AUTOMATED ECONOMIC REASONING WITH QUANTIFIER ELIMINATION

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ABSTRACT

Many theorems in economics can be proven (and hypotheses shown to be false) with "quantifier elimination." Results from real algebraic geometry such as Tarski's quantifier elimination theorem and Collins' cylindrical algebraic decomposition algorithm are applicable because the economic hypotheses, especially those that leave functional forms unspecified, can be represented as systems of multivariate polynomial (sic) equalities and inequalities. The symbolic proof or refutation of economic hypotheses can therefore be achieved with an automated technique that involves no approximation and requires no problem-specific information beyond the statement of the hypothesis itself. This paper also discusses the computational complexity of this kind of automated economic reasoning, its implementation with Mathematica and REDLOG software, and offers several examples familiar from economic theory.

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Economics has been profoundly affected by progress in information technology that has facilitated the collection and processing of vast amounts of data related to economic activity. Information technology has so far assisted less with economic reasoning – deducing conclusions about behavior, markets, or welfare from assumptions or observations about motivation, technology, and market structure – of the sort done by Alfred Marshall, Paul Samuelson, Gary Becker, or Roger Myerson. There are automatic algebraic simplifiers, but simplicity is often in the eye of the beholder and such tools are sparingly used by economic theorists. Computers have already been used for generating numerical examples, but approximation quality is a concern, and more thinking is always needed to appreciate the generality of the results from examples. The purpose of this paper is to show how approximation-free economic reasoning is beginning to be automated, present the mathematical foundations of those procedures, and allow readers of this paper to access a user-friendly tool for automated economic reasoning.

Section I introduces, to an economics audience, quantified systems of polynomial equalities and inequalities, and their quantifier-free equivalents, as defined in real algebraic geometry. Section II shows how a number of hypotheses in economic theory, especially those that leave functional forms unspecified, are isomorphic with those systems. Section III shows how quantifier elimination can be used as a tool for proving hypotheses, detecting inconsistent assumptions, reformulating hypotheses to make them True, measuring the relative strength of alternative assumptions, and generating examples and counterexamples. Results from mathematicians Tarski, Collins, and followers – shown in Section IV – speak to the feasibility of, and algorithms for, eliminating quantifiers from systems of polynomial equality, inequality, and not-equal relations (hereafter "polynomial inequalities") and thereby for confirming or refuting many hypotheses in economic theory. In addition to presenting results from the mathematics literature, Section IV links them with the economic examples, and gives special attention to single-cell decompositions, universal sentences, and existential sentences.

Readers are also pointed to existing software implementations of quantifier-elimination methods and given some indication as to likely progress in this dimension. One of the implementations is in the Wolfram Language/Mathematica, which has a number of other symbolic capabilities such as automated differentiation and various interface options. REDLOG is less familiar and has a more primitive user interface, but typically eliminates quantifiers more quickly than Mathematica does.[1]

I. Sets and hypotheses represented with and without quantifiers

I.A. An example from elementary algebra

As a first example, Figure 1 shows the two dimensional set of coefficients (b,c) of parabolas that have real roots. Equation (1) features two (of many) ways of defining the same set:

$$\{(b,c) \in \mathbb{R}^2 : \exists x(x^2 + bx + c = 0)\} = \\ \{(b,c) \in \mathbb{R}^2 : b^2 \geq 4c\} \tag{1}$$

where b, c and x are scalar real numbers.[2] The first (hereafter, "quantified") definition uses the "existential" quantifier "Exists" ($\exists$) over the quantified variable x in order to represent the parabola property of interest: having a real root. The second "quantifier-free" definition has no quantifiers, but nonetheless describes the same subset of $\mathbb{R}^2$.

Now consider the question of whether a particular parabola (b,c) is in the set featured in (1): whether it has real roots. The question might not be answered in finite time with the quantified definition, taken literally, because a parabola cannot be confirmed to be outside the set without checking all possible values of x. At the same time, the quantifier-free definition provides verification in just one step: verifying whether b and c satisfy the inequality. Ease of verification is why it can be of "enormous" practical value to "eliminate quantifiers" from a set's definition: that is, to take a quantified definition such as the LHS of (1) and transform it into a quantifier-free

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[1] Mathematica calculations have been fast enough for my purposes. See also Bradford, et al. (2016), who find Mathematica to be faster than three other implementations (none of which is REDLOG) and Davenport and England (2015), who find Mathematica to be "exceptionally fast." I have also encountered specific problems that REDLOG processes orders of magnitude more slowly than Mathematica does.

[2] I assume that the coefficient on x^2 is nonzero, and therefore without loss of generality describe roots of quadratic equations by reference to a quadratic equation with a unit coefficient on x^2. Throughout the paper, each variable is assumed to be a scalar real number unless explicitly indicated otherwise.

one such as the RHS of (1).[3] Indeed, some artificial intelligence research equates quantifier elimination with the vernacular concept of "solving" a mathematics problem (Arai, et al. 2014, p. 2).

Real algebraic geometry has a number of results as to (a) what quantified set definitions have an equivalent quantifier-free representation, and (b) algorithms that eliminate quantifiers. The purpose of this paper is to report results from the real algebraic geometry and symbolic computation literatures, show how they permit a significant part of economic theory to be automated, and give more details as to features that are especially relevant for the economics applications.

The quantified definition (1) has one quantified variable, x, and two that are unquantified $\{b,c\}$. Now consider introducing quantifiers over one or two of the parabola coefficients as well. For example, if we introduce the "universal" quantifier "ForAll" ($\forall$) over the linear-term coefficient b, we describe a coefficient set in $\mathbb{R}$:

$$\{c \in \mathbb{R}: \forall b \exists x(x^2 + bx + c = 0)\} = \{c \in \mathbb{R}: c \leq 0\} \quad (2)$$

The quantified definition reads "For all b, there exists a real root" (the order of the quantifiers matters in this case). Again, the quantifier-free definition facilitates verification that any particular parabola is in the set of interest. Introducing a third quantifier, we have:

$$\exists c \forall b \exists x(x^2 + bx + c = 0) = True \quad (3)$$

All three of the variables are quantified in Equation (3). A fully-quantified formula is known as a "sentence," and all sentences are either True or False.[4] Removing the quantifiers from a sentence is known as "deciding" that sentence and, in effect, is a proof of its assertion because the quantifier-free representation of any sentence is either True or False.

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[3] Caviness and Johnson (1998, p. 2).

[4] Another example is the hypothesis that all parabolas have a real root ($\forall\{b,c\}\exists x(x^2 + bx + c = 0)$), which is a False sentence.

Hypotheses that involve assumptions also fit into this framework. Take for example the hypothesis that if any parabola's coefficients satisfy $b^2 \geq 4c$ then it has real roots:

$$\begin{aligned} &\forall\{b,c\}[b^2 \geq 4c \Rightarrow \exists x(x^2+bx+c=0)] \\ &\qquad = \forall\{b,c\}[b^2 < 4c \vee \exists x(x^2+bx+c=0)] = True \end{aligned} \tag{4}$$

The first equality follows from the definition of "Implies" ($\Rightarrow$): that either the implication is True or the assumption is False.[5] The second equality shows the removal of the quantifiers from the sentence, and that the result is True. This example shows how statements of the form "If *X* is true, then so is *Y*" are logically equivalent to Boolean combinations of hypotheses with And ($\wedge$) and Or ($\vee$) and are therefore sentences that would be proven True or False by quantifier elimination.

I.B. The General Framework

I.B.1. The formulated hypothesis

The general framework has $N < \infty$ real scalar variables $x_1, \ldots, x_N$. The formulation *HF* of a hypothesis involves quantifiers on $x_1, \ldots, x_{N-F}$, with the remaining $0 \leq F < N$ of them free (unquantified):

$$\begin{aligned} HF = (Q_1x_1)(Q_2x_2)\ldots(Q_{N-F}x_{N-F})T(x_1,x_2,\ldots,x_N) \\ Q_i \in \{\forall,\exists\}\ \ i=1,\ldots,(N-F) \end{aligned} \tag{5}$$

where the "Tarski formula" *T* by itself is a quantifier-free Boolean combination, with the logical And and Or operators, of a finite number of polynomial (in $x_1, \ldots, x_N$)

[5] Note that, by this definition (also implemented as "Implies" in Mathematica and REDLOG), $\exists c(c < 0 \Rightarrow c > 0) = True$. This is one reason why (a) logicians sometimes argue for alternative definitions of Implies (Priest 2000, p. 53) and (b) when connecting assumptions with implications under the existential quantifier, Mathematica requires that there exists a point in which both the assumption and implication are simultaneously True. See also below on "empty assumption sets."

inequalities.[6] For brevity I also use the Not operator, which merely refers to reversing an inequality (or changing = to ≠), and the Implies operator, which is a shorthand for a Boolean combination of And, Or and Not (see above).

Of particular interest are universal and existential formulations that have the same quantifier on each of the *N–F* variables. In these cases, I show the quantifier only once and list the quantified variables in braces:

$$(Qx_1)(Qx_2)\dots(Qx_{N-F})T(x_1,x_2,\dots,x_N) \equiv Q\{x_1,x_2,\dots,x_{N-F}\}T(x_1,x_2,\dots,x_N) \qquad (6)$$

Hypotheses formulated with only one kind of quantifier, e.g., (6), have the same meaning regardless of the order of the quantifiers. Moreover, every universal formulation can be expressed as an existential formulation, and vice versa:

$$\neg\forall\{x_1,x_2,\dots,x_{N-F}\}T(x_1,x_2,\dots,x_N) = \exists\{x_1,x_2,\dots,x_{N-F}\}\neg T(x_1,x_2,\dots,x_N) \qquad (7)$$

where ¬ is the Not operator.[7] If, for given values of the free variables, the Tarski formula is not True on all of $\mathbb{R}^{N-F}$, then there exists at least one point in $\mathbb{R}^{N-F}$ where the Tarski formula is false, and vice versa. The order-invariance and quantifier-interchangeability properties of universal and existential formulations offer many opportunities for facilitating and verifying computation.

Because of their relationship with proofs, sentences (*F* = 0) are especially useful for automating economic reasoning. This contrasts with previous discussions of quantifier elimination in economic theory, such as Brown and Matzkin (1996), Snyder (2000), Brown and Kubler (2008), Carvajal et al. (2014), and Chambers and Echenique (2016), whose purposes are to derive restrictions on free variables that they associate with "observables." Moreover, with an exception appearing in the appendix of Brown and Matzkin (1996), they do not intend to "carry out" the quantifier elimination but rather be assured that the result of doing so would be a non-empty semi-algebraic set in $\mathbb{R}^F$.

[6] C.W. Brown (2004, 2). For example, putting the middle of (4) in the same format, we have $(\forall b)(\forall c)(\exists x)[b^2 < 4c \vee x^2 + bx + c = 0]$, with the Tarski formula in square brackets.
[7] (7) is known as "De Morgan's law for quantifiers."

I.B.2. An equivalent representation, without quantifiers

As above, we are interested in a quantifier-free, but equivalent, statement HE of a formulated hypothesis HF. Formally,

$$HE = P(x_{N-F+1}, \dots, x_N) \tag{8}$$

where *P* is another Tarski formula (distinct from the *T* appearing in *HF*), and therefore a quantifier-free Boolean combination of a finite number of polynomial inequalities. If there are no free variables ($F = 0$), then *P* is either 1 = 1 (True) or 1 = 0 (False).

Quantifier elimination refers to an algorithmic method that derives *P* from *HF*.[8] We are also interested in the existence and properties of such "automated" method(s), but first we consider some familiar economic hypotheses that fit into this framework and the potential value of such a method as a tool for economic reasoning.

II. Examples of quantifiers in economic analysis

II.A. Concave and quasiconcave production functions

Consider the assertion, adapted from Jehle and Reny (2011), about continuous and differentiable production functions of two inputs *x* and *y*: that all such production functions *f* that are strictly increasing and strictly quasiconcave at a point (*x*,*y*) with positive input quantities are concave functions of their inputs at that point. We can investigate this assertion by formulating a hypothesis within the framework (5), as shown below:

[8] In the quadratic formula example (2), it would be a method that derives the RHS (an inequality restriction on *c*) from the LHS. Ideally, the same method used for (2) could also be used for (1), (3), and (4) and any other application that fits within the general framework (5).

$$\forall\left\{x,y,\frac{\partial f(x,y)}{\partial x},\frac{\partial f(x,y)}{\partial y},\frac{\partial^2 f(x,y)}{\partial x^2},\frac{\partial^2 f(x,y)}{\partial y^2},\frac{\partial^2 f(x,y)}{\partial x\partial y}\right\}$$

$$\Bigg[\Bigg(x>0\wedge y>0\wedge\frac{\partial f(x,y)}{\partial x}>0\wedge\frac{\partial f(x,y)}{\partial y}>0$$

$$\wedge\left(\frac{\partial f(x,y)}{\partial x}\right)^2\frac{\partial^2 f(x,y)}{\partial x^2}+\left(\frac{\partial f(x,y)}{\partial y}\right)^2\frac{\partial^2 f(x,y)}{\partial y^2}<2\frac{\partial f(x,y)}{\partial x}\frac{\partial f(x,y)}{\partial y}\frac{\partial^2 f(x,y)}{\partial x\partial y}\Bigg) \tag{9}$$

$$\Rightarrow\Bigg(\frac{\partial^2 f(x,y)}{\partial x^2}\le 0\wedge\frac{\partial^2 f(x,y)}{\partial y^2}\le 0$$

$$\wedge\frac{\partial^2 f(x,y)}{\partial x^2}\frac{\partial^2 f(x,y)}{\partial y^2}\ge\left(\frac{\partial^2 f(x,y)}{\partial x\partial y}\right)^2\Bigg)\Bigg]=False$$

The hypothesis formulation on the LHS of (9) is a universal sentence with 7 scalar real variables.[9] Its logical form is an assumption with an implication. The first-derivative conditions in the assumption say that, at the arbitrary point (x,y), the production function is increasing in both arguments. The assumption's second-derivative condition says that the production function is strictly quasiconcave at that point. The implication's second-derivative conditions say that the production function is, at the same point, jointly concave in the two inputs. Eliminating the quantifiers from the LHS reveals that the hypothesis is False: there are elements of $\mathbb{R}^7$ in which the assumption is True but the implication is not.

Note that quantifiers in the general framework (5), and in the example (9), are not formulating universal or existential statements about elements of a function space. Rather, (9) refers to values of function arguments and derivatives at a particular point (x,y) and thereby to polynomials in $\mathbb{R}^7$. "ForAll" means all possible values for those seven arguments and derivatives. This isomorphism is essential to broadly applying the results from real algebraic geometry because the latter refer to polynomials in real closed fields (such as the real numbers). Table 1 illustrates the isomorphism more starkly by relabeling the input and derivative values as v_1 through v_7.[10] Every component of the

[9] Two of them, x and y, are not related to the rest of the inequalities, but we use them later as we modify the example.

[10] The variables in (9) are mapped to the generic notation $\{v_1,\ldots,v_7\}$ in alphabetical order (as sorted by Mathematica with variables before functions of variables). Table 1 lists the variables in

assumption and every component of the hypothesis is an equality or inequality comprised of a sum of various products of these variables (with some of the variables appearing more than once in the product).

The ForAll quantifier is powerful because it allows statements about real numbers to generate statements about classes of functions. If, for example, (9) had been True, then it would tell us that any real-valued differentiable two-argument function that is strictly increasing and strictly quasiconcave at a point (with positive inputs) would be concave at that point because any such function has its arguments and derivatives (relevant to concavity and quasiconcavity) at a point described by seven real numbers. (9) is in fact False, which tells us that there exists seven real numbers to assign to those arguments and derivatives that describe, at a point with positive inputs, positive marginal products and strict quasiconcavity but not concavity. Any function with those seven arguments and derivatives would prove by example that not all quasiconcave functions are concave.[11]

It follows that neither (9) nor (5) requires the production function to be a polynomial in x and y alone, as assumed the chapters of Brown and Kubler (2008) that posit semi-algebraic economies. Indeed, as we shall see, an explicitly semi-algebraic production function would restrict the range of analysis and potentially have tremendous computational costs.

II.B. A Monopolist's pass through of marginal costs

Consider a monopolist that faces a demand curve whose inverse is $W(q)$ with $W'(q) < 0$, where q is the quantity sold to consumers. The cost of producing q is $g(q,a)$, which satisfies

order of the complexity of their contribution to the system of polynomial inequalities (C. W. Brown 2004).

[11] Given seven numbers that falsify (9), a entire family of nonconcave but analytic, increasing and quasiconcave functions could be found by using those values to specify the corresponding level and derivative terms in an infinite Taylor-series representation of the production function. It is an entire family because the seven numbers are consistent with any values for the third- and higher-order coefficients in the Taylor series.

$$\frac{\partial g(q,a)}{\partial q} \geq 0\,, \quad \frac{\partial^2 g(q,a)}{\partial q^2} \geq 0\,, \quad \frac{\partial^2 g(q,a)}{\partial q \partial a} > 0 \tag{10}$$

where a is a cost parameter that increases marginal cost. If the monopolist produces and sells q, then he receives $W(q)$ from each unit sold. His optimal quantity is described by:[12]

$$qW(q) - g(q,a) \geq 0 \tag{11}$$

$$\frac{\partial}{\partial q}[qW(q) - g(q,a)] = 0 \tag{12}$$

$$\frac{\partial^2}{\partial q^2}[qW(q) - g(q,a)] < 0 \tag{13}$$

The pass-through rate μ can be defined to be the impact of a on the monopolist's optimal price per unit impact on marginal cost:[13]

$$\mu \equiv \frac{\frac{d}{da}W(q)}{\frac{d}{da}\left[\frac{\partial g(q,a)}{\partial q}\right]} \tag{14}$$

$$\frac{d}{da}\left\{\frac{\partial}{\partial q}[qW(q) - g(q,a)]\right\} = 0 \tag{15}$$

The assumptions (10)-(15) and $W'(q) < 0$ are polynomial inequalities in the 9-dimensional space $\left\{q, \frac{dq}{da}, g(q,a), \frac{\partial g}{\partial q}, \frac{\partial^2 g}{\partial q \partial a}, \frac{\partial^2 g}{\partial q^2}, W(q), W'(q), W''(q)\right\}$. Quantifier elimination in this space can confirm a number of hypotheses about pass-through. For example, a convex demand curve ($W'' > 0$) is necessary but not sufficient for pass-

[12] See also Weyl and Fabinger (2013).
[13] A more familiar derivative would use the partial, rather than total, derivative in the denominator. This paper features the total derivative because it makes the quantifier elimination a bit more complicated and interesting.

through to exceed one ($\mu > 1$) at positive quantities ($q > 0$). Note that both the pass-through implication and the extra assumptions are all (rather trivial) polynomial inequalities in the same space and thereby keep the hypothesis within the general framework (5).

II.C. Laffer curve surprises

Consider the prototype static representative agent economy for examining aggregate consequences of labor income taxation. Specifically, the representative agent has preferences over the amount consumed c and the amount worked n, as represented by the utility function $u(c,n)$. c is a good, n is a bad, and preferences are quasiconcave, in the relevant range. The economy's production set is weakly convex with its boundary described by the monotone increasing production function $f(n)$.

The government levies a constant-rate labor income tax in order to finance a lump sum transfer. Given a labor income tax rate τ, a competitive equilibrium in this economy is a list of five scalars $\{c,n,a,r,w\}$ such that:

(i) Given a, r, w, and τ, the pair (c,n) maximizes the representative worker's utility subject to his budget constraint:

$$(c,n) = \underset{c',n'}{\operatorname{argmax}}\, u(c',n') \quad s.t.\ c' \leq (1-\tau)wn' + r + a \tag{16}$$

(ii) Given w, n maximizes the representative employer's profits a:

$$n = \underset{n'}{\operatorname{argmax}}\, f(n') - wn' \tag{17}$$

$$a = \max_{n'} f(n') - wn' \tag{18}$$

(iii) The government budget constraint balances

$$r = \tau wn \tag{19}$$

There typically is more than one tax rate that is consistent with the same amount of revenue r: for example no revenue could come from $\tau = 0$ or from $\tau = 1$. In this case, the lower tax rate method of obtaining that revenue is associated with more equilibrium utility for the representative agent, but is that a general result? Formally, if we have an equilibrium $\{c_L,n_L,a_L,r_L,w_L\}$ associated with tax rate τ_L and another equilibrium $\{c_H,n_H,a_H,r_H,w_H\}$ associated with tax rate $\tau_H > \tau_L > 0$ and $r_L = r_H > 0$, can we conclude that $u(c_L,n_L) > u(c_H,n_H)$?

Note that, unless τ_L were at the peak of the Laffer curve, we cannot assume that, say, c_H differs from c_L by only a differential change as we did with the first two examples that applied the chain rule of calculus. This Laffer-curve example relates to discrete differences, but is nonetheless amenable to quantifier elimination. Here the assumptions are:

$$0 < \tau_L < \tau_H < 1 \tag{20}$$

$$w_L > 0, w_H > 0, n_L > 0, n_H > 0, c_L > 0, c_H > 0 \tag{21}$$

$$\tau_L w_L n_L = \tau_H w_H n_H \tag{22}$$

$$(c_L - c_H)(n_L - n_H) > 0\,,\ \ (n_L - n_H)w_L \le c_L - c_H \le (n_L - n_H)w_H \tag{23}$$

$$u_L = u(c_L, n_L) > u'_L = u(c'_L, n_H),\ \ c_L - c'_L = (n_L - n_H)(1 - \tau_L)w_L \tag{24}$$

$$u_H = u(c_H, n_H) > u'_H = u(c'_H, n_L),\ \ c_H - c'_H = (n_H - n_L)(1 - \tau_H)w_H \tag{25}$$

$$(u'_L - u_H)(c'_L - c_H) > 0\,, \qquad (u'_H - u_L)(c'_H - c_L) > 0 \tag{26}$$

$$M(c_L, n_L) = (1 - \tau_L)w_L,\ \ M(c_H, n_H) = (1 - \tau_H)w_H \tag{27}$$

The inequalities (20) and (21) define the question of interest (i.e., looking at equilibria with nondegenerate tax rates, positive wage rates, and positive amounts worked).[14] Equation (22) assumes that the two equilibria have the same tax revenue.

[14] For brevity, I also assume that the utility and production functions are continuously differentiable in the neighborhood of each of the two equilibria being considered. This

The inequalities (23) represent the production function assumptions: production is strictly increasing and the marginal product of labor is weakly decreasing.[15] The inequalities (24) define and characterize one of the equilibrium utility levels. In particular, an agent in equilibrium L choses not to work the amount n_H and have the corresponding consumption level c_L'. In equalities (25) do the same with respect to the H equilibrium. The inequalities (26) say that consumption is a good. The equations (27), which include the marginal rate of substitution function M, are the workers' first order condition for each of the two equilibria under consideration.

Formally, the hypothesis of interest is that (20) through (27) imply that $u_L > u_H$. Quantifier elimination in a sixteen-dimensional space shows that more assumptions are required in order to conclude that $u_L > u_H$, because the hypothesis is true in part, but not all, of the sixteen-dimensional space.[16] For example, the additional assumption that consumption and leisure are normal goods – in particular that $(M(c_L,n_L) - M(c_H,n_H))(n_L - n_H) > 0$ – is sufficient. An alternative sufficient assumption is that the demand for labor is elastic ($(w_L n_L - w_H n_H)(n_L - n_H) \geq 0$).

Note that the equilibrium definition (16) through (19) is not in the format (5) and therefore not identical to the Tarski formula that is the conjunction of (20) through (27). However, any pair of a low-tax equilibrium and high-tax equilibrium that satisfies (20) and (21) will have associated with it the sixteen real numbers (delineated in the previous footnote) that satisfy (22) through (27). Anything that is necessarily implied by (20) through (27) must therefore describe any pair of equilibria (satisfying (20) and (21)). In other words, adding any additional implications of (16) through (19) to the assumptions already in the Tarski formula cannot alter a conclusion that is already True.

Conversely, quantifier elimination shows that, without an assumption beyond (20) through (27), it is possible that $u_L < u_H$. Even so, it is not necessarily obvious that there is

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guarantees that different tax rates are associated with different work amounts and permits calculation of marginal rates of substitution and marginal products in the same neighborhoods.

[15] Note that, for brevity, I have eliminated four equations and four variables by making the substitutions $f(n_i) = c_i$, $f'(n_i) = w_i$, $i = L, H$. As a result, c is measuring the levels of both consumption and production, and w is measuring both the wage and the production function slope.

[16] The sixteen variables are $c_L, c_H, c_L', c_H', M(c_L, n_L), M(c_H, n_H), n_L, n_H, u_L, u_H, u_L', u_H', w_L, w_H, \tau_L, \tau_H$. Taking the Tarski formula as (20) – (27) $\Rightarrow u_L > u_H$, the universal sentence is False and the existential sentence is True (i.e., the Tarski formula is True somewhere, but not everywhere).

a pair of equilibria (defined by (16) through (19)) with the property that $u_L < u_H$. Below I show that quantifier elimination can generate necessary conditions for a result, such as $u_L < u_H$, which can then be compared with the equilibrium definition for a possible contradiction. In this Laffer-curve example, it turns out that no other property of equilibrium is relevant for concluding that u_L could be less than u_H. Appendix I proves this by providing specific utility and production functions, and graphing the corresponding Laffer curve, for which $u_L < u_H$ at high tax rates.

It is well known that models with optimization can be examined with local analysis ("first-order conditions" at a point) or revealed preference arguments (e.g., (24) and (25) from the Laffer-curve model), with many results obtainable with either approach.[17] A revealed preference argument can be more versatile because it connects to points away from the optimum, but also more tedious because of the number of inequalities involved. Quantifier elimination alters this tradeoff in favor of reveal preference because the processing of inequalities is embedded in the elimination procedure that, as shown in Section IV, is to be done by computer.

II.D. Concave and quasiconcave production functions revisited

As compared to subsection II.A above, Jehle and Reny (2011) examine more general production functions in that their functions (a) can have more than two inputs and (b) do not have to be differentiable. They relate concavity and quasiconcavity to triples of production inputs, rather than single-point second-derivative restrictions as in equation (9). Quantifier elimination can be applied in the "triples" framework too, as suggested by the subsection II.C's Laffer-curve example. Specifically, it can be shown by quantifier elimination (in the space of real numbers) that a production function f, with any number of inputs, that is positive, homogeneous, and quasiconcave must be concave.

Let V_1 and V_2 be any two vectors of production input and μ denote a scalar. We define two more vectors W_1 and W_2 as well as the scalar λ:

[17] Some of the important contributions to revealed preference theory are Samuelson (1938), Afriat (1967), Brown and Matzkin (1996), and Chambers and Echenique (2016).

$$W_1 = \frac{\mu V_1}{f(\mu V_1)}\,,\quad W_2 = \frac{(1-\mu)V_2}{f\big((1-\mu)V_2\big)} \tag{28}$$

$$\lambda = \frac{f(\mu V_1)}{f(\mu V_1) + f\big((1-\mu)V_2\big)} \tag{29}$$

"Positive, homonogenous, and quasiconcave" are represented algebraically as:

$$f(\mu V_1) > 0\,,\quad f\big((1-\mu)V_2\big) > 0 \tag{30}$$

$$\mu > 0 \Rightarrow f(\mu V_1) = \mu f(V_1)\,,\quad \mu < 1 \Rightarrow f\big((1-\mu)V_2\big) = (1-\mu)f(V_2) \tag{31}$$

$$f(W_1) = f(W_2) = 1 \tag{32}$$

$$f(\mu V_1 + (1-\mu)V_2) = \big[f(\mu V_1) + f\big((1-\mu)V_2\big)\big] f(\lambda W_1 + (1-\lambda)W_2) \tag{33}$$

$$\big(f(W_1) = f(W_2) \wedge 0 < \lambda < 1\big) \Rightarrow f(\lambda W_1 + (1-\lambda)W_2) \geq 1 \tag{34}$$

where (30) specifically refers to *f*'s positive property. The statements (31) refer to both the positive and homogeneous properties. Using the definitions of W_1 and W_2, (32) and (33) refer specifically to the homogeneity of *f*.[18] The statement (34) is the definition of a quasiconcave production function.

The Tarski formula for the "triples" representation is that (29) through (34) together imply that *f* is concave, as represented by (35):

$$0 < \mu < 1 \Rightarrow f(\mu V_1 + (1-\mu)V_2) \geq \mu f(V_1) + (1-\mu)f(V_2) \tag{35}$$

One approach to quantifier elimination from the system (29) - (35) would be to (a) take any *specific value of K*, and (b) apply the algorithm to the (4*K*+10)-dimensional space that includes each of the *K* elements of each of the V_1, V_2, W_1 and W_2 vectors as well as

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[18] Note that, from the definitions of W_1 and W_2, $f(\lambda W_1 + (1-\lambda)W_2) = f\left(\frac{\mu V_1 + (1-\mu)V_2}{f(\mu V_1) + f\left((1-\mu)V_2\right)}\right)$.

values for each of the ten scalars $\{\lambda, \mu, f(V_1), f(V_2), f(W_1), f(W_2), f(\mu V_1), f((1-\mu)V_2), f(\mu V_1 + (1-\mu)V_2), f(\lambda W_1 + (1-\lambda)W_2)\}$.[19] However, the result would apply only to the specific value of K.

A second approach is to leave the dimension of the vectors unspecified, because none of those vectors enter (29) - (35) except through f, which maps vectors into scalars. All of the inequalities in the Tarksi formula are polynomial inequalities in the ten-dimensional space $\{\lambda, \mu, f(V_1), f(V_2), f(W_1), f(W_2), f(\mu V_1), f((1-\mu)V_2), f(\mu V_1 + (1-\mu)V_2), f(\lambda W_1 + (1-\lambda)W_2)\}$. The Tarski formula would be True for all values of this vector if and only Jehle and Reny's Theorem 3.1 – that all positive, homogeneous, and quasiconcave productions are concave – were correct. Quantifier elimination confirms their Theorem.

Note, however, that the quantifier-elimination-based proof, as well as Jehle and Reny's, involves referencing eight different points on the production function f and using the assumed properties to derive relationships between those eight points even though the each definition of concavity and quasiconcavity refers to only three.[20] In this sense, a more comprehensive result from quantifier elimination requires more thought – i.e., "manual" rather than "automated" economic reasoning – as to the setup of the Tarski formula, as compared to the single-point approach (9).

III. Quantifier elimination as a tool for economic reasoning

Quantifier elimination can do more than determine the truth of hypotheses. It can help formulate and understand hypotheses by detecting inconsistent assumptions, calculating necessary and sufficient conditions, and generating examples. Each subsection below explains how this is done and provides examples by reference to the production function, monopolist pricing, and Laffer curve models above.

[19] This approach would also need to include the definitions (28) in its Tarski formula.

[20] Similarly, the Laffer curve problem refers to four points on the utility function (u_L, u_H, u_L', u_H') even though the statement of the hypothesis refers to only two (u_L, u_H).

III.A. Detecting inconsistent assumptions

Anything can be "proven" with an empty assumption set because $A \Rightarrow B$ is identical to $\neg A \vee B$, which is True everywhere that A is false. It is therefore important to know whether assumptions are mutually consistent. Quantifier elimination performs this task too, by beginning with the sentence that there exist values of each of the variables so that the assumptions are simultaneously True. If the quantifier-free representation of that existential sentence is False, then the assumptions are mutually inconsistent.

Take the Laffer curve model, in which the "surprise" result $u_L < u_H$ requires that one of the goods is inferior ($M(c_L,n_L) < M(c_H,n_H)$) at the same time that the demand for labor is relatively inelastic ($(w_L n_L - w_H n_H)(n_L - n_H) < 0$). In order to confirm that the two assumptions are simultaneously compatible with (20) through (27), we eliminate quantifiers from the sentence that says that there exists at least one point in $\mathbb{R}^{16}$ where (20) through (27), $M(c_L,n_L) < M(c_H,n_H)$, and $(w_L n_L - w_H n_H)(n_L - n_H) < 0$ are True.

III.B. Necessary and sufficient conditions: reformulating hypotheses to make them True

The hypothesis (9) is False. Quantifier elimination can show what assumptions could be added so that the reformulated hypothesis is True. For example, consider reformulating the hypothesis without the quantifiers corresponding to the production function's second derivatives, as shown below using the condensed notation of Table 1.

$$\begin{aligned} &\forall\{v_1,v_2,v_3,v_5\}\left[\left(v_1>0 \wedge v_2>0 \wedge v_3>0 \wedge v_5>0 \wedge v_3^2 v_7 + v_5^2 v_4 < 2v_3v_5v_6\right)\right. \\ &\qquad\qquad\left.\Rightarrow \left(v_4 \le 0 \wedge v_7 \le 0 \wedge v_4v_7 \ge v_6^2\right)\right] \\ &= \left\{(v_4,v_6,v_7) \in \mathbb{R}^3 : \left((v_4=0 \vee v_7=0) \wedge v_6=0\right) \vee G(v_4,v_6,v_7)\right\} \end{aligned} \tag{36}$$

As before, there are seven variables, but three of them are unquantified (free) and correspond to the second derivatives of the production function. The LHS of equation (36) is not a sentence. This alone says that the formulated hypothesis might be neither True nor False. Applying quantifier elimination to the formulated hypothesis, we get the RHS of (36), which is a set of restrictions on the free variables $\{v_4,v_6,v_7\}$. The set consists of two subsets described by the blue term and the G term.[21] Each subset's description is providing sufficient conditions to add to the assumptions of (9) to make it True.[22] For example, the blue term says that the hypothesis (9) would be True if its assumptions also included that the production function was a quasilinear function of its inputs, as shown in (37).[23]

$$\forall\{v_1,v_2,v_3,v_4,v_5,v_6,v_7\}$$
$$\Big[\Big(v_1>0\wedge v_2>0\wedge v_3>0\wedge v_5>0\wedge v_3^2v_7+v_5^2v_4<2v_3v_5v_6 \tag{37}$$
$$\wedge\big((v_4=0\vee v_7=0)\wedge v_6=0\big)\Big)\Rightarrow(v_4\leq 0\wedge v_7\leq 0\wedge v_4v_7\geq v_6^2)\Big]=True$$

In other words, quantifier elimination produces the blue term that, when added to the assumptions, makes the formulated hypothesis (9) True.[24] Moreover, some of the byproducts of Cylindrical Algebraic Decomposition (CAD), which is the primary quantifier-elimination algorithm, are further options for algebraically describing restrictions among the of the variables, regardless of whether they are "free." These options facilitate characterizing the intersection of the assumption and hypothesis sets, which goes more directly to the question of what additional assumptions are needed to make a hypothesis True. CAD and other quantifier-elimination algorithms are the subjects of Section IV.

III.C. Necessary and sufficient conditions: distinguishing strong assumptions from weak ones

In many cases there are multiple assumptions that deliver a result. Quantifier elimination can show which, if any, implications are stronger than others, conditional on

[21] G is a more complicated function that is shown in Appendix II.
[22] Together, the two are also necessary.
[23] In the more verbose notation, the blue term is $\left(\left(\frac{\partial^2 f(x,y)}{\partial y^2}=0\vee\frac{\partial^2 f(x,y)}{\partial x^2}=0\right)\wedge\frac{\partial^2 f(x,y)}{\partial x\partial y}=0\right)$.
[24] In other words, because it is not a sentence, equation (36) has a lot in common with the quantifier elimination in Brown and Matzkin (1996) and Chambers and Echenique (2016). However, they do not use CAD, or improvements on it, to actually perform the quantifier elimination.

a set of assumptions A. Specifically, if one of those implications is B and the other C, then the statement that B is a (weakly) stronger implication than C is written as:[25]

$$[A \Rightarrow (B \Rightarrow C)] = [\neg A \vee \neg B \vee C] \tag{38}$$

Reversing B and C in (38) would make the statement that C is a (weakly) stronger implication than B. If both are True, then B and C are equivalent conditional on A $[A \Rightarrow (B \Leftrightarrow C)]$.

In the Laffer curve model, assuming (20) through (27), quantifier elimination proves that a linear production function ($w_L = w_H$) is one with elastic labor demand ($(w_L n_L - w_H n_H)(n_L - n_H) < 0$). Another example: quantifier elimination proves that, conditional on the same assumptions, $n_L > n_H$ is equivalent to $u_L > u_H$.

Of course, irrelevant assumptions do not affect conclusions. Quantifier elimination therefore (a) still obtains the result even when the Tarski formula contains irrelevant assumptions and (b) can be used to identify the irrelevant assumptions. These features of quantifier elimination allow the user to assemble his Tarski formula with less discretion. The proof in the Laffer curve model, for example, requires evaluating utility at four points: $\{u(c_L, n_L), u(c_H, n_H), u(c'_H, n_L), u(c'_L, n_H)\}$, where c'_H (c'_L) is the consumption that would be obtained by working the low-tax (high-tax) amount at the high-tax (low-tax) prices. The first two are obvious, but the user might not be sure whether $\{u(c_L, n'_H), u(c_H, n'_L)\}$ must also be included.[26] These two utility levels, and the revealed-preference restrictions that go with them (for a total of eight utility levels and four revealed-preference restrictions), could be included in the Tarski formula without affecting the conclusions. For that matter, the Tarski formula can be augmented with any additional implication of the equilibrium definition (16) through (19) that is expressible as a polynomial inequality.

[25] To be clear, A, B, and C are Boolean variables.

[26] Any labor amounts with a prime indicates the amount that must be worked in order to obtain the counterfactual consumption.

III.D. Generating examples, counter-examples, and step-by-step proofs

The primary quantifier-elimination algorithm, Cylindrical Algebraic Decomposition (CAD) automatically generates sample points as part of its lifting phase (the phases of CAD are discussed further below). For a hypothesis that is not everywhere False, the CAD can generate an example. For a hypothesis that is not everywhere True, the CAD can generate a counterexample. Take the Laffer curve model, without normal-goods or elastic-demand restrictions. The hypothesis that, conditional on (20) through (27), $u_L > u_H$ is neither everywhere True nor everywhere False. The CAD can therefore generate both an example of $u_L > u_H$ and $u_L \leq u_H$, both of which are consistent with (20) through (27). See Table 2.

A CAD represents a step-by-step proof. In order to see, and qualify, this result, it is necessary to reference some of the theorems from real algebraic geometry and to understand some of the details of CAD construction. This is the purpose of Section IV.

IV. Relevant Theorems from Real Algebraic Geometry

IV.A. Tarski: Quantifier elimination is always possible

Mathematician and logician Alfred Tarski proved that there exists a universal algorithm (that is, one not requiring problem-specific guidance) for quantifier elimination from systems of polynomial inequalities on real closed fields by providing such an algorithm.[27] Because the real numbers are an example of a real closed field, the Tarski result guarantees that there exists a P so that $HF = HE$ and gives an algorithm for finding P. If HF is a sentence, then the quantifier elimination algorithm is a "decision method": a procedure for determining whether HF is True or False.[28]

Tarski's theorem applies to all of the examples above because, when interpreted in the right space, they are special cases of systems of polynomial inequalities. A system with even one transcendental term is not covered by Tarski's theorem (unless there is a change of variables that makes all terms polynomial), although quantifier elimination

[27] Tarksi made the proof in 1930 (Caviness and Johnson 1998, p. 1), but the result was not published until Tarski (1951).
[28] Renegar (1998, p. 221).

may still be possible. There are algorithms for deciding existential sentences that contain transcendental terms, although they are not fully developed (see below).

IV.B. Collins: A more efficient algorithm for quantifier-elimination that defines sets recursively

Although Tarski's method is enough to prove that quantifiers can be eliminated, it is not used in practice due to its "extreme" inefficiency.[29] A major step forward came with the Cylindrical Algebraic Decomposition (CAD) method introduced by mathematician George E. Collins in 1973.[30]

IV.B.1. Properties of CAD

In our setting (5), the CAD method decomposes $\mathbb{R}^N$ into finitely many connect regions, known as "cells," with three properties:

(i) each cell of the CAD is a semi-algebraic set (i.e., it is defined by a finite number of quantifier-free polynomial inequalities).

(ii) The CAD result is cylindrical because the projections of any two of the cells into $\mathbb{R}^k$, $1 \leq k \leq N$, are either identical or disjoint.

(iii) Each cell is adapted to the Tarski formula from which it was derived, which means that none of the polynomials in the Tarski formula T has more than one sign {-1,0,1} in any one of the cells.

Every Tarski formula has such a CAD (Basu, Pollack and Roy (2011, Theorem 5.6)).

The T-adapted (i.e., uniform sign) property of the cells, and the fact that the cells are finite in number, means that any quantified formula can be confirmed in a finite number of steps.[31] The cylindrical property of the decomposition means that the cells have a natural ordering and many times can be processed more than one at a time.

Narrowly speaking, CAD refers to a method, or sometimes an expanded set of polynomials (including, among others, those in the original Tarski formula) obtained by

[29] Arai, et al. (2014). See also Davenport, Siret and Tournier (1988, p. 119), who describe Tarski's method as "completely impractical."
[30] Collins (1973) and Collins (1975).
[31] By construction, the Tarski formula is True at any one point in a cell if and only if it is True everywhere in that cell.

the method. One of result of the CAD method is its cells, described by Cylindrical Algebraic Formulas (CAFs). I follow Kauers (2011), Mathematica, REDLOG, and others by (a) referring to CAD and CAFs interchangeably and (b) displaying CAFs that are simplified to exclude cells where the Tarski formula is false and to unify the remaining cells.[32] For example, I refer to the right-hand sides of (2) and (3) as CADs.

As an example (due to Kauers 2011), take the set of all points in the plane that are outside the origin-centered circle of radius two and inside the hyperbola centered at (1,1):

$$\{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 - 4 > 0 \wedge (x_1 - 1)(x_2 - 1) - 1 < 0\} \tag{39}$$

Figure 2 displays a CAD of the set described in (39), with each of the two-dimensional cells shown as a different color. The CAD partitions the x_1-axis six different ways: $x_1 \leq r_1$, $x_1 \in (r_1, r_2]$, $x_1 \in (r_2,1)$, $x_1 = 1$, $x_1 \in (1,2)$ and $x_1 > 2$, where r_1 and r_2 are the two real roots of $z^4 - 2z^3 - 2z^2 + 8z - 4 = 0$. Each of partition of x_1 defines a cylinder of all points in the plane that would be projected onto that part of the x_1-axis. Each cylinder is itself vertically partitioned where one of the polynomials changes sign, which guarantees that the CAD is adapted to the two polynomials. If the problem were more than two dimensional, then the CAD would proceed further by stacking one-higher dimensioned cylinders on top of the cells in the $[x_1,x_2]$ plane, partitioning those cylinders, stacking one-higher dimensioned cylinders on the cells in the $[x_1,x_2,x_3]$ space, etc.

The CAD shown in Figure 2 has seven two-dimensional cells and two one-dimensional cells. Although not emphasized in Figure 2, the rest of the plane could be decomposed into cells (12 in this case) using the same cylinders. As a result, at most 21 sample points need to be checked – one in each cell – to confirm or deny hypotheses such as "There exists points outside the circle that are inside the hyperbola" or "All points outside the circle are inside the hyperbola." In this way, CAD decides universal and existential sentences in a finite number of steps.

[32] See Strzebonski (2010) and Chen and Maza (2015) for more on the distinction between CAD and CAF.

IV.B.2. CAD Construction

A CAD is obtained in two straightforward, albeit tedious, phases: projection and lifting. In the projection phase, one variable is eliminated at a time in the order that they are quantified in the formulated hypothesis.[33] In order to eliminate a variable, both polynomial intersections and singularities are found.[34] In Figure 2, there are two intersection points, two singularities for the circle ($x_1 = \pm 2$), and one singularity for the hyperbola ($x_1 = 1$) that can be used to make a cylindrical projection.[35] The set described in (39) is two dimensional, so there is only one projection step, but the general case involves one projection step for each variable eliminated, with the exception of the final variable. Also note that higher polynomial degrees can increase the number of intersections and singularities to be processed, as shown by example below.

The following describes the lifting phase, assuming for brevity that x_N is the first variable eliminated, x_{N-1} is the second, etc., until only x_1 is left. In the lifting phase, one scalar value – a sample point – for x_1 is found for each of the cells defined on the x_1-axis by the projection phase. Using those values, sample points in the $[x_1,x_2]$ plane are found for each of the cells in the cylinders above the cells on the x_1–axis. This procedure is repeated ("lifted") through each dimension until there is a sample point in the $[x_1,x_2,\ldots, x_N]$ space for every cell in the CAD. Because the CAD is adapted to the Tarski formula, one sample point is enough to determine which˙ inequalities are satisfied at all points in

[33] This means that CAD can be constructed in at least $N!$ different ways: one for each possible variable-elimination sequence. Although familiar from linear systems, Gaussian elimination is not necessarily a close analogy because polynomial intersections are not the only calculations that may occur as the CAD eliminates a variable (Van den Dries 1988, p. 9).

[34] When viewed as a function of the variable being eliminated, each polynomial in the system has roots that are (a) numbered according to the degree of that variable and (b) potentially a function of the remaining variables. As a function of the remaining variables, the roots need to be identified as real or complex (the source polynomial's singularity points indicate which is the case) and the real ones sorted collectively for the entire system (the polynomial intersection points indicate the sort order). Each sort is associated with conditions on the remaining variables and a partition of the eliminated variable at each root. The exact algebra requires several pages of explanation, for which readers are referred to the literature, especially Arnon, Collins and McCallum (1998) and Basu, Pollack and Roy (2011). I have found the latter to be especially helpful because each subalgorithm is illustrated with an example (see also (Dolzmann, Sturm and Weispfenning 1998)).

[35] Figure 2's projection does not need the $x_1 = -2$ singularity in order to have a cylindrical projection, but it would if the set of interest were the points outside both the circle and hyperbola.

the cell. The quantifier-free formula is the union of all cells where the Tarski formula is True.[36]

Recall the existential quantifier elimination result shown in (1), which has a Tarski formula of $x^2 + bx + c = 0$. If x were eliminated first, and then c, then the full decomposition of $\mathbb{R}^3$ the nine cells below

$$\begin{gathered}(b^2 < 4c) \vee \left(b^2 = 4c \wedge x < -\frac{b}{2}\right) \vee \left(b^2 = 4c \wedge x = -\frac{b}{2}\right) \vee \\ \left(b^2 = 4c \wedge x > -\frac{b}{2}\right) \vee \left(b^2 > 4c \wedge x < -\frac{b}{2} - \frac{\sqrt{b^2 - 4c}}{2}\right) \\ \vee \left(b^2 > 4c \wedge x = -\frac{b}{2} - \frac{\sqrt{b^2 - 4c}}{2}\right) \vee \\ \left(b^2 > 4c \wedge x > -\frac{b}{2} - \frac{\sqrt{b^2 - 4c}}{2} \wedge x < -\frac{b}{2} + \frac{\sqrt{b^2 - 4c}}{2}\right) \\ \vee \left(b^2 > 4c \wedge x = -\frac{b}{2} + \frac{\sqrt{b^2 - 4c}}{2}\right) \\ \vee \left(b^2 > 4c \wedge x > -\frac{b}{2} + \frac{\sqrt{b^2 - 4c}}{2}\right)\end{gathered} \tag{40}$$

Only three of the cells above satisfy the Tarski formula. The (b,c) projections of those three cells are:[37]

$$(b^2 = 4c\ ) \vee (b^2 > 4c\ ) \vee (b^2 > 4c\ ) \tag{41}$$

which simplifies to the RHS of (1).

Note that, for economic applications, CAD construction, and even quantifier elimination by way of CAD, can be fully delegated to either commercial or open-source software packages, much as the standard economics practice for, say, inverting matrices. Mathematica and REDLOG are featured in what follows.

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[36] The lifting phase can be done together with the quantifier elimination, in which case entire cylinders may be discarded (i.e., no sample points calculated).
[37] As C.W. Brown (2003, p. 97) puts it, "the existential quantifier is simply projection."

IV.B.3. Single-cell CADs

In general, CADs can have many cells, especially when the number of variables is large. The three-dimensional version of the circle-hyperbola example has 242 cells (as compared to 9 shown in Figure 2). The four-dimensional version has 3,531.[38] Sometimes the CAD has just one cell, even when there are more than two variables. The single-cell CADs are of special interest because (a) the hypothesis represented by the CAD can be proved recursively in the order in which the CAD projections occurred and (b) the steps of that recursive proof correspond to the components of the single-cell formula generated by the CAD.

Take, for example, a three-dimensional set described by

$$\left\{(x, y, z) \in \mathbb{R}^3 : x > 0 \wedge A_y(x, y) > 0 \wedge A_z(x, y, z) = 0\right\} \tag{42}$$

Where A_y and A_z are quantifier-free functions mapping scalar arguments into $\mathbb{R}^1$. The system of inequalities is triangular in the sense that only $A_z = 0$ contains all three variables and, of the remaining two inequalities, only $A_y > 0$ contains both remaining variables. Eliminating the variables in the order $\{z,y,x\}$, the cylindrical decomposition is:

$$\left\{x > 0 \wedge \left\{y : A_y(x, y) > 0\right\} \wedge \{z : A_z(x, y, z) = 0\}\right\} \tag{43}$$

which is cylindrical because the third atom is conditional on x and y, the second atom is conditional on x, and the first atom can be evaluated without regard for y and z. It has only one cell, as evidenced by the fact that it has no disjunctions ($\vee$). (43) would be a single-cell CAD if the functions A_y and A_z were algebraic (i.e., polynomials).

The monopolist's pass-through model from above is a triangular system with a single-cell CAD, from which a recursive proof can be constructed. Consider the (True) hypothesis that, assuming (11)-(13), (15), a demand curve that is concave and slopes

[38] These CADs (technically, CAFs) were calculated by Mathematica, with cells distinguished by the disjunction operator ($\vee$). The cell counts refer only to the cells where the Tarski formula is True.

down ($W''(q) \leq 0$, $W'(q) < 0$), then marginal costs are not overshifted (ie., $\mu \leq 1$). To see this concisely, we drop the irrelevant variables $\left\{g(q,a), \frac{\partial g(q,a)}{\partial q}, W(q)\right\}$ and the inequalities in which they appear: (11), (12) and the first part of (10).[39] That leaves six variables $\left\{\frac{\partial^2 g}{\partial q \partial a}, \frac{\partial^2 g}{\partial q^2}, \frac{dq}{da}, W''(q), W'(q), q\right\}$ and seven inequalities describing the assumptions of the model.[40] When the six variables are eliminated in this order, the CAD representation of the model's assumptions is:

$$\begin{aligned}&\left\{q > 0 \wedge W'(q) < 0 \wedge W''(q) \leq 0 \wedge \frac{dq}{da} < 0 \wedge \frac{\partial^2 g(q,a)}{\partial q^2} \geq 0\right. \\ &\left.\wedge \frac{\partial^2 g(q,a)}{\partial q \partial a} = \left[2W'(q) + qW''(q) - \frac{\partial^2 g(q,a)}{\partial q^2}\right]\frac{dq}{da}\right\}\end{aligned} \tag{44}$$

which is a single cell (no disjunctions). In order to prove that pass-through cannot exceed one using the CAD (44), we assume the contrary:

$$\frac{\frac{d}{da}W(q)}{\frac{d}{da}\left[\frac{\partial g(q,a)}{\partial q}\right]} = \frac{W'(q)\frac{dq}{da}}{\frac{\partial^2 g(q,a)}{\partial q^2}\frac{dq}{da} + \frac{\partial^2 g(q,a)}{\partial q \partial a}} > 1 \tag{45}$$

We take the last atom of the single-cell CAD, which represents the assumption (15), to eliminate from (45) the first variable (at least) from the elimination list:

$$\frac{W'(q)}{2W'(q) + qW''(q)} > 1 \tag{46}$$

The result (46) contradicts the first three atoms of the CAD, which are the assumptions about a positive quantity and the demand curve's shape. This completes the CAD-inspired proof by contradiction that pass-through cannot exceed one.

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[39] The level of cost $g(q,a)$ appears only in (11) and in doing so does not restrict the remaining variables. With (11) dropped, $W(q)$ appears in only in condition (12), without restricting any of the other variables. With (11) and (12) dropped, marginal cost appears only in the first part of (10), without restricting any of the other variables.

[40] The seven inequalities form a triangular system in the sense that only one of them contains all six variables, a second contains four variables, and the remaining five contain only one variable each.

Note that any CAD eliminating N variables has $N!$ different elimination sequences and thereby up to $N!$ different CADs and up to $N!$ different methods of proving the same result. The CADs may differ in terms of the number of cells, which means the complexity of the proofs they represent may differ.[41] One CAD by itself is enough to decide a universal sentence, but sometimes it may be of interest to examine multiple elimination sequences in order to find a relatively simple proof of that decision.

IV.B.4. Necessary and sufficient conditions revisited with CAD

Each of the $N!$ CADs offers a potentially unique algebraic characterization of the same set. This is useful when we have a hypothesis that does not follow from the assumptions because CAD can offer various algebraic descriptions of the intersection of the assumptions and the hypothesis. Any extra assumption that restricts the model to a subset of that intersection is, together with the original assumptions, sufficient to conclude that the hypothesis is True. Such an extra assumption is readily identified from a CAD.

The model (9) is an example: concave production does not follow from quasiconcave production:

$$\begin{gathered}\forall\{v_3, v_4, v_5, v_6, v_7\} \\ \left[\left(v_3 > 0 \wedge v_5 > 0 \wedge v_3^2 v_7 + v_5^2 v_4 < 2 v_3 v_5 v_6\right) \right. \\ \left. \Rightarrow \left(v_4 \leq 0 \wedge v_7 \leq 0 \wedge v_4 v_7 \geq v_6^2\right)\right] = False\end{gathered} \tag{47}$$

where (49) uses the condensed notation of Table 1 and for brevity omits v_1 and v_2, which do not affect the conclusions. The intersection of the assumptions and the hypothesis is a subset of $\mathbb{R}^5$ that can be characterized with CAD. That description depends on the elimination sequence used during the projection phase. If the variables representing

[41] If variables were eliminated from the pass-through example in reverse order, then the CAD would have four cells rather than the single cell shown in (42). Also note the analogy with Gaussian elimination for full-rank N-dimensional systems of linear equations: there are N! different elimination sequences.

second derivatives (v_4,v_6,v_7) are eliminated last, and otherwise variables are eliminated in reverse alphabetical order, we have:

$$\left\{\begin{matrix} (v_3, v_4, v_5, v_6, v_7) \in \mathbb{R}^5: \\ (v_3 > 0 \wedge v_5 > 0 \wedge v_3^2 v_7 + v_5^2 v_4 < 2 v_3 v_5 v_6) \wedge (v_4 \leq 0 \wedge v_7 \leq 0 \wedge v_4 v_7 \geq v_6^2) \end{matrix}\right\} = \left\{\begin{matrix} (v_3, v_4, v_5, v_6, v_7) \in \mathbb{R}^5: \\ (v_4 = v_6 = 0 \wedge v_7 < 0 \wedge v_3 > 0 \wedge v_5 > 0) \vee \Gamma(v_3, v_4, v_5, v_6, v_7) \end{matrix}\right\} \tag{48}$$

where the right-hand side is the CAD, which consists of two subsets described by the blue term and the Γ term.[42] Each subset is by itself a sufficient condition to add to the assumptions of (49) to make it True.[43] For example, the blue term says that the hypothesis (49) would be True if its assumptions also included that the production function was quasilinear in its second input. In other words, the CAD of the intersection of (49)'s assumption and hypothesis produces the blue term that, when added to the assumptions, makes it True.

Recall that the condition (36) describes a set in just the three free variables because the other two (v_3,v_5) were quantified and then eliminated. In contrast, the CAD (48) is a set in all five dimensions. In other words, CAD can provide a "simple" description of a set by prioritizing variables rather than, or in addition to, eliminating them. In the case of (48), v_3 and v_5 still appear, but restrictions on v_4, v_6, and v_7 are shown without reference to v_3 and v_5: that's that it means for the decomposition to be cylindrical. The CAD approach thereby offers a wider range of descriptions of the necessary and sufficient conditions than does the elimination approach shown in subsection III.B.

IV.C. The complexity of quantifier elimination

Collins' CAD method is not necessarily the most efficient method for quantifier elimination, but it is a good benchmark for understanding the computational complexity of practical problems. CAD's computational complexity (e.g., computing time) is

[42] Γ is a more complicated function that is shown in Appendix II.
[43] Together, the two are also necessary.

polynomial in the number of inequalities and in the maximum degree of their polynomials, but double exponential in the number of variables. In order to anticipate practical experiences with CAD-based quantifier elimination, it helps to see a graph of a double-exponential function, which reflects a kind of "curse of dimensionality." Figures 3a and 3b graph the same function $2^\wedge(2^x)$, but on different domains. Figure 3a's graph is almost a vertical wall: doubling the number of variables (e.g., moving from the middle of the horizontal axis to the right edge) in this range increases the complexity by a factor of more than 4,000. The figure thereby gives the impression that CAD, and perhaps any method for quantifier elimination, may never be practical. Indeed, a number of economics papers discussing quantifier elimination cite the theoretical complexity results and suggest that quantifier elimination is too "computationally demanding" without ever reporting any actual computation times.[44]

But Figure 3b gives a much different impression. It is fairly close to linear, with a doubling of the number of variables less than tripling the complexity. In other words, the feasibility of CAD depends on where the "wall" is located relative to problems of interest and whether those problems can be rephrased to, in effect, move the wall to the right. As Shankar (2002, p. 13) puts it, "Many decision procedures are of exponential, super-exponential, or non-elementary complexity. However, this complexity often does not manifest itself on practical examples." Although CAD and related methods have been used for many problems in geometry (Lasaruk and Sturm 2011), I am not aware of their use on sentences that represent economic problems, which tend to be less symmetric and with lesser polynomial degrees. We must also remember that CAD is not the only method for quantifier elimination, especially in the case of existential sentences.

There is therefore no substitute for actually trying quantifier elimination on economics examples. Each of the examples in this paper, which were chosen for their economic interest – they are interesting enough to appear in textbooks and have journal articles devoted to them – rather than computational simplicity, have been processed in

[44] E.g., Carvajal, et al. (2014). In his lectures at the Cowles Foundation, mathematician Charles Steinhorn (2008, p. 177) conjectures "…quantifier elimination is something that is do-able in principle, but not by any computer that you and I are ever likely to see. Well, I'll retract that last statement because it's probably false." Using the same logic, Anai et al (2014, p. 7) claim – incorrectly, as shown throughout this paper – that "[t]he practical limit to obtain a solution would be at most five variables."

milliseconds. In my own research and teaching, the norm is that problems of this type are processed orders of magnitude faster than the time it takes for me to type the assumptions. Appendix III shows an example, based on a growth model with taxation, whose CAD is in 41 dimensions, yet nonetheless is processed in milliseconds.

With that said, the double-exponential property means that poor judgement and brute force can produce economics problems whose CADs are overwhelmingly complex for today's computers.[45] Take the concave production function example II.A., using quantifier elimination to prove, based on derivatives at a single point, that a homogeneous and quasiconcave production function is a concave production function. This example is a particularly tough test for quantifier-elimination methods because (a) it concerns second-order properties of the model rather than first-order, and thereby more polynomials of higher degree and (b) alternative proof strategies are available that avoid any reference to second derivatives of the production function.[46] Nevertheless, assuming that the production function has two inputs, the quantifier elimination (in seven dimensions) from subsection II.A. takes at most a few dozen milliseconds on a laptop computer (see the first row of Table 3).[47] If three inputs were assumed, twelve variables are quantified and the quantifier elimination occurs in about two minutes with Mathematica, and a fraction of a second with REDLOG. However, neither software package could eliminate quantifiers in the four-input case in less than five days of

[45] Complex in terms of having a large number of cells, and defining the cells with roots of high-degree polynomials.
[46] See subsection II.D.
[47] Specifically, using the condensed notation (36), quantifiers are eliminated from

$$\forall\{v_1,v_2,v_3,v_4,v_5,v_6,v_7\}$$
$$\big[\big(v_1>0\wedge v_2>0\wedge v_3>0\wedge v_5>0\wedge v_3^2v_7+v_5^2v_4<2v_3v_5v_6 \wedge v_2v_4+v_1v_6=0\wedge v_2v_6+v_1v_7=0\big) \Rightarrow \big(v_4\le 0\wedge v_7\le 0\wedge v_4v_7\ge v_6^2\big)\big]$$

where the two equations are the second-order-term restrictions implied by homogeneity and are derived by differentiating Euler's theorem with respect to each input. The quantifier-free equivalent is True (because all homogeneous and quasiconcave production functions are concave).

processing (more than 100 million milliseconds; this case has 18 quantified variables).[48] Here we see the double-exponential property.

The CAD algorithm, especially when applied to universal sentences, is amenable to parallel processing methods that can, in effect, move the "wall" to the right. For example, an *N*-variable universal sentence has *N!* different sequences in which to eliminate variables in the projection phase and there is no known generic formula for determining which of these is the least complex.[49] Parallel processors can be used to, among other things, simultaneously execute different elimination sequences and then terminate all processes that are still running after the first process has completed.[50]

Finally, the CAD algorithm is more general than needed for deciding universal sentences, such as the economic examples provided in this paper. Quantifier-elimination algorithms are being developed specifically for existential sentences, which are equivalent to universal sentences (recall (7)) and known as the "existential theory of the reals." The complexity of the dedicated algorithms are "just" singly exponential in the number of variables even without parallel processing.[51] These advances are not yet fully incorporated into Mathematica and REDLOG software (Passmore & Jackson, 2009; Davenport & England, 2015).[52] As computing power increases and singly-exponential

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[48] Subsection II.D. shows an alternative approach that expresses the general case (i.e., any number of inputs, and a not-necessarily differentiable production function) in the polynomial framework (5). As shown in Table 3, this approach eliminates 10 quantifiers in less than one second.

[49] C.W. Brown (2004). Mathematica and REDLOG have heuristics for guessing an elimination sequence than might economize on computation, but were not used for the results reported in this paper. Also note that the elimination sequence is irrelevant for symmetric a Tarski formula such as (39); in my experience economic hypotheses are not so symmetric.

[50] When consecutive variables have the same quantifier, changing their order does not affect the meaning of the formulated hypothesis, but it does affect features of the CAD, such as the number of cells and the difficulty of the algebraic operations required to obtain it (Brown and Davenport 2007). Table 3 eliminates variables in alphabetical order (as sorted by Mathematica), which generally does not minimize computation time. Algorithms for determining more efficient elimination sequences, which are beyond the scope of this paper, can, for example, find a sequence (and process it) for the four-input model in a few minutes as compared to more than five days for elimination in alphabetical order.

[51] These decision problems are in PSPACE (Canny 1993). See Basu, Pollack and Roy (2011) for a recent theoretical treatment of algorithms for deciding existential sentences.

[52] Separate from the CAD literature, computer scientists have (for the purpose of automatically verifying software programs) developed non-CAD algorithms for deciding existential sentences. Some of them, such as those competing at http://smtcomp.sourceforge.net in the QF-NRA division, deal with nonlinear polynomials. In my experience, Microsoft's Z3 is the most adept at answering the economics questions, but still narrower than the primarily-CAD methods used by

algorithms are put into practice, it is likely that even larger economic problems will be practically processed by quantifier elimination (Passmore, 2011, p. p. 100).

IV.D. The utility of leaving functional forms unspecified

Ironically, specifying assumptions and hypotheses with particular functional forms makes it more difficult to use the quantifier-elimination results. Take, for example, the first derivative restriction $\partial f/\partial x > 0$ in the concave production function example. In this form, it is a (trivial) polynomial inequality in the seven-dimensional space noted above. If, instead, a production function were specified with transcendental marginal product schedules, then we would no longer have a polynomial inequality in x and y. Even a Cobb-Douglas production function with exponents α and β would have its first derivative restriction entered as $\alpha x^{\alpha-1}y^{\beta} > 0$, which is not a polynomial inequality in $\{\alpha,\beta,x,y\}$.[53] Thus, while traditional "pencil-and-paper" proving approaches are many times facilitated with functional-form assumptions, quantifier elimination in the Tarski-Collins tradition is facilitated by avoiding them.

Semi-algebraic economies include Cobb-Douglas utility and production functions as long as their exponent parameters are rational numbers, because then statements about those functions are special cases of polynomial inequalities. Still, for the purposes of implementing quantifier elimination as we have outlined above, their approach is unnecessarily complicated. First, the quantifier-elimination would have to be performed with specific values of the exponent parameters. For example, the production function in the Laffer curve example could be, say, $n^{7/10}$, but then the quantifier-elimination result would refer only to $n^{7/10}$ and not to any other Cobb-Douglas production function, even with rational coefficients.

Mathematica and REDLOG. For example, Z3 does not process the concave production function problem with three production inputs in less than 1000 seconds.

[53] In this case, the marginal-product restrictions could be transformed to be of the form $\alpha > 0$ and $\beta > 0$, which are polynomials in $\{\alpha,\beta\}$, but more complicated statements about Cobb-Douglas production functions cannot be simplified to restrictions on the exponents.

Second, a CAD for hypotheses about a production function having a rational exponent is likely vastly more complicated than the CAD for the same hypothesis expressed in terms of $f(n)$. Take the rather simple (and True) hypothesis that, assuming a Cobb-Douglas production function with a positive exponent, aggregate labor income is increasing in the amount of labor. Expressed in terms of $f(n)$, it is:

$$\Big(w_L = f'(n_L) > 0 \wedge w_H = f'(n_H) > 0 \wedge n_L > n_H > 0$$
$$\wedge f(n_L) > f(n_H) > 0 \wedge \frac{f'(n_L)n_L}{f(n_L)} = \frac{f'(n_H)n_H}{f(n_H)}\Big) \tag{49}$$
$$\Rightarrow w_L n_L > w_H n_H$$

where, as in the Laffer curve example, n denotes labor input and w the wage rate. The second row of assumptions shows the relevant restrictions imposed by Cobb-Douglas: positive output and a constant elasticity of output with respect to input.

Deciding the hypothesis (49) with (an eight-dimensional) CAD takes a fraction of a second because the number of inequalities is low and the polynomial degrees are no more than three.[54] Transforming (49) into a semi-algebraic economy by replacing $f(n)$ with n to a specific rational power reduces the dimensionality of the CAD, but can increase the polynomial degree and thereby increase the decision time by orders of magnitude. Figure 4 shows the relative decision times of the semi-algebraic model with various production-function exponents.[55] The decision time is, for example, five times longer when $f(n)$ is replaced with $n^{5/8}$ (some of the polynomials are of degree 8), and 3,000 times longer when replaced with $n^{23/30}$ (some of the polynamials are of degree 30).

The unnecessary complexity of semi-algebraic production functions is even greater in systems with more inequalities, such as the systems used above to examine the Laffer curve. Table 4 shows some results of deciding the (True) hypothesis that a Cobb-Douglas production function, together with the other assumptions (20) through (26),

[54] The eight dimensions are n_L, n_H, w_L, w_H, $f(n_L)$, $f(n_H)$, $f'(n_L)$, and $f'(n_H)$.

[55] The exponents, when simplified, have denominators ranging from 2 through 30 and numerator equal to the integer making the exponent closest to 7/10 without changing the denominator. E.g., 13/20 is used rather than 14/20, because the latter simplifies to 7/10. Note that the exponent denominator is the degree of the polynomial in the Tarski formula.

guarantees that the lower tax rate is associated with greater utility. The Table's first two rows are the benchmarks with unspecified functional forms. The first row refers to a generic production function $f(n)$ that satisfies (23) and (24) as well as the elasticity restriction $((w_L n_L - w_H n_H)(n_L - n_H) \geq 0)$.[56] As shown in Table 3, the decision time with REDLOG software is 190 milliseconds.[57] The largest power on any one variable in the polynomial system is only two, and no more than three variables are multiplied together at the same time. For additional comparability with Cobb-Douglas, Table 4's second row also imposes that the elasticity is constant (i.e., labor's share of output is the same at allocations L and H). The second row shows a decision time of 380 milliseconds.

The third and fourth rows use the production functions $n^{1/2}$ and $n^{2/3}$, respectively, and results in the shorted decision times because quadratic and cubic formulas can be used. The decision time is about the same with an exponent of 3/4. Several orders of magnitude are added to decision times by using more complicated rational exponents, such as 3/5 or 4/7. The table also shows how a more complicated rational exponent adds to the degree of the polynomial system.

As noted by Brown and Kubler (2008), any specific rational exponent allows Cobb-Douglas production to fit into the Tarski framework. An irrational exponent would not. A symbolic exponent, e.g., α, cannot be processed with Tarski and Collins procedures either, because the degree of every polynomial must be known and specific in order to apply the algorithms. But the generic functional form $f(n)$ fits in the Tarski framework as long as f and its relevant derivatives (at one or more points, as needed) are each treated as a separate variable, because the production function restrictions take the form of polynomials of a degree that is specific, known, and relatively low. This is an example of the computation gains from "rais[ing] the level of abstraction" (Kroening and Strichman 2008, p. v).

[56] The elastic restriction is imposed on the non-semi-algebraic benchmarks because (a) Cobb-Douglas production functions satisfy it and (b) it affects the result (see above).
[57] Mathematica has more overhead computation – e.g., syntax processing and graphic renderings – that make it less reliable for measuring decision times for decisions that are quick (the overhead time is large in comparison to the actual calculations).

V. Conclusions

A number of economic hypotheses are, interpreted in the right space, quantified ("for all") statements about Tarski formulas, each of which is a quantifier-free Boolean combination of polynomial inequalities.[58] In order for an economic hypothesis to fit in this framework, it must be stated in terms of properties of the model that are expressed as a finite number of relationships among real numbers. For example, the hypothesis that, for any supply-demand equilibrium in which the two curves have their usual slopes, a downward supply shift increases the equilibrium quantity and decreases the equilibrium price (Marshall 1895, Book V, Chapter XII) can be expressed in terms of the demand and supply slopes in the neighborhood of an arbitrary equilibrium point, the quantity impact, and the price impact, each of which is a real number. The same hypothesis can alternatively be expressed in terms of relationships between two arbitrary points on the supply curve and two corresponding points on the demand curve, without reference to derivatives (see also Appendix IV). Either way, given that the equilibrium *points* are arbitrary, and that the statements refer to *all possible values* of the real numbers, the truth or falseness of the hypothesis tells us about the properties of supply and demand *functions* on the parts of their domains that satisfy the slope assumptions.

In other words, when Alfred Marshall and other early pioneers of formal economic reasoning made (correct) if-then statements about human behavior, they were implicitly eliminating "for all" quantifiers from a True sentence.[59] The contribution of this paper is to make the quantifier elimination explicit and thereby bringing to bear applicable tools from real algebraic geometry.

Quantifier elimination algorithms automatically decide the truth of such hypotheses in finite time, without approximation or functional-form assumptions. The algorithms, especially Cylindrical Algebraic Decomposition (CAD), can thereby also help formulate and understand hypotheses by detecting inconsistent assumptions,

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[58] See Davenport (2015) and Arai, et al. (2014) for a more systematic measurement of the prevalence of problems that can be posed in this way.

[59] I use the phrase "implicit quantifier elimination" in the same way that Brown and Kubler (2008, p. 4) do, but point out that it applies to if-then sentences, too, and thereby predates Afriat (1967) in the economic literature.

calculating necessary and sufficient conditions, and generating examples. For hypotheses that involve optimization, inequality-intensive revealed preference expressions can, with quantifier elimination, be as easy to process as equation-intensive local analysis, with the added potential of arriving at global conclusions.

These results are not merely hopeful conjectures for the practice of economic theory. Software is already available for automatically eliminating quantifiers, which I have incorporated into an economist-friendly interface running in Mathematica.[60] All of the hypotheses in this paper were refuted or verified merely by entering them as assumptions (e.g., (20) – (27)) and potential implications (e.g, $u_L > u_H$). The interface confirms the mutual consistency of the assumptions, returns True, False, or "True for some [values], False for others," and provides push-button options for further analysis. Figure 5 is a Mathematica screen shot showing the processing of the monopolist's pass-through example.

A wide range of economic hypotheses can be processed in this way, although more work is needed to expand the range, and better understand the practical limits, of economic hypotheses and proofs that can be automated with quantifier elimination. One clear limitation of the methods in this paper is that the number of quantified variables must be finite which, without variable changes or additional verification strategies (e.g., induction, fixed point, or limit arguments), rules out hypotheses that contain integrals or infinite series. As shown by the concave production function example, some economic hypotheses can be represented as quantifier elimination problems in multiple ways, and thereby present a potential tradeoff between user effort (in assembling the Tarski formula) and computational complexity.

The quantifier-elimination methods in this paper deliver conclusions, but a conclusion is not the same as a concise proof. The steps of cylindrical algebraic decomposition construction are themselves a proof, and the CAD can be displayed by quantifier-elimination software. Some CADs have just one cell and thereby immediately show the steps of a concise proof (see subsection IV.B.3), but other CADs are complicated enough that the proof represented by their construction is too lengthy and

[60] The Mathematica package is obtained by executing Get["http://economicreasoning.com"] at a Mathematica prompt. The package has already been used to generate substantive conclusions about the economy (Mulligan and Tsui 2016).

tedious for a human reader to appreciate or practically verify. But even in those cases quantifier elimination could be of tremendous assistance to someone attempting to construct a concise proof by: confirming that a hypothesis is provable, investigating the equivalence of one hypothesis with another, incrementally eliminating or modifying assumptions to see which of them are binding, verifying any number of intermediate results that may serve as one of the steps in the proof, and automatically generating examples.[61]

Human error can result in logically or mathematically erroneous conclusions from economic theory, whether those conclusions were generated with a machine or with pencil and paper. In the latter approach, a diligent reader, editor, or referee, also operating with pencil and paper, has been required to detect and correct publication errors. Human errors could in principle be embedded in quantifier-elimination software (Davenport & England, 2015), although CAD methods can decide any universal sentence with two different methods (i.e., the left- and right-hand sides of (7) involve different CAD software steps) and each of those decisions can be processed in $N!$ different sequences (N is the number of quantified variables). Moreover, multiple software packages are available to perform the same calculation (this paper uses both Mathematica and REDLOG), not to mention the fact that the owners of commercial software packages have both the opportunity and incentive to find and correct software errors.[62] In some instances, such as the hypotheses represented with single-celled CADs, the machine-generated output is itself a practical guide to confirming its conclusion with pencil and paper. Also note that empirical economics publications already include dozens, if not hundreds, of matrix inversions that are never verified with pencil and paper or even with an alternative software package. Perhaps economic theory will follow a similar path.

[61] Some of the non-CAD methods for deciding existential sentences do routinely provide proofs, but those methods so far appear to be more limited in the practical range of economic problems that they can process.

[62] If the machine-generated conclusion is that either an example or counter example exists, then this can readily be verified with pencil and paper because the software provides the example (see also Appendix I).

Appendix I. A non-monotonic Laffer curve

In the Laffer-curve model shown in the main text, it is possible for a higher tax rate to be associated with more utility, holding revenue constant, if leisure is not a normal good and the production function is concave enough in labor. This appendix illustrates this possibility by taking a specific production function that is piecewise linear, and therefore especially concave at the point where the marginal product of labor changes. It also takes a specific utility function for which leisure is an inferior good over some range that includes the kink in the production function.

The utility function is:

$$u(c,n) = \frac{3}{4}\ln c - \frac{n}{50} - \frac{2}{3}(2c-1)^2 n^3 + 1 \tag{50}$$

The production function is (for $n > 0$):

$$f(n) = \min\left\{An + \frac{1}{5}, \frac{4}{5}n\right\} \tag{51}$$

where A is normalized so that the efficient amount of labor is one and greater than the labor at the kink point:

$$-\left.\frac{\partial u/\partial n}{\partial u/\partial c}\right|_{c=A+\frac{1}{5}, n=1} = A < \frac{4}{5} \tag{52}$$

Note that, everywhere along the production function, consumption is a good, labor is a bad, and preferences are convex.

The Laffer curve for this economy can be calculated parametrically by mapping each point (n,c) on the production function with $n \in (0,1)$, except the kink, to a point on the Laffer curve. The tax rate τ at a point is the percentage gap between the marginal rate of substitution and the marginal product w. The transfer amount at the same point is the product τwn.

A range of wage rates, tax rates and transfer amounts are consistent with a competitive equilibrium at the production function kink. All of these points have the same equilibrium work, consumption, and utility.

Utility rises with n along the production function as long as $n < 1$. At the kink point, utility is u_{kink}.[63] Figures 6a and 6b show the resulting Laffer curve with differing degrees of detail, using red to show utility (and therefore labor) that is less than its value at the kink point. Utility at (above) the kink are shown as black and green, respectively. Of course, utility at a zero tax rate (on the green part of the Laffer curve) is greater than it is at a 100 percent tax rate (on the red part). Nevertheless, the fine-detail Figure 6b shows most clearly that the green (low-utility) part of the Laffer curve is sometimes to the left of the red part. This possibility shows why (20) through (27) by themselves are not sufficient to imply that $u_L > u_H$.

Appendix II. The remainder of the sets from subsections III.C and IV.B

Equation (36), using the condensed notation of Table 1 and reproduced below, shows a set representing the (not necessarily True) hypothesis CAD that quasiconcave two-input production functions are concave.

$$\forall\{v_1, v_2, v_3, v_5\}\left[\left(v_1 > 0 \wedge v_2 > 0 \wedge v_3 > 0 \wedge v_5 > 0 \wedge v_3^2 v_7 + v_5^2 v_4 < 2 v_3 v_5 v_6\right) \Rightarrow \left(v_4 \le 0 \wedge v_7 \le 0 \wedge v_4 v_7 \ge v_6^2\right)\right] = \left\{(v_4, v_6, v_7) \in \mathbb{R}^3 : \left((v_4 = 0 \vee v_7 = 0) \wedge v_6 = 0\right) \vee G(v_4, v_6, v_7)\right\} \quad (36)$$

The quantifier-free representation of the set is the disjunction of two terms. The details of the second term are:

$$G(v_4, v_6, v_7) = (v_4 \ge 0 \wedge v_7 \ge 0 \wedge v_6 \le 0) \vee \left(v_4 v_7 \ge v_6^2 \wedge v_4 v_7 > 0\right) \quad (53)$$

Equation (48), reproduced below (and also using the condensed notation from Table 1), is a CAD for the set representing the intersection of the assumption and hypothesis above.

---

[63] Approximately 0.384.

$$
\left\{\begin{matrix}(v_3, v_4, v_5, v_6, v_7) \in \mathbb{R}^5: \\ (v_3 > 0 \wedge v_5 > 0 \wedge v_3^2 v_7 + v_5^2 v_4 < 2v_3 v_5 v_6) \wedge (v_4 \leq 0 \wedge v_7 \leq 0 \wedge v_4 v_7 \geq v_6^2)\end{matrix}\right\} = \left\{\begin{matrix}(v_3, v_4, v_5, v_6, v_7) \in \mathbb{R}^5: \\ (v_4 = v_6 = 0 \wedge v_7 < 0 \wedge v_3 > 0 \wedge v_5 > 0) \vee \Gamma(v_3, v_4, v_5, v_6, v_7)\end{matrix}\right\} \tag{48}
$$

The CAD is a disjunction of two terms. The details of the second term are:

$$
\Gamma(v_3, v_4, v_5, v_6, v_7) = v_3 > 0 \wedge v_4 < 0 \wedge \left\{ [v_5 > 0 \wedge v_6 \geq 0 \wedge v_4 v_7 \geq v_6^2] \vee \left[v_6 < 0 \wedge \left[(v_4 v_7 = v_6^2) \wedge \left(\left(v_5 > 0 \wedge v_5 + \sqrt{\frac{v_3^2}{v_4^2}(v_6^2 - v_4 v_7)} < \frac{v_3 v_6}{v_4} \right) \vee v_4 v_5 + v_4 \sqrt{\frac{v_3^2}{v_4^2}(v_6^2 - v_4 v_7)} - v_3 v_6 < 0 \right) \right] \right] \right\} \tag{54}
$$

Appendix III. A 41-dimensional example from growth theory

This is a two period version of the neoclassical growth model with fiscal policy. Output in each period is an additively separable function of capital k and labor n.[64] Given an initial capital stock k_1, first period output can be used to consume (privately c, or publicly g) or accumulate capital. Second period output and capital are consumed:

$$
f(n_1) + h(k_1) = c_1 + g_1 + (k_2 - k_1) \tag{55}
$$

$$
f(n_2) + h(k_2) + k_2 = c_2 + g_2 \tag{56}
$$

with

[64] A separable, rather than homogeneous, production function adds to the algebraic complexity of the problem. A nonseparable-nonhomogeneous production function would be more complex than either of these.

$$f(n_t) > 0, f'(n_t) > 0, f''(n_t) \leq 0, \frac{\partial}{\partial n_t}[n_t f'(n_t)] > 0 \quad t = 1,2 \tag{57}$$

$$h(k_t) > 0, h'(k_t) > 0 \quad t = 1,2 \tag{58}$$

$$h''(k_2) \leq 0 \tag{59}$$

These resource constraints themselves introduce nineteen variables (from the algebraic perspective): $\{f(n_1), f(n_2), h(k_1), h(k_2), c_1, c_2, g_1, g_2, k_1, k_2, n_1, n_2, f'(n_1), f'(n_2),$ $f''(n_1), f''(n_2), h'(k_1), h'(k_2), h''(k_2)\}$. The representative agent's utility over time series for consumption and labor are:

$$u(c_1, n_1) + \frac{u(c_2, n_2)}{1+\rho} \tag{60}$$

$$\rho > -1\,, \quad \frac{\partial u(c_t, n_t)}{\partial c_t} > 0\,, \quad \frac{\partial u(c_t, n_t)}{\partial n_t} < 0\,, \quad \frac{\partial^2 u(c_t, n_t)}{\partial n_t^2} < 0\,, \quad \frac{\partial^2 u(c_t, n_t)}{\partial c_t^2} < 0 \tag{61}$$

$$\frac{\partial^2 u(c_t, n_t)}{\partial n_t^2}\frac{\partial^2 u(c_t, n_t)}{\partial c_t^2} - \left(\frac{\partial^2 u(c_t, n_t)}{\partial c_t n_t}\right)^2 > 0 \tag{62}$$

$$\frac{\partial M(c_t, n_t)}{\partial c_t} > 0\,, \quad \frac{\partial M(c_t, n_t)}{\partial n_t} > 0 \quad t = 1, 2 \tag{63}$$

where M is the intratemporal marginal rate of substitution function $-\frac{\partial u/\partial n}{\partial u/\partial c}$.[65] Not including (60), the preference restrictions introduce eleven more variables, where are ρ and the various partial derivatives above.

The representative agent chooses consumption, labor, and capital. He pays labor income taxes in each period at constant rates τ_1 and τ_2, respectively. In addition to the resource constraints above, the equilibrium conditions are:

[65] The restrictions (63) are that consumption and leisure are normal goods in the relevant range.

$$M(c_t, n_t) = (1-\tau_t)f'(n_t), \;\; g_t = \tau_t n_t f'(n_t) \quad t = 1, 2 \tag{64}$$

$$(1+\rho)\frac{\partial u(c_1,n_1)}{\partial c_1} / \frac{\partial u(c_2,n_2)}{\partial c_2} = 1 + h'(k_2) \tag{65}$$

These conditions add two more variables, the tax rates, to the previous thirty. In their comparative static form (i.e., total derivatives with respect to g), there are eleven more variables: $\frac{dc_t}{dg}, \frac{dn_t}{dg}, \frac{dk_t}{dg}, \frac{d\tau_t}{dg}, \frac{dg_t}{dg}, t = 1,2$ and $\frac{d\rho}{dg}$. This makes a total of 43 variables, although two of them (k_1 and k_2) are not part of the hypothesis because it involves the total derivative of the resource constraints rather than the resource constraints themselves.

The hypothesis is that – assuming (57)-(59), (61)-(63), $0 < \tau_t < 1$, $c_t > 0$, $n_t > 0$, $g_t \geq 0$, the total derivatives of (55), (56), (64), and (65) – that a permanent increase in government consumption $\left(\frac{dg_1}{dg} = \frac{dg_1}{dg} = 1, \frac{d\rho}{dg} = \frac{dk_1}{dg} = 0\right)$ beginning from an allocation that is not on the upward-sloping part of the Laffer curve $\left(\frac{d\tau_1}{dg} \leq 0, \frac{d\tau_2}{dg} \leq 0\right)$ must increase the amount of labor in period one $\left(\frac{dn_1}{dg} > 0\right)$. Quantifier elimination, taking less than 300 milliseconds (see the final row of Table 3), shows that the hypothesis is True. It also shows that the hypothesis is not always True if either the Laffer curve assumptions or the normal-goods assumptions are removed.[66]

[66] Intuitively, a permanent increase in government consumption has both a wealth effect and a substitution effect. The Laffer curve and normal goods assumptions guarantee that both effects on labor are in the same direction.

Appendix IV. Two Classic Examples

Economics is replete with hypotheses that can be expressed in the real quantifier elimination framework (5). This appendix briefly shows two more of them, drawn from some of the most famous hypotheses in economics: Alfred Marshall's equilibrium comparative statics and Anthony Downs' Median Voter Theorem.

Marshall (1895, Book V, Chapter XII) concluded that, for any supply-demand equilibrium in which the two curves have their usual slopes, a downward supply shift increases the equilibrium quantity q and decreases the equilibrium price p. A local-comparative-statics version of Marshall's result is:

$$\forall\left\{d'(q), s'(q), \frac{\mathbb{d}q}{\mathbb{d}a}, \frac{\mathbb{d}p}{\mathbb{d}a}\right\}$$
$$\left[\left(d'(q) < 0 \wedge s'(q) \geq 0 \wedge \frac{\mathbb{d}}{\mathbb{d}a}[s(q) - a] = \frac{\mathbb{d}p}{\mathbb{d}a} = \frac{\mathbb{d}}{\mathbb{d}a}d(q)\right) \Rightarrow \left(\frac{\mathbb{d}q}{\mathbb{d}a} > 0 \wedge \frac{\mathbb{d}p}{\mathbb{d}a} < 0\right)\right] \tag{66}$$
$$= True$$

where the inverse demand and supply curves are $d(q)$ and $s(q) - a$, respectively (i.e., the parameter a shifts the supply curve down). The first two assumptions are local slope assumptions. The third says that a perturbs the equilibrium. A global version of the result requires nine variables rather than four:

$$\forall\{a, d(q_H), s(q_H), q_H, p_H, d(q_L), s(q_L), q_L, p_L\}$$
$$\big[\big([d(q_L) - d(q_H)](q_L - q_H) < 0 \wedge [s(q_L) - s(q_H)](q_L - q_H) \geq 0 \wedge a > 0 \tag{67}$$
$$\wedge\, s(q_H) = p_H = d(q_H) \wedge s(q_L) - a = p_L = d(q_L)\big) \Rightarrow (q_L > q_H \wedge p_L < p_H)\big] = True$$

The first two assumptions are global slope assumptions. The last three say that the L equilibrium differs from the H equilibrium by a downward supply shift in the amount a.

In Downs' (1957) model, public policy g is one dimensional. Voters are heterogeneous, with single-peaked policy preferences that align them from left to right. The continuous CDF $F(g)$ denotes the fraction of the electorate whose preferred policy is no greater than g. Each of two candidates L, R proposes a policy, which for brevity I take to be distinct with $g_L < g_R$.[67] The proof by quantifier elimination is:

$$\begin{gathered}\forall\{v_L, F(g_L), v_R, F(g_R)\} \\ \Big[(0 \le F(g_L) < F(g_R) \le 1 \wedge v_L + v_R = 1 \wedge [F(g_L) < F(g_R) \Rightarrow F(g_L) < v_L < F(g_R)]) \\ \Rightarrow \left(\left[F(g_L) = \frac{1}{2} \Rightarrow v_L > v_R\right] \wedge \left[F(g_R) = \frac{1}{2} \Rightarrow v_R > v_L\right]\right)\Big] = True\end{gathered} \tag{68}$$

where v_L and v_R denote the vote shares received by the left and right candidates, respectively. The first assumption is a normalization plus the relevant properties of a cumulative density function. The second assumption reflects the arithmetic of vote shares in a two-candidate election. The final assumption is single-peaked preferences: voters choose the candidate whose proposal is closest to their preference (see also (Acemoglu and Robinson 2006, Definition 4.1)).[68] The hypothesis asserts that whichever, if any, candidate's policy is at the median is the election winner.

[67] I leave it to the reader to show how the Tarski formula could be amended to allow for the additional possibility of $g_L = g_R$.

[68] Note that, for the purposes of proving the theorem, we do not have to specify how the voters between g_L and g_R vote, except that some of them vote left and others vote right (hence the strict inequality surrounding v_L).

Figure 1. Parabolas with real roots

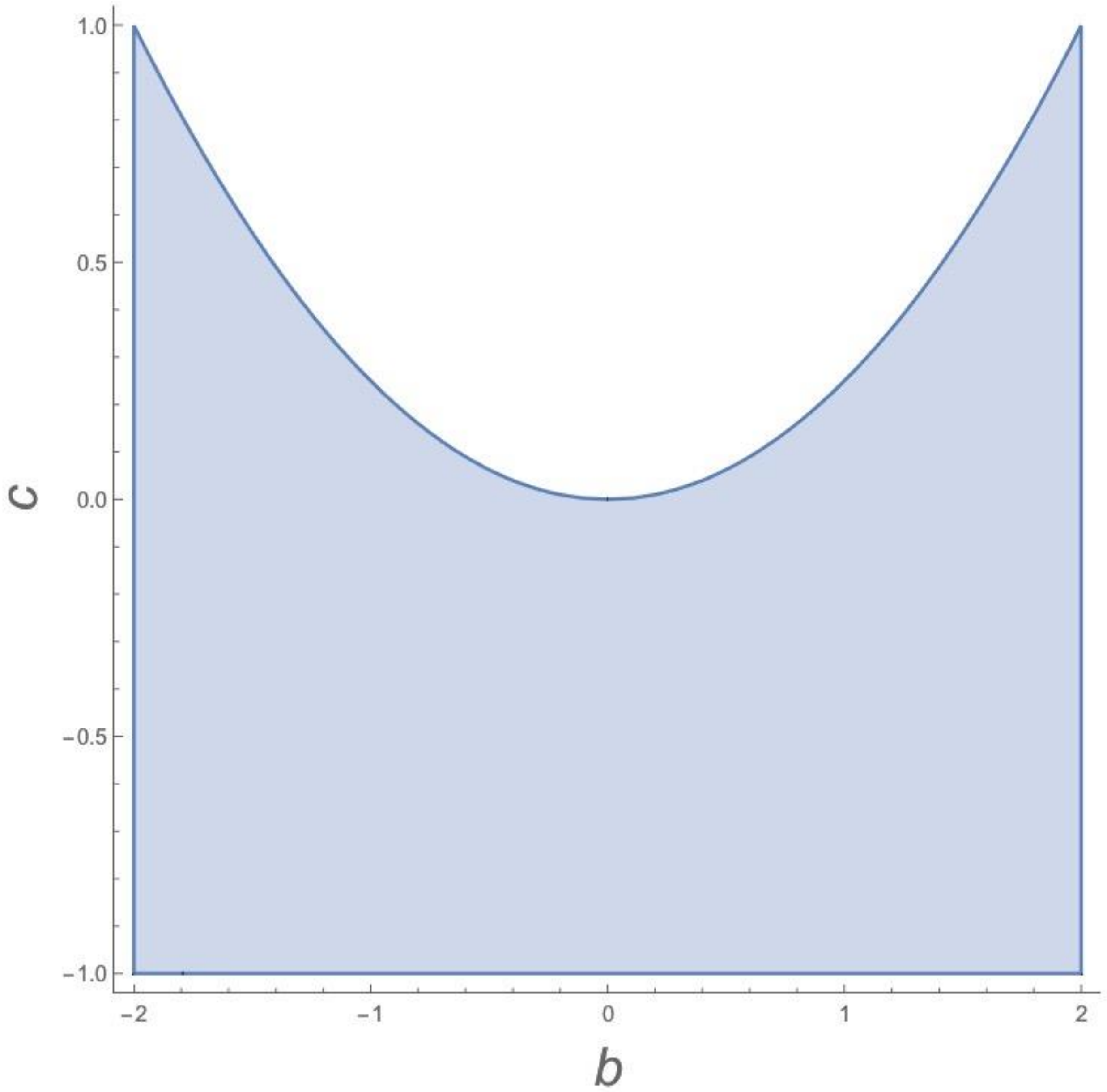


Figure 2. Circle-hyperbola CAD

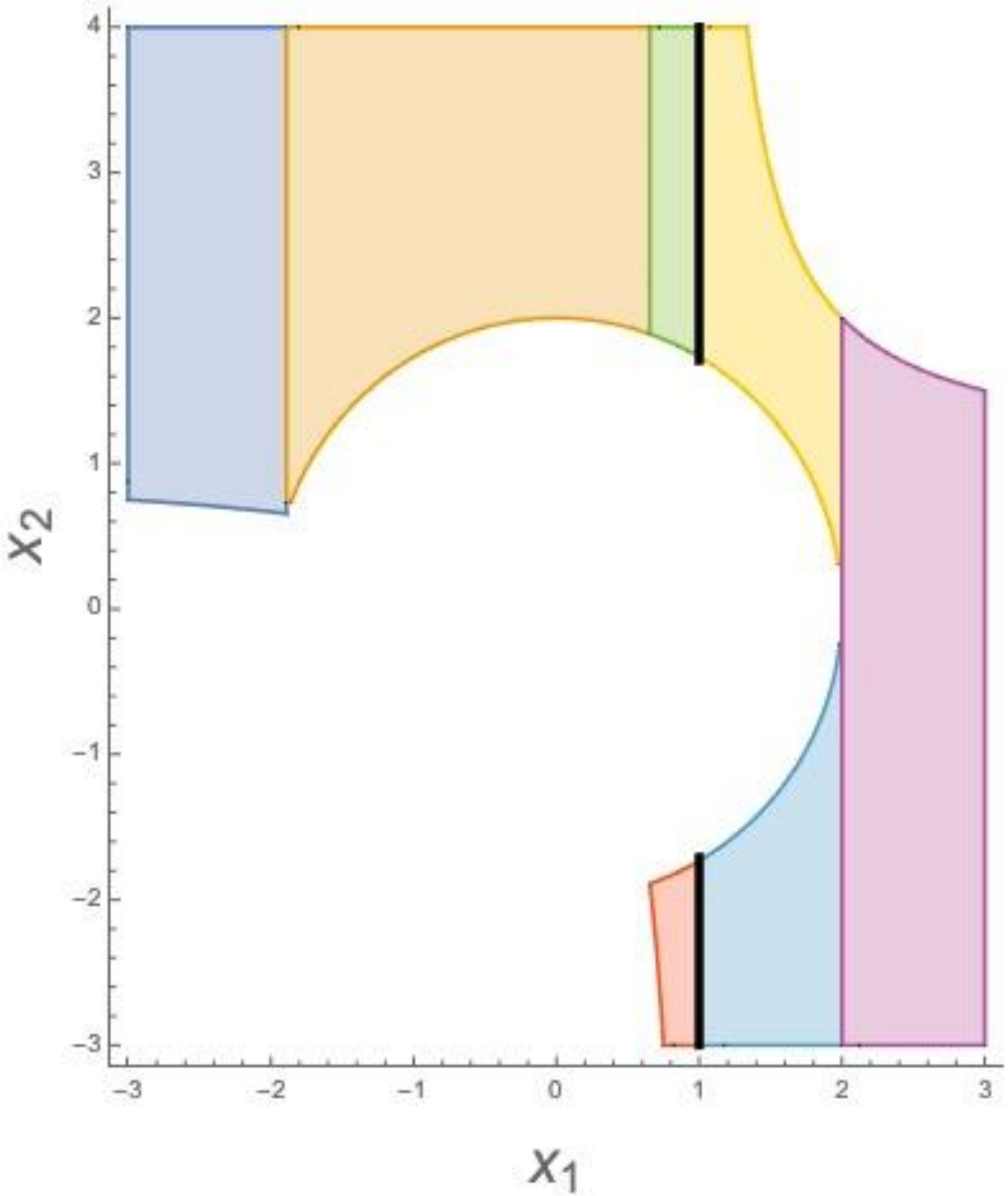


Figure 3a. Double exponential: wide range

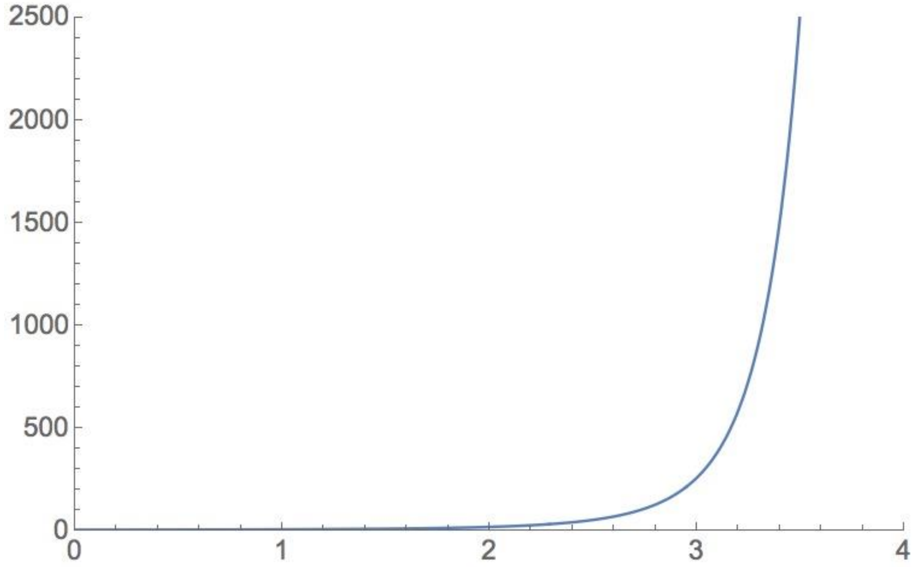


Figure 3b. Double exponential: close range

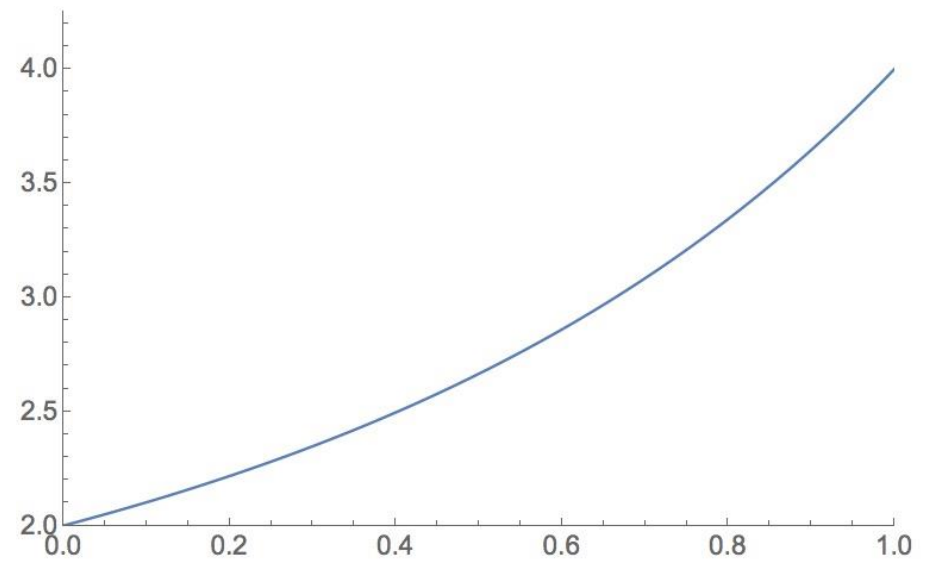
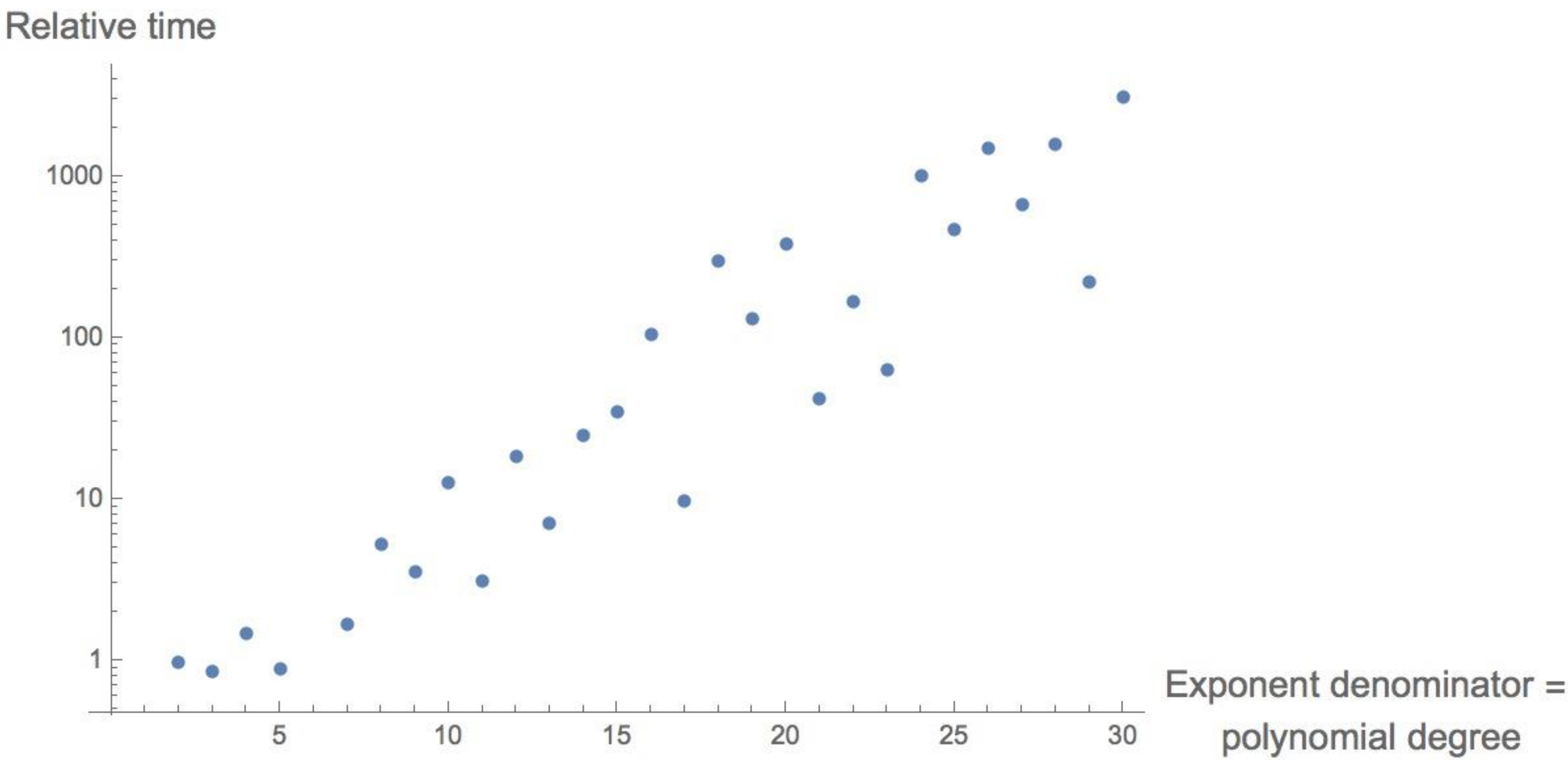


Figure 4. Functional form assumptions unnecessarily complicate quantifier elimination

Times for deciding universal sentences with n^{α},
relative to leaving $f(n)$ unspecified.



Note: Each observation takes α to be a rational number. The Tarski formula has only ten inequalities. Table 4's Tarski formula has twenty-three.

Figure 5. Mathematica screen shot

A monopolist's pass through rate

Setup

In[1]:= `Get["http://economicreasoning.com"]`

In[2]:= `profits[q, a] := q W[q] - g[q, a]`

In[3]:= $\text{ProfitsAreMaximized} = \left\{ \frac{d \frac{\partial \text{profits}(q,a)}{\partial q}}{da} == 0,\ \frac{\partial^2 \text{profits}(q, a)}{\partial q^2} < 0 \right\};$

In[4]:= $\text{BasicAssumptions} = \{ W'(q) < 0$ (* Demand slopes down *),

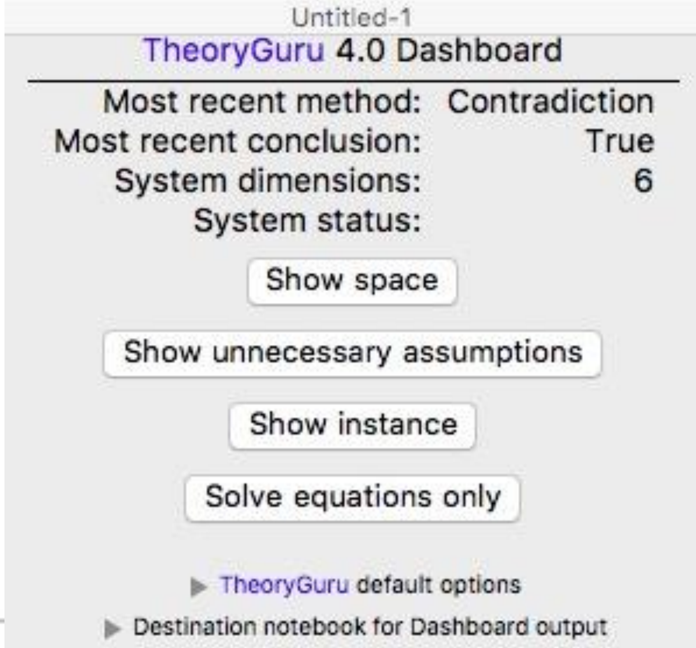
$\frac{\partial^2 g(q, a)}{\partial q^2} \geq 0$ (* MC does not *),

$\frac{\partial^2 g(q, a)}{\partial q \partial a} > 0$ (* a increases MC (a definition) *),

$q > 0\};$

In[5]:= $\text{ConcaveDemand} = W''(q) \leq 0;$

In[6]:= $\text{PassThroughRate} = \frac{dW(q)}{da} \Big/ \frac{d \frac{\partial g(q,a)}{\partial q}}{da};$



Result

In[7]:= `TheoryGuru[{ConcaveDemand, ProfitsAreMaximized, BasicAssumptions}, PassThroughRate ≤ 1]`

Out[7]= `True`

Note: For brevity, the three unnecessary assumptions among (10)-(12) were not entered. As a result, the system is six dimensional rather than nine.

Figure 6a. Laffer curve example: wide range

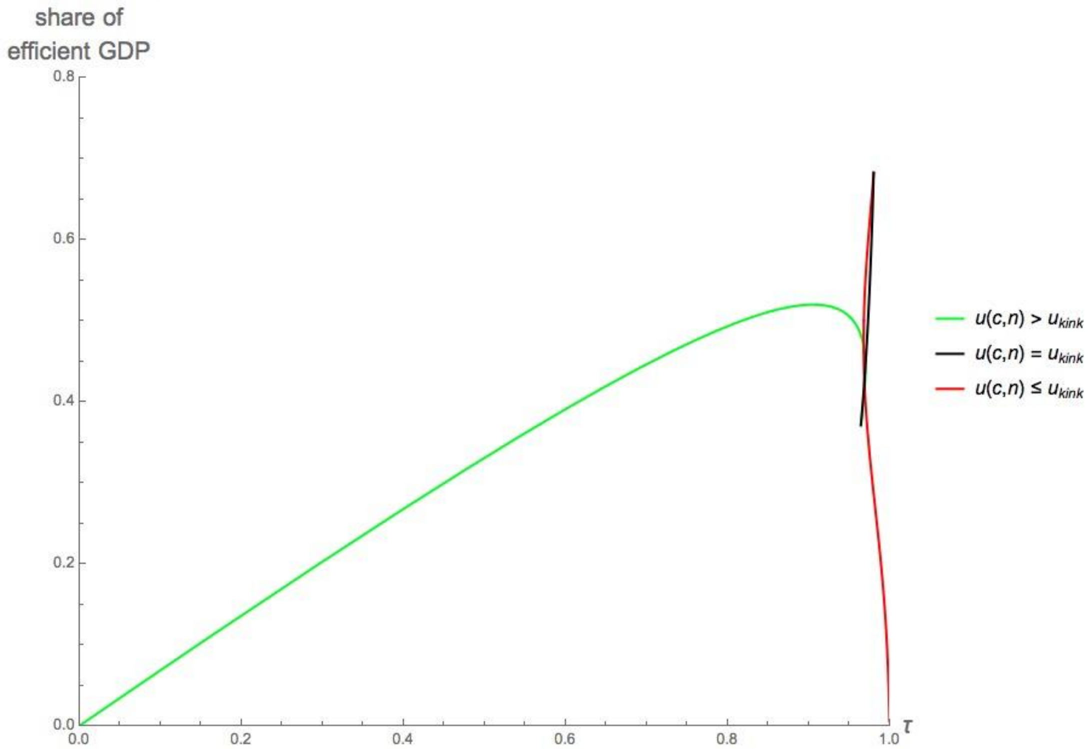


Figure 6b. Laffer curve example: fine detail

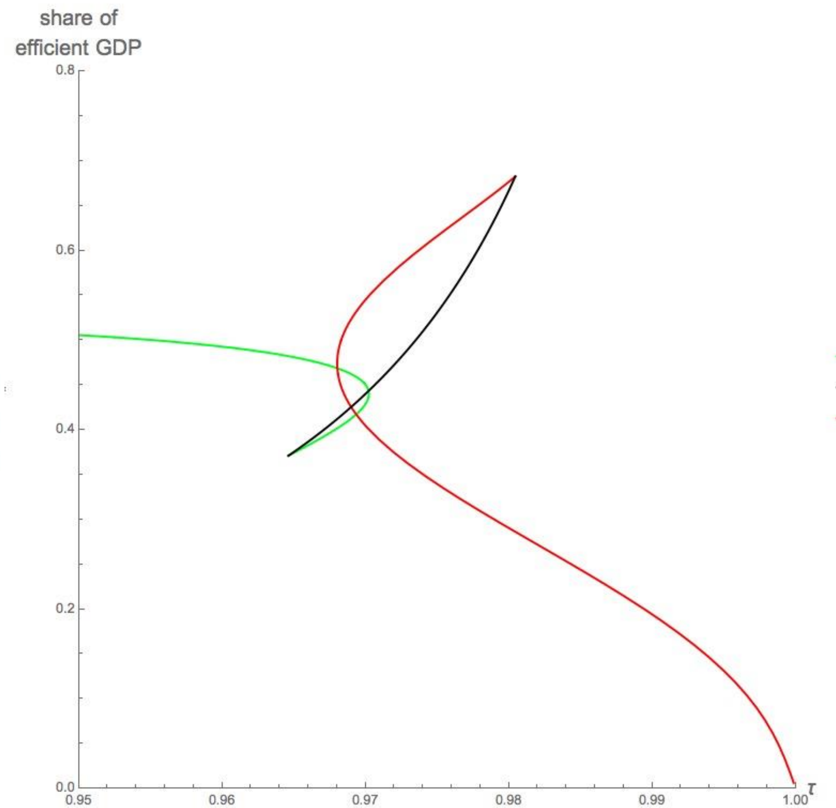


Table 1. Structure of the polynomial system representing the concave production function problem

$$\{v_2 v_4 + v_1 v_6 = 0,\ v_2 v_6 + v_1 v_7 = 0,\ v_1 > 0,\ v_2 > 0,\ v_3 > 0,\ v_5 > 0,\ 2 v_3 v_5 v_6 > v_7 v_3^2 + v_4 v_5^2,\ v_4 v_7 \geq v_6^2,\ v_4 \leq 0,\ v_7 \leq 0\}$$

Polynomial System Complexity Report

7 variables appearing in:
8 inequalities, 2 equations, 0 not equals,
0 other logical relations.

var	a.k.a.	max own deg	max term degree	♯ terms
v_1	x	1	2	3
v_2	y	1	2	3
v_4	$\frac{\partial^2 f(x,y)}{\partial y^2}$	1	3	4
v_7	$\frac{\partial^2 f(x,y)}{\partial x^2}$	1	3	4
v_3	$\frac{\partial f(x,y)}{\partial y}$	2	3	3
v_5	$\frac{\partial f(x,y)}{\partial x}$	2	3	3
v_6	$\frac{\partial^2 f(x,y)}{\partial x \partial y}$	2	3	4

Table 2. Examples and counterexamples
for the Laffer curve model

Variable	$u_L > u_H$	$u_L < u_H$
c_L	$\frac{11}{8}$	$\frac{3}{4}$
c_H	$\frac{7}{8}$	$\frac{17}{16}$
c_L'	$\frac{9}{8}$	1
c_H'	1	1
m_L	1	1
m_H	1	1
n_L	$\frac{3}{2}$	$\frac{3}{4}$
n_H	1	1

Variable	$u_L > u_H$	$u_L < u_H$
u_L	2	$-\frac{1}{2}$
u_H	$\frac{1}{2}$	1
u_L'	1	-1
u_H'	0	0
w_L	1	2
w_H	1	1
τ_L	$\frac{1}{2}$	$\frac{1}{2}$
τ_H	$\frac{3}{4}$	$\frac{3}{4}$

Table 3. Decision times for the economic models
Times for deciding universal sentences, in milliseconds

		Decision time (milliseconds)	
Model	Dimensions represented	Mathematica	REDLOG
Concave production function (derivatives at one point)			
2 inputs	7	51	< 50
3 inputs	12	128,812	< 50
4 inputs	18	> 10^8	> 10^8
Concave production function (three-point comparisons)			
Any number of inputs	10	66	60
Monopolist's pass-through	9	58	60
Laffer-curve surprises			
Basic assumptions only	14	223	180
Normal goods restriction	16	725	200
Elastic labor demand restriction	14	740	190
Growth model processed in 41 dimensions	41	279	230

Note : Universal sentences state a hypotheses to be True for all *N* -dimensional real numbers, where *N* is the number of dimensions needed to represent the model, and typically take longer to decide than existential sentences. Computer time was calculated with Mathematica 10.4 and the PSL version of REDLOG (version 3562) on a Macbook Air Mid 2012 2GHz Interl i7, without using parallel processing or searching for more efficient variable-elimination orders. The variable-elimination order is reverse alphabetical (as sorted by Mathematica), except for the concave production function models, which are alphabetical.

Table 4. Functional form assumptions unnecessarily complicate quantifier elimination

Times for deciding universal sentences, in milliseconds, from REDLOG

Laffer curve model (14 dimensions)

Production function specification	Decision time (milliseconds)	Max polynomial degree own	term
No functional-form assumption			
Elastic labor demand	190	2	3
Constant elastic labor demand	380	2	3
Cobb-Douglas, with exponent of:			
1/2	50	2	3
2/3	120	3	4
3/4	240	4	5
3/5	> 10^8	5	7
Symbolic	Outside the polynomial framework		
Any irrational number	Outside the polynomial framework		

Note : The elastic labor demand specification is (w1n1-w2n2)(n1-n2)>0. The constant elastic labor demand specification also has (w1n1c2=w2n2c1). Max own degree is the maximum exponent on any one variable in the polynomial system. Term degree is the sum of exponents on all variables in a mononomial (max refers to the mononomial with the greatest term degree). Note that equation (27) is not part of these systems.

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