Best Practices for Differentiated Products Demand Estimation with `pyblp`*

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Abstract

Differentiated products demand systems are a workhorse for understanding the price effects of mergers, the value of new goods, and the contribution of products to seller networks. Berry et al. (1995) provide a flexible workhorse model which accounts for the endogeneity of prices and is based on the random coefficients logit. While popular, there exists no standardized generic implementation for the Berry et al. (1995) estimator. This paper reviews and combines several recent advances related to the estimation of BLP-type problems and implements an extensible generic interface via the `pyblp` package. We conduct a number of Monte Carlo experiments and replications which suggest different conclusions than the prior literature: multiple local optima appear to be rare in well-identified problems; it is possible to obtain good performance even in small samples and without exogenous cost-shifters, particularly when "optimal instruments" are employed along with supply-side restrictions.

If Python is installed on your computer, `pyblp` can be installed with the following command:

```
pip install pyblp
```

Up to date documentation for the package is available at `https://pyblp.readthedocs.io`.

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1. Introduction

Empirical models of supply and demand for differentiated products are one of the most important achievements of the New Empirical Industrial Organization (NEIO) literature of the last thirty years. The workhorse model is the Berry et al. (1995) approach, which provides an estimator that allows for flexible substitution patterns across products while also addressing the potential endogeneity of price, and also provides an algorithm for recovering that estimator. It has the advantage that it both scales well for large numbers of potential products, and can be applied to both aggregated and dis-aggregated data. The BLP model and its variants have been used in a wide variety of applications: understanding the value of new goods (Petrin, 2002), the price effects of mergers (Nevo, 2001, 2000a), and two-sided markets (Fan, 2013) and (Lee, 2013). The BLP approach has been applied to a wide number of different questions and industries including hospital demand and negotiations with insurers (Ho and Pakes, 2014), students' choice of schools (Bayer et al., 2007), and reinsurance (Koijen and Yogo, 2016). Moreover, the BLP approach has been extremely influential in the practice of prospective merger evaluation, particularly in recent years.

The model itself is both quite simple to understand, and quite challenging to estimate. At its core, it involves a nonlinear change of variables from the space of observed marketshares to the space of mean utilities for products. After this nonlinear change of variables, the BLP problem is simply either a single linear instrumental variables regression problem, or a two equation (supply and demand) linear IV regression problem. This means that a wide variety of tools for that class of problems are available to researchers.

The main disadvantage of the BLP estimator is that parameters governing the nonlinear change of variables are unknown. This results in a non-linear, non-convex optimization problem with a simulated (or approximated) objective function. The problem must be solved iteratively using non-linear optimization software, and because of the non-convexity, there is no mathematical guarantee that a solution will always be found. This has led to some frustration with the BLP approach (see Knittel and Metaxoglou, 2014). There is also the fear that when estimation is slow or complicated, researchers may cut corners in undesirable ways and sacrifice modeling richness for computational speed.

Despite its popularity this literature lacks a standardized implementation that is sufficiently general to encompass a wide range of potential problems and use cases. Instead, nearly every researcher implements the estimator on their own with problem-specific tweaks and adjustments. This makes replication extremely challenging, and also makes it hard to evaluate various different methodological and statistical improvements to the estimator.

The goal of this paper is to present best practices in the estimation of BLP-type models, some of which are well-known in the literature, others of which are lesser known, and others still are novel to this paper. In addition to presenting these best practices, we provide a common framework,

`pyblp`, which offers a general implementation of the BLP approach as a Python 3 package. This software is general, extensible, and open-source so that it can be modified and extended by scholars as new methodological improvements become available. The hope is that these best practices, along with this standardized and extensible software implementation, reduce some of the barriers to BLP-type estimation, making these techniques accessible to a wider range of researchers and facilitating replication of existing results.

This paper and the accompanying package build upon a growing literature focused on methodological innovations econometric properties of the BLP estimator. In Section 3, we discuss several such improvements and evaluate them via Monte Carlo studies in Section 5. Our objective is to compare practices in the literature and arrive at best practices suitable for a large number of use cases. We then implement these best practices as defaults in `pyblp`. We organize the best practices around several of the tasks in the BLP estimator: solving the fixed point, integration, optimization, and solving counterfactual pricing equilibria.

In addition to best practices, we also provide some novel results. We provide a slightly different characterization of the BLP problem in Section 2 which facilitates estimation with simultaneous supply and demand restrictions. We show how this characterization can be made amenable to large numbers of fixed effects, and in Section 4 we characterize an approximation to the optimal instruments (in the spirit of Amemiya (1977) or Chamberlain (1987)). Our characterization of the problem under the optimal instruments allows us explore parametric identification with supply and demand in a way that parallels Berry and Haile (2014) and makes explicit *cross equation* and *exclusion restrictions*.

On the matter of instruments, our results generally coincide with and build on the existing literature. Gandhi and Houde (2017) construct what they refer to as *differentiation IV*, while Reynaert and Verboven (2014) construct a *feasible approximation to the optimal IV* in the sense of Amemiya (1977) or Chamberlain (1987).[1] We provide routines to construct both sets of instruments. Our results with respect to differentiation IV are mostly consistent with Gandhi and Houde (2017) in that they substantially outperform other simple, yet commonly used forms of the *BLP instruments* (functions of exogenous product characteristics) such as sums or averages. Our results with respect to approximate optimal instruments are broadly similar with those of Reynaert and Verboven (2014) in that the performance gains are substantial both in terms of bias and efficiency.

Our simulations indicate somewhat more positive results than previously observed in the literature when both the approximate optimal instruments and a (correctly specified) supply side are used together. These findings are somewhat different from Reynaert and Verboven (2014) which suggest that once optimal demand side instruments are included, the addition of a supply side has limited benefit.[2] Indeed, our simulations indicate with both a correctly specified supply side and

[1] It is worth mentioning that the actual optimal IV are well-known to be infeasible. See Berry et al. (1995) or Berry et al. (1999).

[2] This is most likely because those authors only examined scenarios with strong cost-shifters.

optimal instruments together, the finite sample performance of the estimator is good even absent strong cost-shifters, which supports the "folklore" around the original Berry et al. (1995) paper: that supply restrictions are valuable in improving the econometric performance of parameters on endogenous variables.

Employing best practices and the `pyblp` software, we are able to revisit recent findings regarding methodological issues and innovations in BLP-type estimators in large-scale Monte Carlo experiments. While many of our results confirm previous findings in the literature, we arrive at different conclusions on several occasions. The findings of Knittel and Metaxoglou (2014) suggest that the BLP problem is often characterized by a large number of local optima, and that these local optima can produce a wide range of potential elasticities and welfare effects. In contrast, our experience is that after implementing best practices parameter estimates and counterfactual predictions are quite stable across choices of optimization software and starting values. Likewise, Armstrong (2016) finds that absent strong cost shifting instruments, as the number of products increases, the "BLP instruments" (characteristics of own and competing products) become weak, and it becomes difficult to reliably estimate demand parameters. Our findings suggest that with a correctly specified supply side and (approximations to the) optimal instruments, parameter estimates can still be estimated precisely even in small samples. In general, we struggle to replicate some of the difficulties found in the previous literature, suggesting that the finite sample performance of BLP estimators may be better than previously thought.

There is also a recent literature of alternative approaches to BLP problems employing different algorithms or statistical estimators, which we do not directly address. This is not meant to suggest that there is anything wrong with these approaches, but merely that they are beyond the scope of this paper. For example, Dubé et al. (2012) propose an alternative estimation algorithm based on the mathematical programming with equilibrium constraints (MPEC) method of Su and Judd (2012) and which Conlon (2017) extends to generalized empirical likelihood estimators. While the MPEC approach has some advantages, we focus on the more popular nested-fixed point problem. Lee and Seo (2015) provide an *approximate BLP* estimator, which is asymptotically similar to the BLP estimator though differs in finite sample. Salanie and Wolak (2018) propose another approximate estimator that can be estimated with linear IV, and is helpful for constructing good starting values. Other common modifications to the BLP model that are beyond the scope of this paper include the *pure characteristics* model of Berry and Pakes (2007) and the inclusion of *micro moments*, as in Petrin (2002) and Berry et al. (2004a).

2. Model

Table 1: Notation

j	products
t	markets
i	"individuals"
f	firms
N	number of products across all markets
T	number of markets
J_t	set/number of products in market t
I_t	set/number of individuals in market t
F_t	set/number of firms in market t
J_{ft}	set/number of products controlled by firm f in market t
θ_1	exogenous linear demand parameters
θ_2	endogenous (nonlinear) parameters including α
$\widetilde{\theta}_2$	endogenous (nonlinear) parameters
θ_3	exogenous linear supply parameters
β	exogenous linear demand parameters excluding fixed effects
α	endogenous linear demand parameter on price
γ	exogenous linear supply parameters excluding fixed effects
s_{jt}	calculated market share of j in market t
$\mathcal{S}_{jt}$	observed market share of j in market t
p_{jt}	price of j in market t
c_{jt}	marginal cost of j in market t
x_{jt}	exogenous (common) characteristics of j in market t
w_{jt}	supply-shifters of j in market t
v_{jt}	demand-shifters of j in market t
u_{ijt}	i's indirect utility of j in market t
δ_{jt}	mean utility of j in market t
μ_{ijt}	i-specific utility of j in market t
ϵ_{ijt}	i's idiosyncratic preferences for j in market t
ξ_{jt}	demand-side structural error
ω_{jt}	supply-side structural error
O	a $J_t \times J_t$ matrix of 0's and 1's where 1 denotes same owners
Δ	a $J_t \times J_t$ matrix of intra-firm (negative) demand derivatives
η_{jt}	markup of j in market t
Z	instruments
g	sample moments
q	objective function
W	weighting matrix

2.1. Demand

Berry et al. (1995) begin with the following problem. An individual i in market $t = 1, \ldots, T$ receives indirect utility from selecting a particular product j:

$$u_{ijt} = \delta_{jt} + \mu_{ijt} + \epsilon_{ijt}. \tag{1}$$

Consumers then choose among $J_t = \{0, 1, \ldots, J_t\}$ discrete alternatives and select exactly one option which provides the most utility (including the outside alternative, denoted $j = 0$):

$$d_{ij} = \begin{cases} 1 & \text{if } u_{ijt} > u_{ikt} \text{ for } j \neq k, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

Aggregate marketshares are given by integrating over heterogeneous consumer choices:

$$s_{jt}(\delta_{\cdot t}) = \int d_{ij}(\delta_{\cdot t}, \mu_{it})\, d\mu_{it}\, d\epsilon_{it}.$$

When ϵ_{ijt} is distributed IID type I extreme value (Gumbel) this becomes:[3]

$$s_{jt}(\delta_{\cdot t}, \theta_2) = \int \frac{\exp[\delta_{jt} + \mu_{ijt}]}{\sum_{k \in J_t} \exp[\delta_{kt} + \mu_{ikt}]} f(\mu_{it} \mid \widetilde{\theta}_2)\, d\mu_{it}. \tag{3}$$

This is often referred to as a *mixed logit* or *random coefficients logit* because each individual i's demands are given by a multinomial logit kernel, while $f(\mu_{it} \mid \widetilde{\theta}_2)$ denotes the *mixing distribution* over the heterogeneous types i and θ_2 parametrizes this heterogeneity. Indeed, θ_2 contains all of the endogenous parameters of the model (the heterogeneous tastes) as well as the endogenous parameter on price α.

The key insight of Berry (1994) or Berry et al. (1995) is that we can perform a nonlinear change of variables (see Berry and Haile, 2014): $\delta_t = D_t^{-1}(\mathcal{S}_t, \theta_2)$ where $\mathcal{S}_t$ denotes the J_t vector of *observed marketshares*. For each market t, equation (3) represents a $J_t \times J_t$ system of equations in $\delta_{\cdot t}$. Given the $\delta_{\cdot t}(\mathcal{S}_t, \widetilde{\theta}_2)$ which solves that system of equations; along with some covariates x_{jt} and v_{jt}, prices p_{jt}, and a structural error ξ_{jt}; under an additivity assumption, one can write the index:

$$\delta_{jt}(\mathcal{S}_t, \widetilde{\theta}_2) = [x_{jt}, v_{jt}]\beta - \alpha p_{jt} + \xi_{jt}. \tag{4}$$

With the addition of some instruments Z_{jt}^D (including the exogenous regressors x_{jt} and v_{jt}), one can construct moment restriction conditions of the form $E[\xi_{jt}' Z_{jt}^D] = 0$.

2.2. Supply

We can derive an additional set of supply moments from the multi-product differentiated Bertrand first order conditions. Consider the profits of firm f which for a single market t controls several

[3]Identification generally requires normalizing one of the options. Often the outside option $u_{i0t} = 0$ is the preferred normalization.

products J_f and sets prices p_j:

$$\underset{p_j\,:\,j\in J_f}{\arg\max} \sum_{j\in J_f} (p_j - c_j)\cdot s_j(\mathbf{p}),$$
$$s_j(p) + \sum_{k\in J_f} \frac{\partial s_k}{\partial p_j}(\mathbf{p})\cdot(p_k - c_k) = 0.$$

It is helpful to write the first order conditions in matrix form so that for a single market t,

$$\begin{aligned}\mathbf{s}(\mathbf{p}) &= \Delta(\mathbf{p})\cdot(\mathbf{p}-\mathbf{c}),\\ (\mathbf{p}-\mathbf{c}) &= \underbrace{\Delta(\mathbf{p})^{-1}\mathbf{s}(\mathbf{p})}_{\eta}.\end{aligned} \tag{5}$$

Here the multi-product Bertrand markup $\eta(\mathbf{p},\mathbf{s},\theta_2)$ depends on Δ, a $J_t \times J_t$ matrix of intra-firm (negative) demand derivatives given by

$$\Delta(\mathbf{p}) \equiv -O \odot \frac{\partial \mathbf{s}}{\partial \mathbf{p}}(\mathbf{p}), \tag{6}$$

which is the element-wise (Hadamard) product $\odot$ of two $J_t \times J_t$ matrices: the matrix of demand derivatives where each (j,k) entry is given by $\frac{\partial s_j}{\partial p_k}$, and the ownership matrix O, given by[4]

$$O_{jk} = \begin{cases} 1 & \text{if } (j,k)\in J_f \text{ for any } f,\\ 0 & \text{otherwise.}\end{cases}$$

This enables us to recover an estimate of marginal costs $c_{jt} = p_{jt} - \eta_{jt}$, which in turn allows us to construct additional *supply side moments*. We can parametrize marginal cost as[5]

$$f(p_{jt} - \eta_{jt}) = f(c_{jt}) = x_{jt}\gamma_1 + w_{jt}\gamma_2 + \omega_{jt} \tag{7}$$

and construct moment conditions of the form $E[\omega'_{jt} Z^S_{jt}] = 0$. The idea is that we can use observed prices, along with information on demand derivatives and firm conduct, to recover markups η_{jt} and then marginal costs c_{jt}. This also imposes a functional form restriction on marginal costs, which depends on both product characteristics x_{jt} and the marginal cost shifters w_{jt} that are excluded from demand.

[4] We can easily consider alternative forms of conduct such as Single Product Oligopoly, Multiproduct Oligopoly, or Multiproduct Monopoly. Miller and Weinberg (2017) consider estimating a single parameter $O(\kappa)$ and Backus et al. (2018) use `pyblp` test various forms of $O(\kappa)$.

[5] The most common choice of $f(\cdot)$ is the identity function, but some authors also consider $f(\cdot) = \log(\cdot)$. In practice this constrains marginal costs to be always positive.

2.3. The Estimator

We can construct a GMM estimator using our supply and demand moments. To do so, we stack their sample analogues to form:[6]

$$g(\theta) = \begin{bmatrix} g^D(\theta) \\ g^S(\theta) \end{bmatrix} = \begin{bmatrix} \frac{1}{N}\sum_{j,t} \xi_{jt} Z^D_{jt} \\ \frac{1}{N}\sum_{j,t} \omega_{jt} Z^S_{jt} \end{bmatrix}$$

and construct a nonlinear GMM estimator for $\theta = [\beta, \alpha, \widetilde{\theta}_2, \gamma]$ with some weighting matrix W:

$$\min_{\theta} q(\theta) \equiv g(\theta)' W g(\theta). \tag{8}$$

For clarity, we partition the parameter space θ into three parts: the $K_1 \times 1$ vector θ_1 contains the demand parameters β, the $K_3 \times 1$ vector θ_3 contains the supply parameters γ, and the remaining parameters, including the price coefficient α, are contained in the $K_2 \times 1$ vector θ_2. The key distinction regarding the parameters in θ_2 is that they are related to the endogenous variables, and thus appear in both equations for demand and supply.

To be explicit we write the entire program as follows:

$$\begin{aligned}
\min_{\theta} q(\theta) &\equiv g(\theta)' W g(\theta) \\
g(\theta) &= \frac{1}{N} \begin{bmatrix} (Z^D)'\xi \\ (Z^S)'\omega \end{bmatrix} \\
\xi_{jt} &= \delta_{jt} - [x_{jt}, v_{jt}]\beta + \alpha p_{jt} \\
\omega_{jt} &= f(p_{jt} - \eta_{jt}) - [x_{jt}, w_{jt}]\gamma \\
\eta_{\cdot t} &= \Delta_t(\tilde{\theta}_2)^{-1}\mathbf{s}_t \\
\mathcal{S}_{jt} = s_{jt}(\delta, \theta_2) &\equiv \int \frac{\exp[\delta_{jt} + \mu_{ijt}]}{\sum_{k \in J_t} \exp[\delta_{kt} + \mu_{ikt}]} f(\mu_{it}|\widetilde{\theta}_2)\, d\mu_{it}.
\end{aligned} \tag{9}$$

This estimator and its econometric properties are discussed in Berry et al. (1995) and Berry et al. (2004b). Our focus is not going to be on the econometric properties of $\hat{\theta}$ but rather various algorithms by which one might obtain $\hat{\theta}$. Technically, we need to solve this program twice. Once to obtain a consistent estimate for $\hat{W}(\hat{\theta})$ and a second time to obtain the efficient GMM estimator.

Many applied papers omit the supply side from (9), and instead estimate $\theta = [\theta_1, \theta_2]$ using the demand moments alone, which is what `pyblp` will do if the user does not provide a supply side. An important justification for not including the supply side is that it may be mis-specified if the researcher does not know (or is not willing to assume) either the functional form of marginal costs

[6] Here $N = \sum_t \dim(J_t)$.

$f(\cdot)$ or firm conduct O.[7] The program without a supply side is as follows:

$$\begin{aligned} \min_{\theta} q^D(\theta) &\equiv g^D(\theta)' W g^D(\theta) \\ g^D(\theta) &= \frac{1}{N}\sum_{j,t} Z_{jt}^{D'} \xi_{j,t} \\ \xi_{jt} &= \delta_{jt} - [x_{jt}, v_{jt}]\beta + \alpha p_{jt} \\ \mathcal{S}_{jt} = s_{jt}(\delta, \theta_2) &\equiv \int \frac{\exp[\delta_{jt} + \mu_{ijt}]}{\sum_{k\in J_t}\exp[\delta_{kt} + \mu_{ikt}]} f(\mu_{it}|\widetilde{\theta}_2)\, d\mu_{it}. \end{aligned} \tag{10}$$

2.4. The Nested Fixed Point Algorithm

In addition to providing an estimator, Berry et al. (1995) provide an algorithm for solving (9) which attempts to simplify the problem. Parameters on exogenous regressors enter the problem linearly; we concentrate out $[\theta_1, \theta_3]$ and perform a nonlinear search over just θ_2 because $[\hat{\theta}_1(\theta_2), \hat{\theta}_3(\theta_2)]$ are implicit functions of other parameters. Our modified algorithm is given below, for each guess of θ_2:

(a) For each market t, solve $\mathcal{S}_{jt} = s_{jt}(\delta_{\cdot t}, \theta_2)$ for $\hat{\delta}_{\cdot t}(\mathcal{S}_{jt}, \theta_2)$.

(b) For each market t, use $\hat{\delta}_{\cdot t}(\mathcal{S}_{jt}, \theta_2)$ to construct $\eta_{\cdot t}(\mathbf{s_t}, \mathbf{p_t}, \hat{\delta}_{\cdot t}(\theta_2), \theta_2)$.

(c) For each market t, recover $\widehat{\eta}_{\cdot t}(\theta_2) = \Delta_t(\hat{\delta}_{\cdot t}(\theta_2), \theta_2)^{-1}\mathbf{s}_t$.

(d) Stack up $\hat{\delta}_{\cdot t}(\mathcal{S}_{jt}, \theta_2)$ and $\hat{c}_{jt}(\hat{\delta}_{\cdot t}(\theta_2), \theta_2) = f(p_{jt} - \widehat{\eta}_{jt}(\widehat{\delta}_{\cdot t}(\theta_2), \theta_2))$ and use linear IV-GMM to recover $[\hat{\theta}_1(\theta_2), \hat{\theta}_3(\theta_2)]$ following the recipe in Appendix A. The following is our somewhat different formulation:

$$\begin{aligned} \hat{\delta}_{jt}(\mathcal{S}_{jt}, \theta_2) + \alpha p_{jt} &= [x_{jt}\ v_{jt}]\beta + \xi_{jt}, \\ f(p_{jt} - \widehat{\eta}_{jt}(\theta_2)) &= [x_{jt}\ w_{jt}]\gamma + \omega_{jt}. \end{aligned} \tag{11}$$

(e) Construct the residuals:

$$\begin{aligned} \hat{\xi}_{jt}(\theta_2) &= \hat{\delta}_{jt}(\theta_2) - [x_{jt}\ v_{jt}]\hat{\beta}(\theta_2) + \alpha p_{jt}, \\ \hat{\omega}_{jt}(\theta_2) &= \hat{c}_{jt}(\theta_2) - [x_{jt}\ w_{jt}]\hat{\gamma}(\theta_2). \end{aligned} \tag{12}$$

(f) Stack the sample moments:

$$g(\theta_2) = \begin{bmatrix} \frac{1}{N}\sum_{jt}\hat{\xi}_{jt}(\theta_2) Z_{jt}^D \\ \frac{1}{N}\sum_{jt}\hat{\omega}_{jt}(\theta_2) Z_{jt}^S \end{bmatrix}. \tag{13}$$

(g) Construct the GMM objective $q(\theta_2) = g(\theta_2)' W g(\theta_2)$.

[7] There are other cases where the supply-side may be mis-specified. Many of these stem from mis-specification around the functional form of $\eta_{\cdot,t}$. One important case might be if firms compete *a la* Cournot rather than differentiated Bertrand oligopoly.

Our setup differs slightly from many BLP applications.[8] A key distinction occurs in step (d) where αp_{jt} appears on the left-hand side of (11). The second distinction occurs in step (e) which requires that we stack the supply and demand equations appropriately (and use the correct weighting GMM weighting matrix). We provide a detailed derivation in Appendix A. Later in Section 3.1, we show how this setup can be adapted to incorporate fixed effects when supply and demand are estimated simultaneously. This requires that the (endogenous) markup term $\eta_{jt}(\mathbf{s_t}, \mathbf{p_t}, \delta_t, \theta_2)$ can be written as a function of only the θ_2 parameters and does not depend on $[\theta_1, \theta_3]$.[9]

We provide analytic gradients for the BLP problem with supply and demand in Appendix B. One advantage of the BLP algorithm is that is performs a nonlinear search over only K_2 *nonlinear parameters*. Consequently, the Hessian matrix is only $K_2 \times K_2$. This implies relatively minimal memory requirements. Also, the IV-GMM regression in step (d) concentrates out the *linear* parameters $[\theta_1, \theta_3]$. This implies that large numbers of linear parameters β can be estimated essentially for free which is important if one includes a large number of fixed effects such as product or market level fixed effects.[10] In fact, other than (a) the remaining steps are computationally trivial. As is well known, (a)-(c) can be performed separately for each market across multiple processors.

The main disadvantage is that all parameters are implicit functions of other parameters, particularly of θ_2. The objective is now a complicated implicit function of θ_2. Once we incorporate any heterogeneous tastes, the resulting optimization problem is non-convex. In general the complexity of this problem grows rapidly with the number of nonlinear θ_2 parameters, while a high number of linear $[\theta_1, \theta_3]$ parameters is more or less for free.

2.5. Nested Logit and RCNL Variants

The random coefficients nested logit (RCNL) model of Brenkers and Verboven (2006) instead places a generalized extreme value (GEV) structure on ϵ_{ijt}. This model is popular in applications where the most important set of characteristics governing substitution is categorical. This has made it popular in studies of alcoholic beverages such as distilled spirits (Conlon and Rao, 2017; Miravete et al., 2018) and beer (Miller and Weinberg, 2017).

Much like the random coefficients model integrates over a heterogeneous distribution where each individual type follows a logit distribution, the RCNL model integrates out over a heterogenous distribution where each individual now follows a *nested logit* (GEV distribution). We expand our definition of the endogenous parameters θ_2 to include the nesting parameter ρ so that $\theta_2 =$

[8] Arguably it is more in line with the original 1995 paper.

[9] Why? We already know we can invert the shares to solve for $\delta_t(\theta_2)$. Because the matrix of demand derivatives $\partial s_{kt}/\partial p_{jt} = -\int \alpha_i \left[s_{ikt} \cdot I_{j=k} - s_{ikt} s_{ijt}\right] f(\mu_{it}, \alpha_i \mid \theta_2)$ again depends only on θ_2 (which contains the price coefficient α).

[10] For example Nevo (2001) and Nevo (2000b) includes product fixed effects.

$[\alpha, \rho, \widetilde{\theta}_2]$:[11]

$$u_{ijt} = \delta_{jt}(\theta_1, \alpha) + \mu_{ijt}(\widetilde{\theta}_2) + \epsilon_{ijt}(\rho).$$

The primary difference from the nested logit is that the inclusive value term for all products in nest J_{gt} now depends on the consumer's type i:

$$\begin{aligned} s_{jt}(\delta_{jt}, \theta_2) &= \int \frac{\exp[(\delta_{jt} + \mu_{ijt}(\widetilde{\theta}_2))/(1-\rho)]}{\exp[I_{ig}/(1-\rho)]} \cdot \frac{\exp[I_{ig}]}{\exp[IV_i]} f(\mu_{ijt}|\widetilde{\theta}_2)\, d\mu_{it}, \\ I_{ig}(\theta_1, \theta_2) &= (1-\rho) \ln \sum_{j \in J_{gt}} \exp\left(\frac{\delta_{jt} + \mu_{ijt}(\widetilde{\theta}_2)}{1-\rho}\right), \quad IV_i(\theta_1, \theta_2) = \ln\left(1 + \sum_{g=1}^{G} \exp I_{ig}\right). \end{aligned} \tag{14}$$

The challenge for estimation is that $\ln \delta_{\cdot t} \leftarrow\!\shortmid \ln \delta_{\cdot t} + \ln \mathcal{S}_{\cdot t} - \ln \mathbf{s}_{\cdot \mathbf{t}}(\delta_{\cdot t}, \theta_2)$ is no longer a contraction. Instead, the contraction must be dampened so that:[12]

$$\ln \delta_{\cdot t} \leftarrow\!\shortmid \ln \delta_{\cdot t} + (1-\rho)\left[\ln \mathcal{S}_{\cdot t} - \ln \mathbf{s}_{\cdot \mathbf{t}}(\delta_{\cdot t}, \theta_2)\right]. \tag{15}$$

This creates an additional challenge where the rate of convergence for the contraction in (15) can become arbitrarily slow as $\rho \to 1$.[13] Thus as more consumers substitute within the nest, this model becomes much harder to estimate. Our simulations will demonstrate that this can be problematic.

As is well known, the relation in (15) has an analytic solution in the absence of random coefficients when $\mu_{ijt} = 0$ for all (i, j, t). This model reduces to the regular nested logit for which the following expression was derived in Berry (1994):

$$\ln \delta_{jt} \equiv \ln \mathcal{S}_{jt} - \ln \mathcal{S}_{0t} - \rho \ln \mathcal{S}_{j|gt}$$

where $\mathcal{S}_{j|gt}$ is the marketshare of j in its nest g.

3. Algorithmic Improvements in `pyblp`

In each subsection, we review several methods from the literature (including some of our own design). While we make many of these methods available as options in `pyblp` our focus is on finding best practices which are fastest and most reliable for the widest array of users. Later, we provide support for these decisions with our Monte Carlo studies.[14]

[11] The nesting parameter can also be indexed as ρ_g so as to vary by group. We support both types of nesting parameters in `pyblp`.

[12] See Grigolon and Verboven (2014) for a derivation. The expression in (15) does not precisely match Grigolon and Verboven (2014) because of a typographical error in their final line.

[13] This can be formalized in terms of the modulus of the contraction mapping or the Lipschitz constant. See Dubé et al. (2012) for more details.

[14] We chose Python over alternatives such as Matlab, Julia, and R for a number of reasons. Python's package management systems are well-established, it has a mature scientific computing ecosystem, and as a general purpose language, Python is conducive to the development of larger projects. Scientific computing in all of these high-level languages is backed by similar implementations of numerical linear algebra routines, such as BLAS and LAPACK.

3.1. Incorporating Many Fixed Effects

There is a long tradition of extending the demand side utility to incorporate product or market fixed effects. For example Nevo (2001, 2000a) allows for product fixed effects ξ_j, so that

$$\delta_{jt} = [x_{jt}, v_{jt}]\beta - \alpha p_{jt} + \xi_j + \Delta\xi_{jt}.$$

These can manageably be incorporated as dummy variables (LSDV) in the linear part of the regression as there are only $J = 24$ products.

Backus et al. (2018) and Conlon and Rao (2017) allow for product (j) store (or chain) (s) specific fixed effects d_{js}. Using weekly UPC-store level Nielsen scanner data it is not uncommon for there to be more that $J = 3,500$ products (in both distilled spirits and ready-to-eat cereal). If one wants to allow for UPC-store fixed effects ξ_{js} this can explode to 100,000 or more fixed effects. Indeed, in Backus et al. (2018) the authors incorporate more than 50,000 such fixed effects. Similarly, there are approximately $T = 500$ weeks t of Nielsen scanner data from 2006-2016. If one wanted to incorporate store-week fixed effects ξ_{st} for only 100 stores this too could reach the order of 50,000 such fixed effects.

Clearly, the *least squares dummy variable* (LSDV) approach will not scale with tens or hundreds of thousands of fixed effects. We might consider the *within transformation* to remove the fixed effects, though we can't directly incorporate both a within transformation and a supply side without re-writing the problem because of endogenous prices p_{jt}. We show how to re-write the problem in Appendix A. Define y_{jt}^D, y_{jt}^S, x_{jt}^D, and x_{jt}^S as follows:

$$\begin{aligned} y_{jt}^D &\equiv \hat{\delta}_{jt}(\theta_2) + \alpha p_{jt} \quad &= [x_{jt} \quad v_{jt}]\beta + \xi_t \equiv x_{jt}^D\beta + \xi_{jt}, \\ y_{jt}^S &\equiv p_{jt} - \hat{\eta}_{jt}(\theta_2) \quad &= [x_{jt} \quad w_{jt}]\gamma + \omega_t \equiv x_{jt}^S\gamma + \omega_{jt}. \end{aligned}$$

Stacking the system across observations yields:

$$\underbrace{\begin{bmatrix} y_D \\ y_S \end{bmatrix}}_{Y} = \underbrace{\begin{bmatrix} X_D & 0 \\ 0 & X_S \end{bmatrix}}_{X} \begin{bmatrix} \beta \\ \gamma \end{bmatrix} + \begin{bmatrix} \xi \\ \omega \end{bmatrix}. \tag{16}$$

After re-arranging terms and re-stacking, this is just a conventional linear IV problem in terms of (Y, X) where the endogenous parameters have been incorporated into Y. This means that the *within transform* can be used to absorb a single dimensional fixed effect.

Consider two dimensions of fixed effects N and T where $N \gg T$:

$$\begin{aligned} \widetilde{y}_{it} &= y_{it} - \overline{y}_{i\cdot} - \overline{y}_{\cdot t}, \\ \widetilde{x}_{it} &= x_{it} - \overline{x}_{i\cdot} - \overline{x}_{\cdot t}. \end{aligned}$$

See Coleman et al. (2018) for a more detailed comparison of various languages.

The simplest approach might be to *iteratively demean*: remove $\overline{x}_{i\cdot}$, update x_{it}, remove $\overline{x}_{\cdot t}$, and repeat this process until $\overline{x}_{i\cdot} = \overline{x}_{\cdot t} = 0$. This *method of alternating projections* (MAP) can be done in a single iteration if $\text{Cov}(\overline{x}_{\cdot t}, \overline{x}_{i\cdot}) = 0$. However, when the two dimensions of fixed effects are correlated this can take many iterations and can be quite burdensome.

The LSDV approach handles the burden of correlation but requires constructing the annihilator matrix to remove all the fixed effects. This approach requires inverting an $(N+T)\times(N+T)$ matrix. Constructing (and inverting) such a large matrix is often infeasible because of memory requirements. Several algorithms have been proposed to deal with this problem. The most popular algorithm is that of Correia (2016), based on the MAP approach of Guimarães and Portugal (2010), which iteratively regresses Y on X in order to residualize Y.[15]

The BLP application is unusual in that we re-run regressions using the X_D and X_S variables many times. However, the left hand side variables $\hat{\delta}_{jt}(\theta_2) + \alpha p_{jt}$ and $\hat{c}(\theta_2)$ change with each guess of θ_2, which means the entire procedure needs to be repeated for each update of θ_2. For two-dimensional fixed effects, `pyblp` uses the algorithm of Somaini and Wolak (2016). This has the advantage that the linear explanatory variables in (X_D, X_S) can be residualized only once, while the left hand side variables (which are $N \times 1$ vectors) still need to be residualized for each guess of θ_2. The savings are particularly large when the dimensions of X_D or X_S are large. For three or more dimensions of fixed effects, `pyblp` uses an iterative de-meaning algorithm similar to the one described above.

3.2. Solving for the Shares

The main challenge of the Nested Fixed Point (NFXP) algorithm is solving the system of market-shares: $\mathcal{S}_{jt} = s_{jt}(\delta, \theta_2)$. In the NFXP approach, θ_2 is treated as fixed, which has the immediate implication that rather than solving a system of N nonlinear equations and N unknowns in δ, we solve T systems of J_t equations and J_t unknowns in $\delta_{\cdot t}$. Independent of how we solve each system of equations, we can solve each market t in parallel.[16]

Consider a single market t, where we search for the J_t vector $\delta_{\cdot t}$ which satisfies

$$\mathcal{S}_{jt} = s_{jt}(\delta_{\cdot t} \mid \theta_2) = \int \frac{\exp[\delta_{jt} + \mu_{ijt}]}{\sum_{k \in J_t} \exp[\delta_{kt} + \mu_{ikt}]} f(\mu_{it} \mid \widetilde{\theta}_2)\, d\mu_{it} \tag{17}$$

While mathematically there is a unique solution, numerically speaking, it is impossible to choose a vector $\delta_{\cdot t}$ which solves (17) exactly. Instead, we must solve the system of equations to some tolerance. We express the tolerance in terms of the log difference in shares:

$$\left\| \ln \mathcal{S}_{jt} - \ln s_{jt}(\delta_{\cdot t}, \theta_2) \right\|_\infty \leq \epsilon^{tol}. \tag{18}$$

[15] This is implemented for the linear IV model in the Stata package `ivreghdfe`.

[16] This same idea provides the sparsity of share constraints in the MPEC approach of Dubé et al. (2012).

Users face a tradeoff with regards to the tolerance of the *inner loop* in (18). If the tolerance is too loose, the numerical error propagates to the rest of the estimation routine.[17] It is also possible to set a tolerance which is too tight and thus can never be satisfied. This is particularly problematic when we sum over a large number of elements. We prefer to set $\epsilon^{tol} \in [10^{-12}, 10^{-14}]$ as the *machine epsilon* for 64-bit doubles is generally $\epsilon^{mach} \in [10^{-16}, 10^{-15}]$.

Jacobian Based Approaches

A direct approach would be to solve the system of J_t equations and J_t unknowns using Newton-type methods. Consider Newton-Raphson iteration below:[18]

$$\delta_{\cdot t}^{h+1} \leftarrow\!\shortmid \delta_{\cdot t}^{h} - \lambda J_s^{-1}(\delta_{\cdot t}^{h}, \widetilde{\theta}_2) \cdot s_t(\delta_{\cdot t}^{h}, \widetilde{\theta}_2) \tag{19}$$

Each Newton-Raphson iteration would require computation of both the J_t vector of marketshares $s_t(\delta_{\cdot t}^{h}, \widetilde{\theta}_2)$, the $J_t \times J_t$ Jacobian matrix $J_s(\delta_{\cdot t}^{h}, \widetilde{\theta}_2) = \frac{\partial s_{\cdot t}}{\partial \delta_{\cdot t}}(\delta_{\cdot t}^{h}, \widetilde{\theta}_2)$, as well as its inverse $J_s^{-1}(\delta_{\cdot t}^{h})$.

There are some alternative *quasi-Newton* methods which solve variants of (19). These variants generally involve modifying the step-size λ or approximating $J_s^{-1}(\delta_{\cdot t}^{h}, \widetilde{\theta}_2)$ in ways that avoid calculating the inverse Jacobian at each step. Quasi-Newton methods are often relatively fast when they work, though they need not converge (they may oscillate, reach a resting point, etc.) and may be sensitive to starting values.[19]

Our experience indicates that the Levenberg-Marquardt Algorithm (LMA) is the fastest and most reliable Jacobian-based solution method.[20] LMA minimizes the following least-squares problem in order to solve the following $J_t \times J_t$ system of nonlinear equations:

$$\min_{\delta_{\cdot t}} \sum_{j=1}^{J} [\mathcal{S}_j - s_j(\delta_{\cdot t}, \theta_2)]^2.$$

The idea is to update our guess of δ_t to to $\delta_t + x$ where x is a $J_t \times 1$ vector. The LM update is given by the solution to the following linear system of equations:

$$\underbrace{[J_s' J_s + \lambda \operatorname{diag}(J_s' J_s)]}_{\text{approximate Hessian}} \cdot x = J_s'[\mathcal{S}_j - s_j(\delta_{\cdot t}, \theta_2)]. \tag{20}$$

[17] Dubé et al. (2012) show how error propagates from (18) to the estimates of $\hat{\theta}$. Lee and Seo (2016) provide a more precise characterization of this error when Newton's method is used.

[18] In practice it is generally faster to solve the linear system: $J_s(\delta_{\cdot t}^{h}, \widetilde{\theta}_2)\left[\delta_{\cdot t}^{h+1} - \delta_{\cdot t}^{h}\right] = -s_t(\delta_{\cdot t}^{h}, \theta_2)$.

[19] Though the BLP problem is a non-convex system of nonlinear equations, there are some features which make it amenable to quasi-Newton methods. The marketshare function is a real-analytic function of $\delta_{\cdot t}$; $s_{jt}(\delta_{\cdot,t} \mid \widetilde{\theta}_2)$ is $\mathbb{C}^\infty$ and bounded between $(0, 1)$ and agrees with its Taylor approximation at any $\delta_{\cdot t}$. This guarantees that at least within some *basin of attraction*, quasi-Newton methods will be quadratically convergent. For an example of a quasi-Newton solution to Section 3.2, see Houde (2012). The other useful property is that with a strictly positive outside good share, when all shares $s_{jt} < 0.5$, the Jacobian $J_s(\delta_{\cdot t}, \widetilde{\theta}_2)$ is strictly diagonally dominant, which guarantees that it is always non-singular.

[20] Specifically we use `scipy.optimize.root`'s `lm` option which calls `MINPACK`'s LM routine (More et al., 1980).

This has the advantage that for $\lambda = 0$ the algorithm takes a full Gauss-Newton step, while for λ large it takes a step in the direction of the gradient (gradient-descent). The additional diagonal term also guarantees the invertibility of approximate Hessian, even when it becomes nearly singular.

As in all iterative solution methods there are two primary costs: the cost per iteration and the number of iterations until convergence. The cost per iteration is driven primarily by the cost of computing (rather than inverting) the Jacobian matrix which involves $J_t \times J_t$ numerical integrals.[21]

Fixed Point Approaches

Berry et al. (1995) also propose a fixed-point approach to solve the $J_t \times J_t$ system of equations in (17). They show that the following is a contraction mapping $f(\delta) = \delta$:

$$f : \delta_{\cdot t}^{h+1} \leftarrow\!\shortmid \delta_{\cdot t}^{h} + \ln \mathcal{S}_{\cdot t} - \ln s_{\cdot t}(\delta_{\cdot t}^{h}, \widetilde{\theta}_2). \tag{21}$$

This kind of contraction mapping is linearly convergent with a rate of convergence that is proportional to $L(\widetilde{\theta}_2)/[1 - L(\widetilde{\theta}_2)]$ where $L(\widetilde{\theta}_2)$ is the Lipschitz constant. Because (21) is a contraction, we know that $L(\widetilde{\theta}_2) < 1$. Dubé et al. (2012) show that for the BLP contraction the Lipschitz constant is defined as $L(\widetilde{\theta}_2) = \max_{\delta \in \Delta} ||I_{J_t} - \frac{\partial \log s_{\cdot t}}{\partial \delta_{\cdot t}}(\delta_{\cdot t}, \widetilde{\theta}_2)||_\infty$.

A smaller Lipschitz constant implies that (21) converges more rapidly. Dubé et al. (2012) show in simulation that all else being equal, a larger outside good share generally implies a smaller Lipschitz constant.[22] Conversely, as the outside good share becomes smaller, the convergence of the fixed point relationship takes increasingly many steps.

Accelerated Fixed Points

Given a fixed point relationship there may be faster ways to obtain a solution to $f(\delta) = \delta$ than direct iteration on the fixed point relation as in (21). There is a large literature on acceleration methods for fixed points. Most of these methods use information from multiple iterations $(\delta^h, \delta^{h+1}, \delta^{h+2}, f(\delta^h), f(f(\delta^h)))$ to approximate $J_{\mathbf{s}}$ or its inverse.[23]

Reynaerts et al. (2012) conduct extensive testing of various fixed point acceleration methods and find that the `SQUAREM` algorithm of Varadhan and Roland (2008) works well on the BLP contraction in (21). The intuition is to form a residual r^h which is determined by the change between the current iteration $\delta_{\cdot t}^h$ and the next iteration $f(\delta_{\cdot t}^h)$, as well as the change in the residual from this iteration to the next v^h to form an estimate of the Jacobian. The residual and the curvature can also be

[21] A typical entry is $\frac{\partial s_{jt}}{\partial \delta_{kt}} = \int [1(j = k) s_{ijt}(\mu_{it}) - s_{ijt}(\mu_{it}) s_{ikt}(\mu_{it})] f(\mu_{it} \mid \widetilde{\theta}_2)\, d\mu_{it}$. The primary cost arises from the numerical integration over heterogeneous types. Even for a large market with $J_t = 1,000$ products, inverting a $1,000 \times 1,000$ matrix is easy relative to J_t^2 numerical integrals.

[22] A simple but novel derivation. Consider the matrix $\frac{\partial \log s_{\cdot t}}{\partial \delta_{\cdot t}} = I_{J_t} - \text{diag}^{-1}(s_{\cdot t})\Gamma(\widetilde{\theta}_2)$ where element (j,k) is $1(j = k) - s_{jt}^{-1} \int s_{ijt}(\mu_{it}) s_{ikt}(\mu_{it}) f(\mu_{it} \mid \widetilde{\theta}_2)\, d\mu_{it}$. This implies that $L(\widetilde{\theta}_2) = \max_\delta [\max_j s_{jt}^{-1} \sum_k |\Gamma_{jk}(\delta_{\cdot t}, \widetilde{\theta}_2)|]$. A rough approximation to the term inside the square braces is $\max_j \sum_k |s_{kt}| \cdot |\rho(s_{ijt}, s_{ikt})| < 1 - s_{0t}$.

[23] Many of these algorithms are special cases of *Anderson Mixing*.

used to construct a step-size α^h. The exact algorithm is described below:[24]

$$\begin{aligned} \delta_{\cdot t}^{h+1} &\leftarrow \delta_{\cdot t}^{h} - 2\alpha^h r^h + (\alpha^h)^2 v^h, \\ \alpha^h = \frac{(v^h)' r^h}{(v^h)' v^h}, \quad r^h &= f(\delta_{\cdot t}^{h}) - \delta_{\cdot t}^{h}, \quad v^h = f(f(\delta_{\cdot t}^{h})) - 2f(\delta_{\cdot t}^{h}) + \delta_{\cdot t}^{h}. \end{aligned} \tag{22}$$

In general the `SQUAREM` method is 4 to 8 times as fast as direct iteration on the BLP contraction in (21). The idea is to take roughly the same number of steps as Newton-Raphson iteration, but to reduce the cost of steps by avoiding calculating the Jacobian directly. In fact, all of the terms in (22) are computed as a matter of course, because these are just iterations of x^h and $f(x^h)$. Unlike direct iteration on (21), there is technically no convergence guarantee as the iteration on (22) is no longer a contraction.

The are alternative acceleration methods in addition to `SQUAREM`. Reynaerts et al. (2012) also consider `DF-SANE` which takes on the form $\delta_{\cdot t}^{h+1} \leftarrow \delta_{\cdot t}^{h} - \alpha^h f(\delta_{\cdot t}^{h})$ with a different choice of the step-size α^h. They find performance is similar to `SQUAREM` though it can be slightly slower and less robust. Consistent with Reynaerts et al. (2012), we find the `SQUAREM` to be the fastest and most reliable fixed-point approach. We find that the Jacobian-based Levenberg-Marquardt approach gives similar reliability and (slightly better) performance.

3.3. Optimization

Optimization of the GMM objective function for the BLP problem can be challenging. The BLP problem itself in (9) represents a non-convex optimization problem,[25] so the Hessian need not be positive semidefinite at all values of θ_2. In practice this means that no optimization routine is guaranteed to find a global minimum in a fixed amount of time. This critique applies both to derivative-based quasi-Newton approaches and to simplex-based Nelder-Mead type approaches. Well-known recommendations involve considering a number of different starting values and optimization routines, verifying that $\hat{\theta}$ satisfies both the first order conditions (the gradient is within a tolerance of zero) and second order conditions (the Hessian matrix has all positive eigenvalues). Both of these are reported by default in `pyblp`.

The optimization interface to `pyblp` is generic in the sense that any optimizer which can be implemented as a Python function should work fine.[26] Though non-derivative based routines such as Nelder-Mead have been frequently used in previous papers, they are not recommended.[27] Indeed,

[24] `pyblp` includes a Python port of the `SQUAREM` package from R.

[25] Absent unobserved heterogeneity or random coefficients the problem is globally convex.

[26] To illustrate this, in the documentation we give an example of such a "custom" routine where we construct a simple brute force solver that searches over a grid of parameter values. This flexibility should allow users to experiment with routines without much difficulty or "upgrade" if better routines are developed.

[27] Both Dubé et al. (2012) and Knittel and Metaxoglou (2014) find that derivative-based routines outperform the simplex-based Nelder-Mead routine both in terms of speed and reliability. Our own tests concur with this prior opinion.

the most important aspect of `pyblp` optimization is that it calculates the analytic gradient for any user-provided model, including models which incorporate both supply and demand moments or fixed effects.[28] Analytic gradients provide both a major speedup in computational time, while also generally improving the chances that the algorithm converges to a valid minimum.

The `pyblp` package uses optimization routines implemented in the open-source SciPy library and also interfaces with commercial routines such as Artleys Knitro. The Knitro software incorporates four different gradient-based search routines, and has been recommended for the BLP problem previously by Dubé et al. (2012). Table 10 lists nine different optimization routines currently supported by `pyblp`. Some optimization routines also allow the user to input constraints on parameters, these can speed up estimation or prevent the optimization routine from considering "unreasonable" values. An important example are *box constraints* on the parameter space, where we restrict components of θ_2 denoted θ_ℓ to $\theta_\ell \in [\underline{\theta}_\ell, \overline{\theta}_\ell]$. Some typical constraints are that demand slopes down and that random coefficients have positive but bounded variances. This is particularly helpful because large values for random coefficients can generate numerical issues which we discuss below.

Consistent with the prior literature, we recommend trying multiple different optimizers and starting values to check for agreement; our simulations indicate that when properly configured, most optimizers arrive at the same parameter estimates. Our recommendation is that researchers try Knitro's `Interior/Direct` algorithm first if available, and then try a `BFGS`-based routine, ideally with constraints on the parameter space such as `L-BFGS-B`.[29]

3.4. Numerical Issues and Tricks

There are several numerical challenges posed by the BLP problem, most of which are related to the exponentiated indirect utilities and the logit denominator: $\sum_j \exp[\delta_{jt} + \mu_{ijt}]$. If some values in this summation are on the order of $\exp(-5) \approx 0.0067$ and others are $\exp(30) > 10^{13}$, their sum may be rounded because the effective precision for most 64-bit doubles is around 15 significant digits. This *loss of precision* arises when taking summations of many numbers with different scales, and depending on the situation, may or may not be problematic. A related problem is *overflow*, which is likely to arise when attempting to compute large values such as $\exp(300)$. This can mean that $s_{jt}(\delta_{\cdot t}, \theta_2) \to 1$ while $s_{kt}(\delta_{\cdot t}, \theta_2) \to 0$ for $k \neq j$, and leads optimization routines to fail.[30]

There are a number of solutions to these problems. One solution is to avoid large values in the argument of $\exp(\cdot)$ by limiting the magnitudes of random coefficients through the *box constraints* described above. Another simple solution involves working market by market and avoiding very large summations. One additional method would be to use a summation algorithm that attempts to

[28] We could not find any examples of simultaneous estimation of supply and demand with analytic gradients in the literature. A likely explanation is that the derivatives of the markup term $\eta_{jt}(\theta_2)$ are quite complicated. See Appendix B for a derivation.

[29] The main disadvantage of Knitro is that it is not freely available—it must be purchased and installed by end-users.

[30] In this case $s_{jt}(\delta_{\cdot t}, \theta_2) = s_{jt}$ cannot be solved for $\delta_{\cdot t}$, thus the inversion for the mean utilities fails.

preserve precision such as *Kahan Summation*.[31] This is substantially slower and is not implemented by default. As suggested in Skrainka (2012b), yet another approach is to use *extended precision arithmetic*. Again, we found this to be substantially slower without improving our statistical performance.[32]

Another way to guarantee overflow safety is to use a protected version of the log-sum-exp (LSE) function: $\text{LSE}(x_1, \ldots, x_K) = \log \sum_k \exp x_k = a + \log \sum_k \exp(x_k - a)$. By choosing $a = \max_k x_k$, use of this function helps ensure overflow safety when evaluating the multinomial logit function. This is a well known trick from the applied mathematics and machine-learning literatures, but there does not appear to be evidence of it in the literature on BLP models. This is implemented by default, as the additional computational cost appears to be trivial.

Implementations of the BLP method have used a number of "tricks" specific to certain software packages (such as MATLAB) in order to speed up the algorithm. One well-known trick involves working with $\exp[\delta_{jt}]$ in place of δ_{jt} in the contraction mapping to avoid transcendental functions like $\exp(\cdot)$. This has little benefit on modern architectures as transcendentals are highly optimized and $\exp(\cdot)$ of a billion numbers takes less than one millionth of a second. Another trick is using $\delta_{\cdot t}$ from a previous guess of θ_2 as a starting value. The `SQUAREM` and `LM` algorithms we suggest for solving the system of equations are relatively insensitive to starting values, so these sorts of speedups are not particularly useful. A MATLAB specific "vectorization" speedup used in Nevo (2000b) and Knittel and Metaxoglou (2014) is to stack all markets together and then construct cumulative sums of $\exp[\delta_{jt} + \mu_{ijt}]$. We find that this is approach is dominated by parallelization across markets T; furthermore, this approach will results in a loss of precision as $T \to \infty$. We provide some analysis of these tricks in Table 4.

3.5. Heterogeneity and Integration

An important aspect of the BLP model is that it incorporates heterogeneity via random coefficients. The challenge is that the integration in (3) needs to be performed numerically:

$$s_{jt}(\delta_t, \theta_2) = \int \frac{\exp[\delta_{jt} + \mu_{ijt}]}{\sum_{k \in J_t} \exp[\delta_{kt} + \mu_{ikt}]} f(\mu_{it}|\theta_2) d\,\mu_{it} \approx \sum_{i=1}^{I_t} w_i \underbrace{\frac{\exp[\delta_{jt} + \hat{\mu}_{ijt}(\nu_{it})]}{\sum_{k \in J_t} \exp[\delta_{kt} + \hat{\mu}_{ikt}(\nu_{it})]}}_{s_{ijt}(\mu_{it}(\theta_2))}. \tag{23}$$

The most common approach is Monte Carlo integration where random draws are taken from some candidate distribution $\nu_{it} \sim f(\nu_{it}|\theta_2)$ and then used to calculate $\mu_{ijt}(\nu_{it}, \theta_2)$. We can then take a weighted sum of individuals $s_{jt}(\delta_t, \theta_2) = \sum_{i=1}^{I_t} w_i s_{ijt}(\delta_t, \mu_{it}(\theta_2))$. The main advantage of Monte-

[31] In Python this is implemented as `math.fsum`.

[32] We are a bit vague on exactly what "extended precision" means because the NumPy implementation of `numpy.longdouble` varies across different architectures. On most Unix platforms, it is implemented with a 128-bit long double. For problems very different from our Monte Carlo designs with very small shares, or for problems with very large numbers of products, additional precision might still be valuable.

Carlo integration (actually pseudo-Monte Carlo integration) is that it avoids the *curse of dimensionality*. While the accuracy for low dimensional integrals increases slowly in the number of points of approximation, the accuracy does not decline quickly as one increases the dimension of integration.

The second approach is the so-called *Gaussian quadrature* approach. Here the integrand is approximated with a polynomial and then integrated exactly as polynomial integration is trivial. In practice this is far simpler as it still amounts to a weighted sum in (23) over a particular choice of (ν_i, w_i). The main choice the user makes is the *polynomial order* of the rule. In effect the user chooses the order of the polynomial used to approximate the integrand. As the order grows, more nodes are required but the accuracy of the approximation improves.

Gaussian quadrature works best when certain conditions are met: that the integrand is bounded and continuously differentiable. Thankfully, the logit kernel in (23) is always bounded on $(0, 1)$ and is infinitely continuously differentiable. This lends itself to quadrature-type approaches. There are a number of different flavors of quadrature rules designed to best approximate integrals under different weighting schemes. The *Gauss-Hermite* family of rules work best when $f(\mu_{it}) \propto \exp[-\mu_{it}^2]$, which (with a change of variables) includes integrals over a normal density. *Gauss-Kronod* rules offer an alternative where for a given level of polynomial accuracy p, they reuse the set of nodes from the $p-1$ polynomial accuracy integration rule. This allows for *adaptive* accuracy without wasting calculations; when the error from numerical integration is large the the polynomial degree can be expanded. Both the advantage and disadvantage of Gaussian quadrature rules is that they do a better job covering the "tail" of the probability distribution. While this increases the accuracy of the approximation, it can also lead to very large values which create *overflow* issues.[33] We prefer to use quadrature rules, and to be careful of potential overflow/underflow issues when computing shares.

The Gaussian quadrature rules apply only to a single dimension. One way to estimate higher dimensional integrals is to construct the product of single dimensional integrals. The disadvantage of *product rules* is that if one needs I_t points to approximate the integral in dimension one, then one needs I_t^d points to approximate the integral in dimension d. This is the so-called *curse of dimensionality*.

The curse of dimensionality is a well-known problem in numerical analysis and several off-the-shelf solutions exist. There are several clever algorithms for improving upon the product rule for higher dimensional integration. Judd and Skrainka (2011) explore *monomial cubature rules* while Heiss and Winschel (2008) use *sparse grid* methods to selectively eliminate nodes from the product rules. One disadvantage of these methods is that they often involve weights which are negative (i.e., $w_i < 0$), which can create problems when trying to decompose the distribution of heterogeneity (particularly for counterfactuals).

Though `pyblp` allows for flexible (user-supplied) distributions of random coefficients, by far

[33] Essentially for some simulated individual i we have that $s_{ijt} \to 1$ and $s_{ikt} \to 0$. This problem has been previously documented by Judd and Skrainka (2011) and Skrainka (2012b).

the most commonly employed choices in the literature are the independent normal and correlated normal distributions for $f(\nu_i \mid \theta_2)$. Here `pyblp` provides some specialized routines to handle these integrals with limited user intervention. There are number of different methods one can use to generate (w_i, ν_i) for the (correlated) normal case:

1. `monte_carlo`: Draw $\nu_{i\ell}$ from the standard normal distribution for each dimension of heterogeneity ℓ. Set $w_i = 1/I_t$ where I_t is the number of simulated "individuals".

2. `product`: Construct the Gauss-Hermite quadrature rule (ν_i, w_i) for a single dimension and build a product rule to obtain a set of weights and nodes in a higher dimension d.

3. `nested_product`: Construct (ν_i, w_i) using a product rule in dimension d but beginning with the basis of Gauss-Kronrod instead of Gauss-Hermite.

4. `grid`: Construct (ν_i, w_i) using a sparse grid rule of Heiss and Winschel (2008) for dimension d with the Gauss-Hermite rule as an initial basis.

5. `nested_grid`: Construct (ν_i, w_i) using a sparse grid rule in dimension d but beginning with the basis of Gauss-Kronrod instead of Gauss-Hermite.

6. Construct (ν_i, w_i) from user-provided data which allows for pseudo-Monte Carlo methods such as *Halton draws* or *Sobol sequences*.

In general, the best practice in low dimensions is probably to use product rules to a relatively high degree of polynomial accuracy. In higher dimensions, sparse grids appear to scale the best both in our own Monte Carlo studies and in those of Judd and Skrainka (2011) and Heiss and Winschel (2008).

3.6. Solving for Pricing Equilibria

Many counterfactuals of BLP-type problems involve perturbing either the market structure, marginal costs, or both, and solving for counterfactual equilibrium prices. Being able to solve for equilibrium prices quickly and accurately is also crucial to generating the optimal instruments in the next section. The Bertrand-Nash first order conditions are defined by (5) for each market t:

$$\mathbf{p} = \mathbf{c} + \underbrace{\Delta(\mathbf{p}, O)^{-1} s(\mathbf{p})}_{\eta(\mathbf{p}, O)}$$

During estimation, in order to recover marginal costs, one need only invert the $J_t \times J_t$ matrix $\Delta(\mathbf{p}, O)$. Solving for counterfactual pricing equilibria is more difficult as now we must solve the $J_t \times J_t$ nonlinear system of equations, often after replacing the ownership matrix with a post-merger counterpart, $O \to O^{post}$:

$$\mathbf{p} = \mathbf{c} + \eta^{post}(\mathbf{p}, O^{post}). \tag{24}$$

In general, solving this problem is hard as it represents a non-convex nonlinear system of equations, where one must simulate in order to compute $\eta(\mathbf{p}, O^{post})$ and its derivatives.[34] Once one incorporates both multi-product firms and arbitrary coefficients into the problem, both existence and uniqueness of an equilibrium become challenging to establish.[35]

One approach might be to solve the system using Newton's method, which requires calculating the $J_t \times J_t$ Jacobian $\frac{\partial \eta(\mathbf{p})}{\partial \mathbf{p}}$. We provide (possibly novel) analytic expressions for this Jacobian in Appendix B.[36] The expression for the Jacobian involves the Hessian matrix of demand $\frac{\partial^2 s_k}{\partial p_j^2}$ as well as tensor products, and can be computationally challenging.[37]

The second, and perhaps most common approach in the literature is treating (24) as a fixed point and iterating on $\mathbf{p} \leftarrow\!\shortmid \mathbf{c} + \eta^{post}(\mathbf{p}, O^{post})$.[38] The problem is that while a fixed point of (24) may represent the Bertrand-Nash equilibrium of (7), it is not necessarily a contraction. In fact, as part of Monte Carlo experiments conducted in Armstrong (2016), the author finds that iterating on (24) does not always lead to a solution and at least some fraction of the time leads to cycles. We were able to replicate this finding for similarly constructed Monte Carlo experiments between 1-5% of the time. Were this relation to be a contraction, we should always be able to iterate on $\mathbf{p} \leftarrow\!\shortmid \mathbf{c} + \eta^{post}(\mathbf{p}, O^{post})$ until we reached the solution.

Our preferred approach borrows from the engineering literature and follows Morrow and Skerlos (2011). This approach does not appear to be well known in the Industrial Organization literature, but in our experiments appears to be highly effective. They reformulate the solution to (5) by breaking up the matrix of demand derivatives into two parts: a $J_t \times J_t$ diagonal matrix Λ, and a $J_t \times J_t$ dense matrix Γ:

$$\begin{aligned}
\frac{\partial \mathbf{s}}{\partial \mathbf{p}}(\mathbf{p}) &= \Lambda(\mathbf{p}) - \Gamma(\mathbf{p}), \\
\Lambda_{jj} &= \int \alpha_i s_{ijt}(\mu_{it}) f(\mu_{it} \mid \widetilde{\theta}_2)\, d\mu_{it}, \\
\Gamma_{jk} &= \int \alpha_i s_{ijt}(\mu_{it}) s_{ikt}(\mu_{it}) f(\mu_{it} \mid \widetilde{\theta}_2)\, d\mu_{it},
\end{aligned} \tag{25}$$

[34] The first Monte Carlo studies which evaluate price and quantity equilibria as part of the data generating process are likely Armstrong (2016), Skrainka (2012a) and Conlon (2017).

[35] Caplin and Nalebuff (1991) and Gallego et al. (2006) have results which apply to single product firms and linear in price utility under logit demands. Konovalov and Sandor (2010) generalizes these to logit demands with linear in price utility and multi-product firms. With the addition of random coefficients, it is possible that the resulting model will violate the quasi-concavity of the profit function that these results require. Morrow and Skerlos (2010) avoid some of these restrictions but place other restrictions on indirect utilities. Existence and uniqueness are beyond the scope of this paper—we instead focus on calculating solutions to the system of first order conditions, assuming such solutions exist.

[36] For example, Knittel and Metaxoglou (2014) do not update $s(p)$ and thus avoid fully solving the system of equations.

[37] An alternative might be to try a Newton-type approach without providing analytic formulas for the Jacobian. This seems slow and ill-advised as there are J_t markups each with J_t derivatives and each derivative involves integration in order to compute both $\Omega(p)$ and $s(p)$.

[38] The "folklore" solution is to dampen this expression and consider $\mathbf{p} \leftarrow\!\shortmid \rho \cdot \mathbf{p} + (1-\rho) \cdot \left[\mathbf{c} + \eta^{post}(\mathbf{p}, O^{post})\right]$. This tends to be slower and more reliable, though we cannot find any theoretical justification for convergence.

where $\alpha_i = \frac{\partial u_{ijt}}{\partial p_{jt}}$, the marginal (dis)-utility of price. Morrow and Skerlos (2011) then reformulate the problem as different fixed point that is specific to mixed logit demands:[39]

$$\mathbf{p} \leftarrow\!\shortmid \mathbf{c} + \zeta(\mathbf{p}), \quad \zeta(\mathbf{p}) = \Lambda(\mathbf{p})^{-1}(O^{post} \odot \Gamma(\mathbf{p}))(\mathbf{p} - \mathbf{c}) - \Lambda(\mathbf{p})^{-1}s(\mathbf{p}). \tag{26}$$

The fixed point in (26) is entirely different from that in (24) and coincides only at resting points. Consistent with results reported in Morrow and Skerlos (2011), we find that (26) is around 3-12 times faster than Newton-type approaches and reliably finds an equilibrium.[40]

Perhaps most consequentially, the ability to solve for a pricing equilibrium rapidly and reliably makes it possible to generate the Amemiya (1977) or Chamberlain (1987) *feasible approximation to the optimal instruments.*

4. Supply and Demand: Optimal Instruments and Overidentifying Restrictions

This section focuses on joint estimation of supply and demand in order to clarify the precise role of overidentifying restrictions. Absent information on firm conduct or the precise functional form for marginal costs, many researchers estimate a demand side only.

As a way to improve performance, we can construct approximations to the optimal instruments in the spirit of Amemiya (1977) or Chamberlain (1987). Approximations to the optimal instruments were featured in both Berry et al. (1995) and Berry et al. (1999) but are not commonly employed in many subsequent studies using the BLP approach in part because they are challenging to construct. Reynaert and Verboven (2014) show approximations to the optimal instruments can improve the econometric performance of the estimator, particularly with respect to the θ_2 parameters. The form we derive is somewhat different from their expression, and incorporates both supply and demand. While the procedure itself is quite involved, the good news is that does not require much in the way of user input, and is fully implemented by the `pyblp` software.

For the GMM problem, Chamberlain (1987) tells us that the optimal instruments are related to the expected Jacobian of the moment conditions where the expectation is taken conditional only on the exogenous regressors z_t. We can write this expectation for each product j and market t as $E[D_j(z_t)\Omega_{jt}^{-1} \mid z_t]$. We use the word "approximation" because the aforementioned expectation

[39] This resembles a well known "folklore" solution to the pricing problem, which is to rescale each equation by its own share s_{jt} (see Skrainka, 2012a). For the plain logit, $\Lambda_{jj}^{-1} = 1/(\alpha s_j)$.

[40] In `pyblp`, iteration is terminated with the *numerical simultaneous stationarity condition* of Morrow and Skerlos (2011) is satisfied: $||\Lambda(\mathbf{p})(\mathbf{p} - \mathbf{c} - \zeta(\mathbf{p}))|| < \epsilon$, where ϵ is a small number and the left hand side is firms' first order conditions.

lacks a closed form. We derive the components of the approximation below:[41]

$$D_j \equiv \underbrace{\begin{bmatrix} \frac{\partial \xi_j}{\partial \beta} & \frac{\partial \omega_j}{\partial \beta} \\ \frac{\partial \xi_j}{\partial \alpha} & \frac{\partial \omega_j}{\partial \alpha} \\ \frac{\partial \xi_j}{\partial \widetilde{\theta}_2} & \frac{\partial \omega_j}{\partial \widetilde{\theta}_2} \\ \frac{\partial \xi_j}{\partial \gamma} & \frac{\partial \omega_j}{\partial \gamma} \end{bmatrix}}_{(K_1+K_2+K_3)\times 2} = \begin{bmatrix} -x_j & 0 \\ -v_j & 0 \\ \frac{\partial \xi_j}{\partial \alpha} & \frac{\partial \omega_j}{\partial \alpha} \\ \frac{\partial \xi_j}{\partial \widetilde{\theta}_2} & \frac{\partial \omega_j}{\partial \widetilde{\theta}_2} \\ 0 & -x_j \\ 0 & -w_j \end{bmatrix}, \quad \Omega \equiv \underbrace{\begin{bmatrix} \sigma_\xi^2 & \sigma_{\xi\omega} \\ \sigma_{\xi\omega} & \sigma_\omega^2 \end{bmatrix}}_{2\times 2}. \tag{27}$$

A little calculation shows that for each market t and each observation j,

$$D_j\Omega^{-1} = \frac{1}{\sigma_\xi^2\sigma_\omega^2 - \sigma_{\xi\omega}^2} \times \begin{bmatrix} -\sigma_\omega^2 x_j & \sigma_{\xi\omega} x_j \\ -\sigma_\omega^2 v_j & \sigma_{\xi\omega} v_j \\ \sigma_\omega^2 \frac{\partial \xi_j}{\partial \alpha} - \sigma_{\xi\omega}\frac{\partial \omega_j}{\partial \alpha} & \sigma_\xi^2 \frac{\partial \omega_j}{\partial \alpha} - \sigma_{\xi\omega}\frac{\partial \xi_j}{\partial \alpha} \\ \sigma_\omega^2 \frac{\partial \xi_j}{\partial \widetilde{\theta}_2} - \sigma_{\xi\omega}\frac{\partial \omega_j}{\partial \widetilde{\theta}_2} & \sigma_\xi^2 \frac{\partial \omega_j}{\partial \widetilde{\theta}_2} - \sigma_{\xi\omega}\frac{\partial \xi_j}{\partial \widetilde{\theta}_2} \\ \sigma_{\xi\omega} x_j & -\sigma_\xi^2 x_j \\ \sigma_{\xi\omega} w_j & -\sigma_\xi^2 w_j \end{bmatrix}. \tag{28}$$

Clearly rows 1 and 5 are co-linear. Let Θ be a conformable matrix of zeros and ones such that

$$(D_j\Omega^{-1}) \odot \Theta = \frac{1}{\sigma_\xi^2\sigma_\omega^2 - \sigma_{\xi\omega}^2} \times \begin{bmatrix} -\sigma_\omega^2 x_{jt} & 0 \\ -\sigma_\omega^2 v_j & \sigma_{\xi\omega} v_j \\ \sigma_\omega^2 \frac{\partial \xi_j}{\partial \alpha} - \sigma_{\xi\omega}\frac{\partial \omega_j}{\partial \alpha} & \sigma_\xi^2 \frac{\partial \omega_j}{\partial \alpha} - \sigma_{\xi\omega}\frac{\partial \xi_j}{\partial \alpha} \\ \sigma_\omega^2 \frac{\partial \xi_j}{\partial \widetilde{\theta}_2} - \sigma_{\xi\omega}\frac{\partial \omega_j}{\partial \widetilde{\theta}_2} & \sigma_\xi^2 \frac{\partial \omega_j}{\partial \widetilde{\theta}_2} - \sigma_{\xi\omega}\frac{\partial \xi_j}{\partial \widetilde{\theta}_2} \\ 0 & -\sigma_\xi^2 x_j \\ \sigma_{\xi\omega} w_j & -\sigma_\xi^2 w_j \end{bmatrix}. \tag{29}$$

We can partition our instrument set by column into "demand" and "supply" instruments:

$$z_j^D \equiv \underbrace{E[(D_j\Omega_{jt}^{-1} \odot \Theta)_{\cdot 1} \mid z]}_{K_1+K_2+(K_3-K_x)}, \quad z_j^S \equiv \underbrace{E[(D_j\Omega_{jt}^{-1} \odot \Theta)_{\cdot 2} \mid z]}_{K_2+K_3+(K_1-K_x)}. \tag{30}$$

In (30), we have $K - K_x$ (where K_x denotes the dimension of common exogenous parameters x_{jt}) instruments for both supply z_j^S and demand z_j^D, though it is evident from (28) that the instruments for the endogenous parameters are not the same.[42] The approximate optimal instruments from the endogenous parameters are *nonlinear* functions of the data, while the remaining optimal instru-

[41] Implicitly we assume that (ξ_{jt}, ω_{jt}) are jointly IID so that $\Omega_{jt} = \Omega$. This is merely a matter of convenient notation, as the extention to heteroskedastic or clustered covariances are straightforward.

[42] This is true except for the knife-edge case where $\frac{\partial \xi_j}{\partial \theta_\ell} / \frac{\partial \omega_j}{\partial \theta_\ell} \propto \frac{\sigma_\xi^2 + \sigma_{\xi\sigma}}{\sigma_\omega^2 + \sigma_{\xi\sigma}}$. In fact, (30) highlights that the set of instruments Z^D and Z^S should never be the same because of the different excluded variables.

ments from the *linear* portions of demand and supply are simply exogenous regressors re-scaled by covariances.

Notice that when we include simultaneous supply and demand moments we have overidentifying restrictions. As shown in (30), we have $2 \times (K - K_x)$ restrictions and K parameters. This gives us $K - 2K_x$ overidentifying restrictions. The first set of K_2 overidentifying restrictions comes from the fact that in (28) we have two restrictions for each of the endogenous (nonlinear) parameters $[\alpha, \widetilde{\theta}_2]$. These are *cross-equation* restrictions that arise from simultaneous estimation of supply and demand.

The other two sets of overidentifying restrictions arise from *exclusion restrictions*, which are made explicit in (30), where w_{jt} and v_{jt} show up in one equation but not the other. There are $K_3 - K_x$ overidentifying restrictions from cost shifters w_{jt} which are excluded from demand. There are $K_1 - K_x$ demand shifters v_{jt} which are excluded from supply.

Remark 1

Our version of the optimal instruments makes explicit the exclusion restrictions in the BLP model. Both the supply and demand moments are used in order to pin down the endogenous parameters θ_2 (including the price parameter α). The role of the exogenous cost-shifters w_{jt} is now explicit: they provide overidentifying restrictions for demand which can be used to pin down α and θ_2. Likewise the role of the exogenous demand-shifters v_{jt} is also made explicit: they provide overidentifying restrictions for supply which can be used to pin down θ_2 as well as the markup term $\eta_{jt}(\theta_2)$, and hence *firm conduct*.

Perhaps most importantly, this tells us precisely where to find exclusion restrictions: *something that enters the other equation*. The idea of using demand shifters as overidentifying restrictions to identify conduct has a long history in industrial organization going back to Bresnahan (1982), and was treated non-parametrically in Berry and Haile (2014). In related work, Backus et al. (2018) show how to use (29) to test for firm conduct.

Remark 2

It is worth pointing out that our derivation of the optimal instruments appears to vary from derivations in the prior literature. Some of the prior literature using optimal instruments for BLP type problems suggests the resulting problem is *just identified* rather than *over identified*. Reynaert and Verboven (2014) appear to construct their version of optimal instruments by summing across the rows of (28) and excluding either the first or third row.[43] This gives them $K = K_1 + K_2 + K_3$ instruments and K unknowns so that the model is just identified. However, because they stack (ξ, ω) they effectively have $2 \times N$ rather than N observations. Conceptually, one way to view their formulation is that it imposes $E[\xi'_{jt} z_{jt}] + E[\omega'_{jt} z_{jt}] = 0$ rather than separately imposing $E[\xi'_{jt} z_{jt}] = 0$

[43] We should caveat that Reynaert and Verboven (2014) do not consider joint supply and demand as their main specification.

and $E[\omega'_{jt} z_{jt}] = 0$.

Remark 3

If one assumes that $\sigma_{\xi\omega} = 0$ or that the supply and demand shocks are uncorrelated, then the expression in (28) clearly shows that the optimal instruments for demand no longer depend on the supply Jacobian $\frac{\partial \xi_j}{\partial \theta_2}$ and likewise that the optimal instruments for supply no longer depend on the demand Jacobian $\frac{\partial \omega_j}{\partial \theta_2}$.[44] We still have overidentifying restrictions in this case as both supply and demand moments provide information on the nonlinear parameters. We caution that there is no *a priori* reason to believe that supply and demand unobservables are uncorrelated with one another.

Constructing Feasible Instruments

The main challenge with implementing the optimal instruments is that the expectations of the Jacobian terms $E[\frac{\partial \xi_j}{\partial \theta_2}, \frac{\partial \omega_j}{\partial \theta_2} \mid z]$ are not directly measured. The most obvious example is that $\frac{\partial \xi_j}{\partial \alpha} = p_j$, but p_j is endogenous, so we must replace it with some estimate of $E[p_j \mid z]$. Here is what Berry et al. (1995) say about optimal instruments:

> *Unfortunately $D_j(z)$ is typically very difficult, if not impossible, to compute. To calculate $D_j(z)$ we would have to calculate the pricing equilibrium for different (ξ_j, ω_j) sequences, take derivatives at the equilibrium prices, and then integrate out over the distribution of such sequences. In addition, this would require an assumption that chooses among multiple equilibria when they exist, and either additional assumptions on the joint distribution of (ξ, ω), or a method for estimating that distribution.*

The appendix of the NBER working paper version of Berry et al. (1999) is even less positive:

> *Calculating a good estimate of $E[p \mid z]$ then requires (i) knowing or estimating the density of the unobservables and (ii) solving at some initial guess for θ the fixed point that defines equilibrium prices for each (ξ, ω) and then integrating this implicit function with respect to the density of the unknown parameters. This process is too complicated to be practical.*

In their main specification, Reynaert and Verboven (2014) suggest computing the approximation to the optimal instruments under an additional assumption of perfect competition, so that $E[p_{jt} \mid z] = E[c_{jt} \mid z] = [x_{jt} \; w_{jt}]\hat{\gamma} + \hat{\omega}_{jt}$. This avoids solving for equilibrium prices (and implicit functions). When they do consider imperfect competition, they construct $E[p_{jt} \mid z_t]$ by regressing the endogenous variable on a series of exogenous regressors (essentially a "first stage" regression).

We follow the possibly more accurate but costly recipe proposed by Berry et al. (1999) and show that with other computational advances in `pyblp` it is feasible to implement. For each market t we can:

[44] In this case the optimal weighting matrix would also have a block diagonal structure with separate blocks for demand and supply moments. The zero covariance restriction is the identification argument in MacKay and Miller (2018).

1. Obtain an initial estimate $\hat{\theta} = [\hat{\beta}, \hat{\alpha}, \hat{\theta}_2, \hat{\gamma}]$.

2. Obtain an initial estimate of $\hat{\Omega}^{-1}$ by computing covariances of $(\hat{\xi}, \hat{\omega})$.

3. Draw the $J_t \times 2$ matrix of structural errors (ξ_t^*, ω_t^*) according to one of the below options.

4. Compute $\hat{y}_{jt}^S = \hat{c}_{jt} = [x_{jt} \; w_{jt}]\hat{\gamma} + \omega_{jt}^*$ and the exogenous portion of utility $\hat{y}_{jt}^D = x_{jt}\hat{\beta} + \xi_{jt}^*$.

5. Use $(\hat{y}_{jt}^D, \hat{y}_{jt}^S, \hat{\alpha}, x_t, w_t)$ to solve for equilibrium prices and quantities $(\hat{p}_t, \hat{s}_t)$ with the ζ-markup approach in Section 3.6. Note that this does not include any endogenous quantities.

6. Treating $(\hat{p}_t, \hat{s}_t, x_t, w_t)$ as data, solve for $\hat{\xi} = \xi(\hat{p}_t, \hat{s}_t, x_t, w_t, \hat{\alpha}, \hat{\theta}_2)$ and $\hat{\omega}_t = \omega(\hat{p}_t, \hat{s}_t, x_t, w_t, \hat{\alpha}, \hat{\theta}_2)$ following Section 2.1.

7. Construct the Jacobian terms $\frac{\partial \hat{\xi}_j}{\partial \theta_2}(\xi_t^*, \omega_t^*)$, $\frac{\partial \hat{\omega}_j}{\partial \theta_2}(\xi_t^*, \omega_t^*)$, and $\hat{D}_j(\xi_t^*, \omega_t^*)$ using the analytic formulas in Appendix B.

8. Average $\hat{D}_j(\xi_t^*, \omega_t^*)$ over several draws of (ξ_t^*, ω_t^*) to construct an estimate of $E[\hat{D}_j \mid z_t]$.

There are three options for generating (ξ_t^*, ω_t^*) suggested by Berry et al. (1999) and `pyblp` makes all three available:

(a) `approximate`: Replace (ξ_t^*, ω_t^*) with their expectation: $(E[\xi_t], E[\omega_t]) = (0, 0)$. This is what Berry et al. (1999) do.

(b) `asymptotic`: Estimate an asymptotic normal distribution for $(\xi_{jt}, \omega_{jt}) \sim N(0, \hat{\Omega})$ and then draw (ξ_t^*, ω_t^*) from that distribution.

(c) `empirical`: Draw (ξ_t^*, ω_t^*) from the joint empirical distribution of $(\hat{\xi}_{jt}, \hat{\omega}_{jt})$.

In general, `asymptotic` and `empirical` are not believed to be computationally feasible, particularly when there are large numbers of draws for (ξ_t^*, ω_t^*). The costly step is Item 5 above which involves solving for a new equilibrium $(\hat{p}_t, \hat{s}_t)$ for each set of draws. These improved optimal instruments are only really feasible because of the advances in Section 3.6, which drastically reduce the amount of time it takes to solve for equilibria. For relatively large problems, constructing optimal instruments may take several minutes. For smaller problems such as Berry et al. (1995) or Nevo (2000b) it takes only several seconds. Our simulations indicate that `approximate` performs as well as the more expensive `asymptotic` or `empirical` formulations. Updating results with optimal instruments in `pyblp` requires only the two lines of code in Figure 1.

Demand Side Only

Most empirical applications of the BLP approach do not include a supply side, but estimate demand alone. This has some advantages and some disadvantages. One important implication is that we lose the *cross-equation* overidentifying restrictions. The absence of the supply side should make it

harder to pin down $\widetilde{\theta}_2$ parameters in particular. However, we still retain the $K_3 - K_x$ *exclusion restrictions* for demand from the cost-shifters w_{jt}. Absent these cost-shifters and without additional restrictions, the endogenous price parameter is not identified. As we show in our simulation results, a correctly specified supply side and the optimal instruments *together* appear to be quite powerful in pinning down parameters, even when other instruments appear weak.

Absent the supply side, we no longer have a model for marginal costs and cannot solve for equilibrium $(\hat{p}_t, \hat{s}_t)$. Instead, the user can supply a vector of expected prices $E[p_t \mid z_t^D]$ or allow `pyblp` to construct the vector with a first-stage regression similar to Reynaert and Verboven (2014). The value of the (approximate) optimal IV now depends on how well price is explained by the exogenous instruments (x_{jt}, w_{jt}).

Remark 4
In our Monte Carlo exercise, we find a substantial additional benefit when incorporating both a supply-side as well as the approximation to the optimal IV. This arises from the difference between the nonlinear form of $E[p_t|z_t^D]$ under the approximation to the optimal IV, and the linear projection of p_t onto z_t^D.

The main advantage of omitting the supply side is that including a mis-specified supply side may be worse than no supply side at all. The most controversial aspect of the supply side is often the *conduct assumption* used to recover the markup $\eta_{jt}(\theta_2)$, which may not be known to the research prior to estimation. The good news is that testing the validity of the supply side moments amounts to a test of over-identifying restrictions. The simplest test involves estimating the full model with supply and demand to obtain $\hat{\theta}$, re-estimating the model using only demand moments to obtain $q^D(\hat{\theta}^D)$, and then comparing GMM objectives in a Hausman manner (see Newey, 1985):

$$LR = Nq(\hat{\theta}) - Nq^D(\hat{\theta}^D) \sim \chi^2_{K-K_x}. \tag{31}$$

There are of course alternatives based on the LM (Score) and Wald tests.

5. Monte Carlo Experiments

Here we provide some Monte Carlo experiments to illustrate some of the best practices laid out in Section 3 and Section 4.[45]

5.1. Monte Carlo Configuration

Our simulation configurations are loosely based on those of Armstrong (2016). The most important is that we solve for equilibrium prices and shares. For each configuration, we construct and solve

[45] We solve our simulations with an Intel Xeon E5-2697 v2 processor operating at 2.70 GHz and DDR3 DIMM operating at 1,866 MHz.

1,000 different synthetic datasets. There are $F = 5$ firms, and each produces a number of products chosen randomly from $J_f \in \{2, 5, 10\}$. There are $T = 20$ markets, and in each market, the number of firms is chosen randomly from $F_t \in \{F-2, F-1, F\} = \{3, 4, 5\}$. This procedure generates variation in the number of firms and products across markets which can be helpful for identification. Sample sizes are generally within $N \in [200, 600]$.

We draw the structural error terms (ξ_{jt}, ω_{jt}) from a mean-zero bivariate normal distribution with $\sigma_\xi^2 = \sigma_\omega^2 = 0.2$ and $\sigma_{\xi\omega} = 0.1$. Linear demand characteristics are $X_1 = [1, x, p]$ and supply characteristics are $X_3 = [1, x, w]$. The one nonlinear characteristic is $X_2 = x$ and heterogeneity is parameterized by $\mu_{ijt} = \sigma_x x_{jt} \nu_i$ where we draw ν_i from the standard normal distribution for 1,000 different individuals in each market. We draw the two exogenous characteristics (x_{jt}, w_{jt}) from the standard uniform distribution and compute the endogenous (p_{jt}, s_{jt}) with the ζ-markup approach of Section 3.6.

Demand-side parameters are $[\beta_0, \beta_1, \alpha]' = [-7, 6, -1]'$ and $\sigma_x = 3$. Other linear parameters were chosen to generate realistic outside shares generally within $s_{0t} \in [0.8, 0.9]$. Supply-side parameters $[\gamma_0, \gamma_1, \gamma_2]' = [2, 1, 0.2]'$ enter into a linear functional form for marginal costs: $c = X_3\gamma + \omega$.

In our different Monte Carlo experiments we modify this baseline problem in a number of ways. In most experiments, we consider three variants:

(a) **Simple** is the baseline problem described above.

(b) **Complex** adds a random coefficient on price: $X_2 = [x, p]$ and $\sigma_p = 0.2$.

(c) **RCNL** adds a nesting parameter $\rho = 0.5$. Each of the J_f products produced by a firm is randomly assigned to one of two nesting groups.

Two broad classes of models are estimated: demand-only models, which are estimated with single equation GMM, and models that also include the supply side, which are estimated with multiple equation GMM.

Instruments are constructed in two stages. The initial instruments consist of all exogenous regressors and BLP instruments (characteristics of other products). We begin with the classic "sums of characteristics" version, and then move to improved versions. For each product j we sum the product characteristics for other products controlled by the same firm f and those controlled by competing firms within the same market t:

$$Z_{jt}^{D,BLP} = Z_{jt}^{S,BLP} = \Big\{1, x_{jt}, w_{jt}, \sum_{k \in J_{ft} \setminus \{j\}} 1, \sum_{k \notin J_{ft}} 1, \sum_{k \in J_{ft} \setminus \{j\}} x_{kt}, \sum_{k \notin J_{ft}} x_{kt}\Big\}. \tag{32}$$

For some specifications we include the *differentiation IV* of Gandhi and Houde (2017). These are a variant on the BLP instruments as they represent a different function of own and rival product characteristics. Pooling products across all markets, for each pair of products (j, k), we construct

the difference $d_{jk} = x_k - x_j$ of the exogenous regressor and then form two types of differentiation IV, which are

$$\begin{aligned} Z_{jt}^{D,Local} = Z_{jt}^{S,Local} &= \Big\{1, x_{jt}, w_{jt}, \sum_{k \in J_{ft} \setminus \{j\}} 1(|d_{jk}| < \sigma_d), \sum_{k \notin J_{ft}} 1(|d_{jk}| < \sigma_d)\Big\}, \\ Z_{jt}^{D,Quad} = Z_{jt}^{S,Quad} &= \Big\{1, x_{jt}, w_{jt}, \sum_{k \in J_{ft} \setminus \{j\}} d_{jk}^2, \sum_{k \notin J_{ft}} d_{jk}^2\Big\}. \end{aligned} \tag{33}$$

The *differentiation IV* come in two flavors: *Local* and *Quadratic*. In both cases, the differentiation IV are constructed by computing the distance in characteristic space between products j and k. The Local measure counts the number of products within a standard deviation of product j, while the Quadratic measure sums up the aggregate distance between j and other products.

Following Gandhi and Houde (2017), for the Complex simulation where there is a random coefficient on price, we construct an additional instrument using fitted values from a regression of endogenous prices onto all exogenous variables (including the constructed instruments above). Further following the suggestions of Berry (1994) or Gandhi and Houde (2017) for discrete variables, for the RCNL simulation, we also include counts of products within the same nest.

Our approximation to the optimal instruments are constructed in three different variants following Section 4. When a supply side is included, the vector of expected prices is computed with the ζ-markup approach of Section 3.6. Absent a supply side, $E[p_{jt} \mid z_t]$ is estimated from a regression of endogenous prices onto all exogenous variables (including constructed instruments).

To numerically integrate choice probabilities, we use Gauss-Hermite product rules that exactly integrate polynomials of degree 17 or less.[46] In some specifications, we compare quadrature with pseudo-Monte Carlo integration and draws from the Halton sequence.

To solve the standard fixed point for $\delta_{\cdot t}$ in each market, we use the `SQUAREM` acceleration method of Varadhan and Roland (2008) with L^∞ tolerance of 10^{-14} and limit the number of contraction evaluations to 1,000. When evaluating the multinomial logit function, we use the log-sum-exp function of Section 3.2 to improve numerical stability. During the first GMM step, `SQUAREM` starts at the solution to the simple logit (or nested logit) model; in the second step, it starts at the estimated first-stage $\hat{\delta}_{\cdot t}$.

To optimize, we supply objective values and analytic gradients to a bounded limited-memory BFGS (`L-BFGS-B`) routine, which is made available by the open-source SciPy library. We use an L^∞ gradient-based tolerance of 10^{-4} and limit the number of major iterations to 1,000.[47] Drawing

[46] In the Simple specification when there is only one nonlinear product characteristic, the product rule is one-dimensional, with $(17+1)/2 = 9$ nodes. In the Complex specification, there are 9^2 nodes. If our simulations were of higher dimension, it would be more efficient to use sparse grid integration. We chose to use product rules because for these small simulation problems the gains from sparse grids are minimal. Sparse grids also have negative weights, which can introduce some estimation problems.

[47] Because of differences in reporting absolute and relative gradient values these appear to be looser than tolerances used in MATLAB solvers reported in the literature. They are not.

different starting values from a uniform distribution with support 50% above and below the true parameter values, we solve each simulation three times and keep the solution with the smallest objective value. For each simulation, we use box constraints 1,000% above and below the true values with the following exceptions: $\sigma_x > 0$, $\sigma_p > 0$, $\rho \in [0, 0.95]$, and when a supply side is estimated, $\alpha < -0.001$.[48]

5.2. Monte Carlo Results

When reporting our Monte Carlo results, we focus on the median bias and median absolute error of the parameter estimates. In Appendices D to F we provide additional results measuring the performance of various counterfactual predictions, such as elasticities, merger effects, and welfare estimates.

Fixed Effects

In Table 2 we compare computational time and memory usage for Monte Carlo experiments where we either absorb fixed effects or include them as dummy variables. As one might expect, absorbing fixed effects dramatically reduces memory requirements (by multiple orders of magnitude) and can speed up computation by as much as five times. As expected, the largest improvements for the two-dimensional case are when one dimension is much larger than the other. Even absorbing relatively small numbers of fixed effects (125 in a single dimension) leads to a substantial reduction in memory usage and computational time. This is a likely advantage of `pyblp` going forward.

Fixed Point Iteration Algorithms

To highlight the effects of different iteration schemes from Section 3.2, we explore several methods of solving the system of share equations for $\delta_{\cdot t}(\theta_2)$. We compare the conventional fixed-point iteration scheme proposed by Berry et al. (1995) to two alternative methods of solving the system of nonlinear equations without the Jacobian: `DF-SANE` and `SQUAREM`. We also compare the Jacobian based methods: Powell's method (implemented in `MINPACK` as `HYBRJ`), and Levenberg-Marquardt (LM, implemented in `MINPACK` as `LMDER`).[49]

We focus mainly on computational burden and report the results in Table 3. For the Simple and Complex simulations we vary the coefficient on the constant term β_0. A smaller value of β_0 leads to a larger share for the outside good. As shown by Dubé et al. (2012), a smaller outside good share implies a larger value for the contraction's Lipschitz constant, which generally increases the number of required iterations. Several patterns emerge. As we shrink the outside good share from $s_0 = 0.91 \to 0.26$ (and increase the Lipschitz constant) the number of required iterations increases approximately by a factor of 5 times for the Simple and Complex simulations. As expected, the

[48] When the optimization software considers $\alpha = 0$, the matrix of demand derivatives Ω becomes singular.

[49] We do not report results for other SciPy root-finding methods such as *Broyden's Method* or *Anderson Acceleration* because we found them too slow and unreliable to be worth considering.

Jacobian-based routines are unaffected by variation of the Lipschitz constant.[50] With the `SQUAREM` iteration scheme the number of iterations increases, but by only 90%.

In general, we find Powell's method reliable, but slightly slower than LM at lower tolerances, and it struggled with a tolerance of 10^{-14}. The `DF-SANE` algorithm performs substantially better than standard fixed point iteration, but less well than our two preferred algorithms: `SQUAREM` (accelerated fixed point iteration) and the Jacobian-based LM.

In terms of other speedups, the effects of the `SQUAREM` iteration scheme are substantial. It reduces the number of iterations between 3-8 times without substantially increasing the cost per iteration. This is because it approximates the Newton step without actually computing the costly Jacobian. It performs well across all settings, but does particularly well in the Berry et al. (1995, 1999) example which includes a supply side.

The LM algorithm reduces the number of iterations, but at a higher cost per iteration. On some of the more difficult problems (those with a small outside good share) it performs as much as 10 times faster than fixed point iteration, and at other times roughly as fast as `SQUAREM`. The speedup is particularly large for the RCNL model, where the contraction is dampened by $(1 - \rho)$. When $\rho \to 1$ the contraction becomes arbitrarily slow. In our experiments, LM performs somewhat better than the quasi-Newton routines in Reynaerts et al. (2012) were reported to perform.[51]

Fixed Point Iteration Tricks

We also consider some commonly employed tricks in the literature to speed up the contraction mapping and report our results in Table 4. In all cases, we employ the `SQUAREM` algorithm. We consider two commonly-employed "tweaks" from the literature: working with $\exp \delta_{jt}$ rather than δ_{jt} to avoid taking logs and eliminating a division in the share computation, and using a "hot-start" where the starting value for the iterative procedure is the value of δ_t^{n-1} which solved the system of equations for the previous guess of θ_2. In addition, we consider the potential cost of our overflow safe modification to the log-sum-exp function.

The results in Table 4 are largely underwhelming. Once we adopt `SQUAREM` acceleration, additional tricks to speed up the problem seem to have little benefit. The "hot-start" approach was able to reduce the number of iterations between by between 10-20% which is possibly worth considering.[52] One clear recommendation is the log-sum-exp trick from Section 3.3, which reduces the chance of overflow problems. This seems relatively low-cost and reduces the possibility that the iteration routine fails to find a solution to the system of equations.

[50] This is consistent with the findings of Dubé et al. (2012) using the MPEC method.

[51] One possible explanation is that Reynaerts et al. (2012) employ a standard *Newton-Raphson* solver, whereas `MINPACK-LMDER` is designed to be more robust to poor starting values.

[52] This introduces a potential numerical problem where the GMM objective $q(\theta^h)$ need not evaluate to precisely the same quantity depending on θ^{h-1}, although for sufficient fixed point tolerances this should not be an issue. In practice, we still recommend a tight tolerance of 10^{-14}.

Numerical Integration

Given extensive past work by Heiss and Winschel (2008), Judd and Skrainka (2011), Skrainka (2012b), and Freyberger (2015), it is not surprising that the choice of integration method has a striking effect on the small sample performance of the estimator. We report the results of our experiments in Table 5. We compare a Gauss-Hermite product rule of degree 17 which has $I_t = 9$ nodes in one dimension and $I_t = 9^2 = 81$ nodes two dimensions with Monte Carlo rules of $I_t = 100$ (roughly the same number of points) and $I_t = 1000$ (roughly the same level of accuracy). With similar numbers of nodes, we see as much as an order of magnitude reduction in bias for the σ parameters and reductions in MAE by a factor of 3-5 times particularly for the random coefficient terms (as one might expect). We see that for the same number of draws, Halton sequences perform better than Monte Carlo integration. When compared to the product rule, ten times as many Monte Carlo or Halton draws provide similar accuracy, but take 3-5 times as long to estimate.

Choice of Instruments

We also consider several different choices of instruments as well as models which include both supply and demand moments or demand moments only. We present a slightly different construction of the Chamberlain (1987) optimal instruments for the BLP problem in Section 4 than those proposed in Reynaert and Verboven (2014).

In Table 6, we present simulation results using the sums of product charateristics version of the BLP instruments from equation (32), both the Local and Quadratic forms of the Gandhi and Houde (2017) differentiation IV from equation (33), and the `approximate` version of the feasible optimal instruments from equation (30).

We find that in most settings the feasible optimal instruments perform best, which is consistent with the findings of Reynaert and Verboven (2014). We also find that the differentiation IV substantially outperform the original BLP instruments, as Gandhi and Houde (2017) suggest.

We also find that the feasible optimal instruments perform much better when the supply side is included than with the demand side alone, while jointly estimating the supply side seems to have a limited impact under the other forms of instruments. In fact, with both optimal instruments and a supply side, the bias is all but eliminated in most of our Monte Carlo experiments, while the MAE (particularly for the random coefficients) is substantially reduced. This does not match Reynaert and Verboven (2014) who find that including the supply side has little effect once feasible optimal instruments are used.[53] We highlight the main theoretical difference in Section 4.[54] The gains of included a correctly specified supply side are also valuable in estimating nonlinear functions of model parameters such as average elasticities and counterfactual simulations such as price effects

[53] A likely important distinction is that their simulations all appear to include "strong cost shifters" in which case all estimators perform quite well, whereas some of our examples include "weak cost shifters".

[54] The punchline is that $E[p_t|z_t]$ can be approximated with a linear projection onto instruments, or the nonlinear estimate from solving for (p^*, s^*) in equilibrium. The addition of the supply side to the optimal IV problem facilitates the latter.

of mergers. We provide additional details in Appendix D and E.

We compute the feasible optimal instruments using the three different techniques from Section 4: the approximate approach employed by Berry et al. (1999) where $(\xi^*, \omega^*) = (0, 0)$ (their unconditional expectation), an asymptotic approach where 100 draws are taken from the estimated normal distribution for $(\xi_{jt}, \omega_{jt}) \sim N(0, \hat{\Omega})$, and an empirical approach where samples are taken with replacement from the joint distribution of $(\hat{\xi}_{jt}, \hat{\omega}_{jt})$. We present results in Table 7. Reassuring for the prior literature, the approximate version of optimal instruments appears to provide nearly all of the benefits of the other versions at much lower cost.

We also show the value of both the supply side restrictions and the optimal instruments is largest when the exogenous cost shifter w_{jt} is weakest. We report results in Table 8 where we reduce the magnitude of the coefficient γ_2, which governs how responsive marginal costs are to the exogenous cost shifter. As we reduce this coefficient it reduces the correlation between p and w. When the cost-shifting instrument is very weak ($\text{Corr}(p, w) \approx 0.05$), this increases both the bias and the variance of the price parameter α consistent with Armstrong (2016). However, if we include the (correctly specified) supply restrictions (with optimal instruments) we are effectively able to eliminate the bias and substantially reduce the variance of the estimates. This finding appears to be novel as Reynaert and Verboven (2014) do not find substantial benefits of including supply side restrictions once optimal instruments are employed.[55]

Consistent with Gandhi and Houde (2017) our recommendation is to start with differentiation IV in the first stage and then compute feasible optimal instruments in the second stage. The substantial small sample benefits of including optimal instruments suggest that they should be employed more widely, particularly when there are multiple random coefficients.

Problem Size

We might also be interested in how the BLP problem scales as we vary it size. We report our results in Table 9. We find that computational time is roughly linear in the number of markets T, while it is grows at a rate closer to $\sqrt{J}$ as we increase the number of products when we use only the demand side. When we include both supply and demand the computational time appears to grow more quickly than J.

Our Monte Carlo exercises are fairly simple, but when using only demand-side restrictions, it appears as if $T = 40$ markets is roughly enough to obtain "reasonable" parameter estimates in terms of bias and efficiency, though our simulation benefit from variation in the number of products per market. These results are somewhat in contrast to those in Armstrong (2016) who finds that when T is small as J becomes large, the instruments become weak and the estimator performs poorly. We find that when we increase the size of J the estimator performs better rather than worse. We

[55] This may have to do with the fact that they construct optimal instruments differently for the simultaneous supply and demand problem so that the model is just identified, whereas our approach constructs them to be overidentified. An important caveat is that simultaneous supply and demand is not the main focus of their work.

attribute this to three main differences: (1) we allow for variation in the number of products per firm across markets; (2) we use the feasible approximation to the optimal instruments and (3) in some specifications we include the additional supply moments.

Optimization Algorithms

We consider a broad array of different optimization algorithms and report our results regarding convergence in Table 10. We report our results for parameter estimates from different algorithms in Appendix Table F.13.

Our preferred algorithms are Knitro's `Interior/Direct` algorithm and `BFGS`-based algorithms in SciPy as they provide the best speed and reliability. We should caution that our simulations are simple enough to be run thousands of times, so it may not be surprising that most optimization software packages appear to work well. It may also be the case that various numerical fixes and improvements to the fixed point iteration problem may have resolved some of the issues with optimization.

Our findings are somewhat different from those in Knittel and Metaxoglou (2014). Whereas those authors found that many different optimization algorithms found a variety of local minima and often failed to converge to a valid minimum, we obtain essentially the opposite result. Using a broad array of optimization routines (3 Knitro algorithms and 6 algorithms implemented in SciPy) we find that more than 99% of all simulation runs converge to a local minimum, which we define as having an L^∞ norm of the gradient sufficiently close to zero and a positive semi-definite Hessian matrix (all eigenvalues weakly positive). Even the non-derivative based Nelder-Mead algorithm, which we do not recommend, appears to do reasonably well at finding an optimum. Direct comparisons with Knittel and Metaxoglou (2014) are difficult and may be a result of our data generating process, and the specific optimization algorithms used (we use primarily derivative based Quasi-Newton solvers, they use a variety of deterministic and stochastic optimizers some of which use derivatives and some of which do not). We provide a more direct comparison in the next section.

6. Replication Exercises

Here we provide replications using `pyblp` for the two best-known BLP applications.

Nevo (2000b)

In our first replication we estimate the model of Nevo (2000b) on the publicly available "fake data." This problem is notable because it includes a combination of observable demographics, unobserved heterogeneity, and product fixed effects. We demonstrate how to construct the problem with `pyblp` in Figure 2.

We estimate the model three times: once following the original Nevo (2000b) example,[56] a

[56] Replication is straightforward because the original instruments are provided along with the data, and the reported estimates are from one-step GMM with the two-stage least squares (2SLS) weighting matrix $(Z'Z)^{-1}$. The data we use are distributed with `pyblp`.

second time where we drop the interaction between the square of income and price, and a third time using the demand side only feasible optimal instruments described above. We report our results in Table 11.[57]

The "Restricted" results have smaller standard errors and lack multicollinearity between income and its square. The "Optimal Instruments" further improve on "Restricted" by reducing standard errors by around 50% on average. We should also mention that because the objective function is nearly zero, a test of overidentifying restrictions fails to reject the model under optimal instruments.

Berry et al. (1995, 1999)

For our second replication, we consider the problem in Berry et al. (1995), which lacks demographic interactions and product fixed effects but adds a supply side and allows the price coefficient to vary with income. We provide the `pyblp` formulation of the problem in Figure 3.

Our configuration for the Berry et al. (1995) problem differs more substantially from the original paper because parts of the original configuration are not included with the data.[58] We estimate the model once with the original paper's BLP instruments, a second time after transforming these instruments with principal component analysis,[59] and a third time with feasible optimal instruments. We report our results in Table 12.

With the exception of price, which we construct with the first-order approximation of Berry et al. (1999), we obtain broadly similar parameter estimates. The optimal instruments suggest somewhat less elastic demand and larger markups than the other instruments. Because the precise form of the instruments and weighting matrix are not included with the data itself, it is difficult to fully replicate the original estimates.

Knittel and Metaxoglou (2014)

Using these configurations, we conduct a more extensive replication exercise meant to mimic Knittel and Metaxoglou (2014) in Appendix C where we employ multiple optimization algorithms and 100 random starting values. We find almost no dispersion in the recovered parameter estimates, objective values, and implied elasticities across optimizers and starting values.

[57] It is well known that the original reported estimates in the code accompanying Nevo (2000b) set the tolerance of the contraction mapping too loose. This explains the discrepancy between our "Replication" results and those originally reported. However, with a GMM objective value of $q(\hat{\theta}) = 4.56$, our replication results are identical to those reported by Dubé et al. (2012) using the MPEC approach.

[58] The data we use are distributed with the `pyblp` package and were obtained from the replication package for Andrews et al. (2017). Following this earlier replication, we start the optimization routine at the published estimates. Also following Berry et al. (1999), we replace the original specification's $\log(y_i - p_j)$ term with its first-order linear approximation p_j/y_i. Otherwise there are individuals for whom $p_j > y_i$ which creates a host of problems. We include the inverse of income, $1/y_i$, as a demographic variable, and configure Π to interact the demographic with prices. Below, we refer to the parameter on this interaction as α and choose an initial value of $\alpha = -10$. Using this approximation means that our price parameter is no longer directly comparable with that of the original paper. Instead of importance sampling, we use 200 pseudo-Monte Carlo draws in each market.

[59] The original instruments are not provided, but $\sum_{j \in J_t} x_{jt}$ are nearly collinear. We follow Conlon (2017) by interacting these BLP instruments up to the second degree and projecting them down to the smallest number of principal components that explain at least 99% of the interactions' variance.

We find that while different sets of instruments result in different estimates of parameters and elasticities, different optimization software and starting values in general do not. In other words, we fail to find the same difficulties obtaining parameter estimates that Knittel and Metaxoglou (2014) do.[60] To demonstrate these results graphically, in Figure 4 we present histograms of mean own-price elasticities obtained from our 9×100 optimizers and starting values alongside the corresponding figures from Knittel and Metaxoglou (2014).

7. Conclusion

Our goal has to be to review recent methodological developments related to the BLP problem, and collect and evaluate them not only in a single paper, but also in a single software package `pyblp`. We have provided a list of best practices with numerical evidence to support our recommendations, and we have implemented them as defaults in `pyblp`. Our hope is that these practices can now be made available to more researchers, and provide a common platform which should facilitate replication.

In addition, we present some methodological results which we believe to be novel. We show how with a slight reformulation of the nested fixed point problem, it is possible to include high dimensional fixed effects in models with simultaneous supply and demand. We also provide a somewhat different expression for optimal instruments than the prior literature (Reynaert and Verboven, 2014) which makes clear the over-identifying restrictions implied by the supply side. Also novel, we find that optimal instruments when combined with a correctly specified supply side are extremely valuable. Consistent with prior work, we find the gains to optimal instruments to be substantial such that they should nearly always be employed. Thankfully, we have made this process extremely straightforward in `pyblp`.

Somewhat reassuringly, we find that under our best practices (including correctly specified supply restrictions and approximations to the optimal instruments) the finite sample of the BLP estimator appears to be pretty good and perhaps better than previously believed.

[60] An important distinction is that our implementation of the BLP problem uses both supply and demand restrictions as in the original Berry et al. (1995) and Berry et al. (1999) papers, whereas Knittel and Metaxoglou (2014) uses only a demand side.

Figure 1: Optimal Instruments Code

```
instrument_results = results.compute_optimal_instruments(method='empirical')
updated_problem = instrument_results.to_problem()
```

This code demonstrates how to use optimal instruments in `pyblp`. Solving a problem with non-optimal instruments gives a `pyblp.ProblemResults` object, which is used to estimate the optimal instruments (here, the `empirical` technique is employed). This gives a `pyblp.OptimalInstrumentResults` object that is converted to another `pyblp.Problem`, which can be solved like any other.

Figure 2: Nevo (2000b) Problem Formulation and Code

```
nevo_problem = pyblp.Problem(
    product_formulations=(
        pyblp.Formulation('0 + prices', absorb='C(product_ids)'),                    # Linear demand
        pyblp.Formulation('1 + prices + sugar + mushy')                              # Nonlinear demand
    ),
    agent_formulation=pyblp.Formulation('0 + income + income_squared + age + child'),  # Demographics
    product_data=pandas.read_csv(pyblp.data.NEVO_PRODUCTS_LOCATION),
    agent_data=pandas.read_csv(pyblp.data.NEVO_AGENTS_LOCATION)
)
print(nevo_problem)

Dimensions:
=======================================
 N     T    K1    K2    D    MD    ED
----  ---  ----  ----  ---  ----  ----
2256  94    1     4     4    20    1
=======================================

Formulations:
=====================================================================
       Column Indices:           0           1           2      3
----------------------------  ------  --------------  -----  -----
 X1: Linear Characteristics    prices
X2: Nonlinear Characteristics    1         prices      sugar  mushy
       d: Demographics         income  income_squared   age   child
=====================================================================
```

This code demonstrates how to construct the problem from Nevo (2000b) in `pyblp`. For convenience, the data comes packaged in `pyblp` and can be accessed via the file paths `pyblp.data.NEVO_PRODUCTS_LOCATION` and `pyblp.data.NEVO_AGENTS_LOCATION`. Names in the product and agent formulations correspond to variable names in the datasets, which are loaded into memory with Python package `pandas`. Printing the problem object displays useful information about the configured problem. When solved, the problem gives the results reported under the "Replication" header in Table 11.

Figure 3: BLP (1995/1999) Problem Formulation and Code

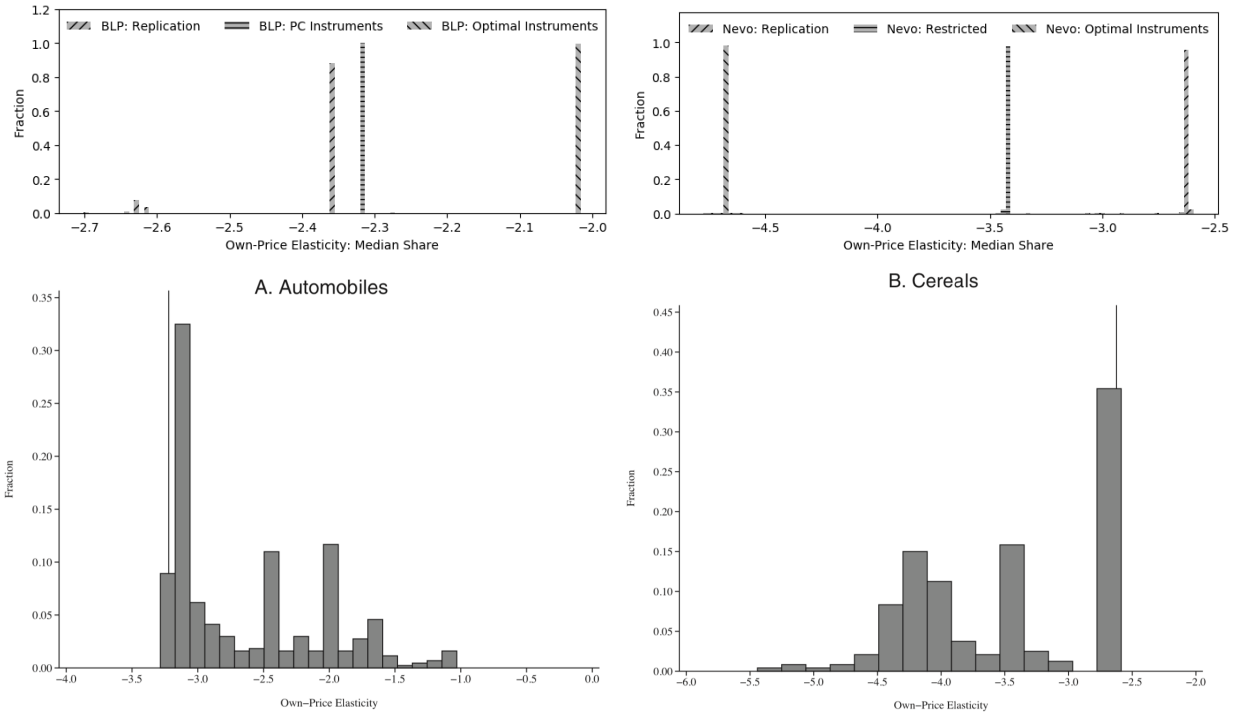
```
blp_problem = pyblp.Problem(
    product_formulations=(
        pyblp.Formulation('1 + hpwt + air + mpd + space'),                      # Linear demand
        pyblp.Formulation('1 + prices + hpwt + air + mpd + space'),             # Nonlinear demand
        pyblp.Formulation('1 + log(hpwt) + air + log(mpg) + log(space) + trend')  # Supply
    ),
    agent_formulation=pyblp.Formulation('0 + I(1 / income)'),                   # Demographics
    product_data=pandas.read_csv(pyblp.data.BLP_PRODUCTS_LOCATION),
    agent_data=pandas.read_csv(pyblp.data.BLP_AGENTS_LOCATION)
)
print(blp_problem)

Dimensions:
=============================================
 N     T    K1    K2    K3    D    MD    MS
----  ---  ----  ----  ----  ---  ----  ----
2217  20    5     6     6     1    11    12
=============================================

Formulations:
================================================================================
       Column Indices:            0          1        2       3          4         5
----------------------------  --------  ---------  -----  --------  ----------  -----
 X1: Linear Characteristics       1        hpwt      air     mpd        space
X2: Nonlinear Characteristics     1       prices    hpwt     air         mpd      space
  X3: Cost Characteristics        1      log(hpwt)   air   log(mpg)  log(space)  trend
       d: Demographics         1/income
================================================================================
```

This code demonstrates how to construct a modified version of the problem from Berry et al. (1995) in `pyblp`. For convenience, the data comes packaged in `pyblp` and can be accessed via the file paths `pyblp.data.BLP_PRODUCTS_LOCATION` and `pyblp.data.BLP_AGENTS_LOCATION`. Names in the product and agent formulations correspond to variable names in the datasets, which are loaded into memory with Python package `pandas`. Printing the problem object displays useful information about the configured problem. When solved, the problem gives the results reported under the "Optimal Instruments" header in Table 12.

Figure 4: Comparison of Histograms for Median Own-Price Elasticities



All figures report the own price elasticitiy for the median product. The top two figures are from `pyblp` where each observation is one trial from 100 starting values and nine optimization algorithms. That exercise is repeated for three different sets of instruments. The bottom two figures are taken from Knittel and Metaxoglou (2014) with a single set of instruments and are reproduced as fair use. For additional details, please consult Appendix C.

Table 2: Fixed Effect Absorption

Problem	FE Dimensions	FE Levels	Absorbed	Seconds	Megabytes
Simple Simulation	1	125	No	20.22	92.55
Simple Simulation	1	125	Yes	13.86	0.70
Simple Simulation	2	125×125	No	27.20	241.75
Simple Simulation	2	125×125	Yes	14.46	1.05
Simple Simulation	2	25×625	No	51.43	642.43
Simple Simulation	2	25×625	Yes	14.42	1.04
Simple Simulation	3	$25 \times 25 \times 25$	No	20.95	37.33
Simple Simulation	3	$25 \times 25 \times 25$	Yes	16.54	1.29
Nevo Example	1	24	No	21.67	0.02
Nevo Example	1	24	Yes	20.46	0.04

This table documents the impact of fixed effect (FE) absorption on estimation speed and memory usage during one GMM step. Reported values are medians across 100 different simulations and 10 identical runs of the Nevo (2000b) example problem. When not absorbed, FEs are included as dummy variables. A single FE is absorbed with de-meaning; two, with the Somaini and Wolak (2016) algorithm; and three, with iterative de-meaning using an infinity norm tolerance of 10^{-14}. To accommodate FEs in the Simple simulation, we first set $T = 625$ and $F_t = J_f = 5$ so that there are $N = 25^3$ products. We then randomly assign $n \in \{1, \ldots, N\}$ to each product and add FEs. The 1-D case has $\beta_{n \bmod 125}$. For the "square" 2-D case, we add $\beta_{n \operatorname{div} 125}$. The "uneven" 2-D case has $\beta_{n \bmod 25}$ and $\beta_{n \operatorname{div} 25}$. The 3-D case has $\beta_{n \bmod 25}$, $\beta_{(n \operatorname{div} 25) \bmod 25}$, and $\beta_{n \operatorname{div} 125}$.

Table 3: Fixed Point Algorithms

Problem	Median s_0	Algorithm	Jacobian	Termination	Mean Milliseconds	Mean Evaluations	Percent Converged
Simple Simulation ($\beta_0 = -7$)	0.91	Iteration	No	Absolute L^∞	5.61	46.72	100.00%
Simple Simulation ($\beta_0 = -7$)	0.91	DF-SANE	No	Absolute L^∞	3.15	17.54	100.00%
Simple Simulation ($\beta_0 = -7$)	0.91	SQUAREM	No	Absolute L^∞	2.38	17.30	100.00%
Simple Simulation ($\beta_0 = -7$)	0.91	SQUAREM	No	Relative L^2	2.49	16.45	100.00%
Simple Simulation ($\beta_0 = -7$)	0.91	Powell	Yes	Relative L^2	3.28	16.33	33.55%
Simple Simulation ($\beta_0 = -7$)	0.91	LM	Yes	Relative L^2	2.32	9.04	100.00%
Simple Simulation ($\beta_0 = -1$)	0.26	Iteration	No	Absolute L^∞	27.10	230.95	100.00%
Simple Simulation ($\beta_0 = -1$)	0.26	DF-SANE	No	Absolute L^∞	6.68	38.16	100.00%
Simple Simulation ($\beta_0 = -1$)	0.26	SQUAREM	No	Absolute L^∞	4.60	34.36	100.00%
Simple Simulation ($\beta_0 = -1$)	0.26	SQUAREM	No	Relative L^2	4.96	33.68	100.00%
Simple Simulation ($\beta_0 = -1$)	0.26	Powell	Yes	Relative L^2	3.45	17.42	12.63%
Simple Simulation ($\beta_0 = -1$)	0.26	LM	Yes	Relative L^2	2.31	8.94	100.00%
Complex Simulation ($\beta_0 = -7$)	0.91	Iteration	No	Absolute L^∞	8.48	47.78	100.00%
Complex Simulation ($\beta_0 = -7$)	0.91	DF-SANE	No	Absolute L^∞	4.35	17.93	100.00%
Complex Simulation ($\beta_0 = -7$)	0.91	SQUAREM	No	Absolute L^∞	3.26	16.72	100.00%
Complex Simulation ($\beta_0 = -7$)	0.91	SQUAREM	No	Relative L^2	3.34	15.98	100.00%

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Problem	Median s_0	Algorithm	Jacobian	Termination	Mean Milliseconds	Mean Evaluations	Percent Converged
Complex Simulation ($\beta_0 = -7$)	0.91	Powell	Yes	Relative L^2	4.07	15.23	32.72%
Complex Simulation ($\beta_0 = -7$)	0.91	LM	Yes	Relative L^2	2.91	8.99	100.00%
Complex Simulation ($\beta_0 = -1$)	0.27	Iteration	No	Absolute L^∞	39.26	228.39	100.00%
Complex Simulation ($\beta_0 = -1$)	0.27	DF-SANE	No	Absolute L^∞	9.05	38.29	100.00%
Complex Simulation ($\beta_0 = -1$)	0.27	SQUAREM	No	Absolute L^∞	6.63	34.28	100.00%
Complex Simulation ($\beta_0 = -1$)	0.27	SQUAREM	No	Relative L^2	6.92	33.63	100.00%
Complex Simulation ($\beta_0 = -1$)	0.27	Powell	Yes	Relative L^2	4.65	17.58	11.00%
Complex Simulation ($\beta_0 = -1$)	0.27	LM	Yes	Relative L^2	2.90	8.95	100.00%
RCNL Simulation ($\rho = 0.5$)	0.91	Iteration	No	Absolute L^∞	31.87	146.57	94.50%
RCNL Simulation ($\rho = 0.5$)	0.91	DF-SANE	No	Absolute L^∞	11.11	39.19	99.25%
RCNL Simulation ($\rho = 0.5$)	0.91	SQUAREM	No	Absolute L^∞	9.77	41.44	100.00%
RCNL Simulation ($\rho = 0.5$)	0.91	SQUAREM	No	Relative L^2	9.76	38.48	100.00%
RCNL Simulation ($\rho = 0.5$)	0.91	Powell	Yes	Relative L^2	5.32	16.33	58.12%
RCNL Simulation ($\rho = 0.5$)	0.91	LM	Yes	Relative L^2	3.82	10.01	100.00%
RCNL Simulation ($\rho = 0.8$)	0.91	Iteration	No	Absolute L^∞	68.73	314.48	93.58%
RCNL Simulation ($\rho = 0.8$)	0.91	DF-SANE	No	Absolute L^∞	17.34	63.27	99.16%
RCNL Simulation ($\rho = 0.8$)	0.91	SQUAREM	No	Absolute L^∞	15.44	65.23	100.00%
RCNL Simulation ($\rho = 0.8$)	0.91	SQUAREM	No	Relative L^2	14.97	60.24	100.00%
RCNL Simulation ($\rho = 0.8$)	0.91	Powell	Yes	Relative L^2	7.46	23.12	62.40%
RCNL Simulation ($\rho = 0.8$)	0.91	LM	Yes	Relative L^2	5.16	13.46	100.00%
Nevo Example	0.54	Iteration	No	Absolute L^∞	12.47	95.70	99.97%
Nevo Example	0.54	DF-SANE	No	Absolute L^∞	5.09	27.34	100.00%
Nevo Example	0.54	SQUAREM	No	Absolute L^∞	3.85	26.15	100.00%
Nevo Example	0.54	SQUAREM	No	Relative L^2	3.97	24.72	100.00%
Nevo Example	0.54	Powell	Yes	Relative L^2	3.76	17.50	30.32%
Nevo Example	0.54	LM	Yes	Relative L^2	2.64	9.50	100.00%
BLP Example	0.89	Iteration	No	Absolute L^∞	14.48	36.82	100.00%
BLP Example	0.89	DF-SANE	No	Absolute L^∞	8.05	17.33	100.00%
BLP Example	0.89	SQUAREM	No	Absolute L^∞	7.50	18.07	100.00%
BLP Example	0.89	SQUAREM	No	Relative L^2	7.11	16.87	100.00%
BLP Example	0.89	Powell	Yes	Relative L^2	17.48	17.13	36.88%
BLP Example	0.89	LM	Yes	Relative L^2	19.66	11.54	100.00%

This table documents the impact of algorithm choice on solving the nested fixed point. Reported values are medians across 100 different simulations and 10 identical runs of the example problems. We report the number of milliseconds and contraction evaluations needed to solve the fixed point, averaged across all markets and one GMM step's objective evaluations. We also report each algorithm's convergence rate (i.e., the percent of times no numerical errors were encountered and a limit of 1,000 iterations was not reached). We configure simple iteration with an absolute L^∞ norm tolerance of 10^{-14} and compare it with `DF-SANE` and `SQUAREM`. The `MINPACK` implementations of algorithms that do require a Jacobian, a modification of the Powell hybrid method and Levenberg-Marquardt, only support relative L^2 norm tolerances, so for comparison's sake we also include `SQUAREM` with the same termination condition. Simulations are configured as in Section 5.1, except for the coefficient on the constant term, β_0, and the nesting parameter, ρ, which we vary to document the effects of decreasing the outside share s_0 and of dampening the contraction in the RCNL model.

Table 4: Fixed Point Tricks

Problem	Median s_0	Overflow Safe	Type	Initial δ	Mean Milliseconds	Mean Evaluations	Percent Converged
Simple Simulation	0.91	Yes	δ	δ^0	2.38	17.30	100.00%
Simple Simulation	0.91	No	δ	δ^0	2.01	17.30	100.00%
Simple Simulation	0.91	Yes	$\exp(\delta)$	δ^0	2.38	17.30	100.00%
Simple Simulation	0.91	No	$\exp(\delta)$	δ^0	2.02	17.30	100.00%
Simple Simulation	0.91	Yes	δ	δ^{n-1}	1.81	15.48	100.00%
Complex Simulation	0.91	Yes	δ	δ^0	3.26	16.72	100.00%
Complex Simulation	0.91	No	δ	δ^0	2.83	16.73	100.00%
Complex Simulation	0.91	Yes	$\exp(\delta)$	δ^0	3.33	16.72	100.00%
Complex Simulation	0.91	No	$\exp(\delta)$	δ^0	2.85	16.73	100.00%
Complex Simulation	0.91	Yes	δ	δ^{n-1}	2.72	14.78	100.00%
RCNL Simulation	0.91	Yes	δ	δ^0	9.77	41.44	100.00%
RCNL Simulation	0.91	No	δ	δ^0	8.30	41.34	100.00%
RCNL Simulation	0.91	Yes	$\exp(\delta)$	δ^0	9.57	41.44	100.00%
RCNL Simulation	0.91	No	$\exp(\delta)$	δ^0	8.46	41.34	100.00%
RCNL Simulation	0.91	Yes	δ	δ^{n-1}	7.59	33.49	100.00%
Nevo Example	0.54	Yes	δ	δ^0	3.85	26.15	100.00%
Nevo Example	0.54	No	δ	δ^0	3.26	26.15	100.00%
Nevo Example	0.54	Yes	$\exp(\delta)$	δ^0	3.84	26.15	100.00%
Nevo Example	0.54	No	$\exp(\delta)$	δ^0	3.26	26.15	100.00%
Nevo Example	0.54	Yes	δ	δ^{n-1}	3.05	20.95	100.00%
BLP Example	0.89	Yes	δ	δ^0	7.50	18.07	100.00%
BLP Example	0.89	No	δ	δ^0	5.67	18.07	100.00%
BLP Example	0.89	Yes	$\exp(\delta)$	δ^0	7.47	18.07	100.00%
BLP Example	0.89	No	$\exp(\delta)$	δ^0	5.73	18.07	100.00%
BLP Example	0.89	Yes	δ	δ^{n-1}	5.42	13.77	100.00%

This table documents the impact of common tricks used in the literature on solving the nested fixed point. Reported values are medians across 100 different simulations and 10 identical runs of the two example problems. We report the number of milliseconds and contraction evaluations needed to solve the nested fixed point, averaged across all markets and objective evaluations of one GMM step. We also report convergence rates (i.e., the percent of times no numerical errors were encountered a limit of 1,000 iterations was not reached). Overflow safe results are those that use the log-sum-exp (LSE) function. The two fixed point types are the standard linear contraction (over δ) and the exponentiated version (over $\exp\delta$). An initial δ^0 means that the contraction always starts at the solution to the logit model, whereas δ^{n-1} is the "hot-start" version where starting values are those that solved the fixed point for the previous guess of θ_2. We use the `SQUAREM` algorithm with an absolute L^∞ norm tolerance of 10^{-14}. Simulations are configured as in 5.1.

Table 5: Alternative Integration Methods

					True Value				Median Bias				Median Absolute Error			
Simulation	Supply	Integration	I_t	Seconds	α	σ_x	σ_p	ρ	α	σ_x	σ_p	ρ	α	σ_x	σ_p	ρ
Simple	No	Monte Carlo	100	1.5	-1	3			0.197	-0.702			0.280	0.708		
Simple	No	Monte Carlo	1,000	3.1	-1	3			0.183	-0.103			0.249	0.203		
Simple	No	Halton	1,000	2.9	-1	3			0.176	0.001			0.246	0.172		
Simple	No	Product Rule	9^1	1.1	-1	3			0.181	-0.031			0.246	0.180		
Simple	Yes	Monte Carlo	100	5.4	-1	3			0.100	-0.671			0.229	0.671		
Simple	Yes	Monte Carlo	1,000	22.1	-1	3			0.031	-0.092			0.177	0.186		
Simple	Yes	Halton	1,000	20.9	-1	3			0.025	0.032			0.170	0.154		
Simple	Yes	Product Rule	9^1	4.0	-1	3			0.012	0.010			0.172	0.156		
Complex	No	Monte Carlo	100	2.7	-1	3	0.2		0.266	-0.838	-0.071		0.308	0.846	0.101	
Complex	No	Monte Carlo	1,000	7.3	-1	3	0.2		0.169	-0.191	-0.006		0.264	0.259	0.122	
Complex	No	Halton	1,000	7.9	-1	3	0.2		0.138	-0.069	0.033		0.243	0.200	0.180	
Complex	No	Product Rule	9^2	3.4	-1	3	0.2		0.058	-0.145	0.085		0.267	0.240	0.177	
Complex	Yes	Monte Carlo	100	9.2	-1	3	0.2		0.107	-0.751	-0.143		0.266	0.786	0.157	
Complex	Yes	Monte Carlo	1,000	31.8	-1	3	0.2		0.020	-0.105	-0.049		0.208	0.230	0.131	
Complex	Yes	Halton	1,000	32.1	-1	3	0.2		-0.043	-0.019	0.034		0.197	0.177	0.150	
Complex	Yes	Product Rule	9^2	10.3	-1	3	0.2		-0.054	-0.043	0.044		0.183	0.179	0.140	
RCNL	No	Monte Carlo	100	8.7	-1	3		0.5	0.229	-0.662		0.033	0.280	0.665		0.050
RCNL	No	Monte Carlo	1,000	35.2	-1	3		0.5	0.179	-0.134		-0.001	0.230	0.207		0.023
RCNL	No	Halton	1,000	35.2	-1	3		0.5	0.168	-0.012		-0.010	0.224	0.172		0.023
RCNL	No	Product Rule	9^1	7.3	-1	3		0.5	0.166	-0.016		-0.008	0.216	0.173		0.022
RCNL	Yes	Monte Carlo	100	28.9	-1	3		0.5	0.083	-0.569		0.036	0.126	0.571		0.043
RCNL	Yes	Monte Carlo	1,000	113.8	-1	3		0.5	0.022	-0.087		0.006	0.107	0.174		0.020
RCNL	Yes	Halton	1,000	115.4	-1	3		0.5	0.020	0.011		-0.000	0.109	0.154		0.017
RCNL	Yes	Product Rule	9^1	18.8	-1	3		0.5	0.007	0.006		0.001	0.105	0.151		0.017

This table documents bias and variance of parameter estimates along with estimation speed over 1,000 simulated datasets for different numerical integration methods and numbers of draws I_t. Monte Carlo integration uses pseudo-random draws from the standard normal distribution, Halton integration uses draws from the deterministic Halton sequence, and we use a degree 17 Gauss Hermite Product Rule. For all problems, we use optimal instruments and `L-BFGS-B`. For more details, refer to Section 5.1.

Table 6: Alternative Instruments

				True Value				Median Bias				Median Absolute Error			
Simulation	Supply	Instruments	Seconds	α	σ_x	σ_p	ρ	α	σ_x	σ_p	ρ	α	σ_x	σ_p	ρ
Simple	No	$[1, x, w]$	0.1	-1	3			0.016	0.015			0.266	0.832		
Simple	No	BLP	0.8	-1	3			0.152	-0.010			0.235	0.403		
Simple	No	Local	0.5	-1	3			0.079	0.021			0.241	0.288		
Simple	No	Quadratic	0.5	-1	3			0.090	-0.026			0.250	0.373		
Simple	No	Optimal	1.1	-1	3			0.181	-0.031			0.246	0.180		
Simple	Yes	$[1, x, w]$	0.6	-1	3			0.016	-0.040			0.269	0.779		
Simple	Yes	BLP	2.5	-1	3			0.044	-0.017			0.237	0.390		
Simple	Yes	Local	1.2	-1	3			0.055	-0.005			0.218	0.296		
Simple	Yes	Quadratic	1.3	-1	3			0.071	-0.079			0.233	0.382		
Simple	Yes	Optimal	4.0	-1	3			0.012	0.010			0.172	0.156		
Complex	No	$[1, x, w]$	0.2	-1	3	0.2		0.001	-0.018	-0.017		0.258	0.822	0.050	
Complex	No	BLP	2.8	-1	3	0.2		0.130	-0.315	-0.116		0.275	0.649	0.200	
Complex	No	Local	1.3	-1	3	0.2		0.009	-0.185	-0.099		0.365	0.364	0.200	
Complex	No	Quadratic	1.1	-1	3	0.2		0.104	0.032	-0.200		0.353	0.406	0.200	
Complex	No	Optimal	3.4	-1	3	0.2		0.058	-0.145	0.085		0.267	0.240	0.177	
Complex	Yes	$[1, x, w]$	1.2	-1	3	0.2		-0.102	-0.055	0.013		0.192	0.763	0.109	
Complex	Yes	BLP	8.9	-1	3	0.2		-0.029	-0.411	0.084		0.242	0.557	0.171	
Complex	Yes	Local	3.6	-1	3	0.2		-0.104	-0.257	0.179		0.245	0.338	0.200	
Complex	Yes	Quadratic	4.1	-1	3	0.2		-0.067	0.152	0.204		0.298	0.375	0.204	
Complex	Yes	Optimal	10.3	-1	3	0.2		-0.054	-0.043	0.044		0.183	0.179	0.140	
RCNL	No	$[1, x, w]$	0.5	-1	3		0.5	0.206	-0.327		0.124	0.349	0.808		0.137
RCNL	No	BLP	5.7	-1	3		0.5	0.182	-0.394		0.014	0.237	0.662		0.037
RCNL	No	Local	3.2	-1	3		0.5	0.126	-0.106		0.009	0.243	0.340		0.047
RCNL	No	Quadratic	3.3	-1	3		0.5	0.142	-0.117		0.011	0.240	0.407		0.047
RCNL	No	Optimal	7.3	-1	3		0.5	0.166	-0.016		-0.008	0.216	0.173		0.022
RCNL	Yes	$[1, x, w]$	2.0	-1	3		0.5	0.013	-0.028		0.005	0.210	0.753		0.108
RCNL	Yes	BLP	11.6	-1	3		0.5	0.057	-0.358		0.015	0.200	0.609		0.036
RCNL	Yes	Local	6.1	-1	3		0.5	0.062	-0.087		0.009	0.173	0.340		0.047
RCNL	Yes	Quadratic	6.5	-1	3		0.5	0.059	-0.112		0.010	0.168	0.403		0.047
RCNL	Yes	Optimal	18.8	-1	3		0.5	0.007	0.006		0.001	0.105	0.151		0.017

This table documents bias and variance of parameter estimates along with estimation speed over 1,000 simulated datasets for different instruments. BLP instruments follow (32). Local and Quadratic instruments follow (33) and Gandhi and Houde (2017). Optimal instruments are the `approximate` version. For all problems, we use a degree 17 Gauss Hermite Product Rule and `L-BFGS-B`. For more details, refer to Section 5.1.

Table 7: Form of Optimal Instruments

				True Value				Median Bias				Median Absolute Error			
Simulation	Supply	Optimality	Seconds	α	σ_x	σ_p	ρ	α	σ_x	σ_p	ρ	α	σ_x	σ_p	ρ
Simple	No	Approximate	1.1	-1	3			0.181	-0.031			0.246	0.180		
Simple	No	Asymptotic	4.1	-1	3			0.180	-0.030			0.246	0.180		
Simple	No	Empirical	4.1	-1	3			0.182	-0.031			0.246	0.180		
Simple	Yes	Approximate	4.0	-1	3			0.012	0.010			0.172	0.156		
Simple	Yes	Asymptotic	19.3	-1	3			0.024	0.012			0.174	0.157		
Simple	Yes	Empirical	19.3	-1	3			0.017	0.012			0.168	0.160		
Complex	No	Approximate	3.4	-1	3	0.2		0.058	-0.145	0.085		0.267	0.240	0.177	
Complex	No	Asymptotic	7.3	-1	3	0.2		0.055	-0.153	0.085		0.267	0.250	0.178	
Complex	No	Empirical	7.3	-1	3	0.2		0.063	-0.156	0.086		0.262	0.250	0.173	
Complex	Yes	Approximate	10.3	-1	3	0.2		-0.054	-0.043	0.044		0.183	0.179	0.140	
Complex	Yes	Asymptotic	36.2	-1	3	0.2		-0.036	-0.045	0.021		0.223	0.225	0.135	
Complex	Yes	Empirical	36.0	-1	3	0.2		-0.038	-0.048	0.023		0.209	0.208	0.125	
RCNL	No	Approximate	7.3	-1	3		0.5	0.166	-0.016		-0.008	0.216	0.173		0.022
RCNL	No	Asymptotic	11.7	-1	3		0.5	0.164	-0.016		-0.008	0.217	0.171		0.021
RCNL	No	Empirical	11.6	-1	3		0.5	0.166	-0.019		-0.009	0.218	0.167		0.023
RCNL	Yes	Approximate	18.8	-1	3		0.5	0.007	0.006		0.001	0.105	0.151		0.017
RCNL	Yes	Asymptotic	59.4	-1	3		0.5	0.009	0.014		0.001	0.107	0.151		0.017
RCNL	Yes	Empirical	58.8	-1	3		0.5	0.008	0.010		0.001	0.105	0.150		0.017

This table documents bias and variance of parameter estimates along with estimation speed over 1,000 simulated datasets for different forms of optimal instruments. The `approximate` form replaces the structural errors with their expectation. The `asymptotic` form estimates an asymptotic normal distribution for the errors and draws from that distribution. The `empirical` form draws from the joint empirical distribution of the estimated errors. For all problems, we use a degree 17 Gauss Hermite Product Rule and `L-BFGS-B`. For more details, refer to Section 5.1.

Table 8: Varying Instrument Strength

					True Value				Median Bias				Median Absolute Error			
Simulation	γ_2	Corr(p, w)	Supply	Seconds	α	σ_x	σ_p	ρ	α	σ_x	σ_p	ρ	α	σ_x	σ_p	ρ
Simple	0.0	0.001	No	1.1	-1	3			0.377	-0.079			0.423	0.197		
Simple	0.0	0.001	Yes	4.2	-1	3			0.009	0.006			0.212	0.164		
Simple	0.1	0.052	No	1.1	-1	3			0.303	-0.060			0.360	0.190		
Simple	0.1	0.052	Yes	4.2	-1	3			0.011	0.012			0.211	0.160		
Simple	0.2	0.102	No	1.1	-1	3			0.181	-0.031			0.246	0.180		
Simple	0.2	0.102	Yes	4.0	-1	3			0.012	0.010			0.172	0.156		
Simple	0.4	0.199	No	1.1	-1	3			0.063	-0.005			0.124	0.169		
Simple	0.4	0.199	Yes	3.9	-1	3			0.009	0.015			0.112	0.146		
Simple	0.8	0.376	No	1.1	-1	3			0.020	0.009			0.062	0.168		
Simple	0.8	0.376	Yes	4.1	-1	3			0.000	0.008			0.060	0.144		
Complex	0.0	0.002	No	3.4	-1	3	0.2		0.189	-0.229	0.105		0.371	0.283	0.184	
Complex	0.0	0.002	Yes	10.5	-1	3	0.2		-0.067	-0.037	0.026		0.232	0.193	0.152	
Complex	0.1	0.054	No	3.4	-1	3	0.2		0.149	-0.193	0.097		0.331	0.271	0.186	
Complex	0.1	0.054	Yes	10.4	-1	3	0.2		-0.058	-0.047	0.039		0.213	0.186	0.144	
Complex	0.2	0.104	No	3.4	-1	3	0.2		0.058	-0.145	0.085		0.267	0.240	0.177	
Complex	0.2	0.104	Yes	10.3	-1	3	0.2		-0.054	-0.043	0.044		0.183	0.179	0.140	
Complex	0.4	0.204	No	3.5	-1	3	0.2		-0.058	-0.134	0.094		0.207	0.252	0.199	
Complex	0.4	0.204	Yes	10.3	-1	3	0.2		-0.062	-0.038	0.049		0.154	0.176	0.138	
Complex	0.8	0.384	No	3.5	-1	3	0.2		-0.102	-0.118	0.082		0.174	0.241	0.191	
Complex	0.8	0.384	Yes	11.4	-1	3	0.2		-0.054	-0.032	0.048		0.126	0.166	0.125	
RCNL	0.0	0.001	No	7.3	-1	3		0.5	0.277	-0.032		-0.014	0.312	0.176		0.025
RCNL	0.0	0.001	Yes	19.2	-1	3		0.5	0.011	0.008		0.001	0.122	0.159		0.018
RCNL	0.1	0.050	No	7.2	-1	3		0.5	0.244	-0.027		-0.013	0.278	0.173		0.023
RCNL	0.1	0.050	Yes	19.0	-1	3		0.5	0.007	0.009		0.001	0.112	0.152		0.017
RCNL	0.2	0.097	No	7.3	-1	3		0.5	0.166	-0.016		-0.008	0.216	0.173		0.022
RCNL	0.2	0.097	Yes	18.8	-1	3		0.5	0.007	0.006		0.001	0.105	0.151		0.017
RCNL	0.4	0.189	No	7.3	-1	3		0.5	0.069	-0.006		-0.004	0.129	0.175		0.020
RCNL	0.4	0.189	Yes	18.8	-1	3		0.5	0.002	0.004		0.001	0.087	0.154		0.018
RCNL	0.8	0.358	No	7.3	-1	3		0.5	0.022	-0.002		-0.001	0.072	0.165		0.020
RCNL	0.8	0.358	Yes	18.9	-1	3		0.5	-0.002	0.003		-0.000	0.063	0.151		0.017

This table documents bias and variance of parameter estimates along with estimation speed over 1,000 simulated datasets for varying instrument strength. To increase the strength of the cost shifter w, we increase its coefficient in the equation for marginal costs and report the increasing correlation between w and prices. For all problems, we use a degree 17 Gauss Hermite Product Rule and `L-BFGS-B`. For more details, refer to Section 5.1.

Table 9: Problem Scaling

					True Value				Median Bias				Median Absolute Error			
Simulation	Supply	T	J_f	Seconds	α	σ_x	σ_p	ρ	α	σ_x	σ_p	ρ	α	σ_x	σ_p	ρ
Simple	No	20	$\{2, 5, 10\}$	1.1	-1	3			0.181	-0.031			0.246	0.180		
Simple	No	40	$\{2, 5, 10\}$	2.3	-1	3			0.103	-0.019			0.174	0.116		
Simple	No	100	$\{2, 5, 10\}$	6.1	-1	3			0.043	0.006			0.104	0.078		
Simple	No	20	$\{4, 10, 20\}$	1.4	-1	3			0.103	0.004			0.167	0.142		
Simple	No	20	$\{10, 25, 50\}$	1.9	-1	3			0.045	-0.018			0.117	0.103		
Simple	Yes	20	$\{2, 5, 10\}$	4.0	-1	3			0.012	0.010			0.172	0.156		
Simple	Yes	40	$\{2, 5, 10\}$	7.8	-1	3			0.018	0.007			0.122	0.116		
Simple	Yes	100	$\{2, 5, 10\}$	20.3	-1	3			0.005	0.010			0.081	0.074		
Simple	Yes	20	$\{4, 10, 20\}$	5.5	-1	3			0.009	0.019			0.150	0.134		
Simple	Yes	20	$\{10, 25, 50\}$	33.1	-1	3			0.009	-0.023			0.113	0.096		
Complex	No	20	$\{2, 5, 10\}$	3.4	-1	3	0.2		0.058	-0.145	0.085		0.267	0.240	0.177	
Complex	No	40	$\{2, 5, 10\}$	7.3	-1	3	0.2		0.059	-0.094	0.040		0.199	0.169	0.118	
Complex	No	100	$\{2, 5, 10\}$	21.3	-1	3	0.2		0.031	-0.035	0.014		0.123	0.097	0.067	
Complex	No	20	$\{4, 10, 20\}$	4.9	-1	3	0.2		0.016	-0.103	0.067		0.227	0.189	0.179	
Complex	No	20	$\{10, 25, 50\}$	7.9	-1	3	0.2		-0.018	-0.082	0.053		0.185	0.137	0.200	
Complex	Yes	20	$\{2, 5, 10\}$	10.3	-1	3	0.2		-0.054	-0.043	0.044		0.183	0.179	0.140	
Complex	Yes	40	$\{2, 5, 10\}$	21.5	-1	3	0.2		-0.019	-0.016	0.020		0.141	0.125	0.097	
Complex	Yes	100	$\{2, 5, 10\}$	62.7	-1	3	0.2		-0.002	-0.013	0.007		0.096	0.078	0.060	
Complex	Yes	20	$\{4, 10, 20\}$	18.7	-1	3	0.2		-0.061	-0.034	0.048		0.182	0.134	0.154	
Complex	Yes	20	$\{10, 25, 50\}$	108.9	-1	3	0.2		-0.028	-0.044	0.023		0.142	0.093	0.132	
RCNL	No	20	$\{2, 5, 10\}$	7.3	-1	3		0.5	0.166	-0.016		-0.008	0.216	0.173		0.022
RCNL	No	40	$\{2, 5, 10\}$	15.5	-1	3		0.5	0.104	-0.014		-0.006	0.159	0.116		0.015
RCNL	No	100	$\{2, 5, 10\}$	46.5	-1	3		0.5	0.045	-0.000		-0.002	0.094	0.072		0.010
RCNL	No	20	$\{4, 10, 20\}$	10.0	-1	3		0.5	0.140	-0.024		-0.001	0.188	0.192		0.021
RCNL	No	20	$\{10, 25, 50\}$	19.0	-1	3		0.5	0.084	-0.053		0.006	0.140	0.239		0.028
RCNL	Yes	20	$\{2, 5, 10\}$	18.8	-1	3		0.5	0.007	0.006		0.001	0.105	0.151		0.017
RCNL	Yes	40	$\{2, 5, 10\}$	42.8	-1	3		0.5	0.001	0.002		0.001	0.074	0.100		0.012
RCNL	Yes	100	$\{2, 5, 10\}$	124.8	-1	3		0.5	-0.004	0.003		0.000	0.050	0.059		0.007
RCNL	Yes	20	$\{4, 10, 20\}$	31.1	-1	3		0.5	0.023	-0.038		0.004	0.117	0.180		0.020
RCNL	Yes	20	$\{10, 25, 50\}$	137.1	-1	3		0.5	0.027	-0.075		0.008	0.120	0.231		0.027

This table documents bias and variance of parameter estimates along with estimation speed over 1,000 simulated datasets for different problem sizes. We separately increase the number of simulated markets T and products per firm J_f. For all problems, we use a degree 17 Gauss Hermite Product Rule and `L-BFGS-B`. For more details, refer to Section 5.1.

Table 10: Optimization Algorithms

							Percent of Runs		Median, First GMM Step			
Simulation	Supply	P	Software	Algorithm	Gradient	Termination	Converged	PSD Hessian	Seconds	Evaluations	$q = g'Wg$	$\|\|\nabla q\|\|_\infty$
Simple	No	1	Knitro	Interior/Direct	Yes	$\|\|\nabla q\|\|_\infty$	100.0%	99.6%	0.5	8	7.17E-15	2.87E-07
Simple	No	1	Knitro	Active Set	Yes	$\|\|\nabla q\|\|_\infty$	100.0%	99.6%	0.4	7	4.43E-15	2.20E-07
Simple	No	1	Knitro	SQP	Yes	$\|\|\nabla q\|\|_\infty$	100.0%	99.7%	0.5	8	4.85E-15	2.29E-07
Simple	No	1	SciPy	Trust-Region	Yes	$\|\|\nabla q\|\|_\infty$	100.0%	99.6%	0.7	10	4.14E-15	2.09E-07
Simple	No	1	SciPy	L-BFGS-B	Yes	$\|\|\nabla q\|\|_\infty$	100.0%	99.6%	0.4	6	2.34E-15	1.56E-07
Simple	No	1	SciPy	BFGS	Yes	$\|\|\nabla q\|\|_\infty$	100.0%	99.7%	0.4	7	2.39E-15	1.60E-07
Simple	No	1	SciPy	TNC	Yes	$\|\|\nabla q\|\|_\infty$	100.0%	99.7%	0.6	9	3.93E-17	1.80E-08
Simple	No	1	SciPy	TNC	Yes	$\|\|\theta^n - \theta^{n-1}\|\|_\infty$	100.0%	99.7%	0.7	11	1.77E-18	2.61E-09
Simple	No	1	SciPy	Nelder-Mead	No	$\|\|\theta^n - \theta^{n-1}\|\|_\infty$	64.1%	99.5%	24.9	123	4.17E-19	1.16E-09
Simple	Yes	2	Knitro	Interior/Direct	Yes	$\|\|\nabla q\|\|_\infty$	100.0%	99.5%	1.9	14	4.08E-01	1.13E-05
Simple	Yes	2	Knitro	Active Set	Yes	$\|\|\nabla q\|\|_\infty$	99.9%	99.7%	1.9	12	4.09E-01	1.46E-05
Simple	Yes	2	Knitro	SQP	Yes	$\|\|\nabla q\|\|_\infty$	99.8%	99.1%	2.2	13	4.07E-01	1.03E-05
Simple	Yes	2	SciPy	Trust-Region	Yes	$\|\|\nabla q\|\|_\infty$	99.1%	99.2%	2.6	18	4.05E-01	2.19E-05
Simple	Yes	2	SciPy	L-BFGS-B	Yes	$\|\|\nabla q\|\|_\infty$	99.9%	99.6%	1.1	9	4.05E-01	1.31E-05
Simple	Yes	2	SciPy	BFGS	Yes	$\|\|\nabla q\|\|_\infty$	99.4%	99.1%	1.3	11	4.07E-01	1.15E-05
Simple	Yes	2	SciPy	TNC	Yes	$\|\|\nabla q\|\|_\infty$	99.4%	99.3%	2.3	18	4.08E-01	9.54E-06
Simple	Yes	2	SciPy	TNC	Yes	$\|\|\theta^n - \theta^{n-1}\|\|_\infty$	99.9%	99.3%	2.0	15	4.07E-01	8.79E-05
Simple	Yes	2	SciPy	Nelder-Mead	No	$\|\|\theta^n - \theta^{n-1}\|\|_\infty$	54.3%	99.1%	48.8	266	4.07E-01	1.98E-07
Complex	No	3	Knitro	Interior/Direct	Yes	$\|\|\nabla q\|\|_\infty$	100.0%	98.7%	2.5	17	2.17E-12	4.65E-06
Complex	No	3	Knitro	Active Set	Yes	$\|\|\nabla q\|\|_\infty$	99.9%	99.1%	1.5	14	1.39E-12	3.94E-06
Complex	No	3	Knitro	SQP	Yes	$\|\|\nabla q\|\|_\infty$	99.9%	98.9%	2.0	16	1.52E-12	4.48E-06
Complex	No	3	SciPy	Trust-Region	Yes	$\|\|\nabla q\|\|_\infty$	99.9%	99.0%	2.4	24	2.23E-12	4.17E-06
Complex	No	3	SciPy	L-BFGS-B	Yes	$\|\|\nabla q\|\|_\infty$	99.9%	98.8%	1.3	11	1.90E-12	4.26E-06
Complex	No	3	SciPy	BFGS	Yes	$\|\|\nabla q\|\|_\infty$	99.9%	98.8%	1.2	13	1.76E-12	4.64E-06
Complex	No	3	SciPy	TNC	Yes	$\|\|\nabla q\|\|_\infty$	99.9%	99.0%	2.2	23	5.64E-13	2.48E-06
Complex	No	3	SciPy	TNC	Yes	$\|\|\theta^n - \theta^{n-1}\|\|_\infty$	99.9%	99.0%	2.0	21	3.35E-11	1.32E-05
Complex	No	3	SciPy	Nelder-Mead	No	$\|\|\theta^n - \theta^{n-1}\|\|_\infty$	56.4%	99.1%	47.0	309	1.27E-18	5.83E-09
Complex	Yes	4	Knitro	Interior/Direct	Yes	$\|\|\nabla q\|\|_\infty$	100.0%	98.6%	6.5	24	6.04E-01	2.15E-05
Complex	Yes	4	Knitro	Active Set	Yes	$\|\|\nabla q\|\|_\infty$	99.7%	98.8%	4.1	20	6.04E-01	2.21E-05
Complex	Yes	4	Knitro	SQP	Yes	$\|\|\nabla q\|\|_\infty$	99.8%	98.9%	6.5	24	6.04E-01	2.03E-05
Complex	Yes	4	SciPy	Trust-Region	Yes	$\|\|\nabla q\|\|_\infty$	97.0%	97.9%	17.1	44	5.49E-01	3.34E-05
Complex	Yes	4	SciPy	L-BFGS-B	Yes	$\|\|\nabla q\|\|_\infty$	99.6%	98.8%	3.1	17	6.07E-01	2.01E-05
Complex	Yes	4	SciPy	BFGS	Yes	$\|\|\nabla q\|\|_\infty$	99.4%	99.0%	3.2	18	6.05E-01	2.04E-05
Complex	Yes	4	SciPy	TNC	Yes	$\|\|\nabla q\|\|_\infty$	98.9%	98.7%	7.3	36	6.03E-01	1.97E-05

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							Percent of Runs		Median, First GMM Step			
Simulation	Supply	P	Software	Algorithm	Gradient	Termination	Converged	PSD Hessian	Seconds	Evaluations	$q = g'Wg$	$\|\|\nabla q\|\|_\infty$
Complex	Yes	4	SciPy	TNC	Yes	$\|\|\theta^n - \theta^{n-1}\|\|_\infty$	99.8%	98.8%	5.2	27	6.06E-01	4.87E-04
Complex	Yes	4	SciPy	Nelder-Mead	No	$\|\|\theta^n - \theta^{n-1}\|\|_\infty$	44.9%	98.9%	84.3	1,001	6.04E-01	3.75E-07
RCNL	No	2	Knitro	Interior/Direct	Yes	$\|\|\nabla q\|\|_\infty$	100.0%	99.0%	4.7	17	9.34E-14	6.36E-06
RCNL	No	2	Knitro	Active Set	Yes	$\|\|\nabla q\|\|_\infty$	100.0%	99.2%	3.3	14	2.40E-14	3.29E-06
RCNL	No	2	Knitro	SQP	Yes	$\|\|\nabla q\|\|_\infty$	100.0%	99.0%	4.5	16	1.13E-14	2.31E-06
RCNL	No	2	SciPy	Trust-Region	Yes	$\|\|\nabla q\|\|_\infty$	99.9%	99.2%	5.9	28	1.96E-12	3.45E-05
RCNL	No	2	SciPy	L-BFGS-B	Yes	$\|\|\nabla q\|\|_\infty$	99.9%	99.0%	2.1	11	7.25E-15	1.62E-06
RCNL	No	2	SciPy	BFGS	Yes	$\|\|\nabla q\|\|_\infty$	99.7%	98.7%	2.5	13	6.23E-15	1.52E-06
RCNL	No	2	SciPy	TNC	Yes	$\|\|\nabla q\|\|_\infty$	99.9%	99.2%	5.1	24	5.36E-16	6.53E-07
RCNL	No	2	SciPy	TNC	Yes	$\|\|\theta^n - \theta^{n-1}\|\|_\infty$	100.0%	99.5%	3.9	19	6.79E-11	8.05E-05
RCNL	No	2	SciPy	Nelder-Mead	No	$\|\|\theta^n - \theta^{n-1}\|\|_\infty$	59.2%	99.2%	100.3	253	4.33E-20	5.89E-09
RCNL	Yes	3	Knitro	Interior/Direct	Yes	$\|\|\nabla q\|\|_\infty$	100.0%	99.1%	9.8	24	5.95E-01	6.70E-05
RCNL	Yes	3	Knitro	Active Set	Yes	$\|\|\nabla q\|\|_\infty$	100.0%	99.5%	7.4	21	5.99E-01	2.85E-05
RCNL	Yes	3	Knitro	SQP	Yes	$\|\|\nabla q\|\|_\infty$	99.8%	99.2%	11.0	24	6.00E-01	2.49E-05
RCNL	Yes	3	SciPy	Trust-Region	Yes	$\|\|\nabla q\|\|_\infty$	99.8%	99.5%	12.2	40	5.99E-01	7.48E-05
RCNL	Yes	3	SciPy	L-BFGS-B	Yes	$\|\|\nabla q\|\|_\infty$	99.7%	99.1%	4.6	15	5.95E-01	1.76E-05
RCNL	Yes	3	SciPy	BFGS	Yes	$\|\|\nabla q\|\|_\infty$	99.8%	99.4%	4.5	15	5.97E-01	1.72E-05
RCNL	Yes	3	SciPy	TNC	Yes	$\|\|\nabla q\|\|_\infty$	99.7%	99.4%	11.4	33	5.95E-01	2.31E-05
RCNL	Yes	3	SciPy	TNC	Yes	$\|\|\theta^n - \theta^{n-1}\|\|_\infty$	99.8%	99.5%	6.9	23	5.95E-01	2.33E-03
RCNL	Yes	3	SciPy	Nelder-Mead	No	$\|\|\theta^n - \theta^{n-1}\|\|_\infty$	43.2%	99.3%	156.2	1,001	5.95E-01	9.30E-07

This table documents optimization convergence statistics along with estimation speed over 1,000 simulated datasets for different optimization algorithms. For comparison's sake, we only perform one GMM step and do not set parameter bounds. We report each algorithm's convergence rate (i.e., the percent of times the algorithm reported that it successfully found an optima and the maximum number of iterations, 1,000, was not reached) and the percent of times the final Hessian matrix was positive semidefinite (i.e., that a local minima was found). We also report medians for the number of seconds needed to solve each problem, the number of objective evaluations, the final GMM objective value $q = \bar{g}'W\bar{g}$, and the L^∞ norm of the final gradient. We compare three `KNITRO` algorithms: `Interior/Direct` (an interior-point mehtod), `Active Set` (sequential linear-quadratic programming), and `SQP` (sequential quadratic programming). From the `scipy.optimize` module, we compare `Trust-Region` (trust-region SQP), `L-BFGS-B` (limited-memory BFGS), `BFGS`, `TNC` (truncated Newton algorithm), and `Nelder-Mead` (Simplex algorithm). All algorithms except for `Nelder-Mead` are configured with a gradient L^∞ norm of 10^{-4}. Since `Nelder-Mead` does not support gradient-based terimation conditions, it is instead configured with a absolute parameter L^∞ norm of 10^{-4}. For comparison's sake, we include a second `TNC` algorithm configured with the same termination condition. For all problems, we use a degree 17 Gauss Hermite Product Rule and optimal instruments. For more details, refer to Section 5.1.

Table 11: Nevo (2000b) Replication

		Original		Replication		Restricted		Optimal Instruments	
Parameters	Variable	Estimate	SE	Estimate	SE	Estimate	SE	Estimate	SE
Means (β)	Price	-32.433	7.743	-62.730	14.803	-32.019	2.304	-25.756	1.475
Standard Deviations (Σ)	Price	1.848	1.075	3.312	1.340	1.803	0.920	2.586	0.617
	Constant	0.377	0.129	0.558	0.163	0.375	0.120	0.221	0.074
	Sugar	0.004	0.012	-0.006	0.014	-0.004	0.012	0.023	0.007
	Mushy	0.081	0.205	0.093	0.185	0.086	0.193	0.290	0.096
Interactions (Π)	Price × Income	16.598	172.334	588.325	270.441	4.187	4.638	-9.699	2.406
	Price × Income^2	-0.659	8.955	-30.192	14.101				
	Price × Child	11.625	5.207	11.055	4.123	11.755	5.197	3.960	2.401
	Constant × Income	3.089	1.213	2.292	1.209	3.101	1.054	6.520	0.468
	Constant × Age	1.186	1.016	1.284	0.631	1.198	1.048	0.029	0.240
	Sugar × Income	-0.193	0.005	-0.385	0.121	-0.190	0.035	-0.271	0.023
	Sugar × Age	0.029	0.036	0.052	0.026	0.028	0.032	0.053	0.017
	Mushy × Income	1.468	0.697	0.748	0.802	1.495	0.648	0.817	0.259
	Mushy × Age	-1.514	1.103	-1.353	0.667	-1.539	1.107	-0.374	0.214
Mean Own-Price Elasticity (ε_{jj})				-3.618		-3.702		-3.679	
Mean Markup ($(p_{jt} - c_{jt})/p_{jt}$)				0.364		0.360		0.365	
Objective ($g'Wg$)		1.49E+01		4.56E+00		1.54E+01		8.27E-14	

This table reports replication results for the model of Nevo (2000b) described in Section 6. Under "Original," we report estimates from the original paper. Under "Replication," we report our replication results. Discrepancies can be explained by the loose contraction mapping tolerance used in an older version of the original code. Under "Restricted," we report additional results for which we remedy a multicollinearity problem with income and its square by excluding the interaction term of the latter. Under "Optimal Instruments," we report results using optimal instruments constructed in the style of Chamberlain (1987) from initial "Restricted" estimates.

Table 12: Berry et al. (1995, 1999) Replication

		Original		Replication		PC Instruments		Optimal Instruments	
Parameters	Variable	Estimate	SE	Estimate	SE	Estimate	SE	Estimate	SE
Means (β)	Constant	-7.061	0.941	-12.595	1.654	-22.947	6.155	-8.538	1.016
	HP / weight	2.883	2.019	-6.978	6.294	4.406	3.201	1.853	2.300
	Air	1.521	0.891	1.847	3.242	0.485	4.659	1.069	1.154
	MP$	-0.122	0.320	-2.130	1.941	0.744	0.207	-1.649	0.798
	Size	3.460	0.610	4.980	0.378	4.763	0.595	1.590	0.784
Standard Deviations (Σ)	Constant	3.612	1.485	9.049	1.405	17.943	6.765	1.319	1.832
	HP / weight	4.628	1.885	17.271	7.199	2.452	5.722	3.315	5.262
	Air	1.818	1.695	-0.438	7.035	2.236	8.015	0.846	1.624
	MP$	1.050	0.272	2.929	2.247	-1.782	0.589	2.071	0.822
	Size	2.056	0.585	-9.996	1.424	-5.228	1.644	1.209	1.066
Term on Price (α)	$\ln(y-p)$	43.501	6.427	8.409	0.936	8.210	0.773	7.962	2.519
Supply Side Terms (γ)	Constant	0.952	0.194	2.499	0.183	2.470	0.182	2.423	0.178
	ln(HP / weight)	0.477	0.056	0.677	0.120	0.662	0.126	0.760	0.198
	Air	0.619	0.038	0.891	0.069	0.902	0.066	0.948	0.093
	ln(MPG)	-0.415	0.055	-0.564	0.116	-0.580	0.117	-0.616	0.153
	ln(Size)	-0.046	0.081	0.010	0.133	-0.006	0.139	0.249	0.184
	Trend	0.019	0.002	0.014	0.003	0.015	0.003	0.016	0.004
Mean Own-Price Elasticity (ε_{jj})				-3.492		-3.408		-2.939	
Mean Markup ($(p_{jt}-c_{jt})/p_{jt}$)				0.368		0.377		0.419	
Objective ($g'Wg$)				3.25E+02		3.63E+02		9.64E+01	

This table reports replication results for the model of Berry et al. (1995, 1999). Under "Original," we report estimates from the original paper. Under the following headers, we report replication results for a modified version of the original model described in Section 6. Under "Replication," we use traditional BLP instruments. Under "PC Instruments," we remedy a multicollinearity problem with the BLP instruments by following Conlon (2017) and interacting them up to a second degree before projecting them down to the smallest number of principal components that explain at least 99% of the interactions' variance. Under "Optimal Instruments," we report results using optimal instruments constructed in the style of Chamberlain (1987) from initial "PC Instruments" estimates.

References

Amemiya, T. (1977): "The Maximum Likelihood and the Nonlinear Three-Stage Least Squares Estimator in the General Nonlinear Simultaneous Equation Model," *Econometrica*, 45, 955–968.

Andrews, I., M. Gentzkow, and J. M. Shapiro (2017): "Measuring the Sensitivity of Parameter Estimates to Estimation Moments," *Quarterly Journal of Economics*, 132, 1553–1592.

Armstrong, T. (2016): "Large Market Asymptotics for Differentiated Product Demand Estimators with Economic Models of Supply," *Econometrica*, 84, 1961–1980.

Backus, M., C. Conlon, and M. Sinkinson (2018): "Common Ownership and Competition in the Ready-To-Eat Cereal Industry," Working Paper.

Bayer, P., F. Ferreira, and R. McMillan (2007): "A Unified Framework for Measuring Preferences for Schools and Neighborhoods," *Journal of Political Economy*, 115, 588–638.

Berry, S. (1994): "Estimating discrete-choice models of product differentiation," *RAND Journal of Economics*, 25, 242–261.

Berry, S., J. Levinsohn, and A. Pakes (1995): "Automobile Prices in Market Equilibrium," *Econometrica*, 63, 841–890.

——— (1999): "Voluntary Export Restraints on Automobiles: Evaluating a Trade Policy," *American Economic Review*, 89, 400–430.

——— (2004a): "Differentiated Products Demand Systems from a Combination of Micro and Macro Data: The New Car Market," *Journal of Political Economy*, 112, 68–105.

Berry, S., O. B. Linton, and A. Pakes (2004b): "Limit Theorems for Estimating the Parameters of Differentiated Product Demand Systems," *Review of Economic Studies*, 71, 613–654.

Berry, S. and A. Pakes (2007): "The Pure Characteristics Demand Model," *International Economic Review*, 48, 1193–1125.

Berry, S. T. and P. A. Haile (2014): "Identification in Differentiated Products Markets Using Market Level Data," *Econometrica*, 82, 1749–1797.

Brenkers, R. and F. Verboven (2006): "Liberalizing a Distribution System: The European Car Market," *Journal of the European Economic Association*, 4, 216–251.

Bresnahan, T. F. (1982): "The oligopoly solution concept is identified," *Economics Letters*, 10, 87 – 92.

Caplin, A. and B. Nalebuff (1991): "Aggregation and Imperfect Competition: On the Existence of Equilibrium," *Econometrica*, 59, 25–59.

Chamberlain, G. (1987): "Asymptotic Efficiency in Estimation with Conditional Moment Restrictions," *Journal of Econometrics*, 34, 305–334.

Coleman, C., S. Lyon, L. Maliar, and S. Maliar (2018): "Matlab, Python, Julia: What to Choose in Economics?" CEPR Discussion Papers 13210, C.E.P.R. Discussion Papers.

Conlon, C. (2017): "The MPEC Approach to Empirical Likelihood Estimation of Demand," Working Paper.

Conlon, C. and N. Rao (2017): "The Price of Liquor is Too Damn High: The Effects of Post and Hold Pricing," Working Paper.

Correia, S. (2016): "Linear Models with High-Dimensional Fixed Effects: An Efficient and Feasible Estimator," Tech. rep., working Paper.

Dubé, J.-P. H., J. T. Fox, and C.-L. Su (2012): "Improving the Numerical Performance of BLP Static and Dynamic Discrete Choice Random Coefficients Demand Estimation," *Econometrica*, 80, 2231–2267.

Fan, Y. (2013): "Ownership Consolidation and Product Characteristics: A Study of the US Daily Newspaper Market," *American Economic Review*, 103, 1598–1628.

Freyberger, J. (2015): "Asymptotic theory for differentiated products demand models with many markets," *Journal of Econometrics*, 185, 162 – 181.

Gallego, G., W. T. Huh, W. Kang, and R. Phillips (2006): "Price Competition with the Attraction Demand Model: Existence of Unique Equilibrium and Its Stability," *Manufacturing & Service Operations Management*, 8, 359–375.

Gandhi, A. and J. Houde (2017): "Measuring Substitution Patterns in Differentiated Products Industries," Working Paper.

Grigolon, L. and F. Verboven (2014): "Nested Logit or Random Coefficients Logit? A Comparison of Alternative Discrete Choice Models of Product Differentiation," *The Review of Economics and Statistics*, 96, 916–935.

Guimarães, P. and P. Portugal (2010): "A simple feasible procedure to fit models with high-dimensional fixed effects," *Stata Journal*, 10, 628–649.

Heiss, F. and V. Winschel (2008): "Likelihood approximation by numerical integration on sparse grids," *Journal of Econometrics*, 144, 62 – 80.

Ho, K. and A. Pakes (2014): "Hospital Choices, Hospital Prices, and Financial Incentives to Physicians," *American Economic Review*, 104, 3841–84.

Houde, J.-F. (2012): "Spatial Differentiation and Vertical Mergers in Retail Markets for Gasoline," *American Economic Review*, 102, 2147–82.

Judd, K. L. and B. Skrainka (2011): "High performance quadrature rules: how numerical integration affects a popular model of product differentiation," CeMMAP working papers CWP03/11, Centre for Microdata Methods and Practice, Institute for Fiscal Studies.

Knittel, C. R. and K. Metaxoglou (2014): "Estimation of Random-Coefficient Demand Models: Two Empiricists' Perspective," *Review of Economics and Statistics*, 96.

Koijen, R. S. and M. Yogo (2016): "Shadow insurance," *Econometrica*, 84, 1265–1287.

Konovalov, A. and Z. Sandor (2010): "On price equilibrium with multi-product firms," *Economic Theory*, 44, 271–292.

Lee, J. and K. Seo (2015): "A computationally fast estimator for random coefficients logit demand models using aggregate data," *The RAND Journal of Economics*, 46, 86–102.

——— (2016): "Revisiting the nested fixed-point algorithm in BLP random coefficients demand estimation," *Economics Letters*, 149.

Lee, R. S. (2013): "Vertical Integration and Exclusivity in Platform and Two-Sided Markets," *American Economic Review*, 103, 2960–3000.

MacKay, A. and N. Miller (2018): "Demand Estimation in Models of Imperfect Competition," Working Paper.

Miller, N. and M. Weinberg (2017): "Understanding the Price Effects of the MillerCoors Joint Venture," *Econometrica*, 85, 1763–1791.

Miravete, E. J., K. Seim, and J. Thurk (2018): "Market Power and the Laffer Curve," *Econometrica*, 86, 1651–1687.

More, J. J., B. S. Garbow, and K. E. Hillstrom (1980): "User guide for MINPACK-1. [In FORTRAN]," .

Morrow, W. R. and S. J. Skerlos (2010): "On the Existence of Bertrand-Nash Equilibrium Prices Under Logit Demand," *CoRR*, abs/1012.5832.

——— (2011): "Fixed-Point Approaches to Computing Bertrand-Nash Equilibrium Prices Under Mixed-Logit Demand," *Operations Research*, 59, 328–345.

Nevo, A. (2000a): "Mergers with differentiated products: the case of the ready-to-eat cereal industry," *RAND Journal of Economics*, 31, 395–421.

——— (2000b): "A Practitioner's Guide to Estimation of Random Coefficients Logit Models of Demand (including Appendix)," *Journal of Economics and Management Strategy*, 9, 513–548.

——— (2001): "Measuring Market Power in the Ready-to-Eat Cereal Industry," *Econometrica*, 69, 307–342.

Newey, W. (1985): "Generalized method of moments specification testing," *Journal of Econometrics*, 29, 229–256.

Petrin, A. (2002): "Quantifying the Benefits of New Products: The Case of the Minivan," *Journal of Political Economy*, 110, 705–729.

Reynaert, M. and F. Verboven (2014): "Improving the performance of random coefficients demand models: The role of optimal instruments," *Journal of Econometrics*, 179, 83–98.

Reynaerts, J., R. Varadhan, and J. C. Nash (2012): "Enhancing the Convergence Properties of the BLP (1995) Contraction Mapping," Vives discussion paper series 35, Katholieke Universiteit Leuven, Faculteit Economie en Bedrijfswetenschappen, Vives, http://ideas.repec.org/p/ete/vivwps/35.html.

Salanie, B. and F. Wolak (2018): "Fast, Robust, and Approximately Correct: Estimating Mixed Demand Systems," Working Paper.

Skrainka, B. (2012a): "Three Essays on Product Differentiation," PhD dissertation, University College London.

Skrainka, B. S. (2012b): "A Large Scale Study of the Small Sample Performance of Random Coefficient Models of Demand," Working Paper.

Somaini, P. and F. Wolak (2016): "An Algorithm to Estimate the Two-Way Fixed Effects Model," *Journal of Econometric Methods*, 5, 143–152.

Su, C.-L. and K. L. Judd (2012): "Constrained optimization approaches to estimation of structural models," *Econometrica*, 80, 2213–2230.

Varadhan, R. and C. Roland (2008): "Simple and Globally Convergent Methods for Accelerating the Convergence of Any EM Algorithm," *Scandinavian Journal of Statistics*, 35, 335–353.

Appendices

A. Concentrating out Linear Parameters

Our objective is to concentrate out $[\hat{\beta}(\theta_2), \hat{\gamma}(\theta_2)]$. Define y_{jt}^D, y_{jt}^S, x_{jt}^D, and x_{jt}^S as follows:

$$\begin{aligned} y_{jt}^D &\equiv \hat{\delta}_{jt}(\theta_2) + \alpha p_{jt} &&= [x_{jt} \quad v_{jt}]\beta + \xi_t \equiv x_{jt}^D \beta + \xi_{jt}, \\ y_{jt}^S &\equiv p_{jt} - \hat{\eta}_{jt}(\theta_2) &&= [x_{jt} \quad w_{jt}]\gamma + \omega_t \equiv x_{jt}^S \gamma + \omega_{jt}. \end{aligned} \tag{A.1}$$

Stacking the system across observations yields:[61]

$$\underbrace{\begin{bmatrix} y_D \\ y_S \end{bmatrix}}_{2N \times 1} = \underbrace{\begin{bmatrix} X_D & 0 \\ 0 & X_S \end{bmatrix}}_{2N \times (K_1+K_3)} \underbrace{\begin{bmatrix} \beta \\ \gamma \end{bmatrix}}_{(K_1+K_3)\times 1} + \underbrace{\begin{bmatrix} \xi \\ \omega \end{bmatrix}}_{2N \times 1}. \tag{A.2}$$

Adding the $N \times q^D$ instruments for demand z^D and the $N \times q^S$ instruments for supply z^S yields $q = q^D + q^S$ moment restrictions:

$$\begin{aligned} g(\beta, \gamma) &= E_n \begin{bmatrix} z_n^{D\prime}(y_n^D - x_n^{D\prime}\beta) \\ z_n^{S\prime}(y_n^S - x_n^{S\prime}\gamma) \end{bmatrix} = 0 \\ \underbrace{g_n}_{q\times 1} &= \underbrace{\frac{1}{N} \underbrace{\begin{bmatrix} Z_D' & 0 \\ 0 & Z_S' \end{bmatrix}}_{q \times 2N} \underbrace{\begin{bmatrix} y_D \\ y_S \end{bmatrix}}_{2N\times 1}}_{\widetilde{Y}} - \underbrace{\frac{1}{N} \underbrace{\begin{bmatrix} Z_D' X_D & 0 \\ 0 & Z_S' X_S \end{bmatrix}}_{q \times (K_1+K_3)} \underbrace{\begin{bmatrix} \beta \\ \gamma \end{bmatrix}}_{(K_1+K_3)\times 1}}_{\widetilde{X}}. \end{aligned} \tag{A.3}$$

Now we can simply perform a GMM regression of $\widetilde{Y}$ on $\widetilde{X}$ where W is the $q \times q$ GMM weighting matrix:[62]

$$\begin{bmatrix} \hat{\beta}(\theta_2) \\ \hat{\gamma}(\theta_2) \end{bmatrix} = (\widetilde{X}' W \widetilde{X})^{-1} \widetilde{X}' W \widetilde{Y}. \tag{A.4}$$

B. Analytic Derivative Calculations

This derivation appears to be novel to the literature for the case of simultaneous estimation of supply and demand.

The gradient of the GMM objective function $q(\theta_2)$ is

$$2G_n(\theta_2)' W g_n(\theta_2)$$

[61] Note: we cannot perform independent regressions unless we are willing to assume that $\text{Cov}(\xi_{jt}, \omega_{jt}) = 0$.

[62] Observe this is the same GMM weighting matrix as for the overall problem. Also note that we require $q > K_1 + K_3$ so that we have overidentifying restrictions, we need at least $K_2 = \dim(\theta_2)$ such restrictions.

The challenging piece here is the Jacobian of the GMM objective

$$G_n(\theta_2) = \frac{1}{N} \underbrace{\begin{bmatrix} Z_D' & 0 \\ 0 & Z_S' \end{bmatrix}}_{q \times 2N} \underbrace{\begin{bmatrix} \frac{\partial \xi}{\partial \theta_2} \\ \frac{\partial \omega}{\partial \theta_2} \end{bmatrix}}_{2N \times K_2},$$

which is a $q \times K_2$ matrix. Because (12) are linear, we can write:

$$\begin{bmatrix} \frac{\partial \xi}{\partial \theta_2} \\ \frac{\partial \omega}{\partial \theta_2} \end{bmatrix} = \begin{bmatrix} \frac{\partial \delta}{\partial \theta_2} \\ -f'(\cdot)\frac{\partial \eta}{\partial \theta_2} \end{bmatrix}. \tag{B.5}$$

For the demand moments, after invoking the implicit function theorem, this has a convenient block structure which can be separated market by market t:[63]

$$\underbrace{\frac{\partial \delta_{\cdot t}}{\partial \theta_2}(\theta_2)}_{J_t \times K_2} = -\underbrace{\left[\frac{\partial s_{\cdot t}}{\partial \delta_{\cdot t}}(\theta_2)\right]^{-1}}_{J_t \times J_t} \times \underbrace{\left[\frac{\partial s_{\cdot t}}{\partial \theta_2}(\theta_2)\right]}_{J_t \times K_2}$$

For the supply moments, the $f'(\cdot)$ in (B.5) comes from (7) where $f(\cdot)$ is typically linear, $f'(\cdot) = 1$, or logarithmic, $f'(c) = 1/c$. Differentiating the supply moments is challenging because demand derivatives $\frac{\partial s_j}{\partial p_k}(\xi(\theta_2), \theta_2)$ in the matrix of intra-firm (negative) demand derivatives $\Delta(\xi(\theta_2), \theta_2)$ depend on both $\xi(\theta_2)$ and θ_2 directly. To avoid tensor product notation, let us consider taking the derivative with respect to an element within θ_2 which we call θ_ℓ:[64]

$$\begin{aligned} \underbrace{\frac{\partial \eta_{\cdot t}}{\partial \theta_\ell}}_{J_t \times 1} &= -\Delta_t^{-1}\frac{\partial \Delta_t}{\partial \theta_\ell}\Delta_t^{-1}s_{\cdot t} - \Delta_t^{-1}\frac{\partial \Delta_t}{\partial \xi}\frac{\partial \xi_{\cdot t}}{\partial \theta_\ell}\Delta_t^{-1}s_{\cdot t} + \Delta_t^{-1}\underbrace{\frac{\partial s_{\cdot t}}{\partial \theta_\ell}}_{0} \\ &= -\underbrace{\Delta_t^{-1}}_{(J_t \times J_t)}\underbrace{\frac{\partial \Delta_t}{\partial \theta_\ell}}_{(J_t \times J_t)}\underbrace{\eta_{\cdot t}}_{(J_t \times 1)} - \underbrace{\Delta_t^{-1}}_{(J_t \times J_t)}\underbrace{\frac{\partial \Delta_t}{\partial \xi_{\cdot t}}}_{(J_t \times J_t \times J_t)}\underbrace{\frac{\partial \xi_{\cdot t}}{\partial \theta_\ell}}_{(J_t \times 1)}\underbrace{\eta_{\cdot t}}_{(J_t \times 1)}. \end{aligned}$$

This is expression is complicated because the supply error $\omega_{\cdot t}$ and the markup $\eta_{\cdot t}$ depend both directly on θ_2 and indirectly on θ_2 through $\xi_{\cdot t}$.

C. Optimization Algorithms and the Example Problems

We extend our comparison of different optimization algorithms and starting values to the two example problems from Section 6. For each algorithm and example problem, we first report convergence

[63] $\frac{\partial \xi}{\partial \delta}$ is the identity matrix, so $\frac{\partial \xi}{\partial \theta_2} = \frac{\partial \delta}{\partial \theta_2}$. The matrix inverse of $\frac{\partial s_{\cdot t}}{\partial \delta_{\cdot t}}(\theta_2)$ is guaranteed by the diagonal dominance of system of equations with respect to δ_{jt}. As long as the outside good has a positive share, we have that for each j, $|\frac{\partial s_{jt}}{\partial \delta_{jt}}| > \sum_{k \neq j} |\frac{\partial s_{kt}}{\partial \delta_{jt}}|$. A square matrix and its transpose have the same eigenvalues and thus are both are nonsingular. In practice, as shares become small, there may still be numerical issues.

[64] In the markup η, $s_{\cdot t}$ is data and thus does not depend on parameters.

statistics across 100 different starting values in Table C.1.[65]

Results are similar to those from solving the simulated problems. For most algorithms, convergence rates are greater than 99% and second order conditions generally always satisfied. The exceptions are `TNC` with a gradient-based norm and Nelder-Mead, which have trouble satisfying their termination conditions. Regardless, all algorithms terminate at essentially the same objective value with a small gradient norm.

In Figure C.1, we plot the distribution of objective values across algorithms and starting values. In Figures C.2 and C.3, we plot how the small amount of variation in objective values translates into differences in post-estimation outputs. Similar to Knittel and Metaxoglou (2014), we exclude values from routines that obviously failed to find a local minima. We consider the mean own-price elasticity and markup, averaged across all products and markups. As in Knittel and Metaxoglou (2014), we also consider the elasticity and markup of the top and median products in terms of marketshare. We find very little variation in post-estimation outputs.

[65] Like the simulations, we draw starting values from a uniform distribution with support 50% above and below the starting values we use in in the replications.

Table C.1: Optimization Algorithms: Example Problems

						Percent of Runs		Median, First GMM Step			
Example	P	Software	Algorithm	Gradient	Termination	Converged	PSD Hessian	Seconds	Evaluations	$q = g'Wg$	$\|\|\nabla q\|\|_\infty$
Nevo: Replication	13	Knitro	Interior/Direct	Yes	$\|\|\nabla q\|\|_\infty$	100.0%	100.0%	183.2	150	4.56E+00	5.19E-04
Nevo: Replication	13	Knitro	Active Set	Yes	$\|\|\nabla q\|\|_\infty$	100.0%	100.0%	69.4	121	4.56E+00	3.90E-04
Nevo: Replication	13	Knitro	SQP	Yes	$\|\|\nabla q\|\|_\infty$	100.0%	100.0%	99.9	94	4.56E+00	4.45E-04
Nevo: Replication	13	SciPy	Trust-Region	Yes	$\|\|\nabla q\|\|_\infty$	100.0%	100.0%	1,928.9	3,085	4.56E+00	7.94E-05
Nevo: Replication	13	SciPy	L-BFGS-B	Yes	$\|\|\nabla q\|\|_\infty$	100.0%	100.0%	999.0	1,642	4.56E+00	7.88E-05
Nevo: Replication	13	SciPy	BFGS	Yes	$\|\|\nabla q\|\|_\infty$	100.0%	100.0%	35.2	60	4.56E+00	3.60E-05
Nevo: Replication	13	SciPy	TNC	Yes	$\|\|\nabla q\|\|_\infty$	7.0%	100.0%	682.6	872	4.56E+00	1.93E-05
Nevo: Replication	13	SciPy	TNC	Yes	$\|\|\theta^n - \theta^{n-1}\|\|_\infty$	100.0%	100.0%	73.8	122	4.56E+00	1.65E-01
Nevo: Replication	13	SciPy	Nelder-Mead	No	$\|\|\theta^n - \theta^{n-1}\|\|_\infty$	0.0%	100.0%	4,107.1	10,001	4.56E+00	3.66E-04
Nevo: Restricted	12	Knitro	Interior/Direct	Yes	$\|\|\nabla q\|\|_\infty$	100.0%	100.0%	172.2	129	1.54E+01	3.49E-04
Nevo: Restricted	12	Knitro	Active Set	Yes	$\|\|\nabla q\|\|_\infty$	100.0%	100.0%	51.6	91	1.54E+01	3.19E-04
Nevo: Restricted	12	Knitro	SQP	Yes	$\|\|\nabla q\|\|_\infty$	100.0%	100.0%	82.0	69	1.54E+01	2.66E-04
Nevo: Restricted	12	SciPy	Trust-Region	Yes	$\|\|\nabla q\|\|_\infty$	100.0%	100.0%	137.3	252	1.54E+01	6.16E-05
Nevo: Restricted	12	SciPy	L-BFGS-B	Yes	$\|\|\nabla q\|\|_\infty$	100.0%	100.0%	433.8	777	1.54E+01	7.94E-05
Nevo: Restricted	12	SciPy	BFGS	Yes	$\|\|\nabla q\|\|_\infty$	100.0%	100.0%	20.3	36	1.54E+01	3.21E-05
Nevo: Restricted	12	SciPy	TNC	Yes	$\|\|\nabla q\|\|_\infty$	1.0%	100.0%	151.2	269	1.54E+01	3.12E-05
Nevo: Restricted	12	SciPy	TNC	Yes	$\|\|\theta^n - \theta^{n-1}\|\|_\infty$	100.0%	100.0%	30.5	55	1.54E+01	3.78E-01
Nevo: Restricted	12	SciPy	Nelder-Mead	No	$\|\|\theta^n - \theta^{n-1}\|\|_\infty$	14.0%	100.0%	3,726.2	10,005	1.54E+01	6.84E-05
Nevo: Optimal Instruments	12	Knitro	Interior/Direct	Yes	$\|\|\nabla q\|\|_\infty$	100.0%	100.0%	177.7	143	6.81E-11	2.78E-04
Nevo: Optimal Instruments	12	Knitro	Active Set	Yes	$\|\|\nabla q\|\|_\infty$	100.0%	100.0%	55.8	96	1.22E-10	2.76E-04
Nevo: Optimal Instruments	12	Knitro	SQP	Yes	$\|\|\nabla q\|\|_\infty$	100.0%	100.0%	89.6	73	4.05E-11	2.20E-04
Nevo: Optimal Instruments	12	SciPy	Trust-Region	Yes	$\|\|\nabla q\|\|_\infty$	100.0%	100.0%	133.6	244	2.10E-09	6.16E-05
Nevo: Optimal Instruments	12	SciPy	L-BFGS-B	Yes	$\|\|\nabla q\|\|_\infty$	100.0%	100.0%	376.4	679	6.36E-10	8.23E-05
Nevo: Optimal Instruments	12	SciPy	BFGS	Yes	$\|\|\nabla q\|\|_\infty$	100.0%	100.0%	18.0	33	2.30E-13	2.51E-05
Nevo: Optimal Instruments	12	SciPy	TNC	Yes	$\|\|\nabla q\|\|_\infty$	98.0%	100.0%	145.7	257	4.76E-15	3.90E-06
Nevo: Optimal Instruments	12	SciPy	TNC	Yes	$\|\|\theta^n - \theta^{n-1}\|\|_\infty$	100.0%	100.0%	25.6	50	1.15E-04	6.34E-01
Nevo: Optimal Instruments	12	SciPy	Nelder-Mead	No	$\|\|\theta^n - \theta^{n-1}\|\|_\infty$	6.0%	100.0%	3,883.1	10,005	2.39E-26	1.08E-11
BLP: Replication	6	Knitro	Interior/Direct	Yes	$\|\|\nabla q\|\|_\infty$	100.0%	100.0%	78.1	28	3.25E+02	3.08E-05
BLP: Replication	6	Knitro	Active Set	Yes	$\|\|\nabla q\|\|_\infty$	100.0%	100.0%	61.8	25	3.25E+02	2.96E-05
BLP: Replication	6	Knitro	SQP	Yes	$\|\|\nabla q\|\|_\infty$	100.0%	100.0%	68.3	25	3.25E+02	3.21E-05
BLP: Replication	6	SciPy	Trust-Region	Yes	$\|\|\nabla q\|\|_\infty$	100.0%	100.0%	185.9	66	3.25E+02	4.18E-05
BLP: Replication	6	SciPy	L-BFGS-B	Yes	$\|\|\nabla q\|\|_\infty$	100.0%	100.0%	83.5	30	3.25E+02	3.71E-05

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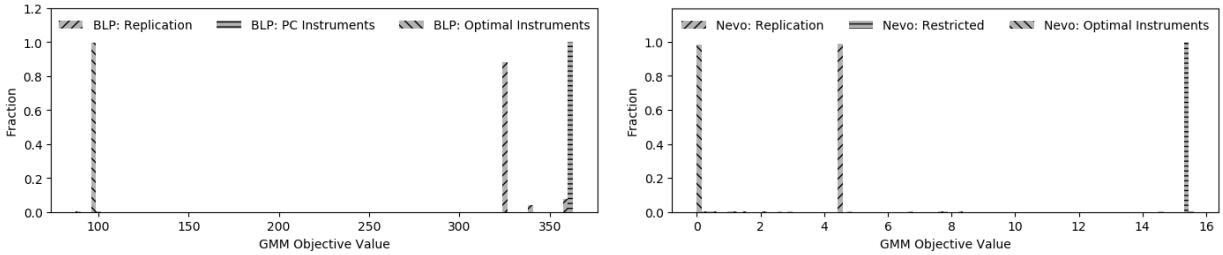
						Percent of Runs		Median, First GMM Step			
Example	P	Software	Algorithm	Gradient	Termination	Converged	PSD Hessian	Seconds	Evaluations	$q = g'Wg$	$\|\|\nabla q\|\|_\infty$
BLP: Replication	6	SciPy	BFGS	Yes	$\|\|\nabla q\|\|_\infty$	100.0%	100.0%	62.2	22	3.25E+02	2.78E-05
BLP: Replication	6	SciPy	TNC	Yes	$\|\|\nabla q\|\|_\infty$	20.0%	95.0%	681.8	215	3.25E+02	3.27E-05
BLP: Replication	6	SciPy	TNC	Yes	$\|\|\theta^n - \theta^{n-1}\|\|_\infty$	100.0%	90.0%	261.1	89	3.25E+02	5.29E-04
BLP: Replication	6	SciPy	Nelder-Mead	No	$\|\|\theta^n - \theta^{n-1}\|\|_\infty$	10.0%	95.0%	3,052.0	10,003	3.25E+02	5.71E-06
BLP: PC Instruments	6	Knitro	Interior/Direct	Yes	$\|\|\nabla q\|\|_\infty$	100.0%	100.0%	70.1	27	3.63E+02	3.38E-05
BLP: PC Instruments	6	Knitro	Active Set	Yes	$\|\|\nabla q\|\|_\infty$	100.0%	100.0%	65.9	26	3.63E+02	3.03E-05
BLP: PC Instruments	6	Knitro	SQP	Yes	$\|\|\nabla q\|\|_\infty$	100.0%	100.0%	69.2	26	3.63E+02	3.56E-05
BLP: PC Instruments	6	SciPy	Trust-Region	Yes	$\|\|\nabla q\|\|_\infty$	100.0%	100.0%	170.4	63	3.63E+02	4.59E-05
BLP: PC Instruments	6	SciPy	L-BFGS-B	Yes	$\|\|\nabla q\|\|_\infty$	100.0%	100.0%	85.0	32	3.63E+02	3.99E-05
BLP: PC Instruments	6	SciPy	BFGS	Yes	$\|\|\nabla q\|\|_\infty$	100.0%	100.0%	70.0	25	3.63E+02	2.77E-05
BLP: PC Instruments	6	SciPy	TNC	Yes	$\|\|\nabla q\|\|_\infty$	63.0%	100.0%	555.6	181	3.63E+02	1.54E-05
BLP: PC Instruments	6	SciPy	TNC	Yes	$\|\|\theta^n - \theta^{n-1}\|\|_\infty$	100.0%	99.0%	233.3	84	3.63E+02	4.53E-04
BLP: PC Instruments	6	SciPy	Nelder-Mead	No	$\|\|\theta^n - \theta^{n-1}\|\|_\infty$	18.0%	100.0%	2,760.1	10,003	3.63E+02	2.18E-06
BLP: Optimal Instruments	6	Knitro	Interior/Direct	Yes	$\|\|\nabla q\|\|_\infty$	100.0%	99.0%	71.6	24	9.64E+01	4.08E-05
BLP: Optimal Instruments	6	Knitro	Active Set	Yes	$\|\|\nabla q\|\|_\infty$	100.0%	100.0%	50.7	21	9.64E+01	3.20E-05
BLP: Optimal Instruments	6	Knitro	SQP	Yes	$\|\|\nabla q\|\|_\infty$	99.0%	99.0%	290.0	21	9.64E+01	2.78E-05
BLP: Optimal Instruments	6	SciPy	Trust-Region	Yes	$\|\|\nabla q\|\|_\infty$	100.0%	100.0%	152.5	52	9.64E+01	4.82E-05
BLP: Optimal Instruments	6	SciPy	L-BFGS-B	Yes	$\|\|\nabla q\|\|_\infty$	100.0%	100.0%	53.6	21	9.64E+01	3.64E-05
BLP: Optimal Instruments	6	SciPy	BFGS	Yes	$\|\|\nabla q\|\|_\infty$	100.0%	100.0%	44.9	17	9.64E+01	3.26E-05
BLP: Optimal Instruments	6	SciPy	TNC	Yes	$\|\|\nabla q\|\|_\infty$	21.0%	100.0%	599.9	202	9.64E+01	3.25E-05
BLP: Optimal Instruments	6	SciPy	TNC	Yes	$\|\|\theta^n - \theta^{n-1}\|\|_\infty$	100.0%	100.0%	184.0	66	9.64E+01	7.71E-04
BLP: Optimal Instruments	6	SciPy	Nelder-Mead	No	$\|\|\theta^n - \theta^{n-1}\|\|_\infty$	17.0%	100.0%	2,067.3	10,003	9.64E+01	2.37E-06

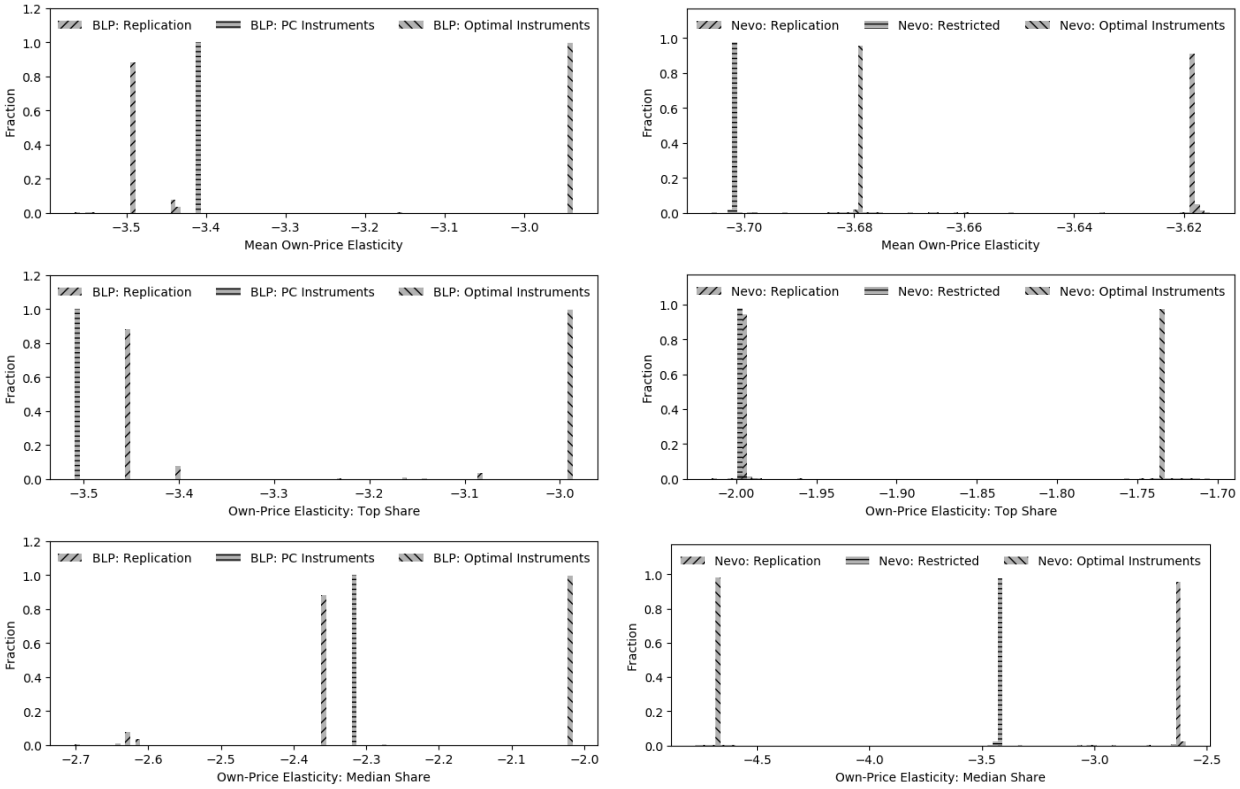
This table documents the same optimization convergence statistics as Table 10, but for the example problems solved with 100 different starting values instead of across different simulated datasets.

Figure C.1: GMM Objective Value Histograms



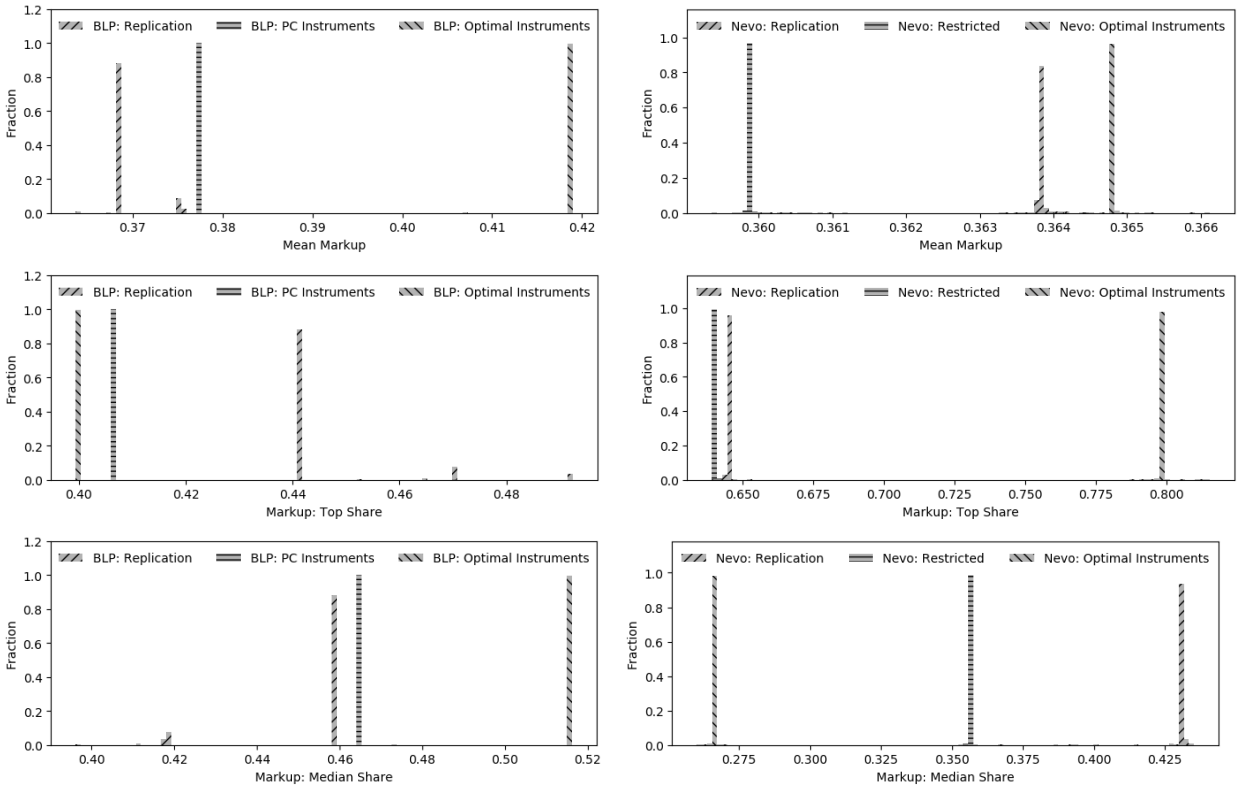
This figure plots the distribution of GMM objective values $q = g'Wg$ from Table C.1. Values for routines that failed to find a local minima are excluded.

Figure C.2: Mean Own-Price Elasticity Histograms



This figure plots distributions of own-price elasticities computed from the results used to create Table C.1. We report elasticities averaged across all products and markets, along with elasticities for the two products considered in Knittel and Metaxoglou (2014): those with the largest and median marketshares across all markets. Values for routines that failed to find a local minima are excluded.

Figure C.3: Mean Markup Histograms



This figure plots distributions of markups computed from the results used to create Table C.1. We report markups averaged across all products and markets, along with markups for the two products considered in Knittel and Metaxoglou (2014): those with the largest and median marketshares across all markets. Values for routines that failed to find a local minima are excluded.

D. Monte Carlo Results for Post-Estimation Outputs

For each set of results used to create the tables in Section 5.2, we report the bias and variance of four post-estimation outputs. Mean own-price elasticities are

$$\bar{\epsilon}_{jj} = \frac{1}{N} \sum_{j,t} \frac{\partial s_{jt}}{p_{jt}} \frac{p_{jt}}{s_{jt}}.$$

Mean aggregate price elasticities are

$$\bar{E} = \frac{1}{N} \sum_{j,t} \frac{s_{jt}(p_{jt} + \Delta p_{jt}) - s_{jt}}{\Delta}$$

where we scale prices by $\Delta = 0.01$. Total producer surplus is

$$\text{PS} = \sum_{j,t} p_{jt} - c_{jt} s_{jt}. \tag{D.6}$$

Total consumer surplus is

$$\text{CS} = \sum_{t,i} w_i \frac{\log(1 + \sum_{j=t}^{J_t} \exp[\delta_{jt} + \mu_{jti}])}{\alpha_i}. \tag{D.7}$$

Table D.2: Alternative Integration Methods: Post-Estimation

					True Median Value				Median Bias				Median Absolute Error			
Simulation	Supply	Integration	I_t	Seconds	$\bar{\varepsilon}_{jj}$	$\bar{E}$	PS	CS	$\bar{\varepsilon}_{jj}$	$\bar{E}$	PS	CS	$\bar{\varepsilon}_{jj}$	$\bar{E}$	PS	CS
Simple	No	Monte Carlo	100	1.5	-3.564	-0.241	2.651	3.653	0.679	0.010	0.402	0.104	0.991	0.063	0.780	0.972
Simple	No	Monte Carlo	1,000	3.1	-3.564	-0.241	2.651	3.653	0.652	0.037	0.540	0.639	0.887	0.057	0.741	0.971
Simple	No	Halton	1,000	2.9	-3.564	-0.241	2.651	3.653	0.630	0.039	0.528	0.626	0.876	0.056	0.711	0.917
Simple	No	Product Rule	9^1	1.1	-3.564	-0.241	2.651	3.653	0.641	0.039	0.560	0.740	0.878	0.055	0.726	0.996
Simple	Yes	Monte Carlo	100	5.4	-3.564	-0.241	2.651	3.653	0.338	-0.009	0.199	-0.074	0.808	0.050	0.585	0.745
Simple	Yes	Monte Carlo	1,000	22.1	-3.564	-0.241	2.651	3.653	0.110	0.004	0.079	0.062	0.628	0.039	0.448	0.599
Simple	Yes	Halton	1,000	20.9	-3.564	-0.241	2.651	3.653	0.080	0.006	0.060	0.040	0.603	0.038	0.443	0.593
Simple	Yes	Product Rule	9^1	4.0	-3.564	-0.241	2.651	3.653	0.045	0.002	0.050	0.079	0.612	0.038	0.450	0.616
Complex	No	Monte Carlo	100	2.7	-3.303	-0.227	2.945	4.500	0.877	0.027	0.908	0.349	1.042	0.064	1.085	1.491
Complex	No	Monte Carlo	1,000	7.3	-3.303	-0.227	2.945	4.500	0.765	0.041	0.786	1.034	0.907	0.055	0.957	1.701
Complex	No	Halton	1,000	7.9	-3.303	-0.227	2.945	4.500	0.778	0.045	0.745	1.025	0.885	0.053	0.888	1.664
Complex	No	Product Rule	9^2	3.4	-3.303	-0.227	2.945	4.500	0.736	0.039	0.676	1.097	0.897	0.052	0.870	2.071
Complex	Yes	Monte Carlo	100	9.2	-3.303	-0.227	2.945	4.500	0.148	-0.026	0.036	-0.666	0.803	0.059	0.716	1.099
Complex	Yes	Monte Carlo	1,000	31.8	-3.303	-0.227	2.945	4.500	0.061	-0.001	0.022	0.047	0.586	0.037	0.540	0.833
Complex	Yes	Halton	1,000	32.1	-3.303	-0.227	2.945	4.500	0.091	0.004	0.032	0.159	0.561	0.034	0.489	0.932
Complex	Yes	Product Rule	9^2	10.3	-3.303	-0.227	2.945	4.500	0.052	0.001	0.013	0.083	0.541	0.033	0.468	0.907
RCNL	No	Monte Carlo	100	8.7	-5.892	-0.195	1.651	3.198	1.054	0.017	0.286	0.309	1.609	0.047	0.464	0.783
RCNL	No	Monte Carlo	1,000	35.2	-5.892	-0.195	1.651	3.198	1.063	0.030	0.337	0.560	1.397	0.042	0.413	0.762
RCNL	No	Halton	1,000	35.2	-5.892	-0.195	1.651	3.198	1.119	0.033	0.342	0.552	1.396	0.041	0.420	0.724
RCNL	No	Product Rule	9^1	7.3	-5.892	-0.195	1.651	3.198	1.082	0.031	0.344	0.666	1.364	0.040	0.415	0.789
RCNL	Yes	Monte Carlo	100	28.9	-5.892	-0.195	1.651	3.198	0.002	-0.011	0.001	-0.137	0.750	0.021	0.192	0.342
RCNL	Yes	Monte Carlo	1,000	113.8	-5.892	-0.195	1.651	3.198	0.056	0.001	0.013	0.016	0.655	0.019	0.173	0.318
RCNL	Yes	Halton	1,000	115.4	-5.892	-0.195	1.651	3.198	0.087	0.002	0.023	-0.007	0.665	0.020	0.182	0.315
RCNL	Yes	Product Rule	9^1	18.8	-5.892	-0.195	1.651	3.198	0.013	0.001	0.013	0.053	0.631	0.018	0.172	0.327

This table reports bias and variance of post-estimation outputs along with estimation speed over 1,000 simulated datasets for different numerical integration methods and numbers of draws I_t. It is created from the same results as Table 5.

Table D.3: Alternative Instruments: Post-Estimation

				True Median Value				Median Bias				Median Absolute Error			
Simulation	Supply	Instruments	Seconds	$\bar{\varepsilon}_{jj}$	$\bar{E}$	PS	CS	$\bar{\varepsilon}_{jj}$	$\bar{E}$	PS	CS	$\bar{\varepsilon}_{jj}$	$\bar{E}$	PS	CS
Simple	No	$[1, x, w]$	0.1	-3.564	-0.241	2.651	3.653	0.051	-0.005	-0.005	0.013	0.946	0.070	0.707	1.031
Simple	No	BLP	0.8	-3.564	-0.241	2.651	3.653	0.540	0.035	0.431	0.636	0.844	0.058	0.697	1.054
Simple	No	Local	0.5	-3.564	-0.241	2.651	3.653	0.285	0.016	0.198	0.314	0.860	0.054	0.656	0.872
Simple	No	Quadratic	0.5	-3.564	-0.241	2.651	3.653	0.319	0.021	0.257	0.437	0.885	0.056	0.670	0.924
Simple	No	Optimal	1.1	-3.564	-0.241	2.651	3.653	0.641	0.039	0.560	0.740	0.878	0.055	0.726	0.996
Simple	Yes	$[1, x, w]$	0.6	-3.564	-0.241	2.651	3.653	0.065	-0.001	0.038	0.092	0.955	0.069	0.710	1.042
Simple	Yes	BLP	2.5	-3.564	-0.241	2.651	3.653	0.153	0.008	0.091	0.237	0.844	0.056	0.621	0.881
Simple	Yes	Local	1.2	-3.564	-0.241	2.651	3.653	0.200	0.008	0.148	0.195	0.764	0.049	0.567	0.775
Simple	Yes	Quadratic	1.3	-3.564	-0.241	2.651	3.653	0.250	0.014	0.184	0.312	0.818	0.052	0.621	0.868
Simple	Yes	Optimal	4.0	-3.564	-0.241	2.651	3.653	0.045	0.002	0.050	0.079	0.612	0.038	0.450	0.616
Complex	No	$[1, x, w]$	0.2	-3.303	-0.227	2.945	4.500	0.021	0.001	-0.022	-0.110	0.925	0.068	0.801	1.512
Complex	No	BLP	2.8	-3.303	-0.227	2.945	4.500	0.746	0.037	0.803	0.945	0.919	0.058	0.965	1.770
Complex	No	Local	1.3	-3.303	-0.227	2.945	4.500	0.587	0.033	0.589	0.443	0.919	0.051	0.880	1.820
Complex	No	Quadratic	1.1	-3.303	-0.227	2.945	4.500	0.665	0.040	0.542	0.840	0.942	0.056	0.861	1.715
Complex	No	Optimal	3.4	-3.303	-0.227	2.945	4.500	0.736	0.039	0.676	1.097	0.897	0.052	0.870	2.071
Complex	Yes	$[1, x, w]$	1.2	-3.303	-0.227	2.945	4.500	-0.000	-0.001	-0.034	-0.514	0.913	0.063	0.765	1.661
Complex	Yes	BLP	8.9	-3.303	-0.227	2.945	4.500	0.340	0.007	0.256	0.544	0.758	0.047	0.683	1.515
Complex	Yes	Local	3.6	-3.303	-0.227	2.945	4.500	0.507	0.022	0.368	1.450	0.791	0.042	0.675	2.078
Complex	Yes	Quadratic	4.1	-3.303	-0.227	2.945	4.500	0.752	0.045	0.582	1.925	1.010	0.057	0.868	3.592
Complex	Yes	Optimal	10.3	-3.303	-0.227	2.945	4.500	0.052	0.001	0.013	0.083	0.541	0.033	0.468	0.907
RCNL	No	$[1, x, w]$	0.5	-5.892	-0.195	1.651	3.198	-0.071	0.029	0.055	0.592	1.831	0.066	0.541	1.169
RCNL	No	BLP	5.7	-5.892	-0.195	1.651	3.198	0.933	0.021	0.281	0.486	1.315	0.041	0.398	0.794
RCNL	No	Local	3.2	-5.892	-0.195	1.651	3.198	0.554	0.019	0.187	0.467	1.341	0.041	0.394	0.743
RCNL	No	Quadratic	3.3	-5.892	-0.195	1.651	3.198	0.646	0.021	0.206	0.509	1.367	0.041	0.382	0.758
RCNL	No	Optimal	7.3	-5.892	-0.195	1.651	3.198	1.082	0.031	0.344	0.666	1.364	0.040	0.415	0.789
RCNL	Yes	$[1, x, w]$	2.0	-5.892	-0.195	1.651	3.198	-0.026	-0.002	-0.008	0.078	1.598	0.046	0.395	0.725
RCNL	Yes	BLP	11.6	-5.892	-0.195	1.651	3.198	0.106	-0.007	0.018	-0.034	1.095	0.038	0.310	0.631
RCNL	Yes	Local	6.1	-5.892	-0.195	1.651	3.198	0.188	0.008	0.071	0.214	1.009	0.030	0.278	0.518
RCNL	Yes	Quadratic	6.5	-5.892	-0.195	1.651	3.198	0.193	0.009	0.070	0.243	0.942	0.029	0.259	0.510
RCNL	Yes	Optimal	18.8	-5.892	-0.195	1.651	3.198	0.013	0.001	0.013	0.053	0.631	0.018	0.172	0.327

This table documents bias and variance of post-estimation outputs along with estimation speed over 1,000 simulated datasets for different instruments. It is created from the same results as Table 6.

Table D.4: Form of Optimal Instruments: Post-Estimation

				True Median Value				Median Bias				Median Absolute Error			
Simulation	Supply	Optimality	Seconds	$\bar{\varepsilon}_{jj}$	$\bar{E}$	PS	CS	$\bar{\varepsilon}_{jj}$	$\bar{E}$	PS	CS	$\bar{\varepsilon}_{jj}$	$\bar{E}$	PS	CS
Simple	No	Approximate	1.1	-3.564	-0.241	2.651	3.653	0.641	0.039	0.560	0.740	0.878	0.055	0.726	0.996
Simple	No	Asymptotic	4.1	-3.564	-0.241	2.651	3.653	0.638	0.039	0.549	0.737	0.876	0.056	0.717	0.995
Simple	No	Empirical	4.1	-3.564	-0.241	2.651	3.653	0.645	0.039	0.559	0.745	0.876	0.055	0.713	0.994
Simple	Yes	Approximate	4.0	-3.564	-0.241	2.651	3.653	0.045	0.002	0.050	0.079	0.612	0.038	0.450	0.616
Simple	Yes	Asymptotic	19.3	-3.564	-0.241	2.651	3.653	0.094	0.004	0.070	0.108	0.623	0.038	0.455	0.616
Simple	Yes	Empirical	19.3	-3.564	-0.241	2.651	3.653	0.059	0.001	0.043	0.075	0.599	0.038	0.453	0.621
Complex	No	Approximate	3.4	-3.303	-0.227	2.945	4.500	0.736	0.039	0.676	1.097	0.897	0.052	0.870	2.071
Complex	No	Asymptotic	7.3	-3.303	-0.227	2.945	4.500	0.740	0.040	0.692	1.247	0.926	0.053	0.888	2.115
Complex	No	Empirical	7.3	-3.303	-0.227	2.945	4.500	0.774	0.041	0.715	1.133	0.933	0.053	0.896	2.073
Complex	Yes	Approximate	10.3	-3.303	-0.227	2.945	4.500	0.052	0.001	0.013	0.083	0.541	0.033	0.468	0.907
Complex	Yes	Asymptotic	36.2	-3.303	-0.227	2.945	4.500	0.084	0.003	0.010	0.096	0.643	0.039	0.549	1.002
Complex	Yes	Empirical	36.0	-3.303	-0.227	2.945	4.500	0.082	0.002	-0.006	0.069	0.600	0.036	0.518	0.985
RCNL	No	Approximate	7.3	-5.892	-0.195	1.651	3.198	1.082	0.031	0.344	0.666	1.364	0.040	0.415	0.789
RCNL	No	Asymptotic	11.7	-5.892	-0.195	1.651	3.198	1.062	0.031	0.337	0.660	1.356	0.040	0.412	0.783
RCNL	No	Empirical	11.6	-5.892	-0.195	1.651	3.198	1.108	0.032	0.350	0.681	1.362	0.039	0.415	0.776
RCNL	Yes	Approximate	18.8	-5.892	-0.195	1.651	3.198	0.013	0.001	0.013	0.053	0.631	0.018	0.172	0.327
RCNL	Yes	Asymptotic	59.4	-5.892	-0.195	1.651	3.198	0.050	0.001	0.012	0.064	0.626	0.019	0.169	0.327
RCNL	Yes	Empirical	58.8	-5.892	-0.195	1.651	3.198	0.051	0.000	0.020	0.054	0.639	0.019	0.168	0.330

This table documents bias and variance of post-estimation outputs along with estimation speed over 1,000 simulated datasets for different forms of optimal instruments. It is created from the same results as Table 7.

Table D.5: Varying Instrument Strength: Post-Estimation

					True Median Value				Median Bias				Median Absolute Error			
Simulation	γ_2	Corr(p,w)	Supply	Seconds	$\bar{\varepsilon}_{jj}$	$\bar{E}$	PS	CS	$\bar{\varepsilon}_{jj}$	$\bar{E}$	PS	CS	$\bar{\varepsilon}_{jj}$	$\bar{E}$	PS	CS
Simple	0.0	0.001	No	1.1	-3.467	-0.244	2.793	3.881	1.303	0.085	1.111	1.546	1.465	0.100	1.705	2.278
Simple	0.0	0.001	Yes	4.2	-3.467	-0.244	2.793	3.881	0.033	0.000	0.024	0.087	0.736	0.049	0.578	0.776
Simple	0.1	0.052	No	1.1	-3.516	-0.243	2.720	3.763	1.051	0.067	0.963	1.286	1.265	0.083	1.260	1.684
Simple	0.1	0.052	Yes	4.2	-3.516	-0.243	2.720	3.763	0.036	0.002	0.033	0.102	0.735	0.047	0.557	0.757
Simple	0.2	0.102	No	1.1	-3.564	-0.241	2.651	3.653	0.641	0.039	0.560	0.740	0.878	0.055	0.726	0.996
Simple	0.2	0.102	Yes	4.0	-3.564	-0.241	2.651	3.653	0.045	0.002	0.050	0.079	0.612	0.038	0.450	0.616
Simple	0.4	0.199	No	1.1	-3.663	-0.237	2.516	3.454	0.237	0.012	0.165	0.241	0.457	0.029	0.333	0.474
Simple	0.4	0.199	Yes	3.9	-3.663	-0.237	2.516	3.454	0.027	-0.001	0.026	0.043	0.409	0.026	0.275	0.390
Simple	0.8	0.376	No	1.1	-3.860	-0.229	2.278	3.082	0.072	0.003	0.046	0.080	0.241	0.017	0.142	0.229
Simple	0.8	0.376	Yes	4.1	-3.860	-0.229	2.278	3.082	0.002	-0.001	0.005	0.026	0.228	0.015	0.139	0.208
Complex	0.0	0.002	No	3.4	-3.224	-0.230	3.088	4.737	1.240	0.071	1.190	1.005	1.412	0.086	1.765	3.937
Complex	0.0	0.002	Yes	10.5	-3.224	-0.230	3.088	4.737	0.028	-0.000	0.020	0.126	0.587	0.038	0.559	1.084
Complex	0.1	0.054	No	3.4	-3.264	-0.228	3.016	4.617	1.069	0.061	1.074	1.266	1.255	0.075	1.425	2.953
Complex	0.1	0.054	Yes	10.4	-3.264	-0.228	3.016	4.617	0.052	-0.000	0.019	0.164	0.591	0.037	0.548	1.062
Complex	0.2	0.104	No	3.4	-3.303	-0.227	2.945	4.500	0.736	0.039	0.676	1.097	0.897	0.052	0.870	2.071
Complex	0.2	0.104	Yes	10.3	-3.303	-0.227	2.945	4.500	0.052	0.001	0.013	0.083	0.541	0.033	0.468	0.907
Complex	0.4	0.204	No	3.5	-3.381	-0.223	2.818	4.281	0.311	0.014	0.220	0.608	0.519	0.029	0.420	1.094
Complex	0.4	0.204	Yes	10.3	-3.381	-0.223	2.818	4.281	0.024	-0.000	-0.004	0.084	0.406	0.024	0.333	0.717
Complex	0.8	0.384	No	3.5	-3.535	-0.215	2.580	3.877	0.111	0.002	0.033	0.234	0.268	0.015	0.193	0.573
Complex	0.8	0.384	Yes	11.4	-3.535	-0.215	2.580	3.877	0.014	-0.001	-0.019	0.116	0.239	0.015	0.178	0.477
RCNL	0.0	0.001	No	7.3	-5.705	-0.197	1.737	3.405	1.639	0.051	0.554	1.091	1.885	0.060	0.718	1.389
RCNL	0.0	0.001	Yes	19.2	-5.705	-0.197	1.737	3.405	0.057	0.001	0.026	0.080	0.678	0.021	0.202	0.376
RCNL	0.1	0.050	No	7.2	-5.798	-0.196	1.692	3.302	1.515	0.045	0.518	1.021	1.746	0.052	0.603	1.179
RCNL	0.1	0.050	Yes	19.0	-5.798	-0.196	1.692	3.302	0.013	0.001	0.014	0.066	0.660	0.020	0.191	0.356
RCNL	0.2	0.097	No	7.3	-5.892	-0.195	1.651	3.198	1.082	0.031	0.344	0.666	1.364	0.040	0.415	0.789
RCNL	0.2	0.097	Yes	18.8	-5.892	-0.195	1.651	3.198	0.013	0.001	0.013	0.053	0.631	0.018	0.172	0.327
RCNL	0.4	0.189	No	7.3	-6.079	-0.192	1.571	3.011	0.515	0.015	0.148	0.306	0.835	0.024	0.220	0.420
RCNL	0.4	0.189	Yes	18.8	-6.079	-0.192	1.571	3.011	0.042	0.001	0.013	0.050	0.531	0.016	0.131	0.255
RCNL	0.8	0.358	No	7.3	-6.457	-0.185	1.431	2.690	0.167	0.006	0.041	0.099	0.443	0.012	0.095	0.182
RCNL	0.8	0.358	Yes	18.9	-6.457	-0.185	1.431	2.690	0.007	0.002	0.006	0.042	0.372	0.010	0.080	0.155

This table documents bias and variance of post-estimation outputs along with estimation speed over 1,000 simulated datasets for varying instrument strength. It is created from the same results as Table 8.

Table D.6: Problem Scaling: Post-Estimation

					True Median Value				Median Bias				Median Absolute Error			
Simulation	Supply	T	J_f	Seconds	$\bar{\varepsilon}_{jj}$	$\bar{E}$	PS	CS	$\bar{\varepsilon}_{jj}$	$\bar{E}$	PS	CS	$\bar{\varepsilon}_{jj}$	$\bar{E}$	PS	CS
Simple	No	20	$\{2, 5, 10\}$	1.1	-3.564	-0.241	2.651	3.653	0.641	0.039	0.560	0.740	0.878	0.055	0.726	0.996
Simple	No	40	$\{2, 5, 10\}$	2.3	-3.566	-0.242	5.330	7.402	0.366	0.021	0.599	0.796	0.620	0.038	0.921	1.285
Simple	No	100	$\{2, 5, 10\}$	6.1	-3.567	-0.242	13.356	18.591	0.155	0.008	0.594	0.897	0.372	0.023	1.424	1.920
Simple	No	20	$\{4, 10, 20\}$	1.4	-3.581	-0.315	3.727	5.920	0.366	0.027	0.389	0.622	0.596	0.051	0.663	1.023
Simple	No	20	$\{10, 25, 50\}$	1.9	-3.591	-0.407	5.474	10.180	0.161	0.024	0.236	0.416	0.419	0.049	0.645	1.214
Simple	Yes	20	$\{2, 5, 10\}$	4.0	-3.564	-0.241	2.651	3.653	0.045	0.002	0.050	0.079	0.612	0.038	0.450	0.616
Simple	Yes	40	$\{2, 5, 10\}$	7.8	-3.566	-0.242	5.330	7.402	0.066	0.002	0.091	0.176	0.435	0.027	0.641	0.844
Simple	Yes	100	$\{2, 5, 10\}$	20.3	-3.567	-0.242	13.356	18.591	0.016	-0.002	0.045	0.153	0.285	0.017	1.016	1.383
Simple	Yes	20	$\{4, 10, 20\}$	5.5	-3.581	-0.315	3.727	5.920	0.033	-0.002	0.025	0.040	0.538	0.046	0.537	0.864
Simple	Yes	20	$\{10, 25, 50\}$	33.1	-3.591	-0.407	5.474	10.180	0.033	0.006	0.057	-0.018	0.406	0.048	0.622	1.165
Complex	No	20	$\{2, 5, 10\}$	3.4	-3.303	-0.227	2.945	4.500	0.736	0.039	0.676	1.097	0.897	0.052	0.870	2.071
Complex	No	40	$\{2, 5, 10\}$	7.3	-3.304	-0.228	5.913	9.104	0.450	0.024	0.785	1.425	0.644	0.037	1.178	2.368
Complex	No	100	$\{2, 5, 10\}$	21.3	-3.304	-0.228	14.780	22.763	0.190	0.010	0.747	1.337	0.395	0.024	1.737	3.108
Complex	No	20	$\{4, 10, 20\}$	4.9	-3.338	-0.298	4.054	7.154	0.465	0.030	0.524	1.044	0.699	0.052	0.829	2.244
Complex	No	20	$\{10, 25, 50\}$	7.9	-3.377	-0.387	5.837	12.031	0.220	0.018	0.304	0.628	0.478	0.050	0.847	2.625
Complex	Yes	20	$\{2, 5, 10\}$	10.3	-3.303	-0.227	2.945	4.500	0.052	0.001	0.013	0.083	0.541	0.033	0.468	0.907
Complex	Yes	40	$\{2, 5, 10\}$	21.5	-3.304	-0.228	5.913	9.104	0.047	0.001	0.051	0.307	0.378	0.022	0.683	1.261
Complex	Yes	100	$\{2, 5, 10\}$	62.7	-3.304	-0.228	14.780	22.763	0.030	0.001	0.088	0.462	0.259	0.016	1.099	2.090
Complex	Yes	20	$\{4, 10, 20\}$	18.7	-3.338	-0.298	4.054	7.154	0.033	-0.002	-0.037	0.349	0.510	0.041	0.593	1.434
Complex	Yes	20	$\{10, 25, 50\}$	108.9	-3.377	-0.387	5.837	12.031	0.042	-0.000	0.009	0.293	0.375	0.042	0.647	1.955
RCNL	No	20	$\{2, 5, 10\}$	7.3	-5.892	-0.195	1.651	3.198	1.082	0.031	0.344	0.666	1.364	0.040	0.415	0.789
RCNL	No	40	$\{2, 5, 10\}$	15.5	-5.889	-0.194	3.269	6.457	0.645	0.018	0.388	0.770	0.975	0.028	0.556	1.002
RCNL	No	100	$\{2, 5, 10\}$	46.5	-5.893	-0.194	8.200	16.167	0.265	0.007	0.372	0.826	0.600	0.017	0.807	1.450
RCNL	No	20	$\{4, 10, 20\}$	10.0	-6.034	-0.232	1.862	4.611	0.840	0.025	0.285	0.732	1.141	0.039	0.376	0.908
RCNL	No	20	$\{10, 25, 50\}$	19.0	-6.132	-0.273	2.141	6.646	0.404	0.013	0.150	0.434	0.738	0.032	0.267	0.807
RCNL	Yes	20	$\{2, 5, 10\}$	18.8	-5.892	-0.195	1.651	3.198	0.013	0.001	0.013	0.053	0.631	0.018	0.172	0.327
RCNL	Yes	40	$\{2, 5, 10\}$	42.8	-5.889	-0.194	3.269	6.457	0.021	-0.001	0.015	0.051	0.478	0.014	0.253	0.472
RCNL	Yes	100	$\{2, 5, 10\}$	124.8	-5.893	-0.194	8.200	16.167	-0.019	-0.002	-0.020	0.065	0.294	0.009	0.402	0.769
RCNL	Yes	20	$\{4, 10, 20\}$	31.1	-6.034	-0.232	1.862	4.611	0.040	-0.003	0.016	0.058	0.704	0.026	0.216	0.514
RCNL	Yes	20	$\{10, 25, 50\}$	137.1	-6.132	-0.273	2.141	6.646	0.029	-0.004	0.010	0.042	0.664	0.030	0.229	0.710

This table documents bias and variance of post-estimation outputs along with estimation speed over 1,000 simulated datasets for different problem sizes. It is created from the same results as Table 9.

E. Monte Carlo Results for Merger Simulation

For each set of results used to create the tables in Section 5.2, we report the bias and variance of three merger simulation outputs.

After a merger of three out of the five firms, we compute post-merger prices and shares with the ζ-markup approach in Section 3.6. We then compute the post-merger change in total producer surplus, PS from (D.6), the change in total consumer surplus, CS from (D.7), and the change in the mean Herfindahl-Hirschman Index,

$$\text{HHI} = \frac{10{,}000}{T} \sum_{f,t} \Big(\sum_{j \in J_{ft}} s_{jt} \Big)^2.$$

Table E.7: Alternative Integration Methods: Merger

					True Median Value			Median Bias			Median Absolute Error		
Simulation	Supply	Integration	I_t	Seconds	$\Delta\overline{\text{HHI}}$	ΔPS	ΔCS	$\Delta\overline{\text{HHI}}$	ΔPS	ΔCS	$\Delta\overline{\text{HHI}}$	ΔPS	ΔCS
Simple	No	Monte Carlo	100	1.5	1,112.227	0.052	-0.168	38.544	-0.013	0.016	38.643	0.019	0.048
Simple	No	Monte Carlo	1,000	3.1	1,112.227	0.052	-0.168	5.182	0.006	-0.026	10.126	0.013	0.042
Simple	No	Halton	1,000	2.9	1,112.227	0.052	-0.168	1.569	0.009	-0.028	8.405	0.013	0.042
Simple	No	Product Rule	9^1	1.1	1,112.227	0.052	-0.168	1.039	0.008	-0.031	8.430	0.013	0.043
Simple	Yes	Monte Carlo	100	5.4	1,112.227	0.052	-0.168	37.555	-0.016	0.028	37.584	0.017	0.038
Simple	Yes	Monte Carlo	1,000	22.1	1,112.227	0.052	-0.168	3.972	-0.002	0.001	9.541	0.009	0.028
Simple	Yes	Halton	1,000	20.9	1,112.227	0.052	-0.168	0.382	0.000	-0.002	7.586	0.010	0.029
Simple	Yes	Product Rule	9^1	4.0	1,112.227	0.052	-0.168	-0.107	-0.001	-0.002	7.969	0.009	0.029
Complex	No	Monte Carlo	100	2.7	1,102.034	0.062	-0.199	55.241	-0.023	0.034	55.558	0.029	0.073
Complex	No	Monte Carlo	1,000	7.3	1,102.034	0.062	-0.199	10.551	0.002	-0.018	17.530	0.020	0.067
Complex	No	Halton	1,000	7.9	1,102.034	0.062	-0.199	2.847	0.007	-0.029	18.205	0.019	0.067
Complex	No	Product Rule	9^2	3.4	1,102.034	0.062	-0.199	4.361	0.000	-0.006	22.742	0.021	0.080
Complex	Yes	Monte Carlo	100	9.2	1,102.034	0.062	-0.199	49.505	-0.024	0.046	50.970	0.025	0.057
Complex	Yes	Monte Carlo	1,000	31.8	1,102.034	0.062	-0.199	6.063	-0.003	0.004	13.494	0.011	0.035
Complex	Yes	Halton	1,000	32.1	1,102.034	0.062	-0.199	-2.820	0.001	-0.005	13.454	0.010	0.033
Complex	Yes	Product Rule	9^2	10.3	1,102.034	0.062	-0.199	-3.620	0.000	-0.004	13.636	0.010	0.032
RCNL	No	Monte Carlo	100	8.7	934.505	0.070	-0.160	18.270	0.000	-0.008	18.662	0.017	0.037
RCNL	No	Monte Carlo	1,000	35.2	934.505	0.070	-0.160	3.473	0.010	-0.026	6.398	0.015	0.035
RCNL	No	Halton	1,000	35.2	934.505	0.070	-0.160	2.865	0.012	-0.026	6.181	0.015	0.036
RCNL	No	Product Rule	9^1	7.3	934.505	0.070	-0.160	0.102	0.013	-0.031	5.500	0.016	0.037
RCNL	Yes	Monte Carlo	100	28.9	934.505	0.070	-0.160	14.081	-0.007	0.009	14.633	0.009	0.018
RCNL	Yes	Monte Carlo	1,000	113.8	934.505	0.070	-0.160	1.374	-0.000	-0.000	4.915	0.007	0.016
RCNL	Yes	Halton	1,000	115.4	934.505	0.070	-0.160	0.903	0.000	-0.001	4.029	0.007	0.016
RCNL	Yes	Product Rule	9^1	18.8	934.505	0.070	-0.160	-1.762	0.001	-0.003	5.334	0.007	0.017

This table reports bias and variance of merger simulation estimates along with estimation speed over 1,000 simulated datasets for different numerical integration methods and numbers of draws I_t. It is created from the same results as Table 5.

Table E.8: Alternative Instruments: Merger

				True Median Value			Median Bias			Median Absolute Error		
Simulation	Supply	Instruments	Seconds	$\Delta\overline{\text{HHI}}$	ΔPS	ΔCS	$\Delta\overline{\text{HHI}}$	ΔPS	ΔCS	$\Delta\overline{\text{HHI}}$	ΔPS	ΔCS
Simple	No	$[1, x, w]$	0.1	1,112.227	0.052	-0.168	0.000	-0.004	0.008	36.980	0.023	0.057
Simple	No	BLP	0.8	1,112.227	0.052	-0.168	0.632	0.006	-0.024	18.846	0.017	0.047
Simple	No	Local	0.5	1,112.227	0.052	-0.168	0.000	0.003	-0.011	13.479	0.013	0.040
Simple	No	Quadratic	0.5	1,112.227	0.052	-0.168	1.648	0.003	-0.014	17.455	0.014	0.042
Simple	No	Optimal	1.1	1,112.227	0.052	-0.168	1.039	0.008	-0.031	8.430	0.013	0.043
Simple	Yes	$[1, x, w]$	0.6	1,112.227	0.052	-0.168	2.343	-0.003	0.004	33.945	0.022	0.053
Simple	Yes	BLP	2.5	1,112.227	0.052	-0.168	1.173	-0.001	-0.005	18.841	0.014	0.042
Simple	Yes	Local	1.2	1,112.227	0.052	-0.168	0.773	0.001	-0.007	13.633	0.012	0.036
Simple	Yes	Quadratic	1.3	1,112.227	0.052	-0.168	3.424	0.001	-0.010	17.441	0.014	0.042
Simple	Yes	Optimal	4.0	1,112.227	0.052	-0.168	-0.107	-0.001	-0.002	7.969	0.009	0.029
Complex	No	$[1, x, w]$	0.2	1,102.034	0.062	-0.199	3.869	-0.008	0.019	36.552	0.027	0.069
Complex	No	BLP	2.8	1,102.034	0.062	-0.199	16.855	-0.005	-0.004	29.910	0.030	0.078
Complex	No	Local	1.3	1,102.034	0.062	-0.199	16.478	-0.007	0.015	29.733	0.027	0.092
Complex	No	Quadratic	1.1	1,102.034	0.062	-0.199	22.426	-0.005	0.005	38.951	0.027	0.082
Complex	No	Optimal	3.4	1,102.034	0.062	-0.199	4.361	0.000	-0.006	22.742	0.021	0.080
Complex	Yes	$[1, x, w]$	1.2	1,102.034	0.062	-0.199	13.536	-0.018	0.042	44.140	0.030	0.077
Complex	Yes	BLP	8.9	1,102.034	0.062	-0.199	7.001	-0.007	-0.002	22.450	0.018	0.050
Complex	Yes	Local	3.6	1,102.034	0.062	-0.199	1.883	-0.008	0.012	33.465	0.023	0.079
Complex	Yes	Quadratic	4.1	1,102.034	0.062	-0.199	813.745	-5,805.879	5,669.888	813.745	6,194.796	6,159.857
Complex	Yes	Optimal	10.3	1,102.034	0.062	-0.199	-3.620	0.000	-0.004	13.636	0.010	0.032
RCNL	No	$[1, x, w]$	0.5	934.505	0.070	-0.160	-12.513	0.011	-0.014	29.579	0.028	0.053
RCNL	No	BLP	5.7	934.505	0.070	-0.160	8.793	0.006	-0.018	21.806	0.017	0.038
RCNL	No	Local	3.2	934.505	0.070	-0.160	-0.813	0.009	-0.021	7.552	0.015	0.035
RCNL	No	Quadratic	3.3	934.505	0.070	-0.160	-0.519	0.010	-0.023	7.577	0.016	0.037
RCNL	No	Optimal	7.3	934.505	0.070	-0.160	0.102	0.013	-0.031	5.500	0.016	0.037
RCNL	Yes	$[1, x, w]$	2.0	934.505	0.070	-0.160	-2.176	-0.002	0.001	33.247	0.019	0.037
RCNL	Yes	BLP	11.6	934.505	0.070	-0.160	7.109	-0.004	0.004	20.759	0.016	0.032
RCNL	Yes	Local	6.1	934.505	0.070	-0.160	-1.576	0.004	-0.011	7.356	0.011	0.025
RCNL	Yes	Quadratic	6.5	934.505	0.070	-0.160	-1.009	0.004	-0.010	7.620	0.011	0.025
RCNL	Yes	Optimal	18.8	934.505	0.070	-0.160	-1.762	0.001	-0.003	5.334	0.007	0.017

This table documents bias and variance of merger simulation estimates along with estimation speed over 1,000 simulated datasets for different instruments. It is created from the same results as Table 6.

Table E.9: Form of Optimal Instruments: Merger

				True Median Value			Median Bias			Median Absolute Error		
Simulation	Supply	Optimality	Seconds	$\Delta\overline{\text{HHI}}$	ΔPS	ΔCS	$\Delta\overline{\text{HHI}}$	ΔPS	ΔCS	$\Delta\overline{\text{HHI}}$	ΔPS	ΔCS
Simple	No	Approximate	1.1	1,112.227	0.052	-0.168	1.039	0.008	-0.031	8.430	0.013	0.043
Simple	No	Asymptotic	4.1	1,112.227	0.052	-0.168	1.004	0.008	-0.030	8.340	0.013	0.043
Simple	No	Empirical	4.1	1,112.227	0.052	-0.168	1.177	0.008	-0.031	8.219	0.013	0.043
Simple	Yes	Approximate	4.0	1,112.227	0.052	-0.168	-0.107	-0.001	-0.002	7.969	0.009	0.029
Simple	Yes	Asymptotic	19.3	1,112.227	0.052	-0.168	-0.125	-0.000	-0.004	7.553	0.009	0.029
Simple	Yes	Empirical	19.3	1,112.227	0.052	-0.168	-0.206	-0.001	-0.003	8.007	0.009	0.028
Complex	No	Approximate	3.4	1,102.034	0.062	-0.199	4.361	0.000	-0.006	22.742	0.021	0.080
Complex	No	Asymptotic	7.3	1,102.034	0.062	-0.199	4.771	-0.000	-0.005	23.247	0.022	0.085
Complex	No	Empirical	7.3	1,102.034	0.062	-0.199	4.683	-0.001	-0.005	22.416	0.021	0.081
Complex	Yes	Approximate	10.3	1,102.034	0.062	-0.199	-3.620	0.000	-0.004	13.636	0.010	0.032
Complex	Yes	Asymptotic	36.2	1,102.034	0.062	-0.199	-0.192	-0.001	0.001	15.022	0.012	0.038
Complex	Yes	Empirical	36.0	1,102.034	0.062	-0.199	-0.779	-0.001	-0.001	14.708	0.011	0.037
RCNL	No	Approximate	7.3	934.505	0.070	-0.160	0.102	0.013	-0.031	5.500	0.016	0.037
RCNL	No	Asymptotic	11.7	934.505	0.070	-0.160	0.051	0.013	-0.030	5.486	0.016	0.038
RCNL	No	Empirical	11.6	934.505	0.070	-0.160	0.057	0.013	-0.030	5.479	0.015	0.037
RCNL	Yes	Approximate	18.8	934.505	0.070	-0.160	-1.762	0.001	-0.003	5.334	0.007	0.017
RCNL	Yes	Asymptotic	59.4	934.505	0.070	-0.160	-1.878	0.001	-0.003	5.152	0.007	0.017
RCNL	Yes	Empirical	58.8	934.505	0.070	-0.160	-1.832	0.001	-0.003	5.355	0.007	0.016

This table documents bias and variance of merger simulation estimates along with estimation speed over 1,000 simulated datasets for different forms of optimal instruments. It is created from the same results as Table 7.

Table E.10: Varying Instrument Strength: Merger

					True Median Value			Median Bias			Median Absolute Error		
Simulation	γ_2	Corr(p, w)	Supply	Seconds	$\Delta\overline{\text{HHI}}$	ΔPS	ΔCS	$\Delta\overline{\text{HHI}}$	ΔPS	ΔCS	$\Delta\overline{\text{HHI}}$	ΔPS	ΔCS
Simple	0.0	0.001	No	1.1	1,114.376	0.056	-0.180	4.224	0.017	-0.066	9.773	0.028	0.102
Simple	0.0	0.001	Yes	4.2	1,114.376	0.056	-0.180	0.361	-0.001	-0.001	7.930	0.012	0.037
Simple	0.1	0.052	No	1.1	1,114.327	0.054	-0.174	2.978	0.015	-0.053	9.163	0.021	0.072
Simple	0.1	0.052	Yes	4.2	1,114.327	0.054	-0.174	0.037	-0.001	-0.002	8.014	0.011	0.035
Simple	0.2	0.102	No	1.1	1,112.227	0.052	-0.168	1.039	0.008	-0.031	8.430	0.013	0.043
Simple	0.2	0.102	Yes	4.0	1,112.227	0.052	-0.168	-0.107	-0.001	-0.002	7.969	0.009	0.029
Simple	0.4	0.199	No	1.1	1,113.231	0.047	-0.158	-0.640	0.002	-0.011	8.171	0.007	0.021
Simple	0.4	0.199	Yes	3.9	1,113.231	0.047	-0.158	-0.745	-0.001	-0.000	7.784	0.006	0.018
Simple	0.8	0.376	No	1.1	1,115.817	0.040	-0.137	-1.692	0.001	-0.004	8.503	0.005	0.012
Simple	0.8	0.376	Yes	4.1	1,115.817	0.040	-0.137	-1.648	0.000	-0.002	7.659	0.004	0.011
Complex	0.0	0.002	No	3.4	1,105.050	0.067	-0.212	17.539	-0.010	0.026	40.247	0.064	0.253
Complex	0.0	0.002	Yes	10.5	1,105.050	0.067	-0.212	-3.808	-0.000	-0.002	13.588	0.012	0.041
Complex	0.1	0.054	No	3.4	1,104.680	0.064	-0.205	12.644	-0.006	0.015	33.654	0.044	0.163
Complex	0.1	0.054	Yes	10.4	1,104.680	0.064	-0.205	-4.410	0.000	-0.007	13.887	0.011	0.038
Complex	0.2	0.104	No	3.4	1,102.034	0.062	-0.199	4.361	0.000	-0.006	22.742	0.021	0.080
Complex	0.2	0.104	Yes	10.3	1,102.034	0.062	-0.199	-3.620	0.000	-0.004	13.636	0.010	0.032
Complex	0.4	0.204	No	3.5	1,105.469	0.057	-0.187	1.164	-0.001	-0.002	18.857	0.012	0.039
Complex	0.4	0.204	Yes	10.3	1,105.469	0.057	-0.187	-4.071	0.001	-0.003	13.095	0.008	0.024
Complex	0.8	0.384	No	3.5	1,107.039	0.049	-0.167	-1.144	-0.001	-0.002	17.379	0.008	0.021
Complex	0.8	0.384	Yes	11.4	1,107.039	0.049	-0.167	-4.252	0.001	-0.003	12.568	0.006	0.015
RCNL	0.0	0.001	No	7.3	940.632	0.075	-0.170	1.572	0.022	-0.054	6.086	0.028	0.069
RCNL	0.0	0.001	Yes	19.2	940.632	0.075	-0.170	-1.237	0.001	-0.004	5.221	0.008	0.020
RCNL	0.1	0.050	No	7.2	938.357	0.073	-0.165	1.008	0.020	-0.048	5.921	0.023	0.055
RCNL	0.1	0.050	Yes	19.0	938.357	0.073	-0.165	-1.593	0.001	-0.004	5.315	0.008	0.018
RCNL	0.2	0.097	No	7.3	934.505	0.070	-0.160	0.102	0.013	-0.031	5.500	0.016	0.037
RCNL	0.2	0.097	Yes	18.8	934.505	0.070	-0.160	-1.762	0.001	-0.003	5.334	0.007	0.017
RCNL	0.4	0.189	No	7.3	937.049	0.065	-0.150	-1.402	0.006	-0.013	5.669	0.008	0.020
RCNL	0.4	0.189	Yes	18.8	937.049	0.065	-0.150	-2.345	0.001	-0.002	5.226	0.006	0.013
RCNL	0.8	0.358	No	7.3	937.049	0.056	-0.134	-2.624	0.002	-0.006	5.444	0.004	0.009
RCNL	0.8	0.358	Yes	18.9	937.049	0.056	-0.134	-3.068	0.001	-0.002	5.314	0.004	0.008

This table documents bias and variance of merger simulation estimates along with estimation speed over 1,000 simulated datasets for varying instrument strength. It is created from the same results as Table 8.

Table E.11: Problem Scaling: Merger

					True Median Value			Median Bias			Median Absolute Error		
Simulation	Supply	T	J_f	Seconds	$\Delta\overline{\text{HHI}}$	ΔPS	ΔCS	$\Delta\overline{\text{HHI}}$	ΔPS	ΔCS	$\Delta\overline{\text{HHI}}$	ΔPS	ΔCS
Simple	No	20	$\{2, 5, 10\}$	1.1	1,112.227	0.052	-0.168	1.039	0.008	-0.031	8.430	0.013	0.043
Simple	No	40	$\{2, 5, 10\}$	2.3	1,113.283	0.102	-0.342	0.975	0.008	-0.035	6.127	0.017	0.058
Simple	No	100	$\{2, 5, 10\}$	6.1	1,070.906	0.256	-0.845	0.264	0.004	-0.037	4.091	0.025	0.087
Simple	No	20	$\{4, 10, 20\}$	1.4	383.076	0.038	-0.090	0.703	0.002	-0.007	1.786	0.006	0.015
Simple	No	20	$\{10, 25, 50\}$	1.9	68.137	0.013	-0.024	-0.009	0.001	-0.001	0.118	0.002	0.003
Simple	Yes	20	$\{2, 5, 10\}$	4.0	1,112.227	0.052	-0.168	-0.107	-0.001	-0.002	7.969	0.009	0.029
Simple	Yes	40	$\{2, 5, 10\}$	7.8	1,113.283	0.102	-0.342	0.110	-0.001	-0.005	5.921	0.012	0.039
Simple	Yes	100	$\{2, 5, 10\}$	20.3	1,070.906	0.256	-0.845	-0.063	-0.006	-0.005	3.703	0.020	0.064
Simple	Yes	20	$\{4, 10, 20\}$	5.5	383.076	0.038	-0.090	0.480	-0.001	0.000	1.729	0.005	0.012
Simple	Yes	20	$\{10, 25, 50\}$	33.1	68.137	0.013	-0.024	-0.004	0.000	-0.000	0.109	0.002	0.003
Complex	No	20	$\{2, 5, 10\}$	3.4	1,102.034	0.062	-0.199	4.361	0.000	-0.006	22.742	0.021	0.080
Complex	No	40	$\{2, 5, 10\}$	7.3	1,102.163	0.122	-0.404	0.000	0.007	-0.035	13.827	0.025	0.090
Complex	No	100	$\{2, 5, 10\}$	21.3	1,060.386	0.304	-1.002	-0.617	0.009	-0.056	7.117	0.034	0.118
Complex	No	20	$\{4, 10, 20\}$	4.9	379.886	0.044	-0.102	0.716	0.001	-0.003	6.181	0.010	0.028
Complex	No	20	$\{10, 25, 50\}$	7.9	67.645	0.015	-0.027	0.060	0.000	-0.000	0.501	0.002	0.005
Complex	Yes	20	$\{2, 5, 10\}$	10.3	1,102.034	0.062	-0.199	-3.620	0.000	-0.004	13.636	0.010	0.032
Complex	Yes	40	$\{2, 5, 10\}$	21.5	1,102.163	0.122	-0.404	-1.463	0.001	-0.010	9.496	0.013	0.043
Complex	Yes	100	$\{2, 5, 10\}$	62.7	1,060.386	0.304	-1.002	-0.309	-0.003	-0.007	5.824	0.023	0.074
Complex	Yes	20	$\{4, 10, 20\}$	18.7	379.886	0.044	-0.102	-0.740	0.000	-0.001	4.357	0.006	0.015
Complex	Yes	20	$\{10, 25, 50\}$	108.9	67.645	0.015	-0.027	-0.021	0.000	-0.000	0.322	0.002	0.003
RCNL	No	20	$\{2, 5, 10\}$	7.3	934.505	0.070	-0.160	0.102	0.013	-0.031	5.500	0.016	0.037
RCNL	No	40	$\{2, 5, 10\}$	15.5	938.052	0.141	-0.324	-0.865	0.016	-0.041	4.183	0.022	0.055
RCNL	No	100	$\{2, 5, 10\}$	46.5	909.372	0.355	-0.813	-1.744	0.014	-0.042	2.805	0.030	0.076
RCNL	No	20	$\{4, 10, 20\}$	10.0	325.245	0.035	-0.063	0.561	0.004	-0.008	1.448	0.006	0.012
RCNL	No	20	$\{10, 25, 50\}$	19.0	58.506	0.008	-0.011	0.080	0.000	-0.001	0.154	0.001	0.001
RCNL	Yes	20	$\{2, 5, 10\}$	18.8	934.505	0.070	-0.160	-1.762	0.001	-0.003	5.334	0.007	0.017
RCNL	Yes	40	$\{2, 5, 10\}$	42.8	938.052	0.141	-0.324	-2.004	0.000	-0.003	3.711	0.011	0.025
RCNL	Yes	100	$\{2, 5, 10\}$	124.8	909.372	0.355	-0.813	-2.365	-0.000	-0.007	2.745	0.017	0.040
RCNL	Yes	20	$\{4, 10, 20\}$	31.1	325.245	0.035	-0.063	0.391	-0.000	-0.000	1.250	0.004	0.007
RCNL	Yes	20	$\{10, 25, 50\}$	137.1	58.506	0.008	-0.011	0.076	-0.000	0.000	0.136	0.001	0.001

This table documents bias and variance of merger simulation estimates along with estimation speed over 1,000 simulated datasets for different problem sizes. It is created from the same results as Table 9.

F. Additional Optimization Algorithm Results

Table 10 documents that the vast majority of run converge to a local optima, regardless of optimization routine. For completeness' sake, in this appendix we replicate the table on BLP instruments instead of optimal instruments (Table F.12), and document the impact of algorithm choice on bias and variance of parameter estimates, post-estimation outputs, and merger simulation estimates. Results are very similar across algorithms.

Table F.12: Optimization Algorithms: BLP Instruments

							Percent of Runs		Median, First GMM Step			
Simulation	Supply	P	Software	Algorithm	Gradient	Termination	Converged	PSD Hessian	Seconds	Evaluations	$q = g'Wg$	$\|\|\nabla q\|\|_\infty$
Simple	No	1	Knitro	Interior/Direct	Yes	$\|\|\nabla q\|\|_\infty$	100.0%	99.2%	0.7	6	4.91E-01	5.56E-06
Simple	No	1	Knitro	Active Set	Yes	$\|\|\nabla q\|\|_\infty$	99.9%	98.8%	0.5	6	4.92E-01	6.15E-06
Simple	No	1	Knitro	SQP	Yes	$\|\|\nabla q\|\|_\infty$	100.0%	99.3%	1.4	6	4.91E-01	5.67E-06
Simple	No	1	SciPy	Trust-Region	Yes	$\|\|\nabla q\|\|_\infty$	98.4%	99.1%	0.7	10	4.91E-01	6.84E-06
Simple	No	1	SciPy	L-BFGS-B	Yes	$\|\|\nabla q\|\|_\infty$	99.8%	99.3%	0.4	6	4.91E-01	5.69E-06
Simple	No	1	SciPy	BFGS	Yes	$\|\|\nabla q\|\|_\infty$	98.4%	99.0%	0.5	7	4.98E-01	6.10E-06
Simple	No	1	SciPy	TNC	Yes	$\|\|\nabla q\|\|_\infty$	98.6%	99.2%	0.9	10	4.91E-01	1.20E-06
Simple	No	1	SciPy	TNC	Yes	$\|\|\theta^n - \theta^{n-1}\|\|_\infty$	99.8%	98.9%	0.8	11	4.92E-01	2.39E-09
Simple	No	1	SciPy	Nelder-Mead	No	$\|\|\theta^n - \theta^{n-1}\|\|_\infty$	78.0%	98.8%	18.4	129	4.91E-01	2.98E-08
Simple	Yes	2	Knitro	Interior/Direct	Yes	$\|\|\nabla q\|\|_\infty$	100.0%	98.8%	1.5	11	1.27E+00	1.99E-05
Simple	Yes	2	Knitro	Active Set	Yes	$\|\|\nabla q\|\|_\infty$	99.2%	98.5%	2.2	10	1.27E+00	1.93E-05
Simple	Yes	2	Knitro	SQP	Yes	$\|\|\nabla q\|\|_\infty$	99.6%	98.7%	2.9	10	1.27E+00	1.86E-05
Simple	Yes	2	SciPy	Trust-Region	Yes	$\|\|\nabla q\|\|_\infty$	98.2%	98.5%	3.2	18	1.27E+00	2.17E-05
Simple	Yes	2	SciPy	L-BFGS-B	Yes	$\|\|\nabla q\|\|_\infty$	99.6%	98.8%	1.6	9	1.27E+00	1.98E-05
Simple	Yes	2	SciPy	BFGS	Yes	$\|\|\nabla q\|\|_\infty$	98.4%	98.4%	1.4	10	1.27E+00	1.71E-05
Simple	Yes	2	SciPy	TNC	Yes	$\|\|\nabla q\|\|_\infty$	98.4%	98.4%	2.2	15	1.27E+00	1.17E-05
Simple	Yes	2	SciPy	TNC	Yes	$\|\|\theta^n - \theta^{n-1}\|\|_\infty$	99.9%	98.6%	1.9	15	1.27E+00	2.52E-05
Simple	Yes	2	SciPy	Nelder-Mead	No	$\|\|\theta^n - \theta^{n-1}\|\|_\infty$	52.9%	98.6%	50.0	265	1.27E+00	8.87E-08
Complex	No	3	Knitro	Interior/Direct	Yes	$\|\|\nabla q\|\|_\infty$	100.0%	95.3%	3.1	20	8.38E-01	1.70E-05
Complex	No	3	Knitro	Active Set	Yes	$\|\|\nabla q\|\|_\infty$	100.0%	94.3%	2.6	18	8.43E-01	1.76E-05
Complex	No	3	Knitro	SQP	Yes	$\|\|\nabla q\|\|_\infty$	99.7%	95.2%	5.1	19	8.38E-01	1.74E-05
Complex	No	3	SciPy	Trust-Region	Yes	$\|\|\nabla q\|\|_\infty$	98.1%	95.1%	5.0	40	8.40E-01	2.73E-05
Complex	No	3	SciPy	L-BFGS-B	Yes	$\|\|\nabla q\|\|_\infty$	99.5%	95.8%	1.9	19	8.42E-01	1.85E-05
Complex	No	3	SciPy	BFGS	Yes	$\|\|\nabla q\|\|_\infty$	98.4%	95.4%	2.0	18	8.47E-01	1.98E-05
Complex	No	3	SciPy	TNC	Yes	$\|\|\nabla q\|\|_\infty$	96.3%	95.6%	4.8	39	8.39E-01	1.75E-05
Complex	No	3	SciPy	TNC	Yes	$\|\|\theta^n - \theta^{n-1}\|\|_\infty$	99.7%	94.4%	3.9	35	8.42E-01	5.11E-05
Complex	No	3	SciPy	Nelder-Mead	No	$\|\|\theta^n - \theta^{n-1}\|\|_\infty$	69.1%	95.4%	40.7	313	8.39E-01	5.44E-07
Complex	Yes	4	Knitro	Interior/Direct	Yes	$\|\|\nabla q\|\|_\infty$	100.0%	98.0%	6.3	24	1.91E+00	2.59E-05
Complex	Yes	4	Knitro	Active Set	Yes	$\|\|\nabla q\|\|_\infty$	99.6%	97.9%	5.1	21	1.91E+00	2.59E-05
Complex	Yes	4	Knitro	SQP	Yes	$\|\|\nabla q\|\|_\infty$	99.8%	97.9%	8.1	24	1.91E+00	2.67E-05
Complex	Yes	4	SciPy	Trust-Region	Yes	$\|\|\nabla q\|\|_\infty$	76.1%	88.3%	91.8	56	2.08E+00	4.82E-05
Complex	Yes	4	SciPy	L-BFGS-B	Yes	$\|\|\nabla q\|\|_\infty$	99.8%	97.8%	4.0	22	1.92E+00	2.85E-05

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							Percent of Runs		Median, First GMM Step			
Simulation	Supply	P	Software	Algorithm	Gradient	Termination	Converged	PSD Hessian	Seconds	Evaluations	$q = g'Wg$	$\|\|\nabla q\|\|_\infty$
Complex	Yes	4	SciPy	BFGS	Yes	$\|\|\nabla q\|\|_\infty$	98.3%	98.2%	3.8	19	1.92E+00	2.91E-05
Complex	Yes	4	SciPy	TNC	Yes	$\|\|\nabla q\|\|_\infty$	92.4%	97.8%	12.9	54	1.91E+00	2.77E-05
Complex	Yes	4	SciPy	TNC	Yes	$\|\|\theta^n - \theta^{n-1}\|\|_\infty$	99.7%	97.8%	6.7	35	1.92E+00	9.70E-04
Complex	Yes	4	SciPy	Nelder-Mead	No	$\|\|\theta^n - \theta^{n-1}\|\|_\infty$	43.4%	97.8%	80.5	1,001	1.91E+00	1.65E-06
RCNL	No	2	Knitro	Interior/Direct	Yes	$\|\|\nabla q\|\|_\infty$	100.0%	98.4%	3.7	12	7.07E-01	3.66E-05
RCNL	No	2	Knitro	Active Set	Yes	$\|\|\nabla q\|\|_\infty$	99.9%	98.8%	3.5	13	7.11E-01	1.55E-05
RCNL	No	2	Knitro	SQP	Yes	$\|\|\nabla q\|\|_\infty$	100.0%	98.4%	4.5	13	7.14E-01	1.11E-05
RCNL	No	2	SciPy	Trust-Region	Yes	$\|\|\nabla q\|\|_\infty$	98.3%	98.7%	9.2	26	7.05E-01	1.29E-04
RCNL	No	2	SciPy	L-BFGS-B	Yes	$\|\|\nabla q\|\|_\infty$	99.7%	98.6%	3.5	14	7.07E-01	1.22E-05
RCNL	No	2	SciPy	BFGS	Yes	$\|\|\nabla q\|\|_\infty$	98.9%	99.1%	2.4	12	7.11E-01	1.12E-05
RCNL	No	2	SciPy	TNC	Yes	$\|\|\nabla q\|\|_\infty$	98.4%	98.4%	5.1	25	7.04E-01	1.54E-05
RCNL	No	2	SciPy	TNC	Yes	$\|\|\theta^n - \theta^{n-1}\|\|_\infty$	99.8%	98.4%	3.9	19	7.07E-01	3.31E-04
RCNL	No	2	SciPy	Nelder-Mead	No	$\|\|\theta^n - \theta^{n-1}\|\|_\infty$	59.9%	98.4%	96.2	265	7.04E-01	3.71E-07
RCNL	Yes	3	Knitro	Interior/Direct	Yes	$\|\|\nabla q\|\|_\infty$	100.0%	98.4%	7.9	20	1.88E+00	4.68E-05
RCNL	Yes	3	Knitro	Active Set	Yes	$\|\|\nabla q\|\|_\infty$	99.7%	98.4%	9.1	18	1.89E+00	3.05E-05
RCNL	Yes	3	Knitro	SQP	Yes	$\|\|\nabla q\|\|_\infty$	99.7%	98.4%	10.6	20	1.87E+00	2.37E-05
RCNL	Yes	3	SciPy	Trust-Region	Yes	$\|\|\nabla q\|\|_\infty$	98.3%	98.4%	16.6	38	1.88E+00	1.09E-04
RCNL	Yes	3	SciPy	L-BFGS-B	Yes	$\|\|\nabla q\|\|_\infty$	99.6%	98.3%	5.2	17	1.88E+00	2.69E-05
RCNL	Yes	3	SciPy	BFGS	Yes	$\|\|\nabla q\|\|_\infty$	98.7%	98.7%	4.3	14	1.89E+00	2.42E-05
RCNL	Yes	3	SciPy	TNC	Yes	$\|\|\nabla q\|\|_\infty$	98.4%	98.4%	9.5	29	1.88E+00	2.34E-05
RCNL	Yes	3	SciPy	TNC	Yes	$\|\|\theta^n - \theta^{n-1}\|\|_\infty$	99.5%	98.4%	6.9	23	1.89E+00	6.46E-04
RCNL	Yes	3	SciPy	Nelder-Mead	No	$\|\|\theta^n - \theta^{n-1}\|\|_\infty$	42.2%	98.4%	151.7	1,001	1.87E+00	4.96E-07

Like Table 10, this table also documents optimization convergence statistics along with estimation speed over 1,000 simulated datasets for different optimization algorithms. Instead of optimal instruments, the problems solved for this table use only the traditional BLP instruments.

Table F.13: Optimization Algorithms: Parameter Estimates

						True Value				Median Bias				Median Absolute Error			
Simulation	Supply	Software	Algorithm	Termination	Seconds	α	σ_x	σ_p	ρ	α	σ_x	σ_p	ρ	α	σ_x	σ_p	ρ
Simple	No	Knitro	Interior/Direct	$\|\|\nabla q\|\|_\infty$	1.3	-1	3			0.180	-0.043			0.246	0.194		
Simple	No	Knitro	Active Set	$\|\|\nabla q\|\|_\infty$	1.2	-1	3			0.180	-0.031			0.246	0.180		
Simple	No	Knitro	SQP	$\|\|\nabla q\|\|_\infty$	1.2	-1	3			0.180	-0.042			0.246	0.195		
Simple	No	SciPy	Trust-Region	$\|\|\nabla q\|\|_\infty$	1.8	-1	3			0.181	-0.031			0.246	0.180		
Simple	No	SciPy	L-BFGS-B	$\|\|\nabla q\|\|_\infty$	1.2	-1	3			0.180	-0.031			0.246	0.180		
Simple	No	SciPy	BFGS	$\|\|\nabla q\|\|_\infty$	1.3	-1	3			0.180	-0.031			0.246	0.180		
Simple	No	SciPy	TNC	$\|\|\nabla q\|\|_\infty$	1.6	-1	3			0.180	-0.031			0.246	0.180		
Simple	No	SciPy	TNC	$\|\|\theta^n - \theta^{n-1}\|\|_\infty$	1.8	-1	3			0.180	-0.031			0.246	0.180		
Simple	No	SciPy	Nelder-Mead	$\|\|\theta^n - \theta^{n-1}\|\|_\infty$	53.4	-1	3			0.181	-0.031			0.246	0.179		
Simple	Yes	Knitro	Interior/Direct	$\|\|\nabla q\|\|_\infty$	4.3	-1	3			0.019	0.001			0.160	0.189		
Simple	Yes	Knitro	Active Set	$\|\|\nabla q\|\|_\infty$	3.9	-1	3			0.018	0.004			0.160	0.184		
Simple	Yes	Knitro	SQP	$\|\|\nabla q\|\|_\infty$	4.1	-1	3			0.018	0.004			0.160	0.188		
Simple	Yes	SciPy	Trust-Region	$\|\|\nabla q\|\|_\infty$	6.1	-1	3			0.018	0.006			0.160	0.184		
Simple	Yes	SciPy	L-BFGS-B	$\|\|\nabla q\|\|_\infty$	3.6	-1	3			0.018	0.003			0.160	0.185		
Simple	Yes	SciPy	BFGS	$\|\|\nabla q\|\|_\infty$	3.8	-1	3			0.017	0.003			0.160	0.184		
Simple	Yes	SciPy	TNC	$\|\|\nabla q\|\|_\infty$	5.6	-1	3			0.018	0.003			0.160	0.185		
Simple	Yes	SciPy	TNC	$\|\|\theta^n - \theta^{n-1}\|\|_\infty$	5.2	-1	3			0.018	0.005			0.159	0.183		
Simple	Yes	SciPy	Nelder-Mead	$\|\|\theta^n - \theta^{n-1}\|\|_\infty$	102.7	-1	3			0.017	0.003			0.160	0.184		
Complex	No	Knitro	Interior/Direct	$\|\|\nabla q\|\|_\infty$	4.9	-1	3	0.2		0.153	-0.103	-0.123		0.255	0.201	0.200	
Complex	No	Knitro	Active Set	$\|\|\nabla q\|\|_\infty$	3.9	-1	3	0.2		0.154	-0.093	-0.149		0.255	0.194	0.200	
Complex	No	Knitro	SQP	$\|\|\nabla q\|\|_\infty$	4.6	-1	3	0.2		0.154	-0.103	-0.104		0.254	0.203	0.200	
Complex	No	SciPy	Trust-Region	$\|\|\nabla q\|\|_\infty$	7.2	-1	3	0.2		0.154	-0.093	-0.148		0.254	0.193	0.200	
Complex	No	SciPy	L-BFGS-B	$\|\|\nabla q\|\|_\infty$	3.7	-1	3	0.2		0.154	-0.090	-0.200		0.255	0.193	0.200	
Complex	No	SciPy	BFGS	$\|\|\nabla q\|\|_\infty$	3.8	-1	3	0.2		0.154	-0.092	-0.088		0.255	0.193	0.200	
Complex	No	SciPy	TNC	$\|\|\nabla q\|\|_\infty$	6.8	-1	3	0.2		0.154	-0.089	0.020		0.255	0.193	0.143	
Complex	No	SciPy	TNC	$\|\|\theta^n - \theta^{n-1}\|\|_\infty$	6.4	-1	3	0.2		0.153	-0.092	0.021		0.255	0.191	0.144	
Complex	No	SciPy	Nelder-Mead	$\|\|\theta^n - \theta^{n-1}\|\|_\infty$	93.6	-1	3	0.2		0.153	-0.093	0.020		0.254	0.192	0.142	
Complex	Yes	Knitro	Interior/Direct	$\|\|\nabla q\|\|_\infty$	13.2	-1	3	0.2		-0.038	-0.049	-0.200		0.181	0.189	0.200	
Complex	Yes	Knitro	Active Set	$\|\|\nabla q\|\|_\infty$	10.0	-1	3	0.2		-0.036	-0.045	-0.085		0.181	0.189	0.200	
Complex	Yes	Knitro	SQP	$\|\|\nabla q\|\|_\infty$	12.8	-1	3	0.2		-0.035	-0.047	-0.081		0.180	0.190	0.200	
Complex	Yes	SciPy	Trust-Region	$\|\|\nabla q\|\|_\infty$	22.3	-1	3	0.2		-0.035	-0.049	-0.078		0.181	0.189	0.200	
Complex	Yes	SciPy	L-BFGS-B	$\|\|\nabla q\|\|_\infty$	9.5	-1	3	0.2		-0.036	-0.049	-0.123		0.181	0.190	0.200	

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						True Value				Median Bias				Median Absolute Error			
Simulation	Supply	Software	Algorithm	Termination	Seconds	α	σ_x	σ_p	ρ	α	σ_x	σ_p	ρ	α	σ_x	σ_p	ρ
Complex	Yes	SciPy	BFGS	$\|\|\nabla q\|\|_\infty$	9.2	-1	3	0.2		-0.036	-0.049	-0.057		0.181	0.190	0.200	
Complex	Yes	SciPy	TNC	$\|\|\nabla q\|\|_\infty$	19.6	-1	3	0.2		-0.036	-0.046	0.011		0.181	0.190	0.149	
Complex	Yes	SciPy	TNC	$\|\|\theta^n - \theta^{n-1}\|\|_\infty$	13.9	-1	3	0.2		-0.035	-0.048	0.007		0.179	0.189	0.149	
Complex	Yes	SciPy	Nelder-Mead	$\|\|\theta^n - \theta^{n-1}\|\|_\infty$	161.4	-1	3	0.2		-0.035	-0.049	0.022		0.181	0.190	0.136	
RCNL	No	Knitro	Interior/Direct	$\|\|\nabla q\|\|_\infty$	9.2	-1	3		0.5	0.166	-0.035		-0.009	0.216	0.189		0.022
RCNL	No	Knitro	Active Set	$\|\|\nabla q\|\|_\infty$	8.0	-1	3		0.5	0.168	-0.016		-0.008	0.215	0.174		0.022
RCNL	No	Knitro	SQP	$\|\|\nabla q\|\|_\infty$	9.6	-1	3		0.5	0.165	-0.016		-0.009	0.216	0.174		0.022
RCNL	No	SciPy	Trust-Region	$\|\|\nabla q\|\|_\infty$	12.8	-1	3		0.5	0.168	-0.016		-0.007	0.215	0.173		0.022
RCNL	No	SciPy	L-BFGS-B	$\|\|\nabla q\|\|_\infty$	6.8	-1	3		0.5	0.168	-0.016		-0.008	0.215	0.173		0.022
RCNL	No	SciPy	BFGS	$\|\|\nabla q\|\|_\infty$	6.6	-1	3		0.5	0.167	-0.016		-0.008	0.218	0.174		0.022
RCNL	No	SciPy	TNC	$\|\|\nabla q\|\|_\infty$	11.7	-1	3		0.5	0.168	-0.016		-0.008	0.216	0.174		0.022
RCNL	No	SciPy	TNC	$\|\|\theta^n - \theta^{n-1}\|\|_\infty$	9.6	-1	3		0.5	0.169	-0.015		-0.008	0.216	0.174		0.022
RCNL	No	SciPy	Nelder-Mead	$\|\|\theta^n - \theta^{n-1}\|\|_\infty$	212.6	-1	3		0.5	0.168	-0.016		-0.008	0.216	0.174		0.022
RCNL	Yes	Knitro	Interior/Direct	$\|\|\nabla q\|\|_\infty$	20.2	-1	3		0.5	0.003	-0.007		-0.000	0.104	0.165		0.020
RCNL	Yes	Knitro	Active Set	$\|\|\nabla q\|\|_\infty$	17.7	-1	3		0.5	0.004	-0.003		0.000	0.104	0.165		0.020
RCNL	Yes	Knitro	SQP	$\|\|\nabla q\|\|_\infty$	20.6	-1	3		0.5	0.004	-0.003		-0.000	0.103	0.164		0.020
RCNL	Yes	SciPy	Trust-Region	$\|\|\nabla q\|\|_\infty$	28.4	-1	3		0.5	0.006	-0.006		0.000	0.104	0.165		0.020
RCNL	Yes	SciPy	L-BFGS-B	$\|\|\nabla q\|\|_\infty$	13.9	-1	3		0.5	0.004	-0.005		-0.000	0.104	0.165		0.020
RCNL	Yes	SciPy	BFGS	$\|\|\nabla q\|\|_\infty$	13.2	-1	3		0.5	0.004	-0.006		0.000	0.104	0.165		0.020
RCNL	Yes	SciPy	TNC	$\|\|\nabla q\|\|_\infty$	23.4	-1	3		0.5	0.005	-0.004		0.000	0.104	0.165		0.020
RCNL	Yes	SciPy	TNC	$\|\|\theta^n - \theta^{n-1}\|\|_\infty$	18.2	-1	3		0.5	0.005	-0.004		0.000	0.104	0.165		0.020
RCNL	Yes	SciPy	Nelder-Mead	$\|\|\theta^n - \theta^{n-1}\|\|_\infty$	302.2	-1	3		0.5	0.004	-0.004		0.000	0.103	0.164		0.020

This table documents bias and variance of parameter estimates along with estimation speed over 1,000 simulated datasets for different optimization algorithms. It is created from the same results as Table 10 after a second GMM step.

Table F.14: Optimization Algorithms: Post-Estimation

						True Median Value				Median Bias				Median Absolute Error			
Simulation	Supply	Software	Algorithm	Termination	Seconds	$\bar{\varepsilon}_{jj}$	$\bar{E}$	PS	CS	$\bar{\varepsilon}_{jj}$	$\bar{E}$	PS	CS	$\bar{\varepsilon}_{jj}$	$\bar{E}$	PS	CS
Simple	No	Knitro	Interior/Direct	$\|\|\nabla q\|\|_\infty$	1.3	-3.564	-0.241	2.651	3.653	0.636	0.038	0.550	0.737	0.877	0.055	0.715	0.983
Simple	No	Knitro	Active Set	$\|\|\nabla q\|\|_\infty$	1.2	-3.564	-0.241	2.651	3.653	0.639	0.039	0.552	0.738	0.878	0.055	0.718	0.983
Simple	No	Knitro	SQP	$\|\|\nabla q\|\|_\infty$	1.2	-3.564	-0.241	2.651	3.653	0.636	0.039	0.552	0.738	0.875	0.055	0.713	0.983
Simple	No	SciPy	Trust-Region	$\|\|\nabla q\|\|_\infty$	1.8	-3.564	-0.241	2.651	3.653	0.641	0.039	0.552	0.738	0.878	0.055	0.718	0.983
Simple	No	SciPy	L-BFGS-B	$\|\|\nabla q\|\|_\infty$	1.2	-3.564	-0.241	2.651	3.653	0.636	0.038	0.552	0.738	0.875	0.055	0.715	0.979
Simple	No	SciPy	BFGS	$\|\|\nabla q\|\|_\infty$	1.3	-3.564	-0.241	2.651	3.653	0.639	0.039	0.556	0.738	0.878	0.055	0.718	0.979
Simple	No	SciPy	TNC	$\|\|\nabla q\|\|_\infty$	1.6	-3.564	-0.241	2.651	3.653	0.639	0.039	0.552	0.738	0.877	0.055	0.718	0.987
Simple	No	SciPy	TNC	$\|\|\theta^n - \theta^{n-1}\|\|_\infty$	1.8	-3.564	-0.241	2.651	3.653	0.639	0.039	0.550	0.737	0.877	0.055	0.718	0.979
Simple	No	SciPy	Nelder-Mead	$\|\|\theta^n - \theta^{n-1}\|\|_\infty$	53.4	-3.564	-0.241	2.651	3.653	0.643	0.039	0.562	0.743	0.878	0.055	0.715	0.979
Simple	Yes	Knitro	Interior/Direct	$\|\|\nabla q\|\|_\infty$	4.3	-3.564	-0.241	2.651	3.653	0.071	0.001	0.053	0.081	0.571	0.036	0.417	0.568
Simple	Yes	Knitro	Active Set	$\|\|\nabla q\|\|_\infty$	3.9	-3.564	-0.241	2.651	3.653	0.068	0.001	0.045	0.077	0.570	0.036	0.417	0.567
Simple	Yes	Knitro	SQP	$\|\|\nabla q\|\|_\infty$	4.1	-3.564	-0.241	2.651	3.653	0.068	0.001	0.053	0.080	0.569	0.036	0.417	0.567
Simple	Yes	SciPy	Trust-Region	$\|\|\nabla q\|\|_\infty$	6.1	-3.564	-0.241	2.651	3.653	0.066	0.001	0.047	0.080	0.573	0.037	0.418	0.568
Simple	Yes	SciPy	L-BFGS-B	$\|\|\nabla q\|\|_\infty$	3.6	-3.564	-0.241	2.651	3.653	0.066	0.001	0.047	0.077	0.573	0.037	0.418	0.569
Simple	Yes	SciPy	BFGS	$\|\|\nabla q\|\|_\infty$	3.8	-3.564	-0.241	2.651	3.653	0.062	0.001	0.044	0.075	0.571	0.036	0.417	0.564
Simple	Yes	SciPy	TNC	$\|\|\nabla q\|\|_\infty$	5.6	-3.564	-0.241	2.651	3.653	0.065	0.001	0.045	0.080	0.571	0.036	0.417	0.565
Simple	Yes	SciPy	TNC	$\|\|\theta^n - \theta^{n-1}\|\|_\infty$	5.2	-3.564	-0.241	2.651	3.653	0.064	0.001	0.049	0.081	0.568	0.037	0.417	0.568
Simple	Yes	SciPy	Nelder-Mead	$\|\|\theta^n - \theta^{n-1}\|\|_\infty$	102.7	-3.564	-0.241	2.651	3.653	0.062	0.001	0.044	0.072	0.571	0.036	0.417	0.565
Complex	No	Knitro	Interior/Direct	$\|\|\nabla q\|\|_\infty$	4.9	-3.303	-0.227	2.945	4.500	0.775	0.045	0.764	1.148	0.889	0.054	0.882	1.638
Complex	No	Knitro	Active Set	$\|\|\nabla q\|\|_\infty$	3.9	-3.303	-0.227	2.945	4.500	0.776	0.045	0.773	1.149	0.887	0.054	0.887	1.633
Complex	No	Knitro	SQP	$\|\|\nabla q\|\|_\infty$	4.6	-3.303	-0.227	2.945	4.500	0.775	0.045	0.764	1.158	0.891	0.054	0.887	1.633
Complex	No	SciPy	Trust-Region	$\|\|\nabla q\|\|_\infty$	7.2	-3.303	-0.227	2.945	4.500	0.775	0.045	0.768	1.178	0.891	0.053	0.878	1.648
Complex	No	SciPy	L-BFGS-B	$\|\|\nabla q\|\|_\infty$	3.7	-3.303	-0.227	2.945	4.500	0.775	0.045	0.772	1.158	0.889	0.054	0.887	1.644
Complex	No	SciPy	BFGS	$\|\|\nabla q\|\|_\infty$	3.8	-3.303	-0.227	2.945	4.500	0.775	0.045	0.764	1.146	0.894	0.054	0.893	1.641
Complex	No	SciPy	TNC	$\|\|\nabla q\|\|_\infty$	6.8	-3.303	-0.227	2.945	4.500	0.776	0.045	0.770	1.171	0.894	0.054	0.888	1.633
Complex	No	SciPy	TNC	$\|\|\theta^n - \theta^{n-1}\|\|_\infty$	6.4	-3.303	-0.227	2.945	4.500	0.774	0.045	0.770	1.171	0.887	0.054	0.891	1.644
Complex	No	SciPy	Nelder-Mead	$\|\|\theta^n - \theta^{n-1}\|\|_\infty$	93.6	-3.303	-0.227	2.945	4.500	0.775	0.045	0.768	1.178	0.894	0.054	0.882	1.648
Complex	Yes	Knitro	Interior/Direct	$\|\|\nabla q\|\|_\infty$	13.2	-3.303	-0.227	2.945	4.500	0.091	0.003	0.042	0.108	0.492	0.030	0.435	0.874
Complex	Yes	Knitro	Active Set	$\|\|\nabla q\|\|_\infty$	10.0	-3.303	-0.227	2.945	4.500	0.094	0.003	0.048	0.110	0.492	0.030	0.437	0.874
Complex	Yes	Knitro	SQP	$\|\|\nabla q\|\|_\infty$	12.8	-3.303	-0.227	2.945	4.500	0.094	0.003	0.050	0.121	0.491	0.030	0.439	0.882
Complex	Yes	SciPy	Trust-Region	$\|\|\nabla q\|\|_\infty$	22.3	-3.303	-0.227	2.945	4.500	0.083	0.003	0.053	0.108	0.493	0.030	0.441	0.870
Complex	Yes	SciPy	L-BFGS-B	$\|\|\nabla q\|\|_\infty$	9.5	-3.303	-0.227	2.945	4.500	0.086	0.003	0.050	0.098	0.496	0.031	0.440	0.877

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						True Median Value				Median Bias				Median Absolute Error			
Simulation	Supply	Software	Algorithm	Termination	Seconds	$\bar{\varepsilon}_{jj}$	$\bar{E}$	PS	CS	$\bar{\varepsilon}_{jj}$	$\bar{E}$	PS	CS	$\bar{\varepsilon}_{jj}$	$\bar{E}$	PS	CS
Complex	Yes	SciPy	BFGS	$\|\|\nabla q\|\|_\infty$	9.2	-3.303	-0.227	2.945	4.500	0.088	0.003	0.048	0.119	0.491	0.030	0.439	0.868
Complex	Yes	SciPy	TNC	$\|\|\nabla q\|\|_\infty$	19.6	-3.303	-0.227	2.945	4.500	0.091	0.003	0.048	0.115	0.499	0.030	0.439	0.874
Complex	Yes	SciPy	TNC	$\|\|\theta^n - \theta^{n-1}\|\|_\infty$	13.9	-3.303	-0.227	2.945	4.500	0.098	0.003	0.051	0.126	0.492	0.030	0.434	0.880
Complex	Yes	SciPy	Nelder-Mead	$\|\|\theta^n - \theta^{n-1}\|\|_\infty$	161.4	-3.303	-0.227	2.945	4.500	0.094	0.003	0.050	0.135	0.492	0.030	0.439	0.870
RCNL	No	Knitro	Interior/Direct	$\|\|\nabla q\|\|_\infty$	9.2	-5.892	-0.195	1.651	3.198	1.103	0.032	0.352	0.682	1.360	0.040	0.419	0.801
RCNL	No	Knitro	Active Set	$\|\|\nabla q\|\|_\infty$	8.0	-5.892	-0.195	1.651	3.198	1.093	0.032	0.352	0.666	1.356	0.039	0.413	0.780
RCNL	No	Knitro	SQP	$\|\|\nabla q\|\|_\infty$	9.6	-5.892	-0.195	1.651	3.198	1.093	0.032	0.351	0.667	1.356	0.040	0.414	0.789
RCNL	No	SciPy	Trust-Region	$\|\|\nabla q\|\|_\infty$	12.8	-5.892	-0.195	1.651	3.198	1.108	0.032	0.352	0.664	1.356	0.039	0.413	0.779
RCNL	No	SciPy	L-BFGS-B	$\|\|\nabla q\|\|_\infty$	6.8	-5.892	-0.195	1.651	3.198	1.113	0.032	0.352	0.666	1.354	0.040	0.413	0.784
RCNL	No	SciPy	BFGS	$\|\|\nabla q\|\|_\infty$	6.6	-5.892	-0.195	1.651	3.198	1.108	0.032	0.351	0.672	1.356	0.040	0.416	0.797
RCNL	No	SciPy	TNC	$\|\|\nabla q\|\|_\infty$	11.7	-5.892	-0.195	1.651	3.198	1.108	0.032	0.352	0.666	1.358	0.039	0.413	0.784
RCNL	No	SciPy	TNC	$\|\|\theta^n - \theta^{n-1}\|\|_\infty$	9.6	-5.892	-0.195	1.651	3.198	1.113	0.032	0.353	0.666	1.356	0.039	0.413	0.779
RCNL	No	SciPy	Nelder-Mead	$\|\|\theta^n - \theta^{n-1}\|\|_\infty$	212.6	-5.892	-0.195	1.651	3.198	1.108	0.032	0.352	0.667	1.356	0.040	0.415	0.792
RCNL	Yes	Knitro	Interior/Direct	$\|\|\nabla q\|\|_\infty$	20.2	-5.892	-0.195	1.651	3.198	0.033	0.001	0.017	0.065	0.630	0.018	0.168	0.310
RCNL	Yes	Knitro	Active Set	$\|\|\nabla q\|\|_\infty$	17.7	-5.892	-0.195	1.651	3.198	0.035	0.001	0.017	0.062	0.627	0.018	0.166	0.305
RCNL	Yes	Knitro	SQP	$\|\|\nabla q\|\|_\infty$	20.6	-5.892	-0.195	1.651	3.198	0.039	0.001	0.019	0.063	0.623	0.018	0.165	0.307
RCNL	Yes	SciPy	Trust-Region	$\|\|\nabla q\|\|_\infty$	28.4	-5.892	-0.195	1.651	3.198	0.040	0.001	0.017	0.063	0.627	0.018	0.167	0.306
RCNL	Yes	SciPy	L-BFGS-B	$\|\|\nabla q\|\|_\infty$	13.9	-5.892	-0.195	1.651	3.198	0.033	0.001	0.017	0.063	0.623	0.018	0.166	0.306
RCNL	Yes	SciPy	BFGS	$\|\|\nabla q\|\|_\infty$	13.2	-5.892	-0.195	1.651	3.198	0.035	0.001	0.016	0.063	0.627	0.018	0.167	0.307
RCNL	Yes	SciPy	TNC	$\|\|\nabla q\|\|_\infty$	23.4	-5.892	-0.195	1.651	3.198	0.038	0.001	0.017	0.063	0.627	0.018	0.167	0.305
RCNL	Yes	SciPy	TNC	$\|\|\theta^n - \theta^{n-1}\|\|_\infty$	18.2	-5.892	-0.195	1.651	3.198	0.033	0.001	0.016	0.063	0.624	0.018	0.166	0.305
RCNL	Yes	SciPy	Nelder-Mead	$\|\|\theta^n - \theta^{n-1}\|\|_\infty$	302.2	-5.892	-0.195	1.651	3.198	0.035	0.001	0.017	0.062	0.627	0.018	0.165	0.304

This table documents bias and variance of post-estimation outputs along with estimation speed over 1,000 simulated datasets for different optimization algorithms. It is created from the same results as Table 10 after a second GMM step.

Table F.15: Optimization Algorithms: Merger

						True Median Value			Median Bias			Median Absolute Error		
Simulation	Supply	Software	Algorithm	Termination	Seconds	$\Delta\overline{\text{HHI}}$	ΔPS	ΔCS	$\Delta\overline{\text{HHI}}$	ΔPS	ΔCS	$\Delta\overline{\text{HHI}}$	ΔPS	ΔCS
Simple	No	Knitro	Interior/Direct	$\|\|\nabla q\|\|_\infty$	1.3	1,112.227	0.052	-0.168	1.087	0.008	-0.030	8.376	0.013	0.043
Simple	No	Knitro	Active Set	$\|\|\nabla q\|\|_\infty$	1.2	1,112.227	0.052	-0.168	1.087	0.008	-0.030	8.430	0.013	0.043
Simple	No	Knitro	SQP	$\|\|\nabla q\|\|_\infty$	1.2	1,112.227	0.052	-0.168	1.026	0.008	-0.030	8.376	0.013	0.043
Simple	No	SciPy	Trust-Region	$\|\|\nabla q\|\|_\infty$	1.8	1,112.227	0.052	-0.168	1.039	0.008	-0.030	8.409	0.013	0.043
Simple	No	SciPy	L-BFGS-B	$\|\|\nabla q\|\|_\infty$	1.2	1,112.227	0.052	-0.168	1.038	0.008	-0.030	8.409	0.013	0.043
Simple	No	SciPy	BFGS	$\|\|\nabla q\|\|_\infty$	1.3	1,112.227	0.052	-0.168	1.142	0.008	-0.030	8.409	0.013	0.043
Simple	No	SciPy	TNC	$\|\|\nabla q\|\|_\infty$	1.6	1,112.227	0.052	-0.168	1.087	0.008	-0.030	8.430	0.013	0.043
Simple	No	SciPy	TNC	$\|\|\theta^n - \theta^{n-1}\|\|_\infty$	1.8	1,112.227	0.052	-0.168	1.142	0.008	-0.030	8.409	0.013	0.043
Simple	No	SciPy	Nelder-Mead	$\|\|\theta^n - \theta^{n-1}\|\|_\infty$	53.4	1,112.227	0.052	-0.168	1.087	0.008	-0.031	8.376	0.013	0.043
Simple	Yes	Knitro	Interior/Direct	$\|\|\nabla q\|\|_\infty$	4.3	1,112.227	0.052	-0.168	0.000	-0.000	-0.001	8.408	0.009	0.027
Simple	Yes	Knitro	Active Set	$\|\|\nabla q\|\|_\infty$	3.9	1,112.227	0.052	-0.168	-0.000	-0.000	-0.001	8.401	0.009	0.027
Simple	Yes	Knitro	SQP	$\|\|\nabla q\|\|_\infty$	4.1	1,112.227	0.052	-0.168	-0.113	-0.000	-0.001	8.401	0.009	0.027
Simple	Yes	SciPy	Trust-Region	$\|\|\nabla q\|\|_\infty$	6.1	1,112.227	0.052	-0.168	-0.000	-0.000	-0.001	8.402	0.009	0.027
Simple	Yes	SciPy	L-BFGS-B	$\|\|\nabla q\|\|_\infty$	3.6	1,112.227	0.052	-0.168	-0.001	-0.001	-0.001	8.401	0.009	0.027
Simple	Yes	SciPy	BFGS	$\|\|\nabla q\|\|_\infty$	3.8	1,112.227	0.052	-0.168	0.000	-0.000	-0.001	8.400	0.009	0.027
Simple	Yes	SciPy	TNC	$\|\|\nabla q\|\|_\infty$	5.6	1,112.227	0.052	-0.168	0.000	-0.000	-0.001	8.401	0.009	0.027
Simple	Yes	SciPy	TNC	$\|\|\theta^n - \theta^{n-1}\|\|_\infty$	5.2	1,112.227	0.052	-0.168	0.000	-0.000	-0.001	8.404	0.009	0.027
Simple	Yes	SciPy	Nelder-Mead	$\|\|\theta^n - \theta^{n-1}\|\|_\infty$	102.7	1,112.227	0.052	-0.168	-0.000	-0.001	-0.001	8.366	0.009	0.027
Complex	No	Knitro	Interior/Direct	$\|\|\nabla q\|\|_\infty$	4.9	1,102.034	0.062	-0.199	2.175	0.006	-0.027	16.382	0.017	0.066
Complex	No	Knitro	Active Set	$\|\|\nabla q\|\|_\infty$	3.9	1,102.034	0.062	-0.199	2.209	0.007	-0.028	16.447	0.017	0.066
Complex	No	Knitro	SQP	$\|\|\nabla q\|\|_\infty$	4.6	1,102.034	0.062	-0.199	2.201	0.006	-0.026	16.265	0.017	0.066
Complex	No	SciPy	Trust-Region	$\|\|\nabla q\|\|_\infty$	7.2	1,102.034	0.062	-0.199	1.963	0.007	-0.027	16.204	0.017	0.066
Complex	No	SciPy	L-BFGS-B	$\|\|\nabla q\|\|_\infty$	3.7	1,102.034	0.062	-0.199	2.209	0.006	-0.027	16.478	0.017	0.065
Complex	No	SciPy	BFGS	$\|\|\nabla q\|\|_\infty$	3.8	1,102.034	0.062	-0.199	2.209	0.007	-0.028	16.325	0.017	0.066
Complex	No	SciPy	TNC	$\|\|\nabla q\|\|_\infty$	6.8	1,102.034	0.062	-0.199	2.142	0.006	-0.027	16.382	0.017	0.066
Complex	No	SciPy	TNC	$\|\|\theta^n - \theta^{n-1}\|\|_\infty$	6.4	1,102.034	0.062	-0.199	2.142	0.007	-0.027	16.325	0.017	0.066
Complex	No	SciPy	Nelder-Mead	$\|\|\theta^n - \theta^{n-1}\|\|_\infty$	93.6	1,102.034	0.062	-0.199	2.347	0.006	-0.027	16.325	0.017	0.066
Complex	Yes	Knitro	Interior/Direct	$\|\|\nabla q\|\|_\infty$	13.2	1,102.034	0.062	-0.199	-2.288	-0.000	-0.005	12.586	0.009	0.032
Complex	Yes	Knitro	Active Set	$\|\|\nabla q\|\|_\infty$	10.0	1,102.034	0.062	-0.199	-2.596	0.000	-0.005	12.612	0.009	0.032
Complex	Yes	Knitro	SQP	$\|\|\nabla q\|\|_\infty$	12.8	1,102.034	0.062	-0.199	-2.507	0.000	-0.005	12.586	0.009	0.032
Complex	Yes	SciPy	Trust-Region	$\|\|\nabla q\|\|_\infty$	22.3	1,102.034	0.062	-0.199	-2.084	0.000	-0.004	12.820	0.009	0.032
Complex	Yes	SciPy	L-BFGS-B	$\|\|\nabla q\|\|_\infty$	9.5	1,102.034	0.062	-0.199	-2.652	0.000	-0.005	12.547	0.010	0.032

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						True Median Value			Median Bias			Median Absolute Error		
Simulation	Supply	Software	Algorithm	Termination	Seconds	$\Delta\overline{\text{HHI}}$	ΔPS	ΔCS	$\Delta\overline{\text{HHI}}$	ΔPS	ΔCS	$\Delta\overline{\text{HHI}}$	ΔPS	ΔCS
Complex	Yes	SciPy	BFGS	$\|\|\nabla q\|\|_\infty$	9.2	1,102.034	0.062	-0.199	-2.634	0.000	-0.005	12.612	0.009	0.031
Complex	Yes	SciPy	TNC	$\|\|\nabla q\|\|_\infty$	19.6	1,102.034	0.062	-0.199	-2.642	0.000	-0.005	12.547	0.010	0.032
Complex	Yes	SciPy	TNC	$\|\|\theta^n - \theta^{n-1}\|\|_\infty$	13.9	1,102.034	0.062	-0.199	-2.560	0.000	-0.005	12.582	0.010	0.032
Complex	Yes	SciPy	Nelder-Mead	$\|\|\theta^n - \theta^{n-1}\|\|_\infty$	161.4	1,102.034	0.062	-0.199	-2.343	0.000	-0.005	12.612	0.009	0.031
RCNL	No	Knitro	Interior/Direct	$\|\|\nabla q\|\|_\infty$	9.2	934.505	0.070	-0.160	0.014	0.013	-0.033	5.506	0.016	0.038
RCNL	No	Knitro	Active Set	$\|\|\nabla q\|\|_\infty$	8.0	934.505	0.070	-0.160	0.102	0.013	-0.032	5.472	0.016	0.037
RCNL	No	Knitro	SQP	$\|\|\nabla q\|\|_\infty$	9.6	934.505	0.070	-0.160	0.043	0.013	-0.032	5.492	0.016	0.037
RCNL	No	SciPy	Trust-Region	$\|\|\nabla q\|\|_\infty$	12.8	934.505	0.070	-0.160	0.104	0.013	-0.031	5.472	0.016	0.037
RCNL	No	SciPy	L-BFGS-B	$\|\|\nabla q\|\|_\infty$	6.8	934.505	0.070	-0.160	0.110	0.013	-0.032	5.488	0.016	0.037
RCNL	No	SciPy	BFGS	$\|\|\nabla q\|\|_\infty$	6.6	934.505	0.070	-0.160	0.069	0.013	-0.032	5.506	0.016	0.037
RCNL	No	SciPy	TNC	$\|\|\nabla q\|\|_\infty$	11.7	934.505	0.070	-0.160	0.110	0.013	-0.032	5.488	0.016	0.037
RCNL	No	SciPy	TNC	$\|\|\theta^n - \theta^{n-1}\|\|_\infty$	9.6	934.505	0.070	-0.160	0.110	0.013	-0.032	5.467	0.016	0.037
RCNL	No	SciPy	Nelder-Mead	$\|\|\theta^n - \theta^{n-1}\|\|_\infty$	212.6	934.505	0.070	-0.160	0.104	0.013	-0.032	5.488	0.016	0.037
RCNL	Yes	Knitro	Interior/Direct	$\|\|\nabla q\|\|_\infty$	20.2	934.505	0.070	-0.160	-1.724	0.001	-0.003	5.314	0.007	0.016
RCNL	Yes	Knitro	Active Set	$\|\|\nabla q\|\|_\infty$	17.7	934.505	0.070	-0.160	-1.651	0.001	-0.003	5.328	0.007	0.016
RCNL	Yes	Knitro	SQP	$\|\|\nabla q\|\|_\infty$	20.6	934.505	0.070	-0.160	-1.637	0.001	-0.004	5.314	0.007	0.016
RCNL	Yes	SciPy	Trust-Region	$\|\|\nabla q\|\|_\infty$	28.4	934.505	0.070	-0.160	-1.637	0.001	-0.003	5.315	0.007	0.016
RCNL	Yes	SciPy	L-BFGS-B	$\|\|\nabla q\|\|_\infty$	13.9	934.505	0.070	-0.160	-1.637	0.001	-0.003	5.320	0.007	0.016
RCNL	Yes	SciPy	BFGS	$\|\|\nabla q\|\|_\infty$	13.2	934.505	0.070	-0.160	-1.677	0.001	-0.003	5.337	0.007	0.016
RCNL	Yes	SciPy	TNC	$\|\|\nabla q\|\|_\infty$	23.4	934.505	0.070	-0.160	-1.651	0.001	-0.004	5.328	0.007	0.016
RCNL	Yes	SciPy	TNC	$\|\|\theta^n - \theta^{n-1}\|\|_\infty$	18.2	934.505	0.070	-0.160	-1.662	0.001	-0.004	5.319	0.007	0.016
RCNL	Yes	SciPy	Nelder-Mead	$\|\|\theta^n - \theta^{n-1}\|\|_\infty$	302.2	934.505	0.070	-0.160	-1.651	0.001	-0.003	5.314	0.007	0.016

This table documents bias and variance of merger simulation estimates along with estimation speed over 1,000 simulated datasets for different optimization algorithms. It is created from the same results as Table 10 after a second GMM step.