Dynamic Robust Utility

P Beissner* F Maccheroni# M Marinacci# S Mukerji@

* ANU #U Bocconi and @QMU

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Robust Dynamic Utilities

- We study utility processes $V(\mathbf{c}) = (V_t(\mathbf{c}))$ that evaluate consumption processes $\mathbf{c} = (\mathbf{c}_t)$
- They are recursively defined as maximal bounded solutions of a forward recursion:

$$V_t(\mathbf{c}) = \mathrm{E}_t\left[\int_t^T f(s, \mathbf{c}_s, V_s(\mathbf{c}), \sigma_s(V(\mathbf{c})))\mathrm{d}s\right]$$

- They are called *robust* because they feature $\sigma_s(V(\mathbf{c}))$, interpreted as a *variability* index for the valuation process $V(\mathbf{c})$

Two noteworthy special cases

- Let
$$f(s, c, y, z) = g(s, c, y) - \kappa_t |z|$$
where κ_t accounts for "maxmin" ambiguity aversion a la Gilboa and Schmeidler (1989)

- We have a specification a la Chen and Epstein (2002), in their leading example of κ-ignorance (more generally, under rectangularity)

Two noteworthy special cases

- Let
$$f(t,c,y,z) = g(t,c,y) - \frac{\upsilon(y)}{2}\varsigma_t^2 z^2$$
where ς_t and υ account for model ambiguity perception and attitude a la Klibanoff et al. (2005)

- We have a specification a la Hansen and Sargent (2011) and Hansen and Miao (2018)

- This is the specification upon which we will delve deeper

Further special cases

- Here the recursion becomes

$$V_t(\mathbf{c}) = \mathrm{E}_t\left[\int_t^T \left(g(s, \mathbf{c}_s, V_s(\mathbf{c})) - \frac{1}{2}\upsilon(V_t(\mathbf{c}))\varsigma_t^2\sigma_s^2(V(\mathbf{c}))\right) \mathrm{d}s\right]$$

- If either υ or ς are zero, it reduces to a stochastic differential utility recursion a la Duffie and Epstein (1992):

$$V_t(\mathbf{c}) = \mathrm{E}_t\left[\int_t^T g(s, c_s, V_s(\mathbf{c}))\, \mathrm{d}s\right]$$

Further special cases

- If, in addition, $g(t, c, y) = u(c) - \beta y$, it further reduces to a traditional expected utility recursion

$$V_t(\mathbf{c}) = \mathrm{E}_t\left[\int_t^T \left(u(\mathbf{c}_s) - \beta_s V_s(\mathbf{c})\right) \mathrm{d}s\right]$$

that is,

$$V_t(\mathbf{c}) = \mathrm{E}_t\left[\int_t^T e^{-\beta(s-t)} u(c_s)\, \mathrm{d}s\right]$$

Heuristic interpretation (sketch)

- DM may posit a finite set

$$\left\{ \mathbb{Q}^{\boldsymbol{\theta}} : \boldsymbol{\theta} \in \Theta \right\}$$

of probability measures equivalent to $\mathbb{P}$, each interpreted as an alternative possible true model

- We can regard Θ as a collection of time-varying parameters $\boldsymbol{\theta} \in L^0$ that parametrize the models $\mathbb{Q}^{\boldsymbol{\theta}}$ via the relation

$$\mathrm{E}_t \left[\frac{\mathrm{d}\mathbb{Q}^{\theta}}{\mathrm{d}\mathbb{P}} \right] = \exp \left(-\frac{1}{2} \int_0^t \theta_s^2 \mathrm{d}s + \int_0^t \theta_s \mathrm{d}B_s \right)$$

Heuristic interpretation (sketch)

- A belief process $\boldsymbol{\pi} = (\pi_t) \subseteq \Delta(\Theta)$ describes the evolution of DM's beliefs about parameters
- The expected parameter process $\vartheta = (\vartheta_t)$ is given by $\vartheta_t = \sum_{\boldsymbol{\theta}} \theta_t \, \pi_t(\boldsymbol{\theta})$
- Process $\varsigma = (\varsigma_t) \in L^2$ of standard deviation is given by

$$\varsigma_t = \sqrt{\sum_{\boldsymbol{\theta}} \theta_t^2 \pi_t(\boldsymbol{\theta}) - \vartheta_t^2}$$

- Process ς accounts for the DM perception of model ambiguity, a subjective feature that depends on the belief process

Heuristic interpretation (sketch)

- Let $\phi : \mathbb{R} \to \mathbb{R}$ be a function that describes attitudes toward model ambiguity (as in the smooth ambiguity model)
- In the robust aggregator

$$f(t, c, y, z) = u(c_t) - \beta y - \frac{1}{2}\varsigma_t^2 v(y) z^2$$

one reads

$$v = \lambda^{\phi} \equiv -\frac{\phi''}{\phi'}$$

- A key caveat detailed in the "real" heuristic

Summing up

- Interpretation of the separable specification:

$$f(t,c,y,z) = u(c_t) - \beta y - \frac{1}{2}\varsigma_t^2 \upsilon(y) z^2$$

- ς indexes the degree of (perceived) model ambiguity over time: the higher ς is, the higher such ambiguity is, with $\varsigma = 0$ iff it is absent

- υ indexes attitudes toward model ambiguity: the (pointwise) higher υ is, the higher is the aversion to such uncertainty, with neutrality iff $\upsilon = 0$

- z^2 describes the impact of the volatility of uncertain streams' valuations

Summing up

- If υ is a constant coefficient (of aversion to model uncertainty), we can write the recursion as

$$V_t(\mathbf{c}) = U_t(\mathbf{c}) - \upsilon \Sigma_t(V(\mathbf{c}))$$

where $U_t(\mathbf{c}) = \mathrm{E}_t\left[\int_t^T e^{-\beta(s-t)} u(\mathbf{c}_s)\, \mathrm{d}s\right]$ and $\Sigma : L^\infty \to L^0$ is given by

$$\Sigma_t(V(\mathbf{c})) = \mathrm{E}_t\left[\int_t^T e^{-\beta(s-t)} \varsigma_s^2 \sigma_s^2(V(\mathbf{c}))\, \mathrm{d}s\right]$$

- Σ can be viewed as a variability index for the dynamic utility operator V

- A mean-variance flavor

In the paper

- We prove that dynamic robust utilities exist, are monotone and dynamically consistent, and can be concave
- We study general decision problems based on dynamic robust utilities
- To illustrate, we study a portfolio problem and derive some asset pricing formulas featuring risk and ambiguity components
- For instance, excess return has the form

$$\mathrm{E}_t\left[\mathrm{d}R_t\right] - r_t \mathrm{d}t \approx \underbrace{\lambda^u(\hat{\mathbf{c}}_t)\sigma_t\left(\hat{\mathbf{c}}\right)\sigma_t\left(R\right)\mathrm{d}t}_{\text{risk term}} + \underbrace{\upsilon\sigma_t(V\left(\hat{\mathbf{c}}\right))\varsigma_t^2\sigma_t\left(R\right)\mathrm{d}t}_{\text{ambiguity term}}$$

At variance

- A first study of aggregators that depend on variability is Lazrak and Quenez (2003). They require, however, Lipschitz conditions that are not germane to our "quadratic analysis"
- But, why including *valuation variability* is a non-trivial issue?
- Back to Principles of Economics

Lotteries

- X is a prize space
- $\Delta(X)$ is the set of lotteries p
- Lotteries are simple prob. measures, with

$$\operatorname{supp} p = \{x \in X : p(x) > 0\}$$

- $u : X \to \mathbb{R}$ is a utility function, with $\operatorname{Im} u = (a, b)$

Ranking lotteries

- Expected utility $\mathbb{E}_u : \Delta(X) \to \mathbb{R}$ is given by

$$\mathbb{E}_u(p) = \sum_{x \in \operatorname{supp} p} u(x)\, p(x)$$

- Standard deviation $\sigma_u : \Delta(X) \to \mathbb{R}$ is given by

$$\sigma_u(p) = \sqrt{\sum_{x \in \operatorname{supp} p} \left(u(x) - \mathbb{E}_u(p)\right)^2 p_i}$$

Expected Utility?

- Expected utility $\mathbb{E}_u$ is the standard (normative) criterion
- A good student might well ask: what about the standard deviation?
- Specifically, given a function $\lambda : (a, b) \to \mathbb{R}$, define

$$M_u(p) = \mathbb{E}_u(p) - \frac{\lambda(\mathbb{E}_u(p))}{2} \sigma_u^2(p)$$

- Why $\mathbb{E}_u$ and not M_u?

Expected Utility?

- Standard answer: the von Neumann-Morgenstern axioms, in particular the Independence Axiom
- If you do not buy this axiom, note that M_u is not even monotone, a basic rationality tenet

Small Variance

- Yet, if σ_u is small enough, monotonicity holds
- Can M_u then be resurrected as an "asymptotic" criterion?

Small Variance

- A function $r : \Delta(X) \to \mathbb{R}$ is $o\left(\sigma_u^2(p)\right)$ if

$$\sigma_u(p_n) \to 0 \Longrightarrow \frac{r(p_n)}{\sigma_u^2(p_n)} \to 0 \qquad \forall \{p_n\} \subseteq \Delta(X)$$

- A function $v : X \to \mathbb{R}$ is ordinally equivalent to u if

$$v = g \circ u$$

for some strictly increasing function $g : (a, b) \to \mathbb{R}$

Expected Utility Redux

- **THM (e.g. de Finetti '52)** There exists $v : X \to \mathbb{R}$, ordinally equivalent to u, such that

$$M_u(p) = \mathbb{E}_v(p) + o\left(\sigma_u^2(p)\right) \qquad \forall p \in \Delta(X)$$

- So, the relevance of the standard deviation σ_u vanishes asymptotically faster than its magnitude

- g is such that $\lambda = -g''/g'$

Continuous Time

- Even if you do not buy the Independence Axiom, this casts some serious doubts on M_u even as an asymptotic criterion
- That said, all this suggests that continuous time might be a key framework where to study *monotone* decision criteria where variability plays a role

Continuous time setup

- Time interval $[0, T]$
- Probability space $(\Omega, \mathcal{F}, \mathbb{P})$
- Brownian motion $B = (B_t)$
- Brownian standard filtration $\mathbb{F} = (\mathcal{F}_t)$
- Filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$

Continuous time setup

- Basic primitive: atemporal utility function $u : C \longrightarrow \mathbb{R}$ over a material consequence $c \in C$, say a consumption good
- The utility function u is bounded, monotone and measurable

Probabilities

- DM faces uncertain consumption streams that depend on exogenous contingencies $\omega \in \Omega$
- Uncertainty is governed by a true *generative mechanism* or *model*
- DM has a *subjective probability* $\mathbb{P}$ on Ω that quantifies beliefs about the relative likelihood of the exogenous contingencies
- $\mathbb{P}$ is DM's best guess of the true model

Dynamic Utility Operators

- Consumption streams are adapted processes, denoted by $\mathbf{c} = (\mathbf{c}_t) \in \mathbf{C}$
- They are evaluated by the DM through the following operator:
- **DEF** A *dynamic utility operator* is a map $V : \mathbf{C} \to L^0$ that associates to each consumption plan $\mathbf{c} = (\mathbf{c}_t)$ a utility process $V(\mathbf{c}) = (V_t(\mathbf{c}))$ such that

$$(u(\mathbf{c}_t)) = (u(\mathbf{c}'_t)) \Longrightarrow V(\mathbf{c}) = V(\mathbf{c}')$$

for all consumption plans $\mathbf{c}, \mathbf{c}' \in \mathbf{C}$

Dynamic Utility Operators

- Each $V_t : \mathbf{C} \to L_t^\infty$ is an intertemporal utility function that, at time t, evaluates consumption streams based on his information $\mathcal{F}_t$ at t.
- From the standpoint of $t = 0$, we can interpret $V_t(\mathbf{c})$ as the continuation value of stream $\mathbf{c}$ at time t.
- A standard separable example is

$$U_t(\mathbf{c}) = \mathrm{E}_t\left[\int_t^T e^{-\beta(s-t)} u(c_s)\,\mathrm{d}s\right] \qquad \forall \mathbf{c} \in \mathbf{C}$$

Dynamic Utility Operators

- The utility process $V(\mathbf{c})$ is assumed to be an Ito process

$$\mathrm{d}V_t(\mathbf{c}) = \mu_t(V(\mathbf{c}))\,\mathrm{d}t + \sigma_t(V(\mathbf{c}))\,\mathrm{d}B_t$$

  for all $\mathbf{c}$

- DM addresses the decision problem

$$\max_{\mathbf{c}} V(\mathbf{c}) \quad \text{sub } \mathbf{c} \in \tilde{\mathbf{C}}$$

  where $\tilde{\mathbf{C}} \subseteq \mathbf{C}$ is a choice set formed by alternative consumption streams among which the decision maker can choose

Aggregators

- **DEF** A map $f : \Omega \times [0, T] \times C \times \mathbb{R}^2 \to \mathbb{R}$ is a *robust* (*stochastic*) *aggregator* if, for all $c, c' \in C$, we have:
  1. for each $t \in [0, T]$, $u(c) \geq u(c') \Longrightarrow f(t, c, 0, 0) \geq f(t, c', 0, 0)$
  2. for each $(t, y, z) \in [0, T] \times \mathbb{R}^2$,
  $$f(t, c, 0, 0) \geq f\left(t, c', 0, 0\right) \Longrightarrow f(t, c, y, z) \geq f\left(t, c', y, z\right)$$
- These conditions ensure that aggregator f is consistent with the posited utility u

Aggregators

- Throughout we assume that:
  1. $f(t,c,\cdot,\cdot):\mathbb{R}^2 \to \mathbb{R}$ is continuous for all $(t,c) \in [0,T] \times \mathbb{R}_+$ and $f(\cdot,\cdot,0,0):[0,T]\times\mathbb{R}_+ \to \mathbb{R}$ is bounded
  2. there exists $k > 0$ such that
     $$|f(t,c,y,z) - f(t,c,0,0)| \leq k\left(1 + |y| + z^2\right)$$
     for all $(t,c,y,z) \in [0,T] \times \mathbb{R}_+ \times \mathbb{R}^2$
  3. $f(\cdot,c,y,z) \in L^0$ for all $(c,y,z) \in [0,\infty) \times \mathbb{R}^2$, i.e., it is an adapted and predictable process
- *Quadratic* setting

Robust Dynamic Utility Operators

- The pair (u, f) is the primitive

- **DEF** A dynamic utility operator $V : L^0_+ \to L^\infty$ is *robust* if it is a maximal bounded solution of the forward recursion

$$V_t(\mathbf{c}) = \mathrm{E}_t\left[\int_t^T f(s, c_s, V_s(\mathbf{c}), \sigma_s(V(\mathbf{c})))\,\mathrm{d}s\right] \qquad \forall \mathbf{c} \in \mathbf{C}$$

  that, for all $t \in [0, T]$, the pair (u, f) induces

- "Robust" because it features $\sigma_s(V(\mathbf{c}))$, interpreted as a variability index for the valuation process $V(\mathbf{c})$

Robust Dynamic Utility Operators

- DM is concerned about such variability because he doubts that his subjective belief is correct
- If DM had no such a doubt, the dynamic utility operator would feature a risk aggregator that does not depend on z, so on the quadratic variation
- Stochastic differential utility a la Duffie and Epstein (1992):

$$V_t(\mathbf{c}) = \mathrm{E}_t\left[\int_t^T f(s, c_s, V_s(\mathbf{c}))\,\mathrm{d}s\right] \qquad \forall \mathbf{c} \in \mathbf{C}$$

- If $f(t, c, y) = u(c_t) - \beta y$, it reduces to

$$V_t(\mathbf{c}) = \mathrm{E}_t\left[\int_t^T e^{-\beta(s-t)} u(c_s)\,\mathrm{d}s\right] \qquad \forall \mathbf{c} \in \mathbf{C}$$

Unique existence

- **PROP** Any pair (u, f) admits a unique robust dynamic utility operator V
- Uniqueness is trivial but not existence
- It relies on an equivalence between forward recursions and quadratic backward stochastic differential equations (BSDE)
- Via this characterization we can establish existence and other properties of robust dynamic utility operators

Engine room

THM A pair $(Y, Z) \in L^\infty \times L^2$ solves the quadratic BSDE

$$Y_t = \xi + \int_t^T g(s, Y_s, Z_s)\mathrm{d}s - \int_t^T Z_s \mathrm{d}B_s \qquad \forall t \in [0, T]$$

if and only if Y is an Ito process that satisfies the forward recursion

$$Y_t = \mathrm{E}_t\left[\xi + \int_t^T g(s, Y_s, \sigma(Y))\mathrm{d}s\right] \qquad \forall t \in [0, T]$$

and $Z = \sigma(Y)$

Monotonicity

- **PROP** Given a robust dynamic utility operator $V : L^0_+ \longrightarrow L^\infty$, at each t we have

  $$\left(u\left(c_s\right)\right)_{s\in[t,T]} \geq \left(u\left(c'_s\right)\right)_{s\in[t,T]} \Rightarrow \left(V_s\left(\mathbf{c}\right)\right)_{s\in[t,T]} \geq \left(V_t\left(\mathbf{c}'\right)\right)_{s\in[t,T]}$$

  for all consumption plans $\mathbf{c}, \mathbf{c}' \in \mathbf{C}$

- A form of consequentialism

- We have monotonicity despite dependence on variability

Other properties

- A robust dynamic utility operator:
  1. satisfies a dynamic consistency property
  2. is concave if its robust aggregator f is concave in (c, y, z), provided some regularity conditions hold

Road ahead (with more time)

- Heuristic interpretation (real)
- Decision problems