asset prices impose bounds on the magnitude of $\tilde{S}$

**Result**

$$L_t \left( \frac{\tilde{S}_{t+1}}{\tilde{S}_t} \right) \geq E_t \log R_{t,t+1} - E_t \log R_{t,t+1}^\infty$$

- $R_{t,t+1}$ — any one-period return
- $R_{t,t+1}^\infty$ — one-period return on the long-horizon bond
- $L_t (Z_{t+1}) = \log E_t [Z_{t+1}] - E_t [\log Z_{t+1}]$ — entropy of random variable $Z_{t+1}$
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Triangular state vector system:

\[
X_{1,t+1} = \Delta_{10} + \Delta_{11}X_{1,t} + \Lambda_{10}W_{t+1}
\]
\[
X_{2,t+1} = \Delta_{20} + \Delta_{21}X_{1,t} + \Delta_{22}X_{2,t} + \Delta_{23} (X_{1,t} \otimes X_{1,t}) + \\
+ \Lambda_{20}W_{t+1} + \Lambda_{21} (X_{1,t} \otimes W_{t+1}) + \Lambda_{23} (W_{t+1} \otimes W_{t+1})
\]

- Stable dynamics if \(\Delta_{11}\) and \(\Delta_{22}\) have stable eigenvalues.
- Structure allows for stochastic volatility through \(X_{1,t} \otimes W_{t+1}\)
Triangular state vector system:

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X_{1,t+1} = \Delta_{10} + \Delta_{11}X_{1,t} + \Lambda_{10}W_{t+1}
\]

\[
X_{2,t+1} = \Delta_{20} + \Delta_{21}X_{1,t} + \Delta_{22}X_{2,t} + \Delta_{23}(X_{1,t} \otimes X_{1,t}) + \\
+ \Lambda_{20}W_{t+1} + \Lambda_{21}(X_{1,t} \otimes W_{t+1}) + \Lambda_{23}(W_{t+1} \otimes W_{t+1})
\]

- Stable dynamics if \(\Delta_{11}\) and \(\Delta_{22}\) have stable eigenvalues.
- Structure allows for stochastic volatility through \(X_{1,t} \otimes W_{t+1}\)

A class of quadratic functions for increments of multiplicative functionals

\[
\log M_{t+1} - \log M_t = \Gamma_0 + \Gamma_1X_t + \Gamma_2(X_{1,t} \otimes X_{1,t}) + \\
+ \Theta_0W_{t+1} + \Theta_1(X_{1,t} \otimes W_{t+1}) + \Theta_2(W_{t+1} \otimes W_{t+1})
\]

- Use to model stochastic discount factors and growth functionals.
The framework allows for quasi-analytical formulas for conditional expectations of multiplicative functionals and for elasticities.

· Start with

\[ \log f(x) = \phi + \Phi x + \frac{1}{2}(x_1)'\Psi(x_1) \]
The framework allows for quasi-analytical formulas for conditional expectations of multiplicative functionals and for elasticities.

· Start with

$$\log f(x) = \phi + \Phi x + \frac{1}{2} (x_1)'\psi(x_1)$$

· Then

$$\log E \left[ \left( \frac{M_{t+1}}{M_t} \right) f(X_{t+1}) | X_t = x \right] = \phi^* + \Phi^* x + \frac{1}{2} (x_1)'\psi^*(x_1) = \log f^*(x)$$

· Multiperiod conditional expectations calculated by iterating on the coefficient mapping.

· Series expansion (perturbation) framework as a special case.
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Value premium

- High book-to-market (B/M) firms (value stocks) have high expected returns relative to low B/M firms (growth stocks) [Fama and French (1992, 1995)]
- Does this reflect intangible risk exposure? [Hansen, Heaton and Li (2005)].
Value premium

- High book-to-market (B/M) firms (value stocks) have high expected returns relative to low B/M firms (growth stocks) [Fama and French (1992, 1995)]
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Use a model by Ai, Croce, Li (2012)

- distinction between physical and intangible capital
- growth firms have more intangible capital that is a source of new investment projects
- new vintages of capital are less exposed to aggregate risk than old vintages
- growth firms are less risky, and thus earn lower expected returns
Aggregate output in the economy

\[ Y_t = (K_t)^\lambda (A_t N_t)^{1-\lambda} = C_t + I_t + I^*_t \]

\[ N_t = 1 \]

Two types of capital

- **physical capital**
  \[ K_{t+1} = (1 - \delta) K_t + B_{t+1} G (I_t, K^*_t) \]

- **intangible capital**
  \[ K^*_{t+1} = [K^*_t - G (I_t, K^*_t)] (1 - \delta^*) + H(I^*_t, K_t) \]

- **production of new physical capital**
  \[ G (I_t, K^*_t) = \left[ \phi (I_t)^{1-\psi} + (1 - \phi) (K^*_t)^{1-\psi} \right]^{1-\psi} \]

Representative household with recursive preferences
*Technology* modeled as permanent (growth rate) shocks

\[ \log A_{t+1} - \log A_t = \Theta + Z_t + \Gamma W_{t+1} \]

where the ‘long-run risk’ component \( Z \) evolves as:

\[ Z_{t+1} = \psi Z_t + \Lambda W_{t+1}^2 \]

Shock to **newly installed capital** is modeled as:

\[ \log B_{t+1} = -\frac{1 - \lambda}{\lambda} \left( Z_t + \Gamma W_{t+1}^1 \right) \]

Neutral technology growth rate and the investment specific shocks are perfectly negatively correlated.
The common component to \( \log A_{t+1} - \log A_t \) and \( \log B_{t+1} \) is \( Z_t + \Gamma W^1_{t+1} \) where the ‘long-run risk’ component \( Z \) evolves as:

\[
Z_{t+1} = \psi Z_t + \Lambda W^2_{t+1}
\]

Two shocks

- **immediate shock**: \( \Gamma W^1_{t+1} \) (rescaled to have a unit sd)
- **long-run risk shock**: \( \Lambda W^2_{t+1} \) (rescaled to have a unit sd)

Impact on \( \log B_{t+1} \) temporary but impact on \( \log A_{t+1} \) is permanent.
The common component to $\log A_{t+1} - \log A_t$ and $\log B_{t+1}$ is $Z_t + \Gamma W^1_{t+1}$ where the ‘long-run risk’ component $Z$ evolves as:

$$Z_{t+1} = \Psi Z_t + \Lambda W^2_{t+1}$$

Two shocks

- **immediate shock**: $\Gamma W^1_{t+1}$ (rescaled to have a unit sd)
- **long-run risk shock**: $\Lambda W^2_{t+1}$ (rescaled to have a unit sd)

Impact on $\log B_{t+1}$ temporary but impact on $\log A_{t+1}$ is permanent.

**Questions**

- What are the sources of risk premia? And what generates the heterogeneity?
  - Which shocks get priced (direct, long-run risk)?
  - At what horizons?
  - Through which channels (neutral, investment-specific)?
• shock-exposure elasticities gradually build (capital accumulation)
• shock-price elasticities are close to flat (recursive preferences)
• shock-exposure elasticities gradually build (capital accumulation)
• shock-price elasticities are close to flat (recursive preferences)
• neutral technology growth rate shock drives the pricing results
Shock-exposure elasticities gradually build (capital accumulation)

Shock-price elasticities are close to flat (recursive preferences)

Neutral technology growth rate shock drives the pricing results

Shock to newly installed capital has only small transitory impact on consumption dynamics \(\rightarrow\) minimal impact on pricing
SHOCK ELASTICITIES FOR CAPITAL VALUATION

- dramatic difference in the responses to the long-run risk shock
- which channel contributes?
SHOCK ELASTICITIES FOR CAPITAL VALUATION

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- which channel contributes?
- responses propagated through the neutral technology channel are similar
SHOCK ELASTICITIES FOR CAPITAL VALUATION

- dramatic difference in the responses to the long-run risk shock
- which channel contributes?
- responses propagated through the neutral technology channel are similar
- differences caused by the exposure to the long-run risk shock propagated through the investment-specific channel
Shock elasticities

- One-period pricing;
- Elasticities over multiple periods;
- Martingale components in stochastic discount factors

Applications

- Exponential-quadratic framework solvable in closed form;
  - Second-order perturbations of DSGE models (Dynare implementation);
  - Model of intangible capital [Ai, Croce and Li (2012)]
  - Model with financial constraints [Gertler and Kiyotaki (2011)]
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A large class of models with frictions in the financial sector

- Households cannot invest in productive capital directly.
- Financing friction in banks casts a wedge between MRS and prices.
- Asset valuation depends on cash flows and ability to relax constraints.
A large class of models with frictions in the financial sector

- Households cannot invest in productive capital directly.
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- Asset valuation depends on cash flows and ability to relax constraints.

**Gertler and Kiyotaki (2011)**

- Households ↔ financial sector ↔ firms
- Occasionally binding constraint.
- Here: Series expansion around the deterministic steady state.
  - Fits the exponential-quadratic framework.
• preferences

\[ U_t (C, L) \equiv E \left[ \sum_{j=0}^{\infty} \left( \log (C_{t+j} - hC_{t+j-1}) - \frac{\chi}{1 + \varepsilon} (L_{t+j})^{1+\varepsilon} \right) \right] \quad | \quad F_t \]
HOUSEHOLD’S PREFERENCES

· preferences

\[ U_t(C, L) \equiv E \left[ \sum_{j=0}^{\infty} \left( \log (C_{t+j} - hC_{t+j-1}) - \frac{\chi}{1 + \varepsilon} (L_{t+j})^{1+\varepsilon} \right) \mid \mathcal{F}_t \right]. \]

· stochastic discount factor

\[ \frac{S_{t+1}^h}{S_t^h} = \beta \frac{U_{c,t+1}}{U_{c,t}} \]

where \( U_c \) is the marginal utility of consumption

\[ U_{c,t} = \frac{1}{C_t - hC_{t-1}} - h\beta E \left[ \frac{1}{C_{t+1} - hC_t} \mid \mathcal{F}_t \right]. \]
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- priced asset: deposit contract

\[ 1 = E \left[ \frac{S_{t+1}^h}{S_t^h} R_{t+1}^d \mid \mathcal{F}_t \right] . \]
· objective

\[
V(k_t, b_t, d_t) = \max_{(k_{t+1}, b_{t+1}, d_{t+1})} \mathbb{E} \left[ \frac{S_{t+1}^h}{S_t^h} \left[ (1 - \sigma) n_{t+1} + \sigma V(k_{t+1}, b_{t+1}, d_{t+1}) \right] \mid F_t \right].
\]
· objective

\[ V(k_t, b_t, d_t) = \max_{(k_{t+1}, b_{t+1}, d_{t+1})} E \left[ \frac{S_{t+1}^h}{S_t^h} \left[ (1 - \sigma) n_{t+1} + \sigma V(k_{t+1}, b_{t+1}, d_{t+1}) \right] | \mathcal{F}_t \right]. \]

· balance-sheet constraint

\[ Q_t k_t = n_t + d_t + b_t \]

· accumulation of net worth

\[ n_{t+1} = R_{t+1}^k Q_t k_t - R_{t+1}^b b_t - R_{t+1}^d d_t \]

· financial constraint

\[ V(k_t, b_t, d_t) \geq \theta^k Q_t k_t - \theta^b b_t \]
SHOCKS

- Neutral productivity shock
  \[ Y_t = A_t K_t^\alpha L_t^{1-\alpha} \]

- Capital-quality shock
  \[ K_{t+1} = \psi_{t+1} [l_t + (1 - \delta) K_t] \]

- Both modeled as stationary
  - but capital-quality shocks gain a lot of persistence through capital accumulation
CASH FLOWS

· Return on capital

\[ R_{t+1}^k = \psi_{t+1} \frac{Z_{t+1} + (1 - \delta) Q_{t+1}}{Q_t} \]
· Return on capital

\[ R_{t+1}^k = \psi_{t+1} \frac{Z_{t+1} + (1 - \delta)Q_{t+1}}{Q_t} \]

· Revenue stream from a unit of capital produced at time \( t \):

\[ G_{t+j} = Z_{t+j} (1 - \delta)^{j-1} \prod_{m=1}^{j} \psi_{t+m}, \quad j \geq 1, \]

· corresponds to a multiplicative functional

\[ \log G_{t+1} - \log G_t = \log (1 - \delta) + \log \psi_{t+1} + \log Z_{t+1} - \log Z_t. \]
CASH FLOWS

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\[ \log G_{t+1} - \log G_t = \log (1 - \delta) + \log \psi_{t+1} + \log Z_{t+1} - \log Z_t. \]

- Profit cash flow of the whole firm

\[ G_t = \psi_t Z_t k_t \]

- Net worth paid to households

\[ G_t = (1 - \sigma) n_t \]
Euler equation

\[ Q_t = E \left[ \frac{S_{t+1}^h}{S_t^h} \left(1 - \sigma + \sigma \mu_{t+1}\right) \frac{1 + \lambda_t}{\mu_t + \lambda_t \theta^k} \psi_{t+1} (Z_{t+1} + (1 - \delta) Q_{t+1}) \mid \mathcal{F}_t \right] \]

- \( \lambda_{t+1} \) Lagrange multiplier on the financial constraint
- \( \mu_{t+1} \) Lagrange multiplier on the balance-sheet constraint
- frictionless economy: \( \lambda_{t+1} = 0, \mu_{t+1} = 1 \)
- ‘stochastic discount factor’ is asset-specific (\( \theta^k \))
SHOCK-EXPOSURE ELASTICITIES

- **Consumption shock-exposure**
- **Net-worth shock-exposure**
- **Unit capital revenue shock-exposure**
- **Profit shock-exposure**

![Graphs showing various shock-exposure elasticities over maturity (quarters)]
SHOCK-PRICE ELASTICITIES

household’s stochastic discount factor

bank’s stochastic discount factor for capital

\[
\frac{S_{t+1}(\theta^h)}{S_t(\theta^h)} = \frac{S_{t+1}^h}{S_t^h} \left(1 - \sigma + \sigma \mu_{t+1}\right) \frac{1 + \lambda_t}{\mu_t + \lambda_t \theta^h}
\]

- shock-price elasticities for \(S(\theta^h)\) are much larger ...
- ... and do not decay to zero.
- state dependence (!!!)
Shock elasticities

- One-period pricing;
- Elasticities over multiple periods;
- Martingale components in stochastic discount factors

Applications

- Exponential-quadratic framework solvable in closed form;
  - Second-order perturbations of DSGE models (Dynare implementation);
  - Model of intangible capital [Ai, Croce and Li (2012)]
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Environment

- State dynamics
  \[ dX_t = \mu (X_t) \, dt + \sigma (X_t) \, dW_t \]
- Multiplicative processes (SDFs, cash flows)
  \[ d \log M_t = \beta (X_t) \, dt + \alpha (X_t) \cdot dW_t \]
CONTINUOUS-TIME BROWNIAN-INFORMATION SETUP

Environment

- State dynamics
  \[ dX_t = \mu (X_t) \, dt + \sigma (X_t) \, dW_t \]
- Multiplicative processes (SDFs, cash flows)
  \[ d \log M_t = \beta (X_t) \, dt + \alpha (X_t) \cdot dW_t \]

Shock elasticities

- Perturbation over a ‘short time interval’ \( dW_0 \)
- Related to the Malliavin derivative

\[ \varepsilon (x, t) = \nu (x) \cdot \left[ \sigma (x)' \left( \frac{\partial}{\partial x} \log E \left[ \frac{M_t}{M_0} \mid X_0 = x \right] \right) + \alpha (x) \right] \]

\[
\begin{align*}
\text{state propagation} & \quad \text{initial impact}
\end{align*}
\]
Conditional expectations of multiplicative functionals

Central object in the analysis is the computation of

$$\phi_t(x) \triangleq E \left[ \frac{M_t}{M_0} \phi_0(X_t) \mid X_0 = x \right].$$
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$$
\phi_t(x) \doteq E \left[ \frac{M_t}{M_0} \phi_0(X_t) \mid X_0 = x \right].
$$

**Differential equation for** $\phi_t(x)$

$$
\frac{\partial}{\partial t} \phi_t = \left( \beta + \frac{1}{2} |\alpha|^2 \right) \phi_t + (\mu + \sigma \alpha) \cdot \frac{\partial \phi_t}{\partial x} + \frac{1}{2} \text{tr} \left[ \sigma \sigma' \frac{\partial^2 \phi_t}{\partial x \partial x'} \right].
$$
Central object in the analysis is the computation of

$$\phi_t(x) \equiv E \left[ \frac{M_t}{M_0} \phi_0(X_t) \mid X_0 = x \right].$$

Differential equation for $\phi_t(x)$

$$\frac{\partial}{\partial t} \phi_t = \left( \beta + \frac{1}{2} |\alpha|^2 \right) \phi_t + (\mu + \sigma \alpha) \cdot \frac{\partial \phi_t}{\partial x} + \frac{1}{2} \text{tr} \left[ \sigma \sigma' \frac{\partial^2 \phi_t}{\partial x \partial x'} \right]$$

Boundary conditions

- terminal: $\phi_0(x) \equiv 1$
- lateral: determined by the economics of the model

Solution

- finite differences, perturbation methods, MC simulations, ...
The Perron–Frobenius problem

\[ E[M_t e(X_t) \mid X_0 = x] = \exp(\eta t) e(x) \]
The Perron–Frobenius problem

\[
E \left[ M_t e(X_t) \mid X_0 = x \right] = \exp(\eta t) e(x)
\]

can be localized

\[
\lim_{t \to 0} \frac{1}{t} \left[ E \left[ M_t e(X_t) \mid X_0 = x \right] - \exp(\eta) e(x) \right] = 0.
\]
The Perron–Frobenius problem

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Resulting PDE

\[ \eta e = \left( \beta + \frac{1}{2} |\alpha|^2 \right) e + (\mu + \sigma \alpha) \cdot \frac{\partial e}{\partial x} + \frac{1}{2} \text{tr} \left[ \sigma \sigma' \frac{\partial^2 e}{\partial x \partial x'} \right] \]

\[ \cdot \text{ Sturm–Liouville problem: finding } (\eta, e) \implies \phi_t (x) = \exp (\eta t) e (x) \]
The Perron–Frobenius problem

\[
E [M_t e (X_t) \mid X_0 = x] = \exp (\eta t) e (x)
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\lim_{t \to 0} \frac{1}{t} [E [M_t e (X_t) \mid X_0 = x] - \exp (\eta) e (x)] = 0.
\]

Resulting PDE

\[
\eta e = \left( \beta + \frac{1}{2} |\alpha|^2 \right) e + (\mu + \sigma \alpha) \cdot \frac{\partial e}{\partial x} + \frac{1}{2} \text{tr} \left[ \sigma \sigma' \frac{\partial^2 e}{\partial x \partial x'} \right]
\]

· Sturm–Liouville problem: finding \((\eta, e) \implies \phi_t (x) = \exp (\eta t) e (x)\)

Martingale factorization

\[
M_t = \exp (\eta t) \tilde{M}_t \frac{e (X_0)}{e (X_t)}
\]
Shock elasticities

- One-period pricing;
- Elasticities over multiple periods;
- Martingale components in stochastic discount factors

Applications

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Two types of agents: households and experts

- Households less efficient in managing capital
- Lending capital to experts is only possible using risk-free bond contracts
  - financial constraints on efficient allocation of capital
Two types of agents: households and experts

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Preferences

- Both agents linear preferences. Experts
  \[ E \left[ \int_0^\infty e^{-\rho t} dC_t^h \mid \mathcal{F}_0 \right] \]
- Households more patient than experts: \( r < \rho \).
- Marginal value of experts’ wealth \( \Theta_t \geq 1 \) due to the financial constraint.
· wealth share of experts:

\[ X_t = \frac{N_t}{Q_t K_t} \]

· aggregate capital

\[ d \log K_t = \beta_k (X_t) \, dt + \alpha_k \, dW_t \]

· drift depends on wealth distribution, since the two agents have different capital accumulation rates
\textbf{EQUILIBRIUM DYNAMICS}

- wealth share of experts:
  \[ X_t = \frac{N_t}{Q_tK_t} \]

- aggregate capital
  \[ d \log K_t = \beta_k (X_t) \, dt + \alpha_k \, dW_t \]

  - drift depends on wealth distribution, since the two agents have different capital accumulation rates

\textbf{State} \( X_t \in (0, \bar{X}] \)

- \( X_t \) large — households invest all capital with firms
- \( X_t \) small — limited intermediation
- \( X_t = \bar{X} \) — marginal value of experts’ wealth declines to one, experts consume
  - reflecting boundary
Evolution of the state

\[ dX_t = \mu (X_t) \, dt + \sigma (X_t) \, dW_t - X_t \, d\zeta_t \]

\cdot \, d\zeta_t \, — \, \text{consumption of experts at } X_t = \bar{x}. 
Evolution of the state

\[ dX_t = \mu (X_t) \, dt + \sigma (X_t) \, dW_t - X_t \, d\zeta_t \]

- \( d\zeta_t \) — consumption of experts at \( X_t = \bar{x} \).

**Stochastic discount factor** of experts

\[ \frac{S_t}{S_0} = e^{-\rho t} \frac{\theta (X_t)}{\theta (X_0)} \]

- \( \Theta_t = \theta (X_t) \) — marginal utility of experts’ wealth
  - not equal to one due to limited risk-sharing
Evolution of the state

\[ dX_t = \mu(X_t) \, dt + \sigma(X_t) \, dW_t - X_t \, d\zeta_t \]

\[ d\zeta_t \] — consumption of experts at \( X_t = \bar{X} \).

**Stochastic discount factor** of experts

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\[ \cdot \] not equal to one due to limited risk-sharing

**Cash flow**: household consumption

\[ d \log C_t = \beta_c(X_t) \, dt + \alpha_c(X_t) \, dW_t \]
• Wealth in hands of experts valuable $\implies$ accumulation, $\mu(x) > 0$
• Experts consume when $\theta(X_t)$ reaches one $\implies$ reflecting right boundary
• Stationary density has two peaks, ‘normal times’ and ‘financial crises’
SHOCK ELASTICITIES IN THE SHORT AND LONG RUN

- Short-horizon consumption response small, even countercyclical.
- Shock-price elasticity is large during ‘financial crises’, zero in the long run.
• shock exposures generally increasing, shock prices decreasing
• Shock-exposure and shock-price elasticities explain the shape of yield differences between risky and risk-free cash flows
  • hump shaped in risky region
  • negative for short horizons in the center of the distribution
Shock elasticities

- One-period pricing;
- Elasticities over multiple periods;
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Applications

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Let $Z_t$ be an $n$-state continuous time Markov chain

- realization a coordinate vector
- intensity matrix $A$
Let $Z_t$ be an $n$-state continuous time Markov chain

- realization a coordinate vector
- intensity matrix $A$

**Multiplicative functional**

$$\log M_t = \sum_{0 \leq u \leq t} (Z_{u-})' \kappa Z_u + \int_0^t (Z_{u-})' \beta du + \int_0^t (Z_{u-})' \alpha dW_t$$

- $\beta$ — elements are conditional (on $Z_{u-}$) growth rates
- $\alpha$ — rows are vectors of conditional exposures to $dW_t$
- $\kappa$ — jump risk matrix, zeros on diagonal
Jump-risk perturbation

\[ \log H_t(r) = \sum_{0 \leq u \leq t} (Z_{u-})' (r \kappa_d) Z_u + \int_0^t (Z_{u-})' \beta_h(r) du \]

- \( \kappa_d \) — perturbation direction (jump combination) that we are interested in
Jump-risk perturbation

$$\log H_t(r) = \sum_{0 \leq u \leq t} (Z_{u-})' (r \kappa_d) Z_u + \int_0^t (Z_{u-})' \beta_h(r) \, du$$

• $\kappa_d$ — perturbation direction (jump combination) that we are interested in

Shock elasticity function

$$\varepsilon(t) = \text{dvec} \left\{ \kappa_d \left( \hat{A} - A \right)' \right\} + \text{dvec} \left\{ [\Xi(t) \times \kappa_d] \hat{A}' \right\}$$

• $\hat{A}$ — intensity matrix distorted by the martingale component of $M$
• $\Xi(t)$ traces out the transitory response
• Consumption modeled as multiplicative functional

\[
\log C_t = \int_0^t (Z_u^e)' \beta c du + \int_0^t (Z_u^e)' \alpha c dW_t
\]

• Agents endowed with Epstein–Zin preferences with \( IES = 1 \)

- Consumption modeled as multiplicative functional
  \[ \log C_t = \int_0^t (Z_{u-})' \beta_c du + \int_0^t (Z_{u-})' \alpha_c dW_t \]

- Agents endowed with Epstein–Zin preferences with \textit{IES} = 1

- Continuation value
  \[ \log V_t = v \cdot Z_t + \log C_t \]

- Even though \( C_t \) does not jump at regime changes, \( V_t \) does:
  \[ \kappa_V = 1_n v' - v1'_n \]

- Compare shock-price elasticities for CRRA and EZ stochastic discount factors
$\kappa_d$ — normalized jump component of the martingale from $\log C$
Substantial information in the term structure of risk premia.

- yields on dividend strips and other maturity specific assets
  - empirical work by Koijen, van Binsbergen and others
  - many models produce counterfactual results
- holding period returns on long-term assets
  - entropy and variance bounds on the martingale component

**Shock elasticities**

- risk exposures and associated compensation
- term structure of risk
Theoretical framework

- continuous time
  - Borovička, Hansen, Hendricks, Scheinkman (2011); Borovička, Hansen (2016)

- discrete time
  - Borovička, Hansen (2014); Nakamura, Sergeyev, Steinsson (2016)

- non-Gaussian framework
  - Borovička, Hansen, Hendricks, Scheinkman (2011); Zviadadze (2016)

- nonlinear impulse response functions
  - Gallant, Rossi and Tauchen (1993); Koop, Pesaran and Potter (1996); Gourieroux and Jasiak (2005); Borovička, Hansen, Scheinkman (2014)
**Long-run pricing**

- martingale decomposition
  - Hansen, Scheinkman (2009, 2012); Borovička, Hansen, Scheinkman (2016); Qin and Linetsky (2014a,b)
- role of the martingale component
  - Alvarez and Jermann (2005); Qin, Linetsky and Nie (2016); Bakshi, Chabi-Yo and Gao (2016); Christensen (2014); ...

**Applications**

- currency returns: Zviadadze (2016)
- long-run risk: Nakamura, Sergeyev, Steinsson (2016)

**Empirical evidence**

- downward sloping term structure of risk